AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **82(1)** (2022), Pages 115–118

The nonexistence of a distance-regular graph with intersection array $\{75, 64, 22, 1; 1, 2, 64, 75\}$

Supalak Sumalroj*

Department of Mathematics
Naresuan University
Phitsanulok, Thailand
`supalaks@nu.ac.th`

Chalermpong Worawannotai

Department of Mathematics
Silpakorn University
Nakhon Pathom, Thailand
`worawannotai_c@silpakorn.edu`

Abstract

We give a new feasibility condition for distance-regular graphs with $c_2 \leq 2$ based on some combinatorial properties of the local graphs. Then we use it to show that there is no distance-regular graph with intersection array $\{75, 64, 22, 1; 1, 2, 64, 75\}$.

In [2, 1] Brouwer, Cohen and Neumaier have compiled a list of arrays that pass certain feasibility conditions to be intersection arrays of some distance-regular graphs. For many of these arrays, the existence of the corresponding distance-regular graphs were not known. Since then some progress has been made and the summary is reported in [3]. In this paper, we give a new feasibility condition for distance-regular graphs with $c_2 \leq 2$ and we use it to rule out the intersection array $\{75, 64, 22, 1; 1, 2, 64, 75\}$ in [2, p. 422].

All graphs we consider are finite, simple and undirected. For any graph Γ, we identify Γ with its vertex set $V(\Gamma)$. We denote the subgraph of Γ induced by a subset S of $V(\Gamma)$ by S itself. For a vertex x in Γ, the subgraph of Γ induced by the neighbors of x is called the *local graph* of Γ with respect to x. A *clique* of a graph Γ is a maximal complete subgraph of Γ. The *eigenvalues* of Γ are the eigenvalues of its adjacency matrix.

* Corresponding author.

ISSN: 2202-3518

Let Γ denote a connected graph with diameter d and a vertex set V. For $x \in V$ and $0 \le i \le d$, let $\Gamma_i(x)$ denote the set of vertices at distance i from x. The graph Γ is called *distance-regular* whenever for all $0 \le i \le d$ and all vertices x and y at distance i, the numbers $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ and $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ depend only on i where $\Gamma_{-1}(x)$ and $\Gamma_{d+1}(x)$ are unspecified. In particular, Γ is a regular graph of degree $k = b_0$ and $c_i + a_i + b_i = k$ for all $0 \le i \le d$. The sequence $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ is called the *intersection array* of Γ.

The following proposition gives an upper bound of the size of a clique of a distance-regular graph in terms of its smallest and largest eigenvalues.

Proposition 1. (See [2, Proposition 4.4.6].) *Let Γ denote a distance-regular graph of diameter $d \ge 2$ with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Then the size of a clique K in Γ is bounded by*

$$|K| \le 1 - k/\theta_d.$$

Observation 2. *In a distance-regular graph with $c_2 = 1$, any two vertices that have at least two common neighbors are adjacent. In particular, the subgraph induced by a cycle is complete.*

Combining Observation 2 with [4, Lemma 3] we have the following result.

Lemma 3. *Let Γ denote a distance-regular graph with $c_2 \le 2$. For a vertex ∞ of Γ, if $\Gamma_1(\infty)$ contains a cycle C of length 4, then the subgraph induced by C is a complete graph K_4.*

The following results are some useful combinatorial properties of a distance-regular graph. The first one generalizes the result [4, Lemma 4].

Theorem 4. *Let Γ denote a distance-regular graph with $c_2 \le 2$ and fix a vertex ∞ of Γ. Then each vertex in $\Gamma_1(\infty)$ is on at least $\left\lceil \frac{1}{2}(a_1^2 + 1 - |\Gamma_1(\infty)|) \right\rceil$ triangles in $\Gamma_1(\infty)$.*

Proof. Let u denote a vertex of $\Gamma_1(\infty)$. Let $u_1, u_2, \ldots, u_{a_1}$ denote the distinct neighbors of u in $\Gamma_1(\infty)$. Let N denote the number of triangles of $\Gamma_1(\infty)$ that contain u. Observe that N is also the number of edges $u_i u_j$ where $1 \le i < j \le a_1$. By Lemma 3, the number of vertices of $\Gamma_1(\infty)$ with distance at most 2 from u is $1 + a_1 + (a_1 - 1)a_1 - 2N = a_1^2 + 1 - 2N \le |\Gamma_1(\infty)|$. Therefore a vertex u is on at least $\left\lceil \frac{1}{2}(a_1^2 + 1 - |\Gamma_1(\infty)|) \right\rceil$ triangles in $\Gamma_1(\infty)$. □

Proposition 5. *Let Γ denote a distance-regular graph with $c_2 = 1$. Then each clique in Γ has size $a_1 + 2$. Moreover, for a vertex ∞ of Γ, the local graph $\Gamma_1(\infty)$ is a disjoint union of complete graphs K_{a_1+1}.*

Proof. By Observation 2, for vertices u, v of Γ, the subgraph $\Gamma_1(u) \cap \Gamma_1(v)$ is complete. Therefore, each clique in Γ has size $a_1 + 2$. For a vertex ∞ of Γ, each clique in $\Gamma_1(\infty)$ has size $a_1 + 1$. By Observation 2 again, two cliques neither share a vertex nor are connected by an edge. Thus, $\Gamma_1(\infty)$ is a disjoint union of complete graphs K_{a_1+1}. □

If we apply Proposition 5 to the intersection arrays from the list in [2, p. 420], we get information about their cliques. Table 1 shows the list of intersection arrays whose sizes of cliques can be determined by the results.

Intersection array	Size of a clique
$\{10, 8, 7; 1, 1, 4\}$	3
$\{14, 12, 12; 1, 1, 3\}$	3
$\{18, 15, 9; 1, 1, 10\}$	4
$\{20, 16, 5; 1, 1, 16\}$	5
$\{24, 18, 9; 1, 1, 16\}$	7
$\{12, 8, 6, 4; 1, 1, 2, 9\}$	5

Table 1: Sizes of cliques of given intersection arrays.

We recall a useful property of binomial coefficients.

Lemma 6. *If $a > b \geq 0$, then* $\binom{a+1}{2} + \binom{b}{2} \geq \binom{a}{2} + \binom{b+1}{2}$.

Proof. Routine calculations. $\square$

Theorem 7. *Let Γ denote a distance-regular graph with $c_2 \leq 2$ and fix a vertex ∞ of Γ. Then each vertex in $\Gamma_1(\infty)$ is on at most*

$$\left\lfloor \frac{a_1}{M-1} \right\rfloor \binom{M-1}{2} + \binom{a_1 - \left\lfloor \frac{a_1}{M-1} \right\rfloor (M-1)}{2}$$

triangles in $\Gamma_1(\infty)$, where M is the cardinality of a largest clique in $\Gamma_1(\infty)$.

Proof. Let u denote a vertex in $\Gamma_1(\infty)$. Let $u_1, u_2, \ldots, u_{a_1}$ denote the distinct neighbors of u in $\Gamma_1(\infty)$. By Lemma 3, no two cliques in $\Gamma_1(\infty)$ have a common edge. Then u is on at most $\left\lfloor \frac{a_1}{M-1} \right\rfloor$ cliques of size M in $\Gamma_1(\infty)$. Consider all cliques containing u. The number of triangles containing u is the number of edges in those cliques that do not have u as their endvertex. Thus, by Lemma 6, the number of triangles containing u is at most $\left\lfloor \frac{a_1}{M-1} \right\rfloor \binom{M-1}{2} + \binom{a_1 - \left\lfloor \frac{a_1}{M-1} \right\rfloor (M-1)}{2}$. $\square$

Theorems 4 and 7 together give us a new feasibility condition for distance-regular graphs with $c_2 \leq 2$. Now we can rule out one new intersection array.

Theorem 8. *A distance-regular graph with intersection array*

$$\{75, 64, 22, 1; 1, 2, 64, 75\}$$

does not exist.

Proof. Suppose that Γ is a distance-regular graph with intersection array $\{75, 64, 22, 1; 1, 2, 64, 75\}$. Then Γ has eigenvalues $75, 15, 3, -5$ and -17. Fix a vertex ∞ of Γ. Then $\Gamma_1(\infty)$ is a 10-regular graph on 75 vertices. By Proposition 1, the size of a clique in Γ is at most 5 and, thus, the size of a clique in $\Gamma_1(\infty)$ is at most 4. By Theorem 4, each vertex in $\Gamma_1(\infty)$ is on at least 13 triangles. By Theorem 7, each vertex in $\Gamma_1(\infty)$ is on at most 9 triangles, a contradiction. $\square$

References

[1] A. E. Brouwer, Corrections and additions to the book "Distance-regular graphs", `http://www.win.tue.nl/~aeb/drg/BCN-ac.ps.gz`. (Accessed July 2021).

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, "Distance-regular graphs", Springer-Verlag, Berlin, Heidelberg, 1989.

[3] E. R. van Dam, J. H. Koolen and H. Tanaka, Distance-regular graphs, *Electron. J. Combin.* #DS22 (2016), 156pp.

[4] S. Sumalroj and C. Worawannotai, The nonexistence of a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$, *Electron. J. Combin.* 23 (1) (2016), #P1.32.

(Received 28 July 2021)