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On a completion problem for Latin arrays

Kevin Akers

Montgomery County Public Schools, Christiansburg, VA 24073, U.S.A.

Stacie Baumann

Auburn University, Auburn, AL 36849, U.S.A.

Sarah Gustafson

Bridgeport High School, Bridgeport, WV 26330, U.S.A.

Jaromy Kuhl

University of West Florida, Pensacola, FL 32514, U.S.A.

Olivia Mosrie   Michael W. Schroeder*

Marshall University, Huntington, WV 25755, U.S.A.

Abstract

An $n \times n$ (partial) Latin array L is an array in which no symbol appears more than once in any row or column; this differs from a (partial) Latin square in that L may have up to n^2 distinct symbols present. We say L is k-completable if there exists a partition of the symbols of L into k parts so that the corresponding induced subarrays are each completable partial Latin squares. In 2015 Kuhl and Schroeder demonstrated the existence of $n \times n$ partial Latin arrays which are not k-completable for each $k < n$, and in this paper, we show that all $n \times n$ partial Latin arrays are n-completable. This addresses a conjecture by Kuhl and Schroeder and also confirms a special case of a conjecture by Häggkvist.

1 Introduction

In this paper we consider a variation on the intricacy of completing partial Latin squares, which was introduced by Daykin and Häggkvist [3] in 1981 and formalized by Opencomb [9] in 1985. Consider the partial Latin square P given in Figure 1. Observe P cannot be completed to a Latin square, however we can partition the

* Corresponding author: `schroederm@marshall.edu`

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$$
\underset{P}{\begin{array}{|c|c|c|}\hline 1 & 2 & \\ \hline 2 & 1 & \\ \hline & & 3 \\ \hline\end{array}}
\quad
\underset{P_1}{\begin{array}{|c|c|c|}\hline 1 & 2 & \\ \hline 2 & & \\ \hline & & \\ \hline\end{array}}
\quad
\underset{P_2}{\begin{array}{|c|c|c|}\hline & & \\ \hline & 1 & \\ \hline & & 3 \\ \hline\end{array}}
\quad
\underset{Q_1}{\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & \\ \hline & & \\ \hline\end{array}}
\quad
\underset{Q_2}{\begin{array}{|c|c|c|}\hline & 2 & \\ \hline 2 & & \\ \hline & & \\ \hline\end{array}}
\quad
\underset{Q_3}{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & 3 \\ \hline\end{array}}
$$

Figure 1: The partial Latin square P has a partition into two completable partial Latin squares P_1 and P_2, demonstrating that completing partial Latin squares has intricacy at least 2. Also, P has a partition into three completable partial Latin squares Q_1, Q_2, and Q_3 whose symbol sets are pairwise disjoint.

entries in P into two partial Latin squares P_1 and P_2, each of which are completable. With this in mind, Daykin and Häggkvist [3] posed the following question:

Question. *Does there exist a minimal positive integer k such that all partial Latin squares of any order can be partitioned into k completable partial Latin squares?*

Daykin and Häggkvist define this integer k as the *intricacy* of completing partial Latin squares. It follows from Ryser's Theorem [10] that such an integer k does exist and is at most 4. As evidenced by the partial Latin square in Figure 1, we have that $k \geq 2$, and no examples are known which demonstrate that $k > 2$. This led Daykin and Häggkvist [3] to conjecture that $k = 2$.

Note that P_1 and P_2 have the symbol 1 in common. In fact, in any partition of P into two completable partial Latin squares, the symbol 1 must appear in both. If we now require that P be partitioned into completable partial Latin squares with disjoint symbol sets, then such a partition must have at least three parts. Furthermore, such a partition is unique and contains the three partial Latin squares Q_1, Q_2, and Q_3 given in Figure 1. All partial Latin squares of order n have such a partition into n parts. This modification gives rise to the following question.

Question. *Does there exist a minimal positive integer k such that all partial Latin squares of any order can be partitioned into k completable partial Latin squares which have pairwise disjoint symbol sets?*

The answer to this question is no; a generalization of the partial Latin squares in Figure 2 demonstrates that there exist partial Latin squares of order n for which a partition into completable partial Latin squares with pairwise disjoint symbol sets requires n parts.

A partial Latin square with only one symbol appearing is completable by Hall's Theorem, which leads to the following observation.

Observation 1.1. *Let n be a positive integer. All partial Latin squares of order n can be partitioned into n completable partial Latin squares with disjoint symbol sets, and n is the smallest such partition size.*

While the above result is trivial for partial Latin squares, its related question to (partial) Latin arrays is not.

1	2	3	
2	3	1	
3	1	2	
			4

1	2	3	4	
2	3	4	1	
3	4	1	2	
4	1	2	3	
				5

1	2	3	4	5	
2	3	4	5	1	
3	4	5	1	2	
4	5	1	2	3	
5	1	2	3	4	
					6

Figure 2: A partial Latin square of order n composed of two disjoint subsquares of orders $n-1$ and 1 cannot be partitioned into fewer than n symbol-disjoint, completable partial Latin squares.

1	2	5
2	1	4
5	4	3

A

1		
	1	

A_1

	2	
2		4
	4	

A_2

		5
5		3

A_3

Figure 3: The Latin array A cannot be partitioned into two completable partial Latin squares with disjoint symbol sets. However A does has a partition into three completable partial Latin squares A_1, A_2, and A_3 whose symbol sets are pairwise disjoint.

Question. *Let n be a positive integer. Does there exist a minimal positive integer $k(n)$, such that all partial Latin arrays of order n can be partitioned into $k(n)$ completable partial Latin squares which have pairwise disjoint symbol sets?*

Consider the Latin array A given in Figure 3. Since A contains P, a partition of A into symbol-disjoint, completable partial Latin squares must have cardinality at least 3, and this is possible; again see Figure 3.

Observation 1.1 implies $k(n) \geq n$, however, the trivial upper bound does not resolve the question for Latin arrays; instead we have that $k(n) \leq n^2$, which is obtained by using a partition in which each partial Latin square has only one symbol appear. We can somewhat easily improve the range for $k(n)$. Using Ryser's Theorem we see that $k(n) \leq 4n$ and, if the earlier conjecture by Daykin and Häggkvist is true, then $k(n) \leq 2n$. Using some techniques that are echoed in this paper, Kuhl and Schroeder [8] proved $k(n) \leq n+1$, and they conjectured that $k(n) = n$.

In this paper, we prove Kuhl's and Schroeder's conjecture using the following strategy. If A is a Latin array of order n with associated partition $\mathcal{A}$ into n completable partial Latin squares, then the average number of filled cells in the elements of $\mathcal{A}$ is n. With this in mind, we first find many completable partial Latin squares in A with disjoint symbol sets that contain more than n filled cells. Then we find an appropriate partition of the rest of A.

In Section 2, we review notation, definitions, and previous results. We also give several results highlighting certain families of completable partial Latin squares. In Section 3 we prove $k(n) = n$ and in Section 4, we apply the result to a problem related to another conjecture by Häggkvist.

It does not seem possible to pose the problem of finding partitions of Latin arrays into symbol-disjoint, completable partial Latin squares as a *construction problem*, as defined by Opencomb [9]. The added condition of symbol-disjointness requires some additional dependency among the parts in a partition, which does not seem compatible with the framework of a construction problem.

2 Background and completability results

We begin this section with the notation and terminology used in this paper. Then we highlight several families of completable partial Latin squares.

2.1 Definitions and Notation

Let n be a positive integer and $[n]$ denote the integer set $\{1, 2, \ldots, n\}$. For a positive integer $k \leq n$, let $[k, n] = \{k, k+1, \ldots, n\}$. A *partial Latin array* of order n (on a symbol set Σ) is an $n \times n$ array for which each cell contains at most one symbol, and no symbol appears more than once in a row or column. We use $\mathrm{PLA}(n)$ to denote the set of all partial Latin arrays of order n.

Let $A \in \mathrm{PLA}(n)$. If each cell of A contains a symbol, we say A is a *Latin array* of order n. We denote the set of Latin arrays of order n as $\mathrm{LA}(n)$. If the cells of A contain at most n distinct symbols, then A is a *partial Latin square* and if, in addition, $A \in \mathrm{LA}(n)$, then A is a *Latin square*. We denote the set of partial Latin squares and Latin squares of order n as $\mathrm{PLS}(n)$ and $\mathrm{LS}(n)$, respectively. We often identify A with a set of ordered triples contained in $[n] \times [n] \times \Sigma$ given by $A = \{(i, j, s) : \text{ cell } (i, j) \text{ of } A \text{ contains } s\}$. The symbol set of A, denoted as Σ_A, is the set of all symbols which appear in A.

For a subset $S \subseteq \Sigma_A$, define $A(S) = \{(i, j, s) \in A : s \in S\}$, which we call the *subarray of A induced by S*. For a partition $\mathcal{S}$ of Σ_A, we call $\{A(S) : S \in \mathcal{S}\}$ a *partition of A induced by $\mathcal{S}$*. If $P \subseteq A$ and there exists a subset $S \subseteq \Sigma_A$ for which $P = A(S)$, we say P is an *induced subarray of A*. Similarly, if $\mathcal{P}$ is a partition of A and there exists a partition $\mathcal{S}$ of Σ_A such that $\mathcal{P} = \{A(S) : S \in \mathcal{S}\}$, we say $\mathcal{P}$ is an *induced partition of A*. In Figure 3, $A \in \mathrm{LA}(3)$ with $\Sigma_A = [5]$, as well as induced partial Latin arrays (and in this case, partial Latin squares) $A_1 = A(\{1\})$, $A_2 = A(\{2, 4\})$, and $A_3 = A(\{3, 5\})$. Note that $\{A_1, A_2, A_3\}$ is an induced partition of A corresponding to $\{\{1\}, \{2, 4\}, \{3, 5\}\}$.

A partial Latin square $P \in \mathrm{PLS}(n)$ is *completable* if there exists $L \in \mathrm{LS}(n)$ such that $P \subseteq L$; in this case L is a *completion* of P. If no such L exists, then P is *incompletable*. For a positive integer k, we say a partial Latin array $A \in \mathrm{PLA}(n)$ is *k-completable* if there exists an induced partition of A consisting of k completable partial Latin squares. Using this language, we want to show that all $A \in \mathrm{PLA}(n)$ are n-completable.

For subsets R and C of $[n]$, let $A(R, C)$ denote the subarray of A at the intersection of rows and columns of A indexed by R and C, respectively; that is $A(R, C) = \{(i, j, s) \in A : i \in R \text{ and } j \in C\}$.

Let $|A|$ denote the number of filled cells of A, which is sometimes referred to as the *volume* of A. For each $x \in \Sigma_A$, let $\sigma_A(x)$ denote the number of times x appears in A; that is $|\{(i, j) \in [n] \times [n] : (i, j, x) \in A\}|$. In particular, we say x is *nearly complete* in A, or simply *nearly complete*, if $\sigma_A(x) = n-1$. We also say x is *complete* in A if $\sigma_A(x) = n$ and x is *incomplete* otherwise. Finally we say x is a *singleton*, *double*, or *triple* in A if $\sigma_A(x)$ is 1, 2, or 3, respectively.

Let $P \in \mathrm{PLA}(n)$. For a symbol x, we say x *extends* in P if there exist $n - \sigma_P(x)$ empty cells in P which, when filled by x, produce a partial Latin array; we call this partial Latin array containing P and the additional $n - \sigma_P(x)$ cells filled with x an *extension* of x in P. Furthermore, for a set of symbols $S = \{x_1, x_2, \ldots, x_k\}$, we say S *extends* in P if there exist $kn - (\sigma_P(x_1) + \cdots + \sigma_P(x_k))$ empty cells in P which, when filled with symbols from S, produce a partial Latin array which we similarly call an *extension* of S in P. The empty set always extends in P, which is used later in inductive constructions. Now suppose that x is a nearly complete symbol in P. Then there exists a row a and a column b in P in which x does not appear. If x extends in P, then (a, b) must be empty in P. We say x *requires* (a, b) in P and, if (a, b) is filled in P with a symbol s distinct from x, we say s *blocks* x in P.

Now suppose that $\Sigma_P \subseteq [n]$. Let $\mathfrak{S}_n$ be the symmetric group acting on $[n]$. For $\pi = (\pi_r, \pi_c, \pi_s) \in \mathfrak{S}_n \times \mathfrak{S}_n \times \mathfrak{S}_n$, let $\pi(P)$ denote the array obtained by permuting the rows, columns, and symbols of P by π_r, π_c, and π_s, respectively. Then π is an *isotopism*, and P and $\pi(P)$ are *isotopic*. Observe that $\pi(P) \in \mathrm{PLS}(n)$ and P is completable if and only if $\pi(P)$ is completable.

For $\alpha \in \mathfrak{S}_3$, let $\alpha(P) = \{(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) : (x_1, x_2, x_3) \in P\}$. Then α is a *conjugation*, and P and $\alpha(P)$ are *conjugates*. Note that α acts on $\mathrm{PLS}(n)$; as such $\alpha(P) \in \mathrm{PLS}(n)$, and P is completable if and only if $\alpha(P)$ is completable. Finally, the *main class* of P is the set of all partial Latin squares which are isotopic to a conjugate of P. Hence completability (and incompletability) is a property of main classes of partial Latin squares.

2.2 Completability results for partial Latin squares

Evans [6] conjectured that all partial Latin squares of order n with volume at most $n-1$ are completable. This was proved true independently by Smetaniuk [11] and Andersen and Hilton [2]. The latter publication also classified the incompletable elements of $\mathrm{PLS}(n)$ with volume n. Additionally, Andersen [1] classified the incompletable elements of $\mathrm{PLS}(n)$ with volume $n+1$. We begin by introducing these results.

For $n \geq 2$ and $k \in [n-1]$, define $B_{k,n}$ as the element of $\mathrm{PLS}(n)$ with volume n given by $\{(i, i, 1) : i \in [k]\} \cup \{(i, k+1, i-k+1) : i \in [k+1, n]\}$, and let $\mathcal{B}_n$ be the union of the main classes of $B_{1,n}, \ldots, B_{n-1,n}$. As an example, we give $B_{2,4}$ and its conjugates

1			
	1		
		2	
		3	

$B_{2,4}$

1	2		
		3	
		4	

$(13)B_{2,4}$

1	2		
		3	
			3

$(23)B_{2,4}$

2	3
3	2

A_1

2	3
4	2

A_2

2	3
4	5

A_3

1	2		
	1		
		3	
			4

T_0

Figure 4: The first three partial Latin squares illustrate $B_{2,4}$ and two of its conjugates. The next three Latin arrays of order 2 and partial Latin square of order 4 are used to describe incompletable partial Latin squares of order n with volume $n+1$, which are summarized in Theorem 2.1.

$(13)B_{2,4}$ and $(23)B_{2,4}$ in Figure 4. Let A_1, A_2, and A_3 be the Latin arrays in LA(2) given in Figure 4. For $n \geq 3$ and $i \in [3]$, define $T_{i,n} = A_i \cup \{(j,j,1) : j \in [3, n-1]\}$, and let $T_0 \in \text{PLS}(4)$ as given in Figure 4. With these objects defined, we may classify the completable partial Latin squares in PLS(n) with volume at most $n+1$.

Theorem 2.1. *Let $n \geq 1$ and $P \in \text{PLS}(n)$.*

(a) *If P has volume at most $n-1$, then P is completable.* [11, 2]

(b) *If P has volume n, then P is incompletable if and only if $P \in \mathcal{B}_n$.* [2]

(c) *If P has volume $n+1$, then P is incompletable if and only if either*

- *a subset of P belongs to $\mathcal{B}_n$,*
- *$n = 3$ and P is in the main class of $T_{1,3}$,*
- *$n = 4$ and P is in the main class of T_0, $T_{1,4}$, or $T_{2,4}$, or*
- *$n \geq 5$ and P is in the main class of $T_{1,n}$, $T_{2,n}$, or $T_{3,n}$.* [1]

Using the language introduced earlier, we highlight the following two results, each of which follow as a result of Theorems 2.1(b) and 2.1(c), respectively.

Lemma 2.2. *Let $P \in \text{PLS}(n)$ with volume $n+1$ such that Σ_P contains at least one singleton and one non-singleton. Then there exists a singleton $\alpha \in \Sigma_P$ such that $P(\Sigma_P \backslash \{\alpha\})$ is a completable partial Latin square with volume n.*

Lemma 2.3. *Let $n \geq 4$ and $P \in \text{PLS}(n)$. If Σ_P consists of a nearly complete symbol α and two singletons a and b, then P is completable if and only if a and b do not block α.*

One of the earliest known completion results we use is attributed to Hall [7], which states that a partial Latin square of order n consisting only of k completed rows is completable. Recall that a partial Latin square in which each symbol appears n times is a conjugate of a Latin square consisting of completed rows. With this in mind, to complete a partial Latin square P, we need only confirm that Σ_P extends in P.

A partial Latin square with only one symbol appearing is trivially completable. So we continue with an observation about partial Latin squares with two symbols. Then we present another observation and two technical lemmas which give sufficient conditions under which other families of partial Latin squares are completable.

Observation 2.4. *Let $n \geq 2$ and $P \in \mathrm{PLS}(n)$ with $\Sigma_P = \{1, 2\}$.*

- **(a)** *Suppose symbols 1 and 2 are both nearly complete in P. Then Σ_P extends in P if and only if neither symbols 1 nor 2 block one another and do not require the same cell in P.*
- **(b)** *Suppose only symbol 1 is nearly complete in P. Then Σ_P extends in P if and only if symbol 2 does not block symbol 1.*
- **(c)** *If neither symbols 1 nor 2 are nearly complete in P, then Σ_P extends in P.*

The observation below is a direct application of Hall's Marriage Theorem and implies the subsequent lemma and corollary.

Observation 2.5. *Let $P \in \mathrm{PLA}(n)$ and suppose that symbol k does not appear in P. Then symbol k extends in P if and only if P does not contain an $a \times b$ filled subarray with $a + b = n + 1$. In particular, symbol k extends in P if*

- **(a)** *at most half of the cells in each row and column of P are filled,*
- **(b)** *n is odd (say $n = 2m - 1$ for an integer m), each row and column of P have at most m filled cells, and P does not contain an $m \times m$ filled subarray,*
- **(c)** *n is even (say $n = 2m - 2$ for an integer m) each row and column of P have at most m filled cells, and P does not contain an $m \times (m - 1)$ or $(m - 1) \times m$ filled subarray, or*
- **(d)** *P has at most one filled cell per column and no row has $n - 1$ filled cells.*

Lemma 2.6. *Let $r \in [n]$ and $P \in \mathrm{PLS}(n)$. Suppose that $x \in \Sigma_P$ and $\sigma_P(x) = r$. Then x extends in P if either*

- **(a)** *$|\Sigma_P| \leq (n - r)/2 + 1$, or*
- **(b)** *$|\Sigma_P| \leq (n - r)/2 + 2$ and $\sigma_P(y) \leq (n - r - 1)/2$ for some $y \in \Sigma_P \backslash \{x\}$.*

Proof. Without loss of generality, assume $(1, 1, x), \dots, (r, r, x) \in P$ and define P' as the subarray $P([r + 1, n], [r + 1, n])$.

First suppose $|\Sigma_P| \leq (n - r)/2 + 1$. Since the number of filled cells in a row or column of P' is at most $|\Sigma_P| - 1 \leq (n - r)/2$, it follows that k extends in P' by Observation 2.5(a). Hence x extends in P.

Now suppose $|\Sigma_P| \leq (n - r)/2 + 2$ and $\sigma_P(y) \leq (n - r - 1)/2$ for some $y \in \Sigma_P$ with $x \neq y$. Assume there exists a filled subarray P'' in P' with a rows and b columns such that $a + b = n - r + 1$. Without loss of generality, assume $a \leq b$. Since each row and column of P' contains at most $|\Sigma_{P'}| - 1 \leq (n - r)/2 + 1$ symbols, it follows that $b = \lfloor (n-r)/2+1 \rfloor$ and $a = \lceil (n-r)/2 \rceil$. Therefore P'' is a Latin rectangle over Σ_P. Then $\sigma_P(y) \geq (n-r)/2$, which is a contradiction. Hence x extends in P' by Observation 2.5, and therefore x extends in P. $\square$

The next corollary follows from an iterative application of Lemma 2.6(a).

Corollary 2.7. *Let $r \in [n]$ and $P \in \mathrm{PLS}(n)$ with $|\Sigma_P| \leq (n-r)/2+1$. If $\sigma_P(x) \leq r$ for all incomplete $x \in \Sigma_P$, then Σ_P extends in P.*

We conclude this section with a technical lemma and corollary that, in later applications, show the completability of some partial Latin squares whose volume exceeds their order, particularly those whose symbols are only doubles and triples.

Lemma 2.8. *Let $n \geq 7$ and $P \in \mathrm{PLS}(n)$. If Σ_P consists of $\lfloor n/2 \rfloor - 1$ complete symbols and two doubles, then P is completable.*

Proof. Denote the doubles in Σ_P as x and y. By Lemma 2.6(b), there exists an extension Q of x in P. If y extends in Q, then P is completable. Now suppose otherwise. In what follows, we produce another extension of x in P through a modest modification of Q in which y does extend. Without loss of generality, assume $(1,1,y)$ and $(2,2,y)$ belong to P.

First suppose $n = 2m+1$. Since each row of $Q([3,2m+1],[3,2m+1])$ contains at most m filled cells, it follows that $Q([3,2m+1],[3,2m+1])$ contains an $m \times m$ filled subarray by Observation 2.5(b). Note that such a subarray, say $Q([m+2,2m+1],[m+2,2m+1])$, is a Latin subsquare on $\Sigma_P \backslash \{y\}$. It follows that $Q([m+2,2m+1],[m+1])$ and $Q([m+1],[m+2,2m+1])$ are empty subarrays, and therefore $Q([2],[m+1])$ and $Q([m+1],[2])$ are both filled.

Since $m \geq 3$, there exist $i,j \in [m+2,2m+1]$ and $k,\ell \in [m+1]$ such that (i,j,x) and (k,ℓ,x) belong to $Q \backslash P$. Observe that (i,ℓ) and (k,j) are empty in Q. Define $R = (Q \backslash \{(i,j,x),(k,\ell,x)\}) \cup \{(i,\ell,x),(k,j,x)\}$, and note that R is also an extension of x in P. Furthermore, $R([m+2,2m+1]\backslash\{i\},[3,m+1])$ and $R([3,m+1],[m+2,2m+1]\backslash\{j\})$ are empty and hence there exist extensions K_1 and K_2 of y in each, respectively. So $R \cup K_1 \cup K_2 \cup \{(i,j,y)\}$ is an extension of y in R, as well as an extension of $\{x,y\}$ in P.

Now suppose $n = 2m$. Since each row of $Q([3,2m],[3,2m])$ again contains at most m filled cells, it follows from Observation 2.5(c) that $Q([3,2m],[3,2m])$ contains an $m \times (m-1)$ or $(m-1) \times m$ filled subarray; without loss of generality we may assume $Q([m+2,2m],[m+1,2m])$ is such a filled subarray. Necessarily $Q([m+2,2m],[m+1,2m])$ is a Latin rectangle in which each symbol of $\Sigma_P \backslash \{y\}$ appears in each row. It follows that $Q([m+2,2m],[m])$ is an empty subarray, and therefore $Q([m+1],[2])$ is filled. Furthermore each row of $Q([m+1],[m+1,2m])$ contains at most $m-2$ filled cells and each column contains exactly one filled cell.

Since $m \geq 4$, there exist $i \in [m+2,2m]$ and $j \in [m+1,2m]$ such that $(i,j,x) \in Q \backslash P$. Additionally, there exist $k,k' \in [m+1]$ and $\ell,\ell' \in [m]$ such that (k,ℓ,x) and (k',ℓ',x) belong to $Q \backslash P$. Observe that not both (k,j) and (k',j) are filled in

Q; without loss of generality, assume that (k, j) is empty in Q. Furthermore , (i, ℓ) is also empty in Q. Again define $R = (Q \backslash \{(i, j, x), (k, \ell, x)\}) \cup \{(i, \ell, x), (k, j, x)\}$, which is also an extension of x in P. Note that $R([m + 2, 2m] \backslash \{i\}, [3, m])$ is empty, which thus has an extension K_1 of y. Additionally there exists an extension K_2 of y in $R([3, m+1], [m+1, 2m] \backslash \{j\})$ by Observation 2.5(d). So $R \cup K_1 \cup K_2 \cup \{(i, j, y)\}$ is an extension of y in R, as well as an extension of $\{x, y\}$ in P. $\square$

This corollary follows from Lemmas 2.6(b) and 2.8.

Corollary 2.9. *Let $r \in [n]$ and $P \in \mathrm{PLS}(n)$ with $\sigma_P(x) \leq r$ for all incomplete $x \in \Sigma_P$. If $|\Sigma_P| \leq (n - r)/2 + 2$ and Σ_P contains at least two doubles, then Σ_P extends in P.*

3 Proofs of n-completability

Many of the results from the previous section (i.e. Lemmas 2.2, 2.6, 2.8, and Corollaries 2.7 and 2.9) had hypotheses which required no knowledge on the location of symbols in a partial Latin square. However, in certain partial Latin squares (in particular those with singletons and nearly complete symbols) we must pay attention to the location of symbols, as outlined in Theorem 2.1 and Observation 2.4, for example.

With this in mind, we begin with results involving partial Latin arrays with no singletons and no nearly complete symbols, then discuss partial Latin arrays which do contain singletons or nearly complete symbols. We then conclude with the main result. Before we begin, we give an observation about completability.

Observation 3.1. *Let $n \geq 1$ and $A \in \mathrm{PLA}(n)$.*

- **(a)** *If A is r-completable, then A is also $(r + 1)$-completable.*
- **(b)** *Suppose $\{B, C\}$ is an induced partition of A where B and C are b- and c-completable, respectively. Then A is $(b + c)$-completable.*

Lemma 3.2. *Let $n \geq 1$ and $A \in \mathrm{PLA}(n)$. Suppose that either*

- **(a)** *$\sigma_A(x) > n/2$ and $\sigma_A(x) \neq n - 1$ for each $x \in \Sigma_A$,*
- **(b)** *$4 \leq \sigma_A(x) \leq n/2$ for each $x \in \Sigma_A$, or*
- **(c)** *$2 \leq \sigma_A(x) \leq 3$ for each $x \in \Sigma_A$ and $n \geq 7$.*

If $|A| > n$, then there exists a completable partial Latin square induced by a subset of Σ_A whose volume exceeds n as well. Furthermore, there exists an integer ℓ and an induced partition $\{B, C\}$ of A such that B is ℓ-completable with $|B| \geq (n + 1)\ell$ and C is completable with $|C| \leq n$.

Proof. First suppose $\sigma_A(x) > n/2$ and $\sigma_A(x) \neq n - 1$ for each $x \in \Sigma_A$. Since $|A| > n$, then $|\Sigma_A| \geq 2$. Let S be any 2-subset of Σ_A. Then $A(S)$ is completable by Observation 2.4(c) and its volume exceeds n.

Next suppose $4 \leq \sigma_A(x) \leq n/2$ for each $x \in \Sigma_A$. If $|\Sigma_A| \leq \lfloor n/4 \rfloor + 1$, then let $S = \Sigma_A$. Otherwise, let S be any subset of Σ_A with cardinality $\lfloor n/4 \rfloor + 1$. Then $A(S)$ is completable by Corollary 2.7 and its volume exceeds n.

For part (c), assume $n \geq 7$ and $2 \leq \sigma_A(x) \leq 3$ for each $x \in \Sigma_A$. Let S be a subset of Σ_A so that $A(S)$ has volume exceeding n, but the volume of $A(S')$ is at most n for any proper subset S' of S. Then either $|A(S)| \in \{n+1, n+2\}$ or $|A(S)| = n+3$ and S contains only triples.

If $|A(S)| = n+1$, $A(S)$ is completable by Theorem 2.1, as $A(S)$ does not belong to the main class of $T_{i,n}$ for any $i \in [3]$ (since $n \geq 7$) and no subset of $A(S)$ belongs to $\mathcal{B}_n$.

Now suppose $|A(S)| = n+2$. If S contains at least two doubles, then $A(S) \leq n/2 \leq (n-3)/2 + 2$, so $A(S)$ is completable by Corollary 2.9 and its volume exceeds n. If S contains exactly one double, then $n \geq 9$ and $|S| = n/3 + 1$; if S contains only triples, then $|S| = (n+2)/3$. So $|S| \leq (n-3)/2 + 1$ in these two cases, hence $A(S)$ is completable by Corollary 2.7 and its volume exceeds n.

Last, suppose that $|A(S)| = n+3$ and S contains only triples. Then $|S| = n/3+1$. Since $n \geq 7$, it follows that $n/3 + 1 \leq \lfloor (n-1)/2 \rfloor$. Therefore $A(S)$ is completable by Corollary 2.7 and its volume exceeds n.

Through an iterative application of this argument, we may produce an induced partition $\{A_1, A_2, \ldots, A_\ell, C\}$ of A for some $\ell \geq 0$ such that each A_i is a completable partial Latin square with volume exceeding n for each $i \in [\ell]$ and C is a partial Latin square whose volume does not exceed n. Since C contains no singletons, we have that C is completable by Theorem 2.1. Furthermore if we let $B = A_1 \cup \cdots \cup A_\ell$, then $\{B, C\}$ is an induced partition of A, B is ℓ-completable, and $|B| \geq (n+1)\ell$. $\square$

Combining the three parts in Lemma 3.2, in conjunction with Observation 3.1(b), gives the following corollary.

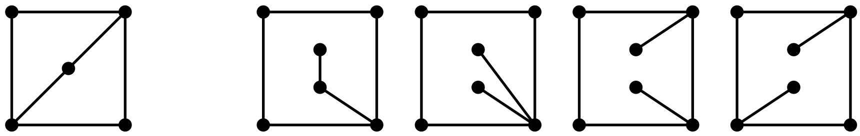
Corollary 3.3. *Let $n \geq 7$ and $A \in \mathrm{PLA}(n)$. Suppose that Σ_A does not contain nearly complete symbols nor singletons. Then there exists an integer $\ell \geq 0$ and an induced partition $\{B, C_1, C_2, C_3\}$ of A such that B is ℓ-completable and $|B| \geq (n+1)\ell$, and C_1, C_2, and C_3 are each completable with volume at most n.*

3.1 Managing Singletons and Nearly Complete Symbols

We begin with a graph theoretic lemma which leads to a completability result involving partial Latin arrays whose symbol sets contain only singletons. Then we present a method for inducing completable partial Latin squares which include nearly complete symbols.

Lemma 3.4. *Let $v \geq 6$ and let G be a simple, bipartite graph with v vertices and v edges. Then G contains two edge-disjoint, 4-vertex paths.*

Proof. Since G is simple, bipartite, and $|V(G)| = |E(G)|$, it follows that G contains



(a) (b)

Figure 5: Subgraphs of G in the proof of Lemma 3.4.

a k-cycle with $k \geq 4$ and k even. If G contains a k-cycle with $k \geq 6$, then the k-cycle (and hence G) contains two edge-disjoint, 4-vertex paths.

Now assume G does not contain a k-cycle with $k \geq 6$, and first suppose G contains two 4-cycles. Each 4-cycle contains a 4-vertex path, and if the 4-cycles are edge-disjoint, then G contains two edge-disjoint 4-vertex paths. Otherwise, G contains a subgraph as given in Figure 5(a), which contains two edge-disjoint 4-vertex paths.

Now suppose G contains only one 4-cycle. Then G is unicyclic and therefore is connected. Since $v \geq 6$, it follows that G contains a subgraph isomorphic to one of the graphs given in Figure 5(b), each of which contains two edge-disjoint 4-vertex paths. □

Lemma 3.5. *Let $n \geq 3$ and $A \in \mathrm{PLA}(n)$, and suppose that Σ_A contains only singletons. If $|A| \leq 2n$, then A is 2-completable. Hence, if $|A| > 2n$, then there exists $A' \subseteq A$ such that $|A'| = n$ and A' is completable.*

Proof. It is sufficient to show that the result holds when $|A| = 2n$. Define $L = \{(1,1,1),(1,2,2),(2,1,3)\}$. If an element $P \in \mathrm{PLS}(n)$ has volume n and contains an isotope of L, then P is completable by Theorem 2.1. Hence, it is sufficient to show that A contains two disjoint isotopes of L.

Let G be the subgraph of $K_{n,n}$ with bipartite sets $\{a_i : i \in [n]\}$ and $\{b_i : i \in [n]\}$ such that $a_i b_j \in E(G)$ if cell (i,j) is filled in A. Then G is simple, bipartite, has $2n$ vertices, and has $2n$ edges. Hence G contains two edge-disjoint 4-vertex paths by Lemma 3.4. Edge-disjoint 4-vertex paths in G correspond to the cell locations of disjoint isotopes of L contained in A, and so A is 2-completable. □

An iterative application of Lemma 3.5 gives the following result.

Lemma 3.6. *Let $n \geq 3$ and $A \in \mathrm{PLA}(n)$, and suppose that Σ_A contains only singletons. If $|A| \leq kn$ for some $k \geq 2$, then A is k-completable.*

Now we focus on partial Latin arrays with nearly complete symbols. We first prove a lemma showing that any Latin array of $\mathrm{LA}(n)$ in which at least half of its symbols are nearly complete is n-completable. The subsequent lemma shows there is some control over how nearly complete symbols may be partitioned to form completable partial Latin squares with volume exceeding their order.

Lemma 3.7. *Suppose $n \geq 4$, and let $A \in \mathrm{LA}(n)$ and N be the set of nearly complete symbols in A. If $|N| \geq |\Sigma_A|/2$, then A is n-completable.*

Proof. Since $|N|(n-1) \leq n^2$, it follows that $|N| \leq n+1$.

First suppose $|N| = n+1$. Then Σ_A consists of the $n+1$ symbols in N and one singleton, which we denote as α. Hence $|\Sigma_A| = n+2$, and thus it is sufficient to find two disjoint 2-subsets which induce completable partial Latin squares, as each of the remaining $n-2$ symbols in Σ_A induces a completable partial Latin square as well.

Without loss of generality, assume $(1, 1, \alpha) \in A$, and let $N' \subseteq N$ be the $n-1$ symbols in row 1 of A. Hence there are two symbols in $N \backslash N'$, say y and z. Then y and z both require cells in row 1 of A. Since $n-1 \geq 3$, there exists a symbol $x \in N'$ which block neither of y and z. As x can only be blocked by one symbol, either y or z does not block x; without loss of generality suppose y does not block x. Finally, let $w \in N'$ be distinct from x. Observe that x and y cannot require the same cell, so $A(\{x, y\})$ is completable by Observation 2.4(a). Furthermore, by Theorem 2.1(b), $A(\{\alpha, w\})$ is completable. Hence A is n-completable.

Next suppose $|N| \leq n$ and let $N^c = \Sigma_A \backslash N$. Define $k = |N^c|$, and since $|N| \geq |\Sigma_A|/2$, it follows that $|\Sigma_A| \leq n+k$ and $k \in [n]$. Similar to the previous case, it is sufficient to find k disjoint 2-subsets of Σ_A which induce completable partial Latin squares and, if $|N| < n$, it is sufficient to find $k-1$ such disjoint 2-subsets.

For each $x \in N^c$, define S_x as the set of symbols in N which are not blocked by x. Further suppose that $(S_x : x \in N^c)$ has an SDR $(s_x \in S_x : x \in N^c)$. Then $\{\{x, s_x\} : x \in N^c\}$ is a set of k disjoint 2-subsets which induce completable partial Latin squares, by Observation 2.4(b), and hence A is n-completable. For the remainder of the proof we assume $(S_x : x \in N^c)$ does not have an SDR.

Observe that $S_x \cup S_y = N$ for any distinct pair $x, y \in N^c$, as every symbol in N is blocked by exactly one symbol in Σ_A. Hence there must be a unique symbol $b \in N^c$ for which b blocks all symbols in N (thus $S_b = \varnothing$) and therefore $S_x = N$ for each $x \in N^c \backslash \{b\}$.

Consider the case when $|N| < n$. Since $|N| > |N^c \backslash \{b\}|$, it follows that $(S_x : x \in N^c \backslash \{b\})$ has an SDR $(s_x \in S_x : x \in N^c \backslash \{b\})$. Hence $\{\{x, s_x\} : x \in N^c \backslash \{b\}\}$ is a set of $k-1$ disjoint 2-subsets which induce completable partial Latin squares, again by Observation 2.4(b), and therefore A is n-completable.

Finally consider the case when $|N| = n$. Observe that $|N^c| \leq n$ and equality is achieved if and only if N^c is comprised of singletons. Since a cell can only be required by at most $n-1$ symbols in N, there exist two symbols in N, say y and z, which require different cells. This implies b is not a singleton and therefore $|N^c| \leq n-1$. As y and z are both blocked by b, it follows that $A(\{y, z\})$ is completable.

For each $x \in N^c \backslash \{b\}$, define S'_x as the set of symbols in $N \backslash \{y, z\}$ which are not blocked by x. Recall that $S_x = N$ for each $x \in N^c \backslash \{b\}$ and so $S'_x = N \backslash \{y, z\}$ for $x \in N^c \backslash \{b\}$. Since $|N^c \backslash \{b\}| \leq n-2 = |N \backslash \{y, z\}|$, it follows that $(S'_x : x \in N^c \backslash \{b\})$ has an SDR $(s'_x \in S'_x : x \in N^c \backslash \{b\})$. Hence $\{\{x, s'_x\} : x \in N^c \backslash \{b\}\} \cup \{y, z\}$ is a set of k disjoint 2-subsets which induce completable partial Latin squares, again by

Observation 2.4(b), and therefore A is n-completable. □

Lemma 3.8. *Let $n \geq 4$ and $A \in \mathrm{LA}(n)$. Let N be the set of nearly complete symbols in A. Suppose that for all $N \subseteq S \subseteq \Sigma_A$, if $A(S)$ is $|N|$-completable, then $|A(S)| < (n+1)|N|$. Then A is n-completable.*

Proof. Let $N^c = \Sigma_A \backslash N$. First suppose that $|N| \geq n$. Then

$$|\Sigma_A| - |N| = |N^c| \leq |A(N^c)| = n^2 - |A(N)| \leq n^2 - (n-1)n = n \leq |N|,$$

so $|N| \geq |\Sigma_A|/2$. Therefore A is n-completable by Lemma 3.7. Now assume $|N| \leq n-1$. If $|N| \geq |N^c|$, then again A is n-completable by Lemma 3.7, so in what follows, we assume $|N| \leq |N^c| - 1$. Next, we show there exists a set $\mathcal{B} = \{B_x : x \in N\}$ of nonempty disjoint subsets of N^c for which $A(\{x\} \cup B_x)$ is completable.

For each $x \in N$, define $C_x = \{y \in N^c : y \text{ does not block } x\}$, and recall that exactly one symbol in Σ_A blocks x, so $|C_x| \geq |N^c| - 1 \geq |N|$. By Hall's Marriage Theorem, $(C_x : x \in N)$ has an SDR $(c_x \in C_x : x \in N)$. By letting $B_x = \{c_x\}$ for each $x \in N$, it follows that $\mathcal{B}$ meets the conditions outlined above.

Now we select $\mathcal{B}$ so that $|\{x \in N : |A(B_x)| = 1\}|$ (which we denote as s) is minimal; that is, $\mathcal{B}$ has a minimum number of subsets consisting of one singleton. Note that s is positive; otherwise $|A(B_x)| \geq 2$ for all $x \in N$ and hence

$$|A(N \cup (\cup\mathcal{B}))| = \sum_{x \in N} |A(\{x\} \cup B_x)| = \sum_{x \in N} (\sigma_A(x) + |A(B_x)|) \geq (n+1)|N|,$$

where $\cup\mathcal{B}$ denotes the union of all sets in $\mathcal{B}$; this contradicts one of our original hypotheses.

Let $x \in N$ such that $|A(B_x)| = 1$, and let $B_x = \{y\}$. Now, define $R = N^c \backslash (\cup\mathcal{B})$. Note that R and $N \cup (\cup\mathcal{B})$ partition Σ_A. Since $A(N \cup (\cup\mathcal{B}))$ is $|N|$-completable (and therefore $(n-1)$-completable), if $A(R)$ is completable, then A is n-completable.

To that end we now show that $A(R)$ is completable. Assume to the contrary. If $|R| = 1$, then $A(R)$ is completable, so $|R| \geq 2$. Then there exists $a \in R$ which does not block x. If a is a singleton, then $A(\{x, y, a\})$ is completable by Lemma 2.3; otherwise $A(\{x, a\})$ is completable by Observation 2.4(b). So if we replace B_x with $\{y, a\}$ when a is a singleton or $\{a\}$ otherwise, then $\mathcal{B}$ would have one fewer subset consisting of one singleton, which contradicts the minimality of s. Hence $A(R)$ is completable and therefore A is n-completable. □

3.2 Main Result

To prove the main result, we somewhat greedily produce an induced partition of a Latin array into completable partial Latin squares with volume exceeding their order, as well as a partial Latin array induced by the *leftover* symbols. We first give a series of lemmas which will be used to handle this leftover partial Latin array. Then we present the proof of the main theorem.

The first lemma is straightforward; it results from being able to partition a relatively sparse partial Latin array into two partial Latin squares which are completable by Theorem 2.1(a).

Lemma 3.9. *Let $n \geq 6$. If $P \in \mathrm{PLA}(n)$ and $|P| \leq n+2$, then P is 2-completable.*

Lemma 3.10. *Let $n \geq 6$ and $P \in \mathrm{PLA}(n)$. Suppose there exists an induced partition $\{P_1, P_2, P_3, S\}$ of P such that P_i is completable and $|P_i| \leq n$ for each $i \in [3]$, and Σ_S is the set of all singletons in P.*

- **(a)** *If $|P| \leq 2n+3$, then P is 3-completable.*
- **(b)** *If $|P| \leq 3n+2$, then P is 4-completable.*
- **(c)** *If $4 \leq k \leq n$ and $3n+3 \leq |P| \leq (k-1)n+k$, then P is k-completable.*

Proof of (a). Without loss of generality, assume $n \geq |P_1| \geq |P_2| \geq |P_3|$. First suppose that $|P_1| = |P_2| = n$. Then $|P_3 \cup S| \leq 3$ and hence $P_3 \cup S$ is completable by Theorem 2.1(a). Since P_1 and P_2 are each completable, it follows that P is 3-completable.

Now suppose $|P_1| = n$ and $|P_2| < n$. If $|S| \leq n-1-|P_2|$, then $P_2 \cup S$ and P_3 are completable by Theorem 2.1(a) and hence P is 3-completable. Otherwise let S' be a $(n-1-|P_2|)$-subset of S. Then $P_2 \cup S'$ is completable and, since $|P_3 \cup (S \backslash S')| \leq 4$, $P_3 \cup (S \backslash S')$ is also completable, each by Theorem 2.1(a). Hence P is 3-completable.

Last, suppose $|P_1| < n$. Since $n \geq 6$, we have that $|P| \leq 3(n-1)$. Hence there exist nonnegative integers s_1, s_2, s_3 such that $|S| = s_1 + s_2 + s_3$ and $s_i \leq n-1-|P_1|$ for each $i \in [3]$. Partition S as $\{S_1, S_2, S_3\}$ with $|S_i| = s_i$ for each $i \in [3]$. Then $P_i \cup S_i$ is completable by Theorem 2.1(a) for each $i \in [3]$, and hence P is 3-completable.

Proof of (b). If $|S| \leq n-1$, then $P_1 \cup P_2 \cup P_3 \cup S$ is 4-completable, as S is completable by Theorem 2.1(a). Now suppose $|S| \geq n$. Partition S as $S' \cup S''$ where $|S'| = n-1$. Then S' is completable by Theorem 2.1(a) and $P_1 \cup P_2 \cup P_3 \cup S''$ is 3-completable by (a). Hence P is 4-completable.

Proof of (c). Since $|P| \geq 3n+3$, there exists a partition $\{S_1, S_2, S_3, S'\}$ of S (S' may be empty) so that $|S_i| = n+1-|P_i|$ for each $i \in [3]$. Then for each $i \in [3]$, there exists a completable partial Latin square $Q_i \subseteq P_i \cup S_i$ which contain P_i and has volume n by Lemma 2.2. Observe that $P \backslash (Q_1 \cup Q_2 \cup Q_3) \subseteq S$ and has volume at most $(k-4)n+k$; therefore $P \backslash (Q_1 \cup Q_2 \cup Q_3)$ is $(k-3)$-completable by Lemma 3.6. So P is k-completable. □

From Lemmas 3.9 and 3.10, along with the observation that all partial Latin squares with volume at most 1 are completable, we have the following corollary.

Corollary 3.11. *Let $n \geq 6$ and $P \in \mathrm{PLA}(n)$. Suppose there exists an induced partition $\{P_1, P_2, P_3, S\}$ of P such that P_i is completable and $|P_i| \leq n$ for each $i \in [3]$, and Σ_S is the set of all singletons in P. If $k \leq n$ and $|P| \leq (k-1)n+k$, then P is k-completable.*

We now conclude this section with the proof of the main theorem.

Theorem 3.12. *Let $n \geq 1$ and $A \in \mathrm{LA}(n)$. Then A is n-completable.*

Proof. The result holds when $n \leq 6$; this was confirmed through tedious case-wise analysis, on which we elaborate in the appendix. So now suppose that $n \geq 7$. Let $N \subseteq \Sigma_A$ be the set of all nearly complete symbols in A.

First suppose there exists an induced partial Latin array Q from A such that $N \subseteq \Sigma_Q$, $|Q| \geq (n+1)|N|$, and Q is $|N|$-completable. Let Z be induced from A such that Σ_Z is the set of singletons in $A \backslash Q$. Let $L = A \backslash (Q \cup Z)$. By Corollary 3.3, there exists an integer ℓ and a partition $\{L_1, L_2, L_3, M\}$ of L such that L_i is completable with $|L_i| \leq n$ for each $i \in [3]$ and M is ℓ-completable with $|M| \geq (n+1)\ell$. Let $P = L_1 \cup L_2 \cup L_3 \cup Z$ and $k = n - \ell - |N|$. Then

$$|P| = n^2 - |Q \cup M| \leq n^2 - (n+1)(n-k) = (k-1)n + k.$$

So P is k-completable by Corollary 3.11. Since $\{M, P, Q\}$ is an induced partition of A, it follows that A is n-completable.

Now suppose no such partial Latin array Q exists. We have that for all subsets S such that $N \subseteq S \subseteq \Sigma_A$ and $A(S)$ is $|N|$-completable, then $|A(S)| < (n+1)|N|$. So A is n-completable by Lemma 3.8. $\square$

4 Applications

In 1980 Häggkvist [4] made the following conjecture:

Conjecture 4.1. *Let n and r be positive integers. Any partial $nr \times nr$ Latin square whose filled cells lie in $n-1$ disjoint $r \times r$ squares can be completed.*

If $r = 1$, this reduces to Theorem 2.1(a). For integers $r, n \geq 1$ and a Latin array $A \in \mathrm{LS}(r)$, Kuhl and Schroeder [8] define a partial Latin square $nA \in \mathrm{PLS}(rn)$ obtained by combining n copies of A in an array of order rn so that the rows, and similarly the columns, of any two copies of A are disjoint. Without loss of generality, we assume the copies of A appear on the main block diagonal of nA. In Figure 6, we give A, $2A$, and $3A$, where $A \in \mathrm{LA}(3)$ is the same as was originally given in Figure 3. Observe that $(n-1)A$, embedded in an $nr \times nr$ array, is a special case of the partial Latin squares addressed by Häggkvist where the blocks are identical and are both row- and column-disjoint.

Kuhl and Schroeder [8] prove the following relationship between these partial Latin squares and their associated Latin arrays.

Lemma 4.2. *Let $A \in \mathrm{LA}(n)$ and $n \geq 1$. If A is n-completable, then nA is completable.*

However, this condition is not necessary. Let $A \in \mathrm{LA}(4)$ as given in Figure 7. If A is 2-completable, then one of the two parts in a partition of A is induced by a symbol

1	2	5
2	1	4
5	4	3

A

1	2	5			
2	1	4			
5	4	3			
			1	2	5
			2	1	4
			5	4	3

$2A$

1	2	5						
2	1	4						
5	4	3						
			1	2	5			
			2	1	4			
			5	4	3			
						1	2	5
						2	1	4
						5	4	3

$3A$

Figure 6: A Latin array and its associated partial Latin squares.

1	4	6	3
5	2	4	1
4	5	3	2
3	1	2	4

A

1	4	6	3				
5	2	4	1				
4	5	3	2				
3	1	2	4				
				1	4	6	3
				5	2	4	1
				4	5	3	2
				3	1	2	4

$2A$

1	4	6	3	2	7	8	5
5	2	4	1	8	3	7	6
4	5	3	2	7	6	1	8
3	1	2	4	6	8	5	7
2	7	8	5	1	4	6	3
8	3	7	6	5	2	4	1
7	6	1	8	4	5	3	2
6	8	5	7	3	1	2	4

Completion of $2A$

Figure 7: A Latin array A which is not 2-completable, but $2A$ is completable.

set containing at least two symbols from $\{1, 2, 3\}$. However such a configuration produces an incompletable partial Latin square. Nevertheless $2A$ is completable.

Using Lemma 4.2 and constructions similar (but weaker) to those found in this paper, Kuhl and Schroeder [8] prove the following.

Theorem 4.3. *Let $n \geq 1$. If $n \geq r + 1$, then A is n-completable (and hence nA is completable) for all $A \in \mathrm{LA}(r)$. If $n \leq r - 1$, there exist $A \in \mathrm{LA}(r)$ such that nA is incompletable.*

The case $n = r$ was left unresolved. However, with the result of this paper, we can give a full classification.

Theorem 4.4. *Let $n \geq 1$. If $n \geq r$, then nA is completable for all $A \in \mathrm{LA}(r)$. If $n \leq r - 1$, there exists $A \in \mathrm{LA}(r)$ which is not n-completable.*

Therefore we have resolved the following special case of Conjecture 4.1.

Theorem 4.5. *Let n and r be positive integers and $n \geq r$. Any partial $nr \times nr$ Latin square whose filled cells lie in $n - 1$ disjoint $r \times r$ squares, each of which are in* *disjoint rows and columns and are contained in some common Latin array of* LA(r), *can be completed.*

Appendix: Small Cases

In this section we describe the manner by which we prove Theorem 3.12 when $n \leq 6$. The result holds when $n = 1$ trivially. When $n = 2$, every partial Latin array is contained in an isotope of one of the three arrays given in Figure 8(a). Observe that A_1 is a Latin square and trivially completable (and hence 2-completable), while A_2 and A_3 have induced partitions $\{A_2(\{1,2\}), A_2\{3\})\}$ and $\{A_3(\{1,2\}), A_3(\{3,4\})\}$ into completable partial Latin squares. Hence Theorem 3.12 holds when $n = 2$.

For the remaining cases, we primarily focus on partitions rather than on Latin arrays. To each Latin array of order n, we associate a partition of n^2 with parts not exceeding n; each part corresponds to the number of times a particular symbol appears in the array. For example, the Latin array of order 6 given in Figure 8(b) has 655443222111 as its associated partition of 36 into parts which do not exceed 6.

A_1:

1	2
2	1

A_2:

1	2
3	1

A_3:

1	2
3	4

(a)

1	2	3	4	6	8
9	1	2	3	4	6
6	9	1	2	3	5
5	7	a	1	2	3
4	5	7	b	1	2
3	4	5	8	c	1

(b)

Figure 8: (a) Main class representatives for all Latin arrays of order 2.
(b) A Latin array with associated partition 655443222111.

To demonstrate this we show Theorem 3.12 is true for $n = 3$ by considering the 12 partitions of 9 into parts of size at most 3. Let $P \in$ LA(3), and thus one of the partitions given in Figure 9 is associated to P. We consider each of the 12 cases. We show in each case that P has an induced partition with at most 3 parts. For most, we simply list the partition, while in some cases we need more exposition.

- **(a)** P (as P is a Latin square in this case)
- **(b)** $P(\{\alpha, \beta, \gamma\})$ and $P(\{\delta\})$
- **(c)** $P(\{\alpha, \beta, \gamma\})$ and $P(\{\delta, \epsilon\})$
- **(d)** First suppose that α can be paired with a double, say β. Then $P(\{\alpha, \beta\})$, $P(\{\gamma\})$, and $P(\{\delta\})$ are each completable. Otherwise, α blocks each double, but since any one cell can be required by at most two doubles, there exist two doubles, say β and γ, which require different cells. Hence $P(\{\alpha\})$, $P(\{\beta, \gamma\})$, and $P(\{\delta\})$ are completable in this case.

Symbol	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
α	3	3	3	3	3	3	3	2	2	2	2	1
β	3	3	3	2	2	2	1	2	2	2	1	1
γ	3	2	1	2	2	1	1	2	2	1	1	1
δ		1	1	2	1	1	1	2	1	1	1	1
ϵ			1		1	1	1	1	1	1	1	1
ζ						1	1		1	1	1	1
η							1			1	1	1
θ											1	1
ι												1

Figure 9: The 12 partitions of 9 into parts of size at most 3. Each column corresponds to a partition, and we identify each part as the number of occurrences of symbols in a Latin array. For example, we associate to the partition in (d) a Latin array with a triple α and doubles β, γ, and δ.

(e) Either δ or ϵ does not block β; without loss of generality, we may assume δ does not block β. Hence $P(\{\beta, \delta\})$, $P(\{\alpha, \epsilon\})$, and $P(\{\gamma\})$ are each completable.

(f) Again without loss of generality, we may assume γ does not block β. Hence $P(\{\beta, \gamma\})$, $P(\{\alpha, \delta\})$, and $P(\{\epsilon, \zeta\})$ are each completable.

(g) $P(\{\beta, \gamma, \delta, \epsilon, \zeta, \eta\})$ is 2-completable by Lemma 3.6; additionally $P(\{\alpha\})$ is completable.

(h) By considering the doubles which share a row or column with ϵ, we may assume without loss of generality that one of the following partial Latin arrays is contained in P:

ϵ	α	β
α		
β		

ϵ	α	β
α		
γ		

ϵ	α	β
γ		
δ		

In the first case, either $(2, 2, \gamma)$ or $(2, 2, \delta)$ belong to P, so without loss of generality assume $(2, 2, \gamma) \in P$. Similarly in the third case, either $(2, 2, \delta)$ or $(2, 2, \beta)$ belong to P, so assume $(2, 2, \delta) \in P$. Hence we may assume that P is one of the following Latin arrays:

ϵ	α	β
α	γ	δ
β	δ	γ

ϵ	α	β
α	δ	γ
γ	β	δ

ϵ	α	β
γ	δ	α
δ	β	γ

In the first and last case, $P(\{\alpha, \beta, \epsilon\})$, $P(\{\gamma\})$, and $P(\{\delta\})$ are each completable. In the second case, $P(\{\alpha, \epsilon\})$, $P(\{\beta, \gamma\})$, and $P(\{\delta\})$ are each completable.

(i) The result holds if there is a way to pair each double with a singleton to produce a completable induced partition. Assume to the contrary. Then by an argument similar to that given in the proof of Lemma 3.7, without loss of generality, we may assume δ blocks each of α, β, and γ. But one location can block at most two doubles, which is a contradiction.

(j) First suppose that α and β induce an intercalate (a subsquare of order 2). Then we may assume P has the form

$$\begin{array}{|c|c|c|} \hline \alpha & \beta & \gamma \\ \hline \beta & \alpha & \delta \\ \hline \epsilon & \zeta & \eta \\ \hline \end{array}.$$

Then $P(\{\alpha,\gamma\})$, $P(\{\beta,\delta\})$, and $P(\{\epsilon,\zeta,\eta\})$ are each completable. Now suppose α and β do not induce an intercalate. Then an occurrence of α appears in either a row with two singletons, or a column with two singletons; without loss of generality, assume an occurrence of α appears in a row with singletons γ and δ. In addition, either ϵ or ζ does not block β; without loss of generality, assume ϵ does not block β. Hence $P(\{\alpha,\delta,\gamma\})$, $P(\{\beta,\epsilon\})$, and $P(\{\zeta,\eta\})$ are each completable.

(k) Without loss of generality, assume β does not block α. Then $P(\{\alpha,\beta\})$ is completable and additionally, $P(\{\gamma,\delta,\epsilon,\zeta,\eta,\theta\})$ is 2-completable by Corollary 3.6.

(l) P is 3-completable by Lemma 3.6.

There are 64, 377, and 2432 partitions of n^2 into parts of size at most n with $n = 4$, 5, and 6, respectively. We confirmed, by analyzing each partition, that any partial Latin array of order n is n-completable when $4 \leq n \leq 6$. Thankfully, we found that many partitions could be argued through similar methods. In some cases, with the assistance of a computer, we were able to reduce the number of cases we needed to consider by hand.

We use two additional criteria for building completable partial Latin squares using only doubles and triples when $n = 5$ and $n = 6$. It follows from Theorem 2.1(c) that if $P \in \mathrm{PLS}(5)$ is incompletable and Σ_P consists of three doubles, then P contains an intercalate. A double can form an intercalate with at most one other double, so if $P \in \mathrm{PLA}(5)$ and Σ_P contains at least 5 doubles, then there exist three doubles which induce a partial Latin square without an intercalate, which is therefore completable.

Similarly, a result of R. Euler and Oleksik [5] implies that if $P \in \mathrm{PLS}(6)$ is incompletable and Σ_P consists of three triples, then P contains an intercalate. A triple can form an intercalate with at most 3 other triples, so if $P \in \mathrm{PLA}(6)$ and Σ_P contains at least 9 triples, then there exist three triples which induce a partial Latin square without an intercalate, which is therefore completable.

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