AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **83(1)** (2022), Pages 109–128

Optimal orientations of vertex-multiplications of Cartesian products of graphs

H.W. Willie Wong* E.G. Tay

National Institute of Education
Nanyang Technological University
Singapore

Abstract

Koh and Tay proved a fundamental classification of G vertex-multiplications into three classes $\mathscr{C}_0$, $\mathscr{C}_1$ and $\mathscr{C}_2$. In this paper, we prove that vertex-multiplications of Cartesian products of graphs $G \times H$ lie in $\mathscr{C}_0$ (respectively, $\mathscr{C}_0 \cup \mathscr{C}_1$) if $G^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_1$), $d(G) \geq 2$ and $d(G \times H) \geq 4$, providing further support for a conjecture by Koh and Tay. We also focus on Cartesian products involving trees, paths and cycles and show that most of them lie in $\mathscr{C}_0$.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only graphs with no loops or parallel edges. For any vertices $v, x \in V(G)$, the *distance* from v to x, $d_G(v, x)$, is defined as the length of a shortest path from v to x. For $v \in V(G)$, its *eccentricity* $e_G(v)$ is defined as $e_G(v) := \max\{d_G(v, x) \mid x \in V(G)\}$. A vertex x is called an *eccentric vertex* of v if $d_G(v, x) = e_G(v)$. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) := \max\{e_G(v) \mid v \in V(G)\}$ while the *radius* of G, denoted by $r(G)$, is defined as $r(G) := \min\{e_G(v) \mid v \in V(G)\}$. The above notions are defined similarly for a digraph D; and we refer the reader to [1] for any undefined terminology. For a digraph D, a vertex x is said to be *reachable* from another vertex v if $d_D(v, x) < \infty$. The *outset* and *inset* of a vertex $v \in V(D)$ are defined to be $O_D(v) := \{x \in V(D) \mid v \to x\}$ and $I_D(v) := \{y \in V(D) \mid y \to v\}$ respectively. If there is no ambiguity, we shall omit the subscript for the above notation.

An *orientation* D of a graph G is a digraph obtained from G by assigning a direction to every edge $e \in E(G)$. An orientation D of G is said to be *strong* if every two vertices in $V(D)$ are mutually reachable. An edge $e \in E(G)$ is a *bridge* if $G - e$ is disconnected. Robbins' well-known One-way Street Theorem [17] states that a connected graph G has a strong orientation if and only if G is bridgeless.

* Corresponding author; `williewong088@gmail.com`

ISSN: 2202-3518 ©The author(s). Released under the CC BY-ND 4.0 International License

Given a connected and bridgeless graph G, let $\mathscr{D}(G)$ be the family of strong orientations of G. The *orientation number* of G is defined as

$$\bar{d}(G) := \min\{d(D) \mid D \in \mathscr{D}(G)\}.$$

Any orientation D in $\mathscr{D}(G)$ with $d(D) = \bar{d}(G)$ is called an *optimal orientation* of G. The general problem of finding the orientation number of a connected and bridgeless graph is very difficult. Moreover, Chvátal and Thomassen [3] proved that it is NP-hard to determine whether a graph admits an orientation of diameter 2. Hence, it is natural to focus on special classes of graphs. The orientation number was evaluated for various classes of graphs, such as the complete graphs [2, 14, 16] and complete bipartite graphs [4, 19].

In 2000, Koh and Tay [11] introduced a new family of graphs, G vertex-multiplications, and extended the results on the orientation number of complete n-partite graphs. Let G be a given connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any sequence of n positive integers (s_i), a G *vertex-multiplication*, denoted by $G(s_1, s_2, \ldots, s_n)$, is the graph with vertex set $V^* = \bigcup_{i=1}^{n} V_i$ and edge set E^*, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, for $i = 1, 2, \ldots, n$; and for any $u, v \in V^*$, $uv \in E^*$ if and only if $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $v_iv_j \in E(G)$. For instance, if $G \cong K_n$, then the graph $G(s_1, s_2, \ldots, s_n)$ is a complete n-partite graph with partite sizes $s_1, s_2, \ldots, s_n$. Also, we say G is a *parent graph* of a graph H if $H \cong G(s_1, s_2, \ldots, s_n)$ for some sequence (s_i) of positive integers.

For $i = 1, 2, \ldots, n$, we denote the k-th vertex in V_i by (k, v_i), i.e. $V_i = \{(k, v_i) \mid k = 1, 2, \ldots, s_i\}$. Hence, two vertices (k, v_i) and (l, v_j) in V^* are adjacent in $G(s_1, s_2, \ldots, s_n)$ if and only if $i \neq j$ and $v_iv_j \in E(G)$. We will loosely use the two notations of a vertex, for example, if $v_i = j$, then $v_i = v_j$ and $s_i = s_j$. For convenience, we write $G^{(s)}$ in place of $G(s, s, \ldots, s)$ for any positive integer s, and it is understood that the number of s is equal to the order n of G. Thus, $G^{(1)}$ is simply the graph G itself.

G vertex-multiplications are a natural generalisation of complete multipartite graphs. Optimal orientations minimising the diameter can also be used to solve a variant of the Gossip Problem on a graph G. The Gossip Problem attributed to Boyd by Hajnal et al. [6] is stated as follows:

> "There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?"

The Problem has been the source of many papers that have studied the spread of information by telephone calls, conference calls, letters and computer networks. One can imagine a network of people modelled by a G vertex-multiplication where the parent graph is G and persons within a partite set are not allowed to communicate directly with each other, for perhaps secrecy or disease containment reasons.

The following theorem by Koh and Tay [11] provides a fundamental classification on G vertex-multiplications.

Theorem 1.1. (Koh and Tay [11]) *Let G be a connected graph of order $n \geq 3$. If $s_i \geq 2$ for $i = 1, 2, \ldots, n$, then $d(G) \leq \bar{d}(G(s_1, s_2, \ldots, s_n)) \leq d(G) + 2$.*

In view of Theorem 1.1, all graphs of the form $G(s_1, s_2, \ldots, s_n)$, with $s_i \geq 2$ for all $i = 1, 2, \ldots, n$, can be classified into three classes $\mathscr{C}_j$, where

$$\mathscr{C}_j = \{G(s_1, s_2, \ldots, s_n) \mid \bar{d}(G(s_1, s_2, \ldots, s_n)) = d(G) + j\},$$

for $j = 0, 1, 2$. Henceforth, we assume $s_i \geq 2$ for $i = 1, 2, \ldots, n$. The following lemma was found useful in proving Theorem 1.1.

Lemma 1.2. (Koh and Tay [11]) *Let s_i, t_i be integers such that $s_i \leq t_i$ for $i = 1, 2, \ldots, n$. If the graph $G(s_1, s_2, \ldots, s_n)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then $\bar{d}(G(t_1, t_2, \ldots, t_n)) \leq \max\{m, d(F)\}$.*

Koh and Tay [11] made the following conjecture and proved it for some families of graphs, including cycles.

Conjecture A. (Koh and Tay [11]) If G is a graph such that $d(G) \geq 3$ and $s_i \geq 2$ for $i = 1, 2, \ldots, n$, then $G(s_1, s_2, \ldots, s_n) \notin \mathscr{C}_2$.

These results and conjecture were generalised to digraphs by Gutin et al. [5]. Ng and Koh [15] examined cycle vertex-multiplications and Koh and Tay [13] investigated tree vertex-multiplications. Since trees with diameter at most 2 are parent graphs of complete bipartite graphs and are completely solved, they considered trees of diameter at least 3 and proved Conjecture A for trees.

Theorem 1.3. (Koh and Tay [13])
If T is a tree of order n and $3 \leq d(T) \leq 5$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

Theorem 1.4. (Koh and Tay [13])
If T is a tree of order n and $d(T) \geq 6$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

In [20], Wong and Tay proved a characterisation for vertex-multiplications of trees with diameter 5 in $\mathscr{C}_0$ and $\mathscr{C}_1$. In [21], they almost completely characterised vertex-multiplications of trees with diameter 4.

In this paper, we examine vertex-multiplications of Cartesian products of graphs and provide further support for Conjecture A. Our main approach is a nimble application of Lemma 1.2 via some elementary orientations (see Definition 2.1) leveraging on the neat structure enjoyed by Cartesian products of graphs. We also focus on Cartesian products involving trees, paths and cycles and show that most of them lie in $\mathscr{C}_0$. The Cartesian product of two graphs G and H is denoted by $G \times H$, where $V(G \times H) = \{\langle u, x\rangle \mid u \in V(G), x \in V(H)\}$ and $E(G) = \{\langle u, x\rangle\langle v, y\rangle \mid u = v$ and $xy \in E(H)$, or $uv \in E(G)$ and $x = y\}$. Since Cartesian products of disconnected graphs are disconnected, we concern ourselves with only connected graphs. We shall denote a path (respectively, cycle, complete graph) of order n as P_n (respectively, C_n, K_n) while T_d represents a tree of diameter d. Since the orientation number of complete bipartite graphs $K(p, q)$ has been characterised by Šoltés [19] and Gutin [4] and $P_2 \times P_2 \cong K(2, 2)$, we shall exclude $P_2 \times P_2$ from our discussion. In Section 2, we consider Cartesian products of graphs in the general setting.

Theorem 1.5. *Let G and H be connected graphs with order at least 2. If $d(G) \geq 2$ and $G^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_1$), then $(G \times H)^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_0 \cup \mathscr{C}_1$).*

Corollary 1.6. *Let G and H be connected graphs with order at least 2. If $d(G\times H) \geq 4$, $d(G) \geq 2$ and $G^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_1$), then $(G \times H)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$ (respectively, $\mathscr{C}_0 \cup \mathscr{C}_1$).*

In Section 3, we prove that the vertex-multiplications of Cartesian products of two trees are mostly in $\mathscr{C}_0$.

Theorem 1.7. *If $\lambda \geq 2$ and $\mu \geq 3$, then $(T_\lambda \times T_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.*

For trees with diameter 2, the same conclusion holds if both trees are paths, i.e. $P_3 \times P_3$.

Theorem 1.8.
(a) $(P_3 \times P_2)^{(2)} \in \mathscr{C}_1$.
(b)

$$(P_\lambda \times P_\mu)(s_1, s_2, \ldots, s_n) \in \begin{cases} \mathscr{C}_0, & \text{if } \lambda \geq 4, \mu = 2, \text{ or } \lambda \geq \mu \geq 3, \\ \mathscr{C}_0 \cup \mathscr{C}_1, & \text{if } (\lambda, \mu) = (3, 2). \end{cases}$$

We also prove an analogue on the hypercube graph $Q_\lambda = \overbrace{K_2 \times K_2 \times \cdots \times K_2}^{\lambda}$, $\lambda \in \mathbb{Z}^+$.

Proposition 1.9.
(a) $Q_3^{(2)} \in \mathscr{C}_0$ and $Q_3(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.
(b) For $\lambda \geq 4$, $Q_\lambda(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

In Sections 4 and 5, we examine the Cartesian products of a tree and a cycle, and two cycles respectively.

Theorem 1.10. *If $\lambda \geq 2$ and $\mu \geq 4$ or $\lambda = \mu = 3$, then $(T_\lambda \times C_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.*

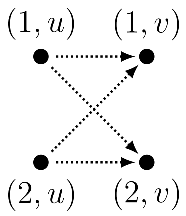
Theorem 1.11.
(a) For $\lambda \geq 4, \mu \geq 3$, $(C_\lambda \times C_\mu)^{(2)} \in \mathscr{C}_0$.
(b)

$$(C_\lambda \times C_\mu)(s_1, s_2, \ldots, s_n) \in \begin{cases} \mathscr{C}_0, & \text{if } \lambda \geq \mu \geq 4, \\ \mathscr{C}_0 \cup \mathscr{C}_1, & \text{if } (\lambda, \mu) = (3, 3) \text{ or } (4, 3). \end{cases}$$

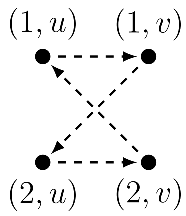
2 General results

In defining an orientation, we use the following notation to write succinctly. For any orientation D, $\tilde{D}$ denotes the orientation satisfying $u \to v$ in $\tilde{D}$ if and only if $v \to u$ in D.

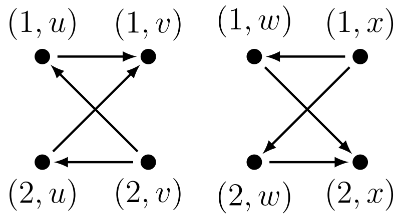
Definition 2.1. Suppose uv and wx are edges of a graph G and D is an orientation of $G^{(2)}$. We denote
(a) $u \rightrightarrows v$ if $\{(1,u),(2,u)\} \to \{(1,v),(2,v)\}$ in D (see Figure 1(a)).
(b) $u \rightsquigarrow v$ if $(1,u) \to (1,v) \to (2,u) \to (2,v) \to (1,u)$ in D (see Figure 1(b)).
(c) $u \overset{1}{\twoheadrightarrow} v$ if $(2,v) \to \{(1,u),(2,u)\} \to (1,v)$ and $w \overset{2}{\twoheadrightarrow} x$ if $(1,x) \to \{(1,w),(2,w)\} \to (2,x)$ in D (see Figure 1(c)).



(a) $u \rightrightarrows v$.



(b) $u \rightsquigarrow v$.



(c) $u \overset{1}{\twoheadrightarrow} v$ and $w \overset{2}{\twoheadrightarrow} x$.

Figure 1: Notation for orientations.

Proof of Theorem 1.5: Since $G^{(2)} \in \mathscr{C}_0$, there exists an orientation D of $G^{(2)}$ such that $d(D) = d(G)$. Define an orientation D^* of $(G \times H)^{(2)}$ as follows:
For any $u, v \in V(G)$, any $x \in V(H)$ and any $p, q = 1, 2$,

$$(p, \langle u, x\rangle) \to (q, \langle v, x\rangle) \iff (p, u) \to (q, v) \text{ in } D, \tag{2.1}$$

i.e. each copy of $G^{(2)}$ is oriented similarly to D.
For any $u \in V(G)$ and any $x, y \in V(H)$,

$$\langle u, x\rangle \rightsquigarrow \langle u, y\rangle \iff xy \in E(H). \tag{2.2}$$

We remark that the definition in (2.2) is arbitrary since $xy \in E(H)$ is equivalent to $yx \in E(H)$. However, this does not affect the following argument. For a well-defined orientation, one may linearly order the vertices in $V(H)$ before applying (2.2) with the condition x precedes y.

We claim that $d_{D^*}((p, \langle u, x\rangle), (q, \langle v, y\rangle)) \le d(G \times H) = d(G) + d(H)$ for $p, q = 1, 2$. If $x = y$, then by (2.1), $d_{D^*}((p, \langle u, x\rangle), (q, \langle v, x\rangle)) \le d(D) = d(G) < d(G) + d(H)$.

Suppose $x \neq y$. In view of (2.2), there exists a $(p, \langle u, x\rangle) - (r, \langle u, y\rangle)$ path of length $d_{D^*}((p, \langle u, x\rangle), (r, \langle u, y\rangle)) \le d_H(x, y) \le d(H)$ for some $r = 1, 2$. If $(q, \langle v, y\rangle) =$

$(r, \langle u, y\rangle)$ (i.e. $q = r$, $u = v$), then we are done. If $(q, \langle v, y\rangle) = (3 - r, \langle u, y\rangle)$, then

$$\begin{aligned} d_{D^*}((p, \langle u, x\rangle), (q, \langle v, y\rangle)) &\le d_{D^*}((p, \langle u, x\rangle), (r, \langle u, y\rangle)) + 2 \\ &\le d(H) + 2 \\ &\le d(H) + d(G) \end{aligned}$$

by (2.2). Finally, if $u \neq v$, then

$$\begin{aligned} d_{D^*}((p, \langle u, x\rangle), (q, \langle v, y\rangle)) &\le d_{D^*}((p, \langle u, x\rangle), (r, \langle u, y\rangle)) + d_{D^*}((r, \langle u, y\rangle), (q, \langle v, y\rangle)) \\ &\le d(H) + d(D) \\ &= d(H) + d(G). \end{aligned}$$

The proof is similar if $G^{(2)} \in \mathscr{C}_1$. □

Proof of Corollary 1.6: Since every vertex lies in a directed C_4 in the orientation D^* because of (2.2), it follows from Lemma 1.2 that $(G \times H)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$. The proof is similar if $G^{(2)} \in \mathscr{C}_1$. □

Corollary 2.2. *For all $i \in \mathbb{Z}^+$, let G_i be a connected graph with order at least two. If $(G_1 \times G_2)^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_1$), then*
(a) for $j \geq 3$, $(\prod_{i=1}^{j} G_i)^{(2)} \in \mathscr{C}_0$ (respectively, $\mathscr{C}_0 \cup \mathscr{C}_1$), and
(b) for $k \geq 4$, $(\prod_{i=1}^{k} G_i)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$ (respectively, $\mathscr{C}_0 \cup \mathscr{C}_1$).

Proof: (a) Since $d(G_1 \times G_2) \geq 2$, the result follows from Theorem 1.5.
(b) Since $d(\prod_{i=1}^{k} G_i) \geq 4$, $d(G_1 \times G_2) \geq 2$, the result follows from Corollary 1.6. □

Corollary 2.3. *Let G be a connected graph with order at least two.*
(a) If $3 \leq d \leq 5$, then $(T_d \times G)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.
(b) If $d \geq 6$, then $(T_d \times G)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

Proof: Since $d(T_d \times G) \geq 4$ and by Corollary 1.6, (a) and (b) follow from Theorems 1.3 and 1.4 respectively. □

3 Cartesian product of trees $T_\lambda \times T_\mu$

In this section, we shall show that Corollary 2.3(a) can be further improved in the case of $T_\lambda \times T_\mu$. Before that, we introduce a notation for trees T_d with $d \leq 5$. Whenever we speak of a tree with even diameter d, we denote by $\mathtt{c}$ the unique central vertex of T_d, i.e. $e_{T_d}(\mathtt{c}) = r(T_d)$, and the neighbours of $\mathtt{c}$ by $[i]$, i.e. $N_{T_d}(\mathtt{c}) = \{[i] \mid i = 1, 2, \ldots, \deg_{T_d}(\mathtt{c})\}$. For each $i = 1, 2, \ldots, \deg_{T_d}(\mathtt{c})$, we further denote the neighbours of $[i]$, excluding $\mathtt{c}$, by $[\alpha, i]$, i.e. $N_{T_d}([i]) - \{\mathtt{c}\} = \{[\alpha, i] \mid \alpha = 1, 2, \ldots, \deg_{T_d}([i]) - 1\}$.

If d is odd, we let c_1 and c_2 be the two central vertices of T_d, i.e. $e_{T_d}(\mathsf{c}_k) = r(T_d)$ for $k = 1, 2$. For $k = 1, 2$, denote the neighbours of c_k, excluding c_{3-k}, by $[i]_k$. i.e. $N_{T_d}(\mathsf{c}_k) - \{\mathsf{c}_{3-k}\} = \{[i]_k \mid i = 1, 2, \ldots, \deg_{T_d}(\mathsf{c}_k) - 1\}$. For each $i = 1, 2, \ldots, \deg_{T_d}(\mathsf{c}_k) - 1$, we denote the neighbours of $[i]_k$, excluding c_k, by $[\alpha, i]_k$. i.e. $N_{T_d}([i]_k) - \{\mathsf{c}_k\} = \{[\alpha, i]_k \mid \alpha = 1, 2, \ldots, \deg_{T_d}([i]_k) - 1\}$. Figures 2 and 3 illustrate the use of this notation.

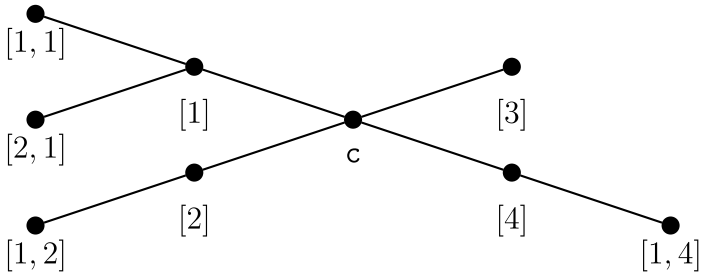


Figure 2: Labelling vertices in a T_4

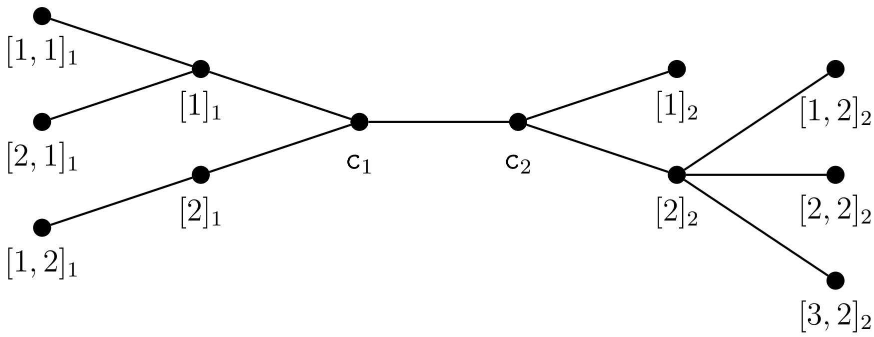


Figure 3: Labelling vertices in a T_5

With this, we prove Theorem 1.7.

Proof of Theorem 1.7: Let $G := T_\lambda \times T_\mu$. By Corollary 2.3(b), it suffices to consider $\lambda, \mu \leq 5$. Define an orientation $D_{(\lambda,\mu)}$ for $G^{(2)}$ as follows:

Case 1. λ is even and μ is odd, i.e. $\lambda = 2, 4$ and $\mu = 3, 5$.

$$\langle \mathsf{c}, \mathsf{c}_2 \rangle \rightrightarrows \langle [y], \mathsf{c}_2 \rangle \rightrightarrows \langle [y], \mathsf{c}_1 \rangle \rightrightarrows \langle \mathsf{c}, \mathsf{c}_1 \rangle \rightrightarrows \langle \mathsf{c}, \mathsf{c}_2 \rangle \tag{3.1}$$

for all $[y] \in N_{T_\lambda}(\mathsf{c})$. Excluding the edges defined above, for each $[i]_1 \in N_{T_\mu}(\mathsf{c}_1) - \{\mathsf{c}_2\}$, each $\alpha = 1, 2, \ldots, \deg_{T_\mu}([i]_1) - 1$, each $[j]_2 \in N_{T_\mu}(\mathsf{c}_2) - \{\mathsf{c}_1\}$, and each $\beta = 1, 2, \ldots, \deg_{T_\mu}([j]_2) - 1$,

$$\langle x, [\alpha, i]_1 \rangle \rightsquigarrow \langle x, [i]_1 \rangle \rightsquigarrow \langle x, \mathsf{c}_1 \rangle, \langle x, [\beta, j]_2 \rangle \rightsquigarrow \langle x, [j]_2 \rangle \rightsquigarrow \langle x, \mathsf{c}_2 \rangle \rightsquigarrow \langle x, \mathsf{c}_1 \rangle, \tag{3.2}$$

for all $x \in V(T_\lambda)$, and

$$\left.\begin{array}{l}\langle[\gamma,k],[\alpha,i]_1\rangle \rightsquigarrow \langle[k],[\alpha,i]_1\rangle \rightsquigarrow \langle \mathsf{c},[\alpha,i]_1\rangle,\\ \langle[\gamma,k],[i]_1\rangle \rightsquigarrow \langle[k],[i]_1\rangle \rightsquigarrow \langle \mathsf{c},[i]_1\rangle,\\ \langle[\gamma,k],[\beta,j]_2\rangle \rightsquigarrow \langle[k],[\beta,j]_2\rangle \rightsquigarrow \langle \mathsf{c},[\beta,j]_2\rangle,\\ \langle[\gamma,k],[j]_2\rangle \rightsquigarrow \langle[k],[j]_2\rangle \rightsquigarrow \langle \mathsf{c},[j]_2\rangle,\\ \langle[\gamma,k],\mathsf{c}_t\rangle \rightsquigarrow \langle[k],\mathsf{c}_t\rangle \text{ for } t=1,2,\end{array}\right\} \tag{3.3}$$

for all $[k] \in N_{T_\lambda}(\mathsf{c})$ and $\gamma = 1, 2, \dots, \deg_{T_\lambda}([k])-1$. (See Figure 4 when $G = T_4 \times T_5$.)

We claim that $d(D_{(\lambda,\mu)}) = \lambda + \mu = d(G)$. Let $u, v \in V(G)$ and $P := w_0 w_1 \dots w_l$ be a shortest $u - v$ path in G with $u = w_0$ and $v = w_l$. If $d_G(u, v) \le d(G) - 2$ and P satisfies

$$w_i \rightsquigarrow w_{i+1} \text{ or } w_{i+1} \rightsquigarrow w_i \text{ for all } i = 0, 1, \dots, l-1, \tag{3.4}$$

then $d_{D_{(\lambda,\mu)}}((p, u), (q, v)) \le d_G(u, v) + 2 \le d(G)$ for $p, q = 1, 2$. In particular, this holds for $u = \langle[\gamma_1, k], y_1\rangle$, $v = \langle[\gamma_2, k], y_2\rangle$ with $\gamma_1 \ne \gamma_2$ in $T_4 \times T_\mu$. So, by symmetry of (3.1)-(3.3), we may assume without loss of generality that c has two eccentric vertices in T_λ, i.e. $T_\lambda = P_3$ if $\lambda = 2$, and $T_\lambda = P_5$ if $\lambda = 4$. Furthermore, by symmetry of (3.2), we may assume c_i has two eccentric vertices for $i = 1, 2$, in T_μ.

For the pairs of u, v that do not satisfy (3.4), we claim that there exists a path P with length at most $d(G)$ that satisfies

$$w_i \rightrightarrows w_{i+1} \text{ for some } i = 0, 1, \dots, l-1 \text{ and } w_{j+1} \rightrightarrows w_j \text{ for none of } j = 0, 1, \dots, l-1.$$

Then, we can conclude $d_{D_{(\lambda,\mu)}}((p, u), (q, v)) \le d(G)$ for $p, q = 1, 2$.

First, consider $(\lambda, \mu) = (2, 3)$. We list these paths P while omitting symmetric scenarios. For $i = 1, 2$, and $j = 1, 2$,

$$\begin{aligned}
P^1 &= \langle[1],[1]_1\rangle\langle[1],\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle\langle\mathsf{c},[j]_1\rangle\langle[2],[j]_1\rangle.\\
P^2 &= \langle[1],[1]_1\rangle\langle[1],\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[i],\mathsf{c}_2\rangle\langle[i],[j]_2\rangle.\\
P^3 &= \langle[1],[1]_1\rangle\langle[1],\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle\mathsf{c},[j]_2\rangle.\\
P^4 &= \langle[1],[1]_2\rangle\langle[1],\mathsf{c}_2\rangle\langle[1],\mathsf{c}_1\rangle\langle[1],[j]_1\rangle\langle\mathsf{c},[j]_1\rangle\langle[2],[j]_1\rangle.\\
P^5 &= \langle[1],[1]_2\rangle\langle\mathsf{c},[1]_2\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[2],\mathsf{c}_2\rangle\langle[2],\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle.\\
P^6 &= \langle[1],[1]_2\rangle\langle\mathsf{c},[1]_2\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[2],\mathsf{c}_2\rangle\langle[2],[j]_2\rangle.\\
P^7 &= \langle\mathsf{c},[1]_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[i],\mathsf{c}_2\rangle\langle[i],[j]_2\rangle\langle\mathsf{c},[j]_2\rangle.\\
P^8 &= \langle\mathsf{c},[1]_2\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[i],\mathsf{c}_2\rangle\langle[i],\mathsf{c}_1\rangle\langle[i],[j]_1\rangle\langle\mathsf{c},[j]_1\rangle.\\
P^9 &= \langle\mathsf{c},[1]_2\rangle\langle\mathsf{c},\mathsf{c}_2\rangle\langle[i],\mathsf{c}_2\rangle\langle[i],\mathsf{c}_1\rangle\langle\mathsf{c},\mathsf{c}_1\rangle.
\end{aligned}$$

To prove for $(\lambda, \mu) = (2, 5)$, note that $D_{(2,3)}$ is a subdigraph of $D_{(2,5)}$. Moveover, for any $(p, u) \in V(D_{(2,5)}) - V(D_{(2,3)})$, there exists a vertex $(r, x) \in V(D_{(2,3)})$ such that $u \rightsquigarrow x$ or $x \rightsquigarrow u$. Hence, if $(p, u) \in V(D_{(2,5)}) - V(D_{(2,3)})$ and $(q, v) \in V(D_{(2,3)})$, then $\max\{d_{D_{(2,5)}}((p, u), (q, v)),\ d_{D_{(2,5)}}((q, v),\ (p, u))\} \le 1 + d(D_{(2,3)}) \le 7$ for $p, q = 1, 2$. Similarly, if $(p, u), (q, v) \in V(D_{(2,5)}) - V(D_{(2,3)})$, then $d_{D_{(2,5)}}((p, u), (q, v)) \le 2 + d(D_{(2,3)}) \le 7$ for $p, q = 1, 2$.

A similar argument can be made for $(\lambda, \mu) = (4, 3)$ since $D_{(2,3)}$ is a subdigraph of $D_{(4,3)}$; and thereafter for $(\lambda, \mu) = (4, 5)$ since $D_{(4,3)}$ is a subdigraph of $D_{(4,5)}$.

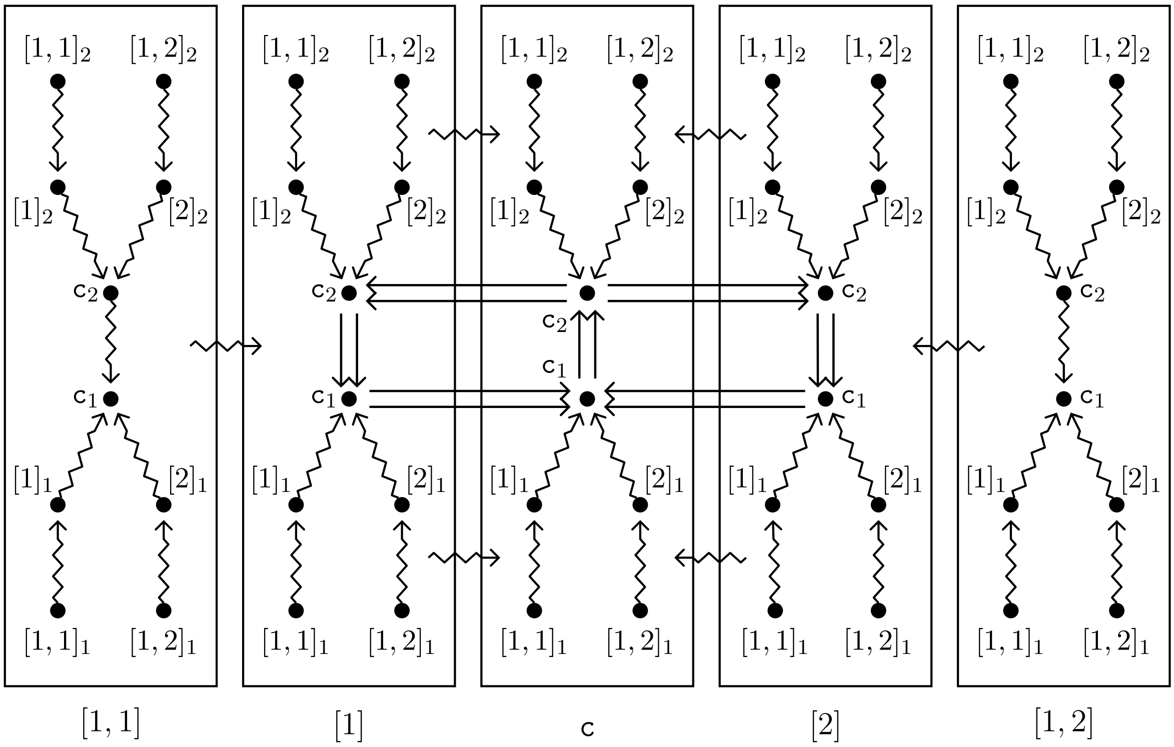


Figure 4: Orientation $D_{(4,5)}$ of $(T_4 \times T_5)^{(2)}$, where $d(D_{(4,5)}) = 9$.

Note: For $u = [1,1], [1], \mathsf{c}, [2], [1,2]$, the u-copy of T_5 is contained in a rectangle and the vertex $\langle u, x\rangle$ is simply labelled as x for clarity. For example, the bottom leftmost vertex is $\langle [1,1], [1,1]_1 \rangle$.

We can use a similar strategy as in Case 1 to prove $d(D_{(\lambda,\mu)}) = \lambda + \mu = d(G)$ for Cases 2 and 3. Hence, we state only the orientations and refer the interested reader to [22].

Case 2. λ and μ are both even, i.e. $\lambda = 2, 4$ and $\mu = 4$.

For each $[i] \in N_{T_\lambda}(\mathsf{c})$, and each $\alpha = 1, 2, \ldots, \deg_{T_\lambda}([i]) - 1$ and each $[j] \in N_{T_\mu}(\mathsf{c})$, and each $\beta = 1, 2, \ldots, \deg_{T_\mu}([j]) - 1$,

$$\langle [i], \mathsf{c}\rangle \rightrightarrows \langle \mathsf{c}, \mathsf{c}\rangle \text{ and } \langle \mathsf{c}, [j]\rangle \rightrightarrows \langle [i], [j]\rangle;$$

excluding the edges defined above,

$$\langle [\alpha, i], y\rangle \rightsquigarrow \langle [i], y\rangle \rightsquigarrow \langle \mathsf{c}, y\rangle$$

for all $y \in V(T_\mu)$, and

$$\langle x, [\beta, j]\rangle \rightsquigarrow \langle x, [j]\rangle \rightsquigarrow \langle x, \mathsf{c}\rangle$$

for all $x \in V(T_\lambda)$.

Case 3. λ and μ are both odd, i.e. $\lambda, \mu = 3, 5$.

For each $[i]_1 \in N_{T_\mu}(\mathsf{c}_1) - \{\mathsf{c}_2\}$, each $\alpha = 1, 2, \ldots, \deg_{T_\mu}([i]_1) - 1$, each $[j]_2 \in N_{T_\mu}(\mathsf{c}_2) - \{\mathsf{c}_1\}$, each $\beta = 1, 2, \ldots, \deg_{T_\mu}([j]_2) - 1$,

$$\langle \mathsf{c}_1, [i]_1 \rangle \rightrightarrows \langle \mathsf{c}_1, \mathsf{c}_1 \rangle \rightrightarrows \langle \mathsf{c}_2, \mathsf{c}_1 \rangle \rightrightarrows \langle \mathsf{c}_2, [i]_1 \rangle \rightrightarrows \langle \mathsf{c}_1, [i]_1 \rangle,$$
$$\langle \mathsf{c}_1, [j]_2 \rangle \rightrightarrows \langle \mathsf{c}_1, \mathsf{c}_2 \rangle \rightrightarrows \langle \mathsf{c}_2, \mathsf{c}_2 \rangle \rightrightarrows \langle \mathsf{c}_2, [j]_2 \rangle \rightrightarrows \langle \mathsf{c}_1, [j]_2 \rangle;$$

excluding the edges defined above,

$$\langle x, [\alpha, i]_1 \rangle \rightsquigarrow \langle x, [i]_1 \rangle \rightsquigarrow \langle x, \mathsf{c}_1 \rangle, \langle x, [\beta, j]_2 \rangle \rightsquigarrow \langle x, [j]_2 \rangle \rightsquigarrow \langle x, \mathsf{c}_2 \rangle \rightsquigarrow \langle x, \mathsf{c}_1 \rangle,$$

for all $x \in V(T_\lambda)$, and

$$\langle [\gamma, k]_1, y \rangle \rightsquigarrow \langle [k]_1, y \rangle \rightsquigarrow \langle \mathsf{c}_1, y \rangle, \langle [\theta, l]_2, y \rangle \rightsquigarrow \langle [l]_2, y \rangle \rightsquigarrow \langle \mathsf{c}_2, y \rangle \rightsquigarrow \langle \mathsf{c}_1, y \rangle,$$

for each $[k]_1 \in N_{T_\lambda}(\mathsf{c}_1) - \{\mathsf{c}_2\}$, each $\gamma = 1, 2, \ldots, \deg_{T_\lambda}([k]_1) - 1$, each $[l]_2 \in N_{T_\lambda}(\mathsf{c}_2) - \{\mathsf{c}_1\}$, each $\theta = 1, 2, \ldots, \deg_{T_\lambda}([l]_2) - 1$ and each $y \in V(T_\mu)$. $\square$

Next, we shall prove two lemmas for the investigation of the rectangular grid $P_\lambda \times P_\mu$. For P_n (respectively, C_n), we shall use the natural labelling of vertices where $E(P_n) = \{(i, i+1) \mid i = 1, 2, \ldots, n-1\}$ (respectively, $E(C_n) = E(P_n) \cup \{(n, 1)\}$).

Lemma 3.1. *Let G be a graph and D be an orientation of $G^{(2)}$. If $u_0u_1u_2$ is a unique shortest $u_0 - u_2$ path in G and $d_D((p, u_0), (q, u_2)) = d_D((p, u_2), (q, u_0)) = 2$ for all $p, q = 1, 2$, then $u_0 \overset{1}{\twoheadrightarrow} u_1 \overset{2}{\twoheadleftarrow} u_2$ or $u_0 \overset{2}{\twoheadrightarrow} u_1 \overset{1}{\twoheadleftarrow} u_2$.*

Proof: Suppose $(1, u_1) \to (1, u_2)$. Now, for $p = 1, 2$, since $d_D((1, u_2), (p, u_0)) = 2$, it follows that $(1, u_2) \to (2, u_1) \to (p, u_0)$. Since $d_D((p, u_0), (q, u_2)) = 2$ for $p, q = 1, 2$, it follows that $(p, u_0) \to (1, u_1) \to (q, u_2)$ must hold. It is now necessary from $d_D((2, u_2), (1, u_0)) = 2$ that $(2, u_2) \to (2, u_1)$. Thus, $u_0 \overset{1}{\twoheadrightarrow} u_1 \overset{2}{\twoheadleftarrow} u_2$. Similarly, an argument reversing all arcs will give $u_0 \overset{2}{\twoheadrightarrow} u_1 \overset{1}{\twoheadleftarrow} u_2$ if $(1, u_2) \to (1, u_1)$. $\square$

Lemma 3.2. *Let G be a graph and D be an orientation of $G^{(2)}$. Suppose $v_0v_1 \ldots v_k$, $k \geq 2$, is a shortest $v_0 - v_k$ path of length k in G and D satisfies*

(a) $v_i \overset{1}{\twoheadrightarrow} v_{i+1} \overset{2}{\twoheadleftarrow} v_{i+2}$ for some i, $0 \leq i \leq k-2$, and

(b) if $j \notin \{i, i+1\}$, then either $v_j \rightsquigarrow v_{j+1}$ or $v_{j+1} \rightsquigarrow v_j$.

Then, $d_D((p, v_0), (q, v_k)) = d_D((p, v_k), (q, v_0)) = k$ for $p, q = 1, 2$.

Proof: Assume $v_j \rightsquigarrow v_{j+1}$ for all $j \notin \{i, i+1\}$; the proof is similar otherwise. Note that $(p, v_0) \to (p, v_1) \to \cdots \to (p, v_i)$, $\{(1, v_i), (2, v_i)\} \to (1, v_{i+1}) \to \{(1, v_{i+2}), (2, v_{i+2})\}$ and $(p, v_{i+2}) \to (p, v_{i+3}) \to \cdots \to (p, v_k)$ for all $p = 1, 2$. Thus, for $p, q = 1, 2$, $d_D((p, v_0), (q, v_k)) = d_G(v_0, v_k) = k$. By symmetry, we have $d_D((p, v_k), (q, v_0)) = d_G(v_k, v_0) = k$ for $p, q = 1, 2$. $\square$

Proof of Theorem 1.8: Let $G := P_\lambda \times P_\mu$ and note that $d(G) = \lambda + \mu - 2$.

Case 1. $\lambda = 3$ and $\mu = 2$.

We first prove $\bar{d}(G^{(2)}) = 4$. Suppose there exists an orientation D of $G^{(2)}$ such that $d(D) = 3$. Since $d_D((p, \langle 1, 2\rangle), (q, \langle 3, 2\rangle)) = d_D((q, \langle 3, 2\rangle), (p, \langle 1, 2\rangle)) = 2$ for all $p, q = 1, 2$, we may assume from Lemma 3.1 that $\langle 1, 2\rangle \overset{1}{\twoheadrightarrow} \langle 2, 2\rangle \overset{2}{\twoheadleftarrow} \langle 3, 2\rangle$. Similarly, we assume $\langle 1, 1\rangle \overset{1}{\twoheadrightarrow} \langle 2, 1\rangle \overset{2}{\twoheadleftarrow} \langle 3, 1\rangle$ (the case $\langle 1, 1\rangle \overset{2}{\twoheadrightarrow} \langle 2, 1\rangle \overset{1}{\twoheadleftarrow} \langle 3, 1\rangle$ is similar). Since $d_D((1, \langle 1, 1\rangle), (2, \langle 2, 2\rangle)) \leq 3$, it follows that $(1, \langle 2, 1\rangle) \to (2, \langle 2, 2\rangle)$. However, we have $d_D((1, \langle 3, 2\rangle), (1, \langle 2, 1\rangle)) > 3$, which contradicts $d(D) = 3$. Hence, $\bar{d}(G^{(2)}) \geq 4$.

Define an orientation $D_{(3,2)}$ for $G^{(2)}$ as follows:

$$\langle 1, j\rangle \overset{1}{\twoheadrightarrow} \langle 2, j\rangle \overset{2}{\twoheadleftarrow} \langle 3, j\rangle \text{ for } j = 1, 2, \text{ and } \langle i, 1\rangle \rightsquigarrow \langle i, 2\rangle \text{ for } i = 1, 2, 3.$$

It is easy to verify $d(D_{(3,2)}) = 4$. Hence, $G^{(2)} \in \mathscr{C}_1$ and we are done for (a).

Case 2. $\lambda \geq 4$ and $\mu = 2$.

Define an orientation $D_{(\lambda,2)}$ for $G^{(2)}$ as follows: (See Figure 5 when $\lambda = 4$.)

$$\begin{aligned}
&\langle 1, 2\rangle \overset{1}{\twoheadrightarrow} \langle 2, 2\rangle \overset{2}{\twoheadleftarrow} \langle 3, 2\rangle,\ \langle \lambda - 2, 1\rangle \overset{1}{\twoheadrightarrow} \langle \lambda - 1, 1\rangle \overset{2}{\twoheadleftarrow} \langle \lambda, 1\rangle,\\
&\langle i, 1\rangle \rightsquigarrow \langle i, 2\rangle \text{ for } i = 1, 2, \ldots, \lambda,\\
&\langle j, 1\rangle \rightsquigarrow \langle j + 1, 1\rangle \text{ for } j = 1, 2, 3, \ldots, \lambda - 3, \text{ and}\\
&\langle k, 2\rangle \rightsquigarrow \langle k + 1, 2\rangle \text{ for } k = 3, 4, \ldots, \lambda - 1.
\end{aligned}$$

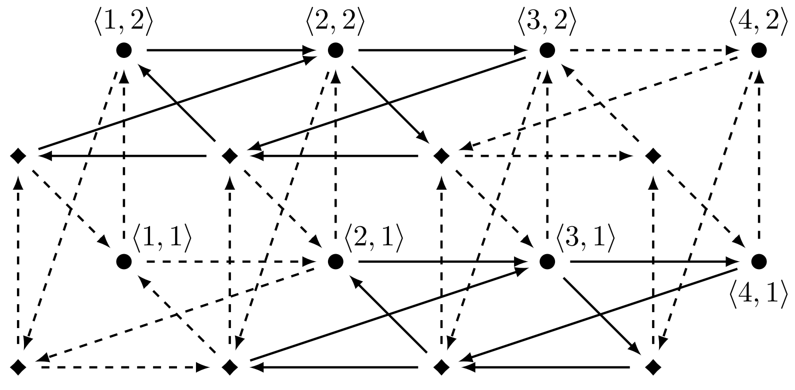


Figure 5: Orientation $D_{(4,2)}$ of $(P_4 \times P_2)^{(2)}$, where $d(D_{(4,2)}) = 4$.

Note: The vertices $(p, \langle u, x\rangle)$, for $p = 1, 2$, are represented by • and ⧫ respectively. The vertex $(1, \langle u, x\rangle)$ is simply labelled as $\langle u, x\rangle$ for clarity. For example, the bottom leftmost • and ⧫ are $(1, \langle 1, 1\rangle)$ and $(2, \langle 1, 1\rangle)$ respectively. The same simplification is used for Figures 6 and 7.

We claim that $d(D_{(\lambda,2)}) = d(G)$. Let $u, v \in V(G)$, where $d_G(u, v) \leq d(G) - 2$. By the definition of $D_{(\lambda,2)}$, we have $d_{D_{(\lambda,2)}}((p, u), (q, v)) \leq d_G(u, v) + 2 \leq d(G)$ for $p, q =$

$1, 2$. Hence, it suffices to consider vertices $u, v \in V(G)$, where $d_G(u, v) = d(G) - 1$ or $d_G(u, v) = d(G)$. We illustrate this for u being the 'top left' and v being the 'bottom right' vertices in Figure 5 and the other cases can be proved similarly. That is, for $(u, v) = (\langle 1, 2\rangle, \langle \lambda - 1, 1\rangle), (\langle 1, 2\rangle, \langle \lambda, 2\rangle), (\langle 1, 2\rangle, \langle \lambda, 1\rangle), (\langle 2, 2\rangle, \langle \lambda, 1\rangle)$, the claim follows by invoking Lemma 3.2 on their respective shortest paths:

$$\begin{aligned}
&P^1 = \langle 1, 2\rangle\langle 2, 2\rangle\langle 3, 2\rangle \ldots \langle \lambda - 1, 2\rangle\langle \lambda - 1, 1\rangle.\\
&P^2 = \langle 1, 2\rangle\langle 2, 2\rangle\langle 3, 2\rangle \ldots \langle \lambda - 1, 2\rangle\langle \lambda, 2\rangle.\\
&P^3 = P^2 \text{ with } \langle \lambda, 1\rangle.\\
&P^4 = \langle 2, 2\rangle\langle 2, 1\rangle\langle 3, 1\rangle \ldots \langle \lambda - 2, 1\rangle\langle \lambda - 1, 1\rangle\langle \lambda, 1\rangle.
\end{aligned}$$

Case 3. $\lambda \geq \mu \geq 3$.

Define an orientation $D_{(\lambda,\mu)}$ for $G^{(2)}$ as follows:

$$\begin{aligned}
&\left\langle \left\lceil \frac{\lambda}{2} \right\rceil - 1, \left\lceil \frac{\mu}{2} \right\rceil \right\rangle \overset{1}{\twoheadrightarrow} \left\langle \left\lceil \frac{\lambda}{2} \right\rceil, \left\lceil \frac{\mu}{2} \right\rceil \right\rangle \overset{2}{\twoheadleftarrow} \left\langle \left\lceil \frac{\lambda}{2} \right\rceil + 1, \left\lceil \frac{\mu}{2} \right\rceil \right\rangle \text{ and}\\
&\left\langle \left\lceil \frac{\lambda}{2} \right\rceil, \left\lceil \frac{\mu}{2} \right\rceil - 1 \right\rangle \overset{1}{\twoheadrightarrow} \left\langle \left\lceil \frac{\lambda}{2} \right\rceil, \left\lceil \frac{\mu}{2} \right\rceil \right\rangle \overset{2}{\twoheadleftarrow} \left\langle \left\lceil \frac{\lambda}{2} \right\rceil, \left\lceil \frac{\mu}{2} \right\rceil + 1 \right\rangle.
\end{aligned}$$

Except for the edges defined above,

$$\langle i, j\rangle \rightsquigarrow \langle i + 1, j\rangle, \text{ and } \langle i, j\rangle \rightsquigarrow \langle i, j + 1\rangle \text{ for all } 1 \leq i \leq \lambda - 1 \text{ and } 1 \leq j \leq \mu - 1.$$

Similar to Case 2, it can be proved that $d(D_{(\lambda,\mu)}) = d(G)$. Hence, $G^{(2)} \in \mathscr{C}_0$ for Cases 2 and 3. To complete (b), observe that every vertex lies in a directed C_4 in each orientation $D_{(\lambda,\mu)}$ of all three cases and invoke Lemma 1.2. $\square$

We end the section with a result on the hypercube graph.

Proof of Proposition 1.9: We shall prove $\bar{d}(Q_3^{(2)}) = 3 = d(Q_3)$. Denote the vertices of the two disjoint copies of C_4 in Q_3 by $1, 2, 3, 4$, and $5, 6, 7, 8$. Define an orientation D of $Q_3^{(2)}$ as follows:

$$\begin{aligned}
&i \rightsquigarrow i + 1 \rightsquigarrow i + 2 \rightsquigarrow i + 3 \text{ and } i \rightsquigarrow i + 3 \text{ for } i = 1, 5,\\
&4 \rightrightarrows 8, 2 \rightrightarrows 6, 5 \rightrightarrows 1, \text{ and } 7 \rightrightarrows 3.
\end{aligned}$$

It is easy to verify that $d(D) = 3$. Hence, $Q_3^{(2)} \in \mathscr{C}_0$. Now, by Theorem 1.5, $Q_\lambda^{(2)} \in \mathscr{C}_0$ for $\lambda \geq 3$. Since every vertex lies in a directed C_4, it follows from Lemma 1.2 that $Q_3(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$ and $Q_\lambda(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$ for $\lambda \geq 4$. $\square$

4 Cartesian product of trees with cycles $T_\lambda \times C_\mu$

In this section, we consider the Cartesian product of trees with cycles.

Proof of Theorem 1.10:

Case 1. $\lambda \geq 2$ and $\mu \geq 4$.

Let (V_1, V_2) be a bipartition of $V(T_\lambda)$, i.e. V_1 and V_2 are independent sets. Let F be a strong orientation of C_μ, say $1 \to 2 \to \cdots \to \mu \to 1$, and define an orientation D for $(T_\lambda \times C_\mu)^{(2)}$ as follows:

$$\langle u, x\rangle \rightrightarrows \langle u, y\rangle \iff x \to y \text{ in } F$$

for any $u \in V_1$ and any $x, y \in V(C_\mu)$, i.e. the copy $C_\mu^{(2)}$ is oriented similarly to F.

$$\langle u, x\rangle \rightrightarrows \langle u, y\rangle \iff y \to x \text{ in } F$$

for any $u \in V_2$ and any $x, y \in V(C_\mu)$, i.e. the copy $C_\mu^{(2)}$ is oriented similarly to $\tilde{F}$.

$$\langle u, x\rangle \rightsquigarrow \langle v, x\rangle$$

for any $u, v \in V(T_\lambda)$ with $uv \in E(T_\lambda)$ and any $x \in V(C_\mu)$.

We claim that $d_D((p, \langle u, x\rangle), (q, \langle v, y\rangle)) \leq \lambda + \lfloor \frac{\mu}{2} \rfloor = d(T_\lambda \times C_\mu)$ for any $\langle u, x\rangle$, $\langle v, y\rangle \in V(T_\lambda \times C_\mu)$, and $p, q = 1, 2$. Suppose $u = v \in V_1$. Note that either $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$ or $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$. So, there exist paths P and P' in D, each of length at most $\lfloor \frac{\mu}{2} \rfloor$, from $\{(1, \langle u, x\rangle), (2, \langle u, x\rangle)\}$ to $\{(1, \langle u, y\rangle), (2, \langle u, y\rangle)\}$ and from $\{(1, \langle w, x\rangle), (2, \langle w, x\rangle)\}$ to $\{(1, \langle w, y\rangle), (2, \langle w, y\rangle)\}$, where $w \in V_2$ is some vertex adjacent to u in T_λ respectively. In the former case where $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, P suffices and we are done. In the latter case where $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, we shall further assume $\langle u, x\rangle \rightsquigarrow \langle w, x\rangle$ for simplicity; the proof is similar otherwise. Then, $(p, \langle u, x\rangle)(p, \langle w, x\rangle)$, P' and $(3 - q, \langle w, y\rangle)(q, \langle u, y\rangle)$ form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $2 + \lfloor \frac{\mu}{2} \rfloor \leq \lambda + \lfloor \frac{\mu}{2} \rfloor$. A similar proof follows if $u = v \in V_2$.

Suppose $u \neq v$. Let $uw_1w_2 \ldots w_lv$ be the unique shortest $u - v$ path in T_λ. For simplicity, we shall assume $\langle u, x\rangle \rightsquigarrow \langle w_1, x\rangle \rightsquigarrow \cdots \rightsquigarrow \langle v, x\rangle$; the proof is similar otherwise. If $x = y$, then $(p, \langle u, x\rangle)(p, \langle w_1, x\rangle) \ldots (p, \langle v, x\rangle)(3-p, \langle w_l, x\rangle)(3-p, \langle v, x\rangle)$ guarantees a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $\lambda + 2 \leq \lambda + \lfloor \frac{\mu}{2} \rfloor$.

Next, suppose $x \neq y$. Futhermore, we shall assume $v \in V_1$ (and hence $w_l \in V_2$); the proof is similar if $v \in V_2$. Again, consider the cases $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$ or $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$. So, there exist paths Q and Q' in D, each of length at most $\lfloor \frac{\mu}{2} \rfloor$, from $\{(1, \langle v, x\rangle), (2, \langle v, x\rangle)\}$ to $\{(1, \langle v, y\rangle), (2, \langle v, y\rangle)\}$ and from $\{(1, \langle w_l, x\rangle), (2, \langle w_l, x\rangle)\}$ to $\{(1, \langle w_l, y\rangle), (2, \langle w_l, y\rangle)\}$ respectively. In the former case where $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, $(p, \langle u, x\rangle)(p, \langle w_1, x\rangle) \ldots (p, \langle v, x\rangle)$ and Q form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $\lambda + \lfloor \frac{\mu}{2} \rfloor$. In the latter case where $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, $(p, \langle u, x\rangle)(p, \langle w_1, x\rangle) \ldots (p, \langle w_l, x\rangle)$ with Q' and $(q, \langle w_l, y\rangle)(q, \langle v, y\rangle)$ form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ of length at most $\lambda + \lfloor \frac{\mu}{2} \rfloor$. Hence, $(T_\lambda \times C_\mu)^{(2)} \in \mathscr{C}_0$.

Case 2. $\lambda = \mu = 3$.

Define an orientation D for $(T_3 \times C_3)^{(2)}$ as follows: (See Figure 6.) For all

$[i]_1 \in N_T(\mathsf{c}_1) - \{\mathsf{c}_2\}$ and all $[j]_2 \in N_T(\mathsf{c}_2) - \{\mathsf{c}_1\}$,

$$\langle \mathsf{c}_1, 1\rangle \rightrightarrows \langle \mathsf{c}_1, 2\rangle \rightrightarrows \langle \mathsf{c}_1, 3\rangle \rightrightarrows \langle \mathsf{c}_1, 1\rangle,\ \langle \mathsf{c}_2, 3\rangle \rightrightarrows \langle \mathsf{c}_2, 2\rangle \rightrightarrows \langle \mathsf{c}_2, 1\rangle \rightrightarrows \langle \mathsf{c}_2, 3\rangle,$$
$$\langle [i]_1, y\rangle \overset{1}{\twoheadrightarrow} \langle \mathsf{c}_1, y\rangle \overset{2}{\twoheadleftarrow} \langle \mathsf{c}_2, y\rangle,\text{ and } \langle \mathsf{c}_2, y\rangle \rightsquigarrow \langle [j]_2, y\rangle \text{ for all } y = 1, 2, 3,$$
$$\langle [i]_1, 1\rangle \rightsquigarrow \langle [i]_1, 2\rangle \rightsquigarrow \langle [i]_1, 3\rangle \rightsquigarrow \langle [i]_1, 1\rangle,\text{ and } \langle [j]_2, 1\rangle \rightsquigarrow \langle [j]_2, 2\rangle \rightsquigarrow \langle [j]_2, 3\rangle \rightsquigarrow \langle [j]_2, 1\rangle.$$

It is straightforward to verify that $d(D) = 4$. In view of the symmetry of D, it suffices to check D for $(T_3 \times C_3)^{(2)}$ where c_i has two end-vertex neighbours $[1]_i, [2]_i$ for $i = 1, 2$ in T_3. We remark that the checking includes the distance from any vertex in the $[1]_1$-copy (respectively, $[1]_2$-copy) of $C_3^{(2)}$ to any vertex in the $[2]_1$-copy (respectively, $[2]_2$-copy) of $C_3^{(2)}$, although only one $[i]_1$-copy (respectively, $[j]_2$-copy) is shown in Figure 6 for brevity. Hence, $(T_3 \times C_3)^{(2)} \in \mathscr{C}_0$.

Since every vertex lies in a directed C_4 in D of both cases, it follows from Lemma 1.2 that $(T_\lambda \times C_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$. □

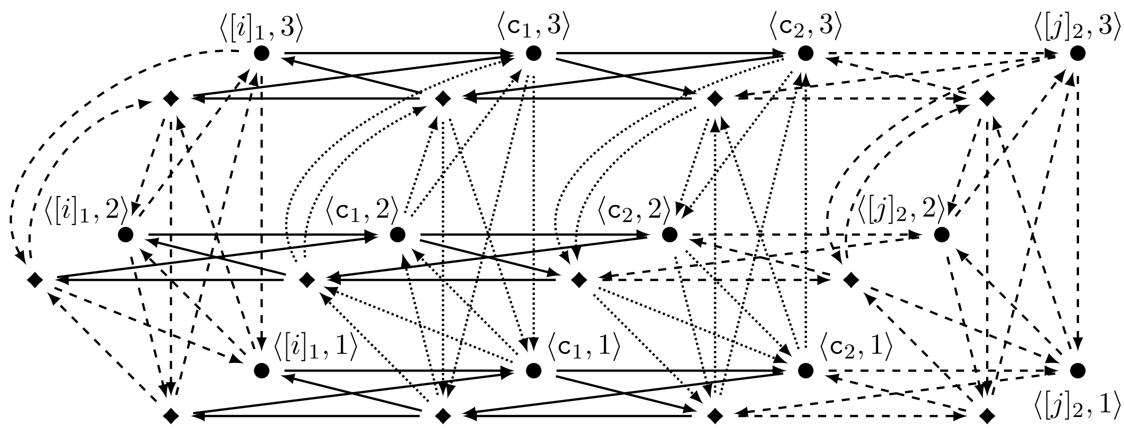


Figure 6: Partial orientation D of $(T_3 \times C_3)^{(2)}$,
where $[i]_1 \in N_T(\mathsf{c}_1) - \{\mathsf{c}_2\}$ and $[j]_2 \in N_T(\mathsf{c}_2) - \{\mathsf{c}_1\}$ and $d(D) = 4$.

Next, we want to consider $T_2 \times C_3$ and $P_2 \times C_3$. Instead, we shall prove more general results involving K_μ, $\mu \geq 3$, in place of C_3. For $T_2 \times K_\mu$, we split into cases of $\deg_{T_2}(\mathsf{c}) = 2$ (i.e. $T_2 = P_3$) and $\deg_{T_2}(\mathsf{c}) > 2$.

Proposition 4.1. *For $\mu \geq 3$,*
(a) if $\deg_{T_2}(\mathsf{c}) = 2$, then $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.
(b) if $\deg_{T_2}(\mathsf{c}) > 2$, then $(T_2 \times K_\mu)^{(2)} \in \mathscr{C}_1$ and $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

Proof: Note that $d(T_2 \times K_\mu) = 3$. Define an orientation D for $(T_2 \times K_\mu)^{(2)}$ as follows:

$$\langle [1], j\rangle \overset{1}{\twoheadrightarrow} \langle \mathsf{c}, j\rangle \overset{2}{\twoheadleftarrow} \langle [i], j\rangle \tag{4.1}$$

for all $[i] \in N_{T_2}(\mathsf{c}) - \{[1]\}$ and $j = 1, 2, \ldots, \mu$.

$$\left.\begin{array}{l}\langle v, j_1\rangle \rightsquigarrow \langle v, j_2\rangle \text{ whenever } 2 \le j_1 < j_2 \le \mu, \text{ and} \\ \langle v, j\rangle \rightsquigarrow \langle v, 1\rangle \rightsquigarrow \langle v, 2\rangle \text{ for } j = 3, 4, \ldots, \mu,\end{array}\right\} \tag{4.2}$$

for all $v \in V(T_2)$.

(a) Suppose $\deg_{T_2}(\mathsf{c}) = 2$. It is easy to check that the subdigraph of D induced by the set of vertices $\{(p, \langle v, j\rangle) \mid p = 1, 2; v \in V(T_2); j = 1, 2, 3\}$ has diameter 3. Next, note for all $v \in T_2$, and all $j = 4, 5, \ldots, \mu$, that $\langle v, j\rangle$ really plays the same role as $\langle v, 3\rangle$ in view of (4.2). Hence, it remains to check that the distance of any two vertices in each copy of $K_\mu^{(2)}$ is at most 3. This follows since $u \rightsquigarrow v$ or $v \rightsquigarrow u$ for all u, v in each copy of K_μ. Hence, $(T_2 \times K_\mu)^{(2)} \in \mathscr{C}_0$. Since every vertex lies in a directed C_3, it follows from Lemma 1.2 that $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

(b) Now, consider the case $\deg_{T_2}(\mathsf{c}) > 2$. Suppose there exists an orientation F of $(T_2 \times K_\mu)^{(2)}$ with $d(F) = 3$. By Lemma 3.1, $\langle [1], 1\rangle \overset{1}{\twoheadrightarrow} \langle \mathsf{c}, 1\rangle \overset{2}{\twoheadleftarrow} \langle [2], 1\rangle$ and $\langle [1], 1\rangle \overset{1}{\twoheadrightarrow} \langle \mathsf{c}, 1\rangle \overset{2}{\twoheadleftarrow} \langle [3], 1\rangle$. However, this contradicts $\langle [3], 1\rangle \overset{1}{\twoheadrightarrow} \langle \mathsf{c}, 1\rangle \overset{2}{\twoheadleftarrow} \langle [2], 1\rangle$. Thus, $(T_2 \times K_\mu)^{(2)} \in \mathscr{C}_1$.

To show $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$, we need to verify $d(D) = 4$. In view of (a) and its symmetry among the vertices $\langle [i], j\rangle$ for $[i] \in N_{T_2}(\mathsf{c}) - \{[1]\}$ by (4.1), it suffices to check $d_D((p, \langle [2], j\rangle), (q, \langle [3], j\rangle)) \le 4$ for $j = 1, 2, \ldots, \mu$, and $p, q = 1, 2$. That is, the partial orientation in Figure 7 has diameter 4, which is easy to check. Since every vertex lies in a directed C_3, it follows from Lemma 1.2 that $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$. ☐

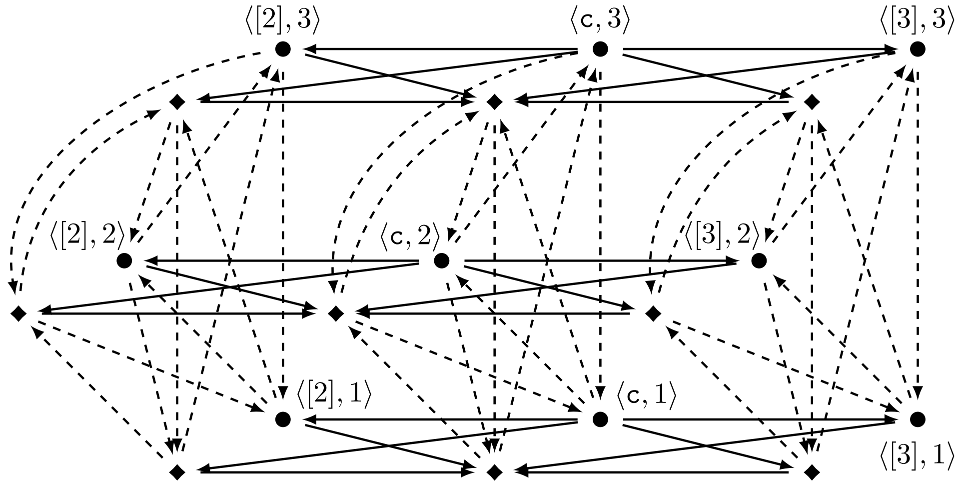


Figure 7: Partial orientation D of $(T_2 \times K_3)^{(2)}$ when $\deg_{T_2}(\mathsf{c}) > 2$, where $d(D) = 4$.

In Proposition 4.3, we generalise the sufficient condition in Proposition 4.1(b), "$\deg_{T_2}(\mathsf{c}) > 2$", for the vertex-multiplication of $T_2 \times K_\mu$ to be in $\mathscr{C}_1$. To this end, recall the classical theorem of Sperner.

Theorem 4.2. (Sperner [18]) *Let $n \in \mathbb{Z}^+$ and $\mathscr{A}$ be an antichain of $\mathbb{N}_n := \{1, 2, \ldots, n\}$ (i.e. $A \nsubseteq B$ for all $A, B \in \mathscr{A}$). Then $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ with equality holding if and only if all members in $\mathscr{A}$ have the same size, $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. (The two sizes coincide if n is even.)*

Recall that we loosely use the two notations of a vertex, i.e. if $v_i = j$, then $v_i = v_j$ and $s_i = s_j$. So $s_{\langle \mathsf{c}, v\rangle}$ corresponds to the vertex $\langle \mathsf{c}, v\rangle$ in the next proposition.

Proposition 4.3. *Let $\mu \geq 3$ and $m = \min\{s_{\langle \mathsf{c}, v\rangle} \mid v \in V(K_\mu)\}$. If $\deg_{T_2}(\mathsf{c}) > \binom{m}{\lfloor m/2 \rfloor}$, then $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_1$.*

Proof: Suppose D is an orientation of $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n)$ with $d(D) = 3 = d(T_2 \times K_\mu)$. In view of parity, $d_D((p, \langle [i], v\rangle), (q, \langle [j], v\rangle)) = 2$ for any $p = 1, 2, \ldots, s_{\langle [i], v\rangle}$, $q = 1, 2, \ldots, s_{\langle [j], v\rangle}$ and all $[i], [j] \in N_{T_2}(\mathsf{c})$ with $[i] \neq [j]$. For any $(p, \langle [i], v\rangle) \in V(D)$, define $O^{\langle \mathsf{c}, v\rangle}((p, \langle [i], v\rangle)) = O((p, \langle [i], v\rangle)) \cap \{(r, \langle \mathsf{c}, v\rangle) \mid r = 1, 2, \ldots, s_{\langle \mathsf{c}, v\rangle)}\}$. Since $\deg_{T_2}(\mathsf{c}) > \binom{m}{\lfloor m/2 \rfloor}$, there exists some $v^* \in V(K_\mu)$ such that $\deg_{T_2}(\mathsf{c}) > \binom{s_{\langle \mathsf{c}, v^*\rangle}}{\lfloor s_{\langle \mathsf{c}, v^*\rangle}/2 \rfloor}$. By Sperner's Theorem, for some $p^* = 1, 2, \ldots, s_{\langle [i^*], v^*\rangle}$, some $q^* = 1, 2, \ldots, s_{\langle [j^*], v^*\rangle}$ and some $[i^*], [j^*] \in N_{T_2}(\mathsf{c})$ with $[i^*] \neq [j^*]$, we have

$$O^{\langle \mathsf{c}, v^*\rangle}((p^*, \langle [i^*], v^*\rangle)) \subseteq O^{\langle \mathsf{c}, v^*\rangle}((q^*, \langle [j^*], v^*\rangle)).$$

Hence, it follows that $d_D((p^*, \langle [i^*], v^*\rangle), (q^*, \langle [j^*], v^*\rangle)) > 2$, a contradiction. Hence $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \notin \mathscr{C}_0$. By Proposition 4.1 (b), $(T_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_1$. □

Remark 4.4. The same proof and notation as Proposition 4.3 shows that if $\deg_{T_2}(\mathsf{c}) > \binom{m}{\lfloor m/2 \rfloor}$, then $(T_2 \times K_2)(s_1, s_2, \ldots, s_n) \notin \mathscr{C}_0$.

Proposition 4.5. *For $\mu \geq 3$, $(P_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_1$.*

Proof: Suppose F is an orientation of $(P_2 \times K_\mu)(s_1, s_2, \ldots, s_n)$ with $d(F) = 2 = d(P_2 \times K_\mu)$. It follows from $d_F((p, \langle u, x\rangle), (q, \langle v, x\rangle)) \leq 2$ that $(p, \langle u, x\rangle) \to (q, \langle v, x\rangle)$ for $u, v \in V(P_2)$, $x \in V(K_\mu)$, $p = 1, 2, \ldots, s_{\langle u, x\rangle}$, $q = 1, 2, \ldots, s_{\langle v, x\rangle}$. Then,

$$d_F((q, \langle v, x\rangle), (p, \langle u, x\rangle)) > 2,$$

a contradiction. Hence, $(P_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \notin \mathscr{C}_0$.

Define an orientation D of $(P_2 \times K_\mu)^{(2)}$ as follows:

$$\begin{aligned}
&\langle 2, 1\rangle \rightrightarrows \langle 1, 1\rangle, \langle 1, 2\rangle \rightrightarrows \langle 2, 2\rangle, \langle 1, i\rangle \rightsquigarrow \langle 2, i\rangle \text{ for } i = 3, 4, \ldots, \mu.\\
&\langle k, j_1\rangle \rightsquigarrow \langle k, j_2\rangle \text{ whenever } 2 \leq j_1 < j_2 \leq \mu, \text{ and}\\
&\langle k, j\rangle \rightsquigarrow \langle k, 1\rangle \rightsquigarrow \langle k, 2\rangle \text{ for } j = 3, 4, \ldots, \mu,
\end{aligned}$$

for $k = 1, 2$.

It can be verified easily that $d(D) = 3$. Hence, $(P_2 \times K_\mu)^{(2)} \in \mathscr{C}_1$. Furthermore, since every vertex lies in a directed C_3, it follows from Lemma 1.2 that $(P_2 \times K_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_1$. □

5 Cartesian product of two cycles $C_\lambda \times C_\mu$

In this section, we prove Theorem 1.11.

Proposition 5.1. *If $\lambda \geq \mu \geq 4$, then $(C_\lambda \times C_\mu)(s_1, s_2, \dots, s_n) \in \mathscr{C}_0$.*

Proof: We shall use a similar strategy as in Theorem 1.10. Partition $V(C_\lambda)$ into $V_1 = \{v \mid v \text{ is odd}\}$ and $V_2 = \{v \mid v \text{ is even}\}$. Let F be a strong orientation of C_μ, say $1 \to 2 \to \dots \to \mu \to 1$, and define an orientation D for $(C_\lambda \times C_\mu)^{(2)}$ as follows:

$$\langle u, x\rangle \rightrightarrows \langle u, y\rangle \iff x \to y \text{ in } F$$

for any $u \in V_1$, and any $x, y \in V(C_\mu)$, i.e. the copy $C_\mu^{(2)}$ is oriented similarly to F.

$$\langle u, x\rangle \rightrightarrows \langle u, y\rangle \iff y \to x \text{ in } F$$

for any $u \in V_2$, and any $x, y \in V(C_\mu)$, i.e. the copy $C_\mu^{(2)}$ is oriented similarly to $\tilde{F}$.

$$\langle u, x\rangle \rightsquigarrow \langle u+1, x\rangle \text{ (addition is taken modulo } \lambda)$$

for any $u \in V(C_\lambda)$ and any $x \in V(C_\mu)$.

We claim that $d_D((p, \langle u, x\rangle), (q, \langle v, y\rangle)) \leq \lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor = d(C_\lambda \times C_\mu)$ for any $\langle u, x\rangle, \langle v, y\rangle \in V(C_\lambda \times C_\mu)$, and $p, q = 1, 2$. Suppose $u = v \in V_1$. Note that either $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$ or $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$. So, there exist paths P and P' in D, each of length at most $\lfloor \frac{\mu}{2} \rfloor$, from $\{(1, \langle u, x\rangle), (2, \langle u, x\rangle)\}$ to $\{(1, \langle v, y\rangle), (2, \langle v, y\rangle)\}$ and from $\{(1, \langle w, x\rangle), (2, \langle w, x\rangle)\}$ to $\{(1, \langle w, y\rangle), (2, \langle w, y\rangle)\}$ where $w \in V_2$ is some vertex adjacent to u in C_λ respectively. In the former case where $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, P suffices and we are done. In the latter case where $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, we shall further assume $w = u + 1 \pmod{\lambda}$ for simplicity; the proof is similar if $u = w + 1 \pmod{\lambda}$. Then, $(p, \langle u, x\rangle)(p, \langle w, x\rangle)$ with P' and $(3-q, \langle w, y\rangle)(q, \langle u, y\rangle)$ form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $2 + \lfloor \frac{\mu}{2} \rfloor \leq \lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$. A similar proof follows if $u = v \in V_2$.

Suppose $u \neq v$. For simplicity, we shall assume $u(u+1)\dots(u+l)v$ to be a shortest $u - v$ path in C_λ; the proof is similar if the shortest path is $u(u-1)\dots(u-l)v$. If $x = y$, then $(p, \langle u, x\rangle)(p, \langle u+1, x\rangle)\dots(p, \langle v, x\rangle)(3-p, \langle u+l, x\rangle)(3-p, \langle v, x\rangle)$ guarantees a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $\lfloor \frac{\lambda}{2} \rfloor + 2 \leq \lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$.

Next, suppose $x \neq y$. Furthermore, we shall assume $v \in V_1$; the proof is similar if $v \in V_2$. Again, consider the cases $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$ or $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$. In the former case where $d_F(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, there is a path Q of length at most $\lfloor \frac{\mu}{2} \rfloor$ from $\{(1, \langle v, x\rangle), (2, \langle v, x\rangle)\}$ to $\{(1, \langle v, y\rangle), (2, \langle v, y\rangle)\}$ in D. So, $(p, \langle u, x\rangle)(p, \langle u+1, x\rangle)\dots(p, \langle u+l, x\rangle)(p, \langle v, x\rangle)$ and Q form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ path of length at most $\lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$. In the latter case where $d_{\tilde{F}}(x, y) \leq \lfloor \frac{\mu}{2} \rfloor$, unless $u = \lambda$ is odd and $v = 1$, there exists some $i \in \{0, 1\}$ such that $u + i \in V_2$. Moreover, there is a path Q' of length at most $\lfloor \frac{\mu}{2} \rfloor$ from $\{(1, \langle u+i, x\rangle), (2, \langle u+i, x\rangle)\}$ to $\{(1, \langle u+i, y\rangle), (2, \langle u+i, y\rangle)\}$ in D so that $(p, \langle u, x\rangle)(p, \langle u+1, x\rangle)\dots(p, \langle u+i, x\rangle)$ with Q' and $(q, \langle u+i, y\rangle)(q, \langle u+i+1, y\rangle)\dots(q, \langle v, y\rangle)$ form a $(p, \langle u, x\rangle) - (q, \langle v, y\rangle)$ of length at most $\lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$.

Finally, if $u = \lambda$ is odd, $v = 1$, and $y - x \leq \lfloor \frac{\mu}{2} \rfloor + 1 \pmod{\mu}$, then $(p, \langle \lambda, x \rangle) \to (p, \langle 1, x \rangle) \to \{(1, \langle 1, x+1 \rangle), (2, \langle 1, x+1 \rangle)\} \to \{(1, \langle 1, x+2 \rangle), (2, \langle 1, x+2 \rangle)\} \to \cdots \to \{(1, \langle 1, y \rangle), (2, \langle 1, y \rangle)\}$ ensures a path of length at most $1 + \lfloor \frac{\mu}{2} \rfloor + 1 \leq \lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$. If $u = \lambda$ is odd, $v = 1$, and $y - x > \lfloor \frac{\mu}{2} \rfloor + 1 \pmod{\mu}$, then $(p, \langle \lambda, x \rangle) \to (3 - p, \langle \lambda - 1, x \rangle) \to \{(1, \langle \lambda - 1, x - 1 \rangle), (2, \langle \lambda - 1, x - 1 \rangle)\} \to \cdots \to \{(1, \langle \lambda - 1, y \rangle), (2, \langle \lambda - 1, y \rangle)\}$ and $(q, \langle \lambda - 1, y \rangle)(q, \langle \lambda, y \rangle)(q, \langle 1, y \rangle)$ form a $(p, \langle \lambda, x \rangle) - (q, \langle 1, y \rangle)$ path of length at most $3 + \lceil \frac{\mu}{2} \rceil - 2 \leq \lfloor \frac{\lambda}{2} \rfloor + \lfloor \frac{\mu}{2} \rfloor$.

Since every vertex lies in a directed C_4, it follows from Lemma 1.2 that $(C_\lambda \times C_\mu)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$. □

Corollary 5.2. $(C_3 \times C_3)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

Proof: We claim that $d(D) = 3 = d(C_3 \times C_3) + 1$ where D is as defined in Proposition 5.1. For any $\langle u, x \rangle, \langle v, y \rangle \in V(C_\lambda \times C_\mu)$, observe that either $\langle u, x \rangle \rightsquigarrow \langle v, x \rangle$ or $\langle v, x \rangle \rightsquigarrow \langle u, x \rangle$ and $\langle v, x \rangle \rightrightarrows \langle v, x+1 \rangle \rightrightarrows \langle v, x+2 \rangle$ or $\langle v, x \rangle \rightrightarrows \langle v, x-1 \rangle \rightrightarrows \langle v, x-2 \rangle$, where the addition is taken modulo 3, proves the claim. Hence, $(C_3 \times C_3)^{(2)} \in \mathscr{C}_0 \cup \mathscr{C}_1$. Since every vertex lies in a directed C_3, it follows from Lemma 1.2 that $(C_3 \times C_3)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$. □

Proposition 5.3. *$(C_4 \times C_3)^{(2)} \in \mathscr{C}_0$ and $(C_4 \times C_3)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.*

Proof: Define an orientation D for $(C_4 \times C_3)^{(2)}$ as follows:

$$\begin{aligned}
&\langle 2, i \rangle \rightsquigarrow \langle 1, i \rangle \text{ and } \langle 3, i \rangle \rightsquigarrow \langle 4, i \rangle \text{ for } i = 1, 2, 3. \\
&\langle 1, 2 \rangle \rightrightarrows \langle 1, 1 \rangle \rightrightarrows \langle 4, 1 \rangle \rightrightarrows \langle 4, 2 \rangle \rightrightarrows \langle 1, 2 \rangle \rightrightarrows \langle 1, 3 \rangle \rightrightarrows \langle 4, 3 \rangle \rightrightarrows \langle 4, 2 \rangle, \\
&\langle 3, 2 \rangle \rightrightarrows \langle 3, 1 \rangle \rightrightarrows \langle 2, 1 \rangle \rightrightarrows \langle 2, 2 \rangle \rightrightarrows \langle 3, 2 \rangle \rightrightarrows \langle 3, 3 \rangle \rightrightarrows \langle 2, 3 \rangle \rightrightarrows \langle 2, 2 \rangle, \\
&\langle 1, 3 \rangle \rightrightarrows \langle 1, 1 \rangle, \langle 4, 1 \rangle \rightrightarrows \langle 4, 3 \rangle, \langle 3, 3 \rangle \rightrightarrows \langle 3, 1 \rangle, \text{ and } \langle 2, 1 \rangle \rightrightarrows \langle 2, 3 \rangle.
\end{aligned}$$

It is easy to check $d(D) = 3 = d(C_4 \times C_3)$. Since every vertex lies in a directed C_4, it follows from Lemma 1.2 that $(C_4 \times C_3)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$. □

6 Concluding remarks

In this paper, we considered primarily vertex-multiplications of Cartesian products involving trees, paths and cycles as they are some special families of graphs studied for orientations (see [7–10]). We refer the interested reader to a good survey on orientations of graphs [12] by Koh and Tay.

It can be shown that $(T_2 \times T_2)(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$. We believe its characterisation likely involves notions and techniques of Extremal Set Theory such as antichains. This is akin to Proposition 4.3 and vertex-multiplications of trees with diameter 4 (see [21]). Hence, we conclude by proposing the following problem.

Problem 6.1. Characterise the vertex-multiplications $(T_2 \times T_2)(s_1, s_2, \ldots, s_n)$ that belong to $\mathscr{C}_0$.

Acknowledgements

The first author would like to thank the National Institute of Education, Nanyang Technological University of Singapore, for the generous support of the Nanyang Technological University Research Scholarship.

We would also like to thank the editors and referees for their helpful comments.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, algorithms and applications*, Springer Verlag, London, (2009).

[2] F. Boesch and R. Tindell, Robbin's theorem for mixed multigraphs, *Amer. Math. Monthly* 87 (1980), 716–719.

[3] V. Chvátal and C. Thomassen, Distances in orientations of graphs, *J. Combin. Theory Ser. B* 24 (1978), 61–75.

[4] G. Gutin, m-sources in complete multipartite digraphs, *Vestsi Acad. Navuk BSSR Ser. Fiz-Mat* 5 (1989), 101–106; (in Russian).

[5] G. Gutin, K. M. Koh, E. G. Tay and A. Yeo, Almost minimum diameter orientations of semicomplete multipartite and extended digraphs, *Graphs Combin.* 18 (2002), 499–506.

[6] A. Hajnal, E. C. Milner and E. Szemeredi, A cure for the telephone disease, *Canad. Math. Bull.* 15 (1972), 447–450.

[7] K. M. Koh and K. T. Lee, Optimal orientations of products of odd cycles and paths, preprint (1998).

[8] K. M. Koh and E. G. Tay, Optimal orientations of products of paths and cycles, *Discrete Appl. Math.* 78 (1997), 163–174.

[9] K. M. Koh and E. G. Tay, On optimal orientations of Cartesian products of even cycles and paths, *Networks* 30 (1997), 1–7.

[10] K. M. Koh and E. G. Tay, On optimal orientations of Cartesian products of trees, *Graphs Combin.* 17 (2001), 79–97.

[11] K. M. Koh and E. G. Tay, On optimal orientations of G vertex-multiplications, *Discrete Math.* 219 (2000), 153–171.

[12] K. M. Koh and E. G. Tay, On optimal orientations of graphs and digraphs: A survey, *Graphs Combin.* 18 (2002), 745–756.

[13] K. M. Koh and E. G. Tay, On optimal orientations of tree vertex-multiplications, *Australas. J. Combin.* 34 (2006), 69–87.

[14] S. B. Maurer, The king chicken theorems, *Math. Mag.* 53 (1980), 67–80.

[15] K. L. Ng and K. M. Koh, On optimal orientation of cycle vertex multiplications, *Discrete Math.* 297 (2005), 104–118.

[16] J. Plesník, Diametrically critical tournaments, *Časopis pro pěstovánái matematiky* 100 (1975), 361–370.

[17] H. E. Robbins, A theorem on graphs, with an application to a problem of traffic control, *Amer. Math. Monthly* 46 (1939), 281–283.

[18] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* 27 (1928), 544–548.

[19] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, *Math. Slovaca* 36 (1986), 289–296.

[20] W. H. W. Wong and E. G. Tay, A complete characterisation of vertex-multiplications of trees with diameter 5, *Theory Appl. Graphs* 8 (2021), Art. 6.

[21] W. H. W. Wong and E. G. Tay, Application of some techniques in Sperner Theory: Optimal orientations of vertex-multiplications of trees with diameter 4, arXiv:2110.09003 [math.CO], (2021).

[22] W. H. W. Wong and E. G. Tay, Optimal orientations of vertex-multiplications of Cartesian products of graphs, arXiv:2010.02792 [math.CO], (2020).

(Received 31 May 2021; revised 24 Feb 2022)