AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **83(1)** (2022), Pages 141–162

Gallai-Ramsey numbers for graphs with five vertices and chromatic number four

Qinghong Zhao* Bing Wei†

Department of Mathematics
University of Mississippi
University, MS 38677, U.S.A.
`qzhao1@olemiss.edu` `bwei@olemiss.edu`

Abstract

Given a graph H and an integer $k \geq 1$, the Gallai-Ramsey number $GR_k(H)$ is defined to be the minimum integer n such that every k-edge coloring of the complete graph K_n contains either a rainbow (all different colors) triangle or a monochromatic copy of H. In this paper, we determine the Gallai-Ramsey numbers for all connected graphs with five vertices and chromatic number 4. With our results, the Gallai-Ramsey numbers for all isolated-free graphs with five vertices except K_5 are determined.

1 Introduction

In this paper, we only deal with finite, simple and undirected graphs. Given a graph G and the vertex set $V(G)$, let $|G|$ denote the number of vertices of G and $G[W]$ denote the subgraph of G induced by a set $W \subseteq V(G)$. Given disjoint vertex sets $X, Y \subseteq V(G)$, if each vertex in X is adjacent to all vertices in Y and all the edges between X and Y are colored with the same color, then we say that X is *mc-adjacent* to Y, that is, X is *blue-adjacent* to Y if all the edges between X and Y are colored with blue. We use P_n and K_n to denote the path and complete graph on n vertices, respectively. We define $[k] = \{1, \ldots, k\}$ for any integers $k \geq 1$.

The complete graphs under edge coloring without a rainbow triangle usually have pretty interesting and somehow beautiful structures. In 1967, Gallai studied the structure under the guise of transitive orientations and obtained the following result [7] which was restated in [10] in the terminology of graphs.

* Corresponding Author. Supported by the Summer Graduate Research Assistantship Program of Graduate School at the University of Mississippi.

† Supported in part by CLA Faculty Research and Creative Achievement Grant at the University of Mississippi and by NSFC (No. 11701530).

ISSN: 2202-3518

Theorem 1.1 ([7, 10])**.** *For a complete graph G under any edge coloring without a rainbow triangle, there exists a partition of $V(G)$ (called a Gallai-partition) with parts $V_1, V_2, \dots, V_\ell$, $\ell \geq 2$, such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In honor of Gallai's result, the edge coloring of a complete graph without a rainbow triangle is called *Gallai coloring*. We use (G, c) to denote a complete graph G under the Gallai coloring $c : E(G) \to [k]$. Given graphs $H_1, \dots, H_k$ and an integer $k \geq 1$, the *Ramsey number* $R(H_1, \dots, H_k)$ is defined to be the minimum integer n such that every k-edge coloring of K_n contains a monochromatic copy of H_i in color $i \in [k]$, and the *Gallai-Ramsey number* $GR(H_1, \dots, H_k)$ is defined to be the minimum integer n such that every Gallai k-coloring of K_n contains a monochromatic copy of H_i in color $i \in [k]$. We simply write $R_k(H)$ and $GR_k(H)$ when $H = H_1 = \cdots = H_k$, and $R((k-r)K, rH)$ and $GR_k((k-r)K, rH)$ when $H = H_1 = \cdots = H_r$ and $K = H_{r+1} = \cdots = H_k$. Similar to the notation $GR_k((k-s-r)F, sK, rH)$ when $F = H_1 = \cdots = H_{k-s-r}$, $K = H_{k-s-r+1} = \cdots = H_{k-r}$ and $H = H_{k-r+1} = \cdots = H_k$. Given a Gallai-partition $V_1, V_2, \dots, V_\ell$ of a complete graph G, we define $\mathcal{G} = G[\{v_1, \dots, v_\ell\}] = K_\ell$ as a *reduced graph* of G, where $v_i \in V_i$ for all $i \in [\ell]$. Obviously, there exists a monochromatic copy of H in $\mathcal{G}$ if $\ell \geq R_2(H)$, which leads to a monochromatic copy of H in G. Clearly, $GR_2(K, H) = R(K, H)$ and $GR_k(H) \leq R_k(H)$ for $k \geq 1$. The exact values of $R_2(K_t)$ for $t \geq 5$ are not determined so far. Similarly, determining the exact value of $GR_k(H)$ for a graph H is far from trivial sometimes, even for a small graph. The general behavior of $GR_k(H)$ for all graphs H was established in [9].

Theorem 1.2 ([9])**.** *Let H be a fixed graph with no isolated vertices. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

In 2015, Fox, Grinshpun and Pach [4] posed a conjecture for $GR_k(K_t)$. It is worth mentioning that the cases $t = 3, 4$ were proved in [1],[9] and [13]. Recently, Magnant and Schiermeyer [14] have made a breakthrough for the case $t = 5$ while there are not any results for the cases $t \geq 6$. More information on Gallai-Ramsey number can be found in [6, 21, 22].

Let $\mathscr{G}$ denote a class of isolated-free graphs with five vertices (see Figure 1). It is worth mentioning that the Gallai-Ramsey numbers for all the graphs in $\mathscr{G}$ with chromatic number 2 have been determined for G_1–G_4 in [3], for G_5 in [5, 21], for G_6 in [12] and for G_7 in [20]. Recently, the Gallai-Ramsey numbers for all the graphs in $\mathscr{G}$ with chromatic number 3 were determined by [15] for G_8, [16, 18] for G_9, [24] for G_{10}, [12] for G_{11}–G_{14}, [19] for G_{15} and G_{16} and [23] for G_{11}–G_{19}. In this paper, we obtain the exact values of $GR_k(H)$ for $H \in \{G_{20}, G_{21}, G_{22}\}$, as follows. With our results, the Gallai-Ramsey numbers for all graphs in $\mathscr{G} \setminus \{G_{23}\}$ are determined.

Theorem 1.3. *Let $H \in \{G_{20}, G_{21}, G_{22}\}$. For all $k \geq 1$, we have*

$$GR_k(H) = \begin{cases} (R_2(H) - 1) \cdot 17^{(k-2)/2} + 1, & \text{if } k \text{ is even,} \\ 4 \cdot 17^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

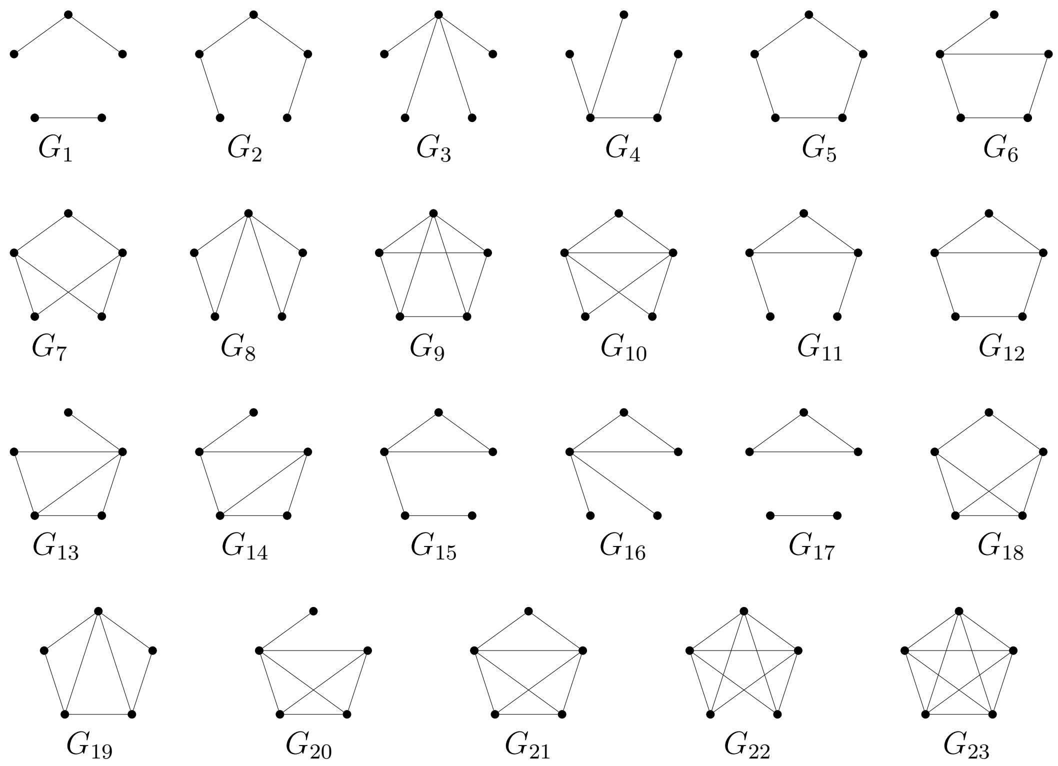

Figure 1: All isolated-free graphs with five vertices.

In order to prove Theorem 1.3, we give a unified proof for the following two results.

Theorem 1.4. *Let $H \in \{G_{20}, G_{21}\}$. For all $k \geq 1$ and r with $0 \leq r \leq k$,*

$$GR_k((k-r)K_3, rH) =$$

$$\begin{cases} 5^{(k-r)/2} \cdot 17^{r/2} + 1, & \text{if } (k-r) \text{ and } r \text{ are both even,} \quad (a_1) \\ 2 \cdot 5^{(k-r-1)/2} \cdot 17^{r/2} + 1, & \text{if } (k-r) \text{ is odd and } r \text{ is even,} \quad (a_2) \\ 8 \cdot 5^{(k-r-1)/2} \cdot 17^{(r-1)/2} + 1, & \text{if } (k-r) \text{ and } r \text{ are both odd,} \quad (a_3) \\ 4 \cdot 5^{(k-r)/2} \cdot 17^{(r-1)/2} + 1, & \text{if } (k-r) \text{ is even and } r \text{ is odd.} \quad (a_4) \end{cases}$$

Theorem 1.5. *For all $k \geq 1$, s and r with $0 \leq s \leq k$ and $0 \leq r \leq k$,*

$$GR_k((k-s-r)P_3, sK_3, rG_{22}) =$$

$$\begin{cases} 5^{s/2} \cdot \lfloor 21 \cdot 17^{(r-2)/2} \rfloor + 1, & \text{if } s, r \text{ are both even and } s + r = k, \quad (b_1) \\ 2 \cdot 5^{s/2} \cdot 17^{r/2} + 1, & \text{if } s, r \text{ are both even and } s + r < k, \quad (b_2) \\ 5^{(s-1)/2} \cdot \lfloor 42 \cdot 17^{(r-2)/2} \rfloor + 1, & \text{if } s \text{ is odd, } r \text{ is even and } s + r = k, \quad (b_3) \\ 4 \cdot 5^{(s-1)/2} \cdot 17^{r/2} + 1, & \text{if } s \text{ is odd, } r \text{ is even and } s + r < k, \quad (b_4) \\ 10 \cdot 5^{(s-1)/2} \cdot 17^{(r-1)/2} + 1, & \text{if } s, r \text{ are both odd and } s + r = k, \quad (b_5) \\ 16 \cdot 5^{(s-1)/2} \cdot 17^{(r-1)/2} + 1, & \text{if } s, r \text{ are both odd and } s + r < k, \quad (b_6) \\ 4 \cdot 5^{s/2} \cdot 17^{(r-1)/2} + 1, & \text{if } s \text{ is even, } r \text{ is odd and } s + r = k, \quad (b_7) \\ \lfloor 32 \cdot 5^{(s-2)/2} \rfloor \cdot 17^{(r-1)/2} + 1, & \text{if } s \text{ is even, } r \text{ is odd and } s + r < k. \quad (b_8) \end{cases}$$

2 Preliminaries

For ease of notation, let $GR_k((k-r)K_3, rH) = w(k,r)+1$ with $H \in \{G_{20}, G_{21}\}$ and $GR_k((k-s-r)P_3, sK_3, rG_{22}) = f(k,s,r)+1$. We start this section by using $w(k,r)$, $f(k,s,r)$ and the functions listed in the Appendix to derive the following two tables based on the cases a_1–a_4 and b_1–b_8.

By applying the functions listed in the Appendix and Tables 1 and 2, we can get the following inequalities that will be used in Section 3. One can check the inequalities (a)–(e), (h)–(n) and (q) by easy computations. In order to be convenient to check the rest of the inequalities, we refer the readers to Table 3 for more details.

Case	a_1	a_2	a_3	a_4
$\frac{w(k-1,r)}{w(k,r)}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$
$\frac{w(k-2,r)}{w(k,r)}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$\frac{w(k,r-1)}{w(k,r)}$	$\frac{8}{17}$	$\frac{10}{17}$	$\frac{5}{8}$	$\frac{1}{2}$
$\frac{w(k-1,r-1)}{w(k,r)}$	$\frac{4}{17}$	$\frac{4}{17}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{w(k-2,r-1)}{w(k,r)}$	$\frac{8}{85}$	$\frac{2}{17}$	$\frac{1}{8}$	$\frac{1}{10}$
$\frac{w(k,r-2)}{w(k,r)}$	$\frac{5}{17}$	$\frac{5}{17}$	$\frac{5}{17}$	$\frac{5}{17}$
$\frac{w(k-1,r-2)}{w(k,r)}$	$\frac{2}{17}$	$\frac{5}{34}$	$\frac{5}{34}$	$\frac{2}{17}$
$\frac{w(k-2,r-2)}{w(k,r)}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$

Table 1: The ratios of corresponding functions to $w(k,r)$.

Case	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$\frac{f(k-1,s-1,r)}{f(k,s,r)}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$s=1:\frac{3}{8}$ $s\geq 3:\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{f(k-2,s-2,r)}{f(k,s,r)}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$s=2:\frac{3}{16}$ $s\geq 4:\frac{1}{5}$
$\frac{f(k,s+1,r-1)}{f(k,s,r)}$	$\frac{10}{21}$	$\frac{8}{17}$	$\frac{10}{21}$	$\frac{8}{17}$	$r=1:\frac{1}{2}$ $r\geq 3:\frac{21}{34}$	$\frac{5}{8}$	$r=1:\frac{1}{2}$ $r\geq 3:\frac{21}{34}$	$s=0:\frac{2}{3}$ $s\geq 2:\frac{5}{8}$
$\frac{f(k-1,s,r-1)}{f(k,s,r)}$	$\frac{4}{21}$	$s=0:\frac{3}{17}$ $s\geq 2:\frac{16}{85}$	$\frac{5}{21}$	$\frac{4}{17}$	$r=1:\frac{1}{5}$ $r\geq 3:\frac{21}{85}$	$\frac{1}{4}$	$r=1:\frac{1}{4}$ $r\geq 3:\frac{21}{68}$	$s=0:\frac{1}{3}$ $s\geq 2:\frac{5}{16}$
$\frac{f(k,s,r-1)}{f(k,s,r)}$	$s=0:\frac{2}{7}$ $s\geq 2:\frac{32}{105}$	$s=0:\frac{3}{17}$ $s\geq 2:\frac{16}{85}$	$\frac{8}{21}$	$\frac{4}{17}$	$\frac{2}{5}$	$\frac{1}{4}$	$\frac{1}{2}$	$s=0:\frac{1}{3}$ $s\geq 2:\frac{5}{16}$
$\frac{f(k-1,s-1,r-1)}{f(k,s,r)}$	$\frac{16}{105}$	$\frac{8}{85}$	$s=1:\frac{1}{7}$ $s\geq 3:\frac{16}{105}$	$s=1:\frac{3}{34}$ $s\geq 3:\frac{8}{85}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{1}{8}$
$\frac{f(k-2,s-1,r-1)}{f(k,s,r)}$	$\frac{2}{21}$	$\frac{8}{85}$	$\frac{2}{21}$	$s=1:\frac{3}{34}$ $s\geq 3:\frac{8}{85}$	$r=1:\frac{1}{10}$ $r\geq 3:\frac{21}{170}$	$\frac{1}{8}$	$r=1:\frac{1}{10}$ $r\geq 3:\frac{21}{170}$	$\frac{1}{8}$
$\frac{f(k,s+2,r-2)}{f(k,s,r)}$	$r=2:\frac{5}{21}$ $r\geq 4:\frac{5}{17}$	$\frac{5}{17}$	$r=2:\frac{5}{21}$ $r\geq 4:\frac{5}{17}$	$\frac{5}{17}$	$\frac{5}{17}$	$\frac{5}{17}$	$\frac{5}{17}$	$s=0:\frac{16}{51}$ $s\geq 2:\frac{5}{17}$
$\frac{f(k,s+1,r-2)}{f(k,s,r)}$	$\frac{4}{21}$	$\frac{2}{17}$	$\frac{5}{21}$	$\frac{5}{34}$	$\frac{16}{85}$	$\frac{2}{17}$	$\frac{4}{17}$	$s=0:\frac{8}{51}$ $s\geq 2:\frac{5}{34}$
$\frac{f(k-1,s+1,r-2)}{f(k,s,r)}$	$r=2:\frac{2}{21}$ $r\geq 4:\frac{2}{17}$	$\frac{2}{17}$	$r=2:\frac{5}{42}$ $r\geq 4:\frac{5}{34}$	$\frac{5}{34}$	$\frac{2}{17}$	$\frac{2}{17}$	$\frac{5}{34}$	$s=0:\frac{8}{51}$ $s\geq 2:\frac{5}{34}$
$\frac{f(k-2,s,r-2)}{f(k,s,r)}$	$r=2:\frac{1}{21}$ $r\geq 4:\frac{1}{17}$	$\frac{1}{17}$	$r=2:\frac{1}{21}$ $r\geq 4:\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$
$\frac{f(k-1,s,r-2)}{f(k,s,r)}$ $=\frac{f(k,s,r-2)}{f(k,s,r)}$	$\frac{2}{21}$	$\frac{1}{17}$	$\frac{2}{21}$	$\frac{1}{17}$	$\frac{8}{85}$	$\frac{1}{17}$	$s=0:\frac{3}{34}$ $s\geq 2:\frac{8}{85}$	$\frac{1}{17}$

Table 2: The ratios of corresponding functions to $f(k,s,r)$.

For all $k \geq 3$ and $r \geq 1$, we have $w(k,r)+1 >$

$$\begin{cases} 4w(k-1,r-1) \geq w(k-1,r-1)+k+1, \text{ where } w(k-1,r-1) \geq k+1, & (a) \\ 5w(k-2,r) \geq 2w(k-1,r) \geq w(k-1,r)+k, \text{ where } w(k-1,r) \geq k, & (b) \\ w(k,r-1)+w(k-1,r-1) \geq w(k,r-1)+r+4, & \\ \qquad\qquad\qquad\qquad \text{where } w(k-1,r-1) \geq r+4, & (c) \\ 8w(k-2,r-1) \geq 5w(k-2,r-1)+r \geq r+6, \text{ where } w(k-2,r-1) \geq r, & (d) \\ 17w(k-2,r-2) \geq 12w(k-2,r-2)+r, \text{ where } w(k-2,r-2) \geq r, & (e) \\ w(k-1,r)+2w(k-1,r-1), & (f) \\ 2w(k-1,r-2)+w(k,r-1)+w(k-2,r-2). & (g) \end{cases}$$

For all $k \geq 3$ and $s+r \geq 1$, we have $f(k,s,r)+1 >$

$$\begin{cases} 2f(k,s,r-1) \geq f(k,s,r-1)+s+r+1, \text{ where } f(k,s,r-1) \geq s+r+1, & (h) \\ 2f(k-1,s-1,r) \geq f(k-1,s-1,r)+s+r, & \\ \qquad\qquad\qquad\qquad \text{where } f(k-1,s-1,r) \geq s+r, & (i) \\ 3f(k-1,s,r-1) \geq 2f(k-1,s,r-1)+r, \text{ where } f(k-1,s,r-1) \geq r, & (j) \\ 8f(k-2,s-1,r-1) \geq 5f(k-2,s-1,r-1)+r, & \\ \qquad\qquad\qquad\qquad \text{where } f(k-2,s-1,r-1) \geq r, & (k) \\ f(k-1,s,r-1)+f(k,s+1,r-1), & (l) \\ 5f(k-2,s-2,r), & (m) \\ 5f(k-1,s-1,r-1), & (n) \\ 2f(k-1,s,r-1)+3f(k-2,s-1,r-1), & (o) \\ f(k-1,s,r-1)+2f(k-1,s-1,r-1)+2f(k-2,s-1,r-1). & (p) \end{cases}$$

For abbreviation, we use $\{\alpha;\beta\}+\{\gamma;\theta\}$ and $\alpha+\beta+\{\gamma;\theta;\varphi\}$ to denote $\{\alpha+\gamma,\alpha+\theta,\beta+\gamma,\beta+\theta\}$ and $\{\alpha+\beta+\gamma,\alpha+\beta+\theta,\alpha+\beta+\varphi\}$ in the following inequalities, respectively. For all $k \geq 3$ and $r \geq 2$, we have $f(k,s,r)+1 >$

$$\begin{cases} 17f(k-2,s,r-2) \geq 14f(k-2,s,r-2)+r, \text{ where } f(k-2,s,r-2) \geq r, & (q) \\ f(k,s+2,r-2)+2f(k-1,s,r-1), & (r) \\ 4f(k-1,s,r-2)+f(k,s+1,r-1), & (s) \\ \{f(k,s+1,r-2)+f(k,s,r-1); f(k-1,s+1,r-2)+f(k,s+1,r-1)\} & \\ \qquad\qquad +\{3f(k-2,s,r-2); 2f(k-1,s,r-2)\}, & (t) \\ 3f(k,s+1,r-2)+f(k-2,s,r-2)+2f(k-1,s,r-2), & (u) \\ 6f(k-1,s+1,r-2)+f(k-2,s,r-2), & (v) \\ 3f(k-1,s+1,r-2)+f(k,s+1,r-2)+\{4f(k-2,s,r-2); & \\ \qquad\qquad f(k-1,s+1,r-2)+2f(k-2,s,r-2); & \\ \qquad\qquad 2f(k-1,s,r-2)+f(k-2,s,r-2)\}. & (w) \end{cases}$$

We next list many known results that shall be applied in the proofs of Theorems 1.4 and 1.5.

Theorem 2.1 ([1,9]). *For all $k \geq 1$,*

$$GR_k(K_3) = \begin{cases} 5^{k/2}+1, & \text{if } k \text{ is even,} \\ 2\cdot 5^{(k-1)/2}+1, & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 2.2 ([2]). *$R(K_3,G_{20}) = R(K_3,G_{21}) = 9$ and $R(K_3,G_{22}) = 11$.*

Theorem 2.3 ([3]). *For all $k \geq 1$, $GR_k(P_3) = 3$.*

Theorem 2.4 ([8]). *$R_2(K_3) = 6$, $R(K_3,K_4) = 9$ and $R_2(K_4) = 18$.*

Theorem 2.5 ([11]). *$R_2(G_{20}) = R_2(G_{21}) = 18$ and $R_2(G_{22}) = 22$.*

Theorem 2.6 ([17]). *$R(P_3,K_3) = 5$ and $R(P_3,G_{22}) = 7$.*

3 Proofs of Theorems 1.4 and 1.5

We first show that $GR_k((k-r)K_3, rH) \geq w(k,r)+1$ with $H \in \{G_{20}, G_{21}\}$ and $GR_k((k-s-r)P_3, sK_3, rG_{22}) \geq f(k,s,r)+1$ for $k \geq 1$ and s, r with $0 \leq s \leq k$ and $0 \leq r \leq k$ by construction. We iteratively construct a k-edge colored complete graph I_k for all $k \geq 1$ which contains neither a rainbow triangle nor an appropriately colored monochromatic copy of $H \in \{G_{20}, G_{21}, G_{22}\}$ or K_3 or P_3 by the following iterative procedures. In other words, we first distribute the first r colors to $H \in \{G_{20}, G_{21}, G_{22}\}$ for Theorems 1.4 and 1.5, then we distribute the last $k-r$ colors to K_3 for Theorem 1.4; the middle s colors to K_3 and the last $k-s-r$ colors to P_3 for Theorem 1.5.

Let $i \geq 0$ be an even number in each of the following iterative procedures. By Theorems 2.2–2.4, let G^1 be a 2-edge colored K_{17} with colors $i+1$ and $i+2$ which contains no a monochromatic copy of K_4, G^2 be a 2-edge colored K_{10} with colors r and $r+1$ which contains neither a monochromatic copy of G_{22} in color r nor a monochromatic copy of K_3 in color $r+1$, G^3 be a 2-edge colored K_8 with colors r and $r+1$ which contains neither a monochromatic copy of K_4 in color r nor a monochromatic copy of K_3 in color $r+1$, G^4 be a 2-edge colored K_8 with colors r and $r+1$ which contains neither a monochromatic copy of $H \in \{G_{20}, G_{21}\}$ in color r nor a monochromatic copy of K_3 in color $r+1$, G^5 be a 2-edge colored K_5 with colors i and $i+1$ which contains no a monochromatic copy of K_3, G^6 be a 2-edge colored K_5 with colors $i+1$ and $i+2$ which contains no a monochromatic copy of K_3 and G^7 be a 1-edge colored K_2 with colors in $[k] \setminus [s+r]$ which contains no a monochromatic copy of P_3. Let I_0 be a single vertex. Suppose we have constructed I_{i-1} or I_i for some $i < k$ whenever running an iterative procedure. We distinguish two cases.

(I) $s + r = k$. (respectively, $k - r + r = k$ for the constructions of Theorem 1.4)

1. If $i \leq r-2$, then we construct I_{i+2} by replacing each vertex of G^1 with a copy of I_i which begins with I_2 for all even $r \geq 4$ and I_0 for all odd $r \geq 3$, where I_2 is a 2-edge colored K_{21} with colors 1 and 2 which contains no a monochromatic copy of G_{22}. (respectively, we construct I_{i+2} by replacing each vertex of G^1 with a copy of I_i which begins with I_0.)

2. If $i = r-1$, then we construct I_{i+2} by replacing each vertex of I_i with a copy of G^2 (respectively, G^4) when s (respectively, $k-r$) is odd and construct I_{i+1} by replacing each vertex of I_i with a monochromatic copy of K_4 in color r when s (respectively, $k-r$) is even.

3. If $r \leq i \leq s+r-2$ (respectively, $r \leq i \leq k-2$), then we construct I_{i+1} by replacing each vertex of G^5 with a copy of I_{i-1} when s (respectively, $k-r$) is even and r is odd, otherwise we construct I_{i+2} by replacing each vertex of G^6 with a copy of I_i.

4. If $r \leq i$ and $i = s+r-1$ (respectively, $r \leq i$ and $i = k-1$), then we construct I_{i+1} by replacing each vertex of I_{i-1} with a copy of G^5 when s (respectively,

$k - r$) is even and r is odd, and construct I_{i+1} by connecting two copies of I_i with all new edges in color k when s (respectively, $k - r$) is odd and r is even.

One can obtain the desired construction I_k for cases a_1 and b_1 from procedures 1 and 3, cases a_2 and b_3 from procedures 1, 3 and 4, cases a_3 and b_5 from procedures 1–3 and cases a_4 and b_7 from procedures 1–4.

(II) $s + r < k$.

5. If $i \leq r - 2$, then we construct I_{i+2} by replacing each vertex of G^1 with a copy of I_i.

6. If $i = r - 1$, then we construct I_{i+1} by replacing each vertex of I_i with a copy of K_3 in color r when $s = 0$ and construct I_{i+2} by replacing each vertex of I_i with a copy of G^3 when $s \geq 1$.

7. If $r \leq i \leq s + r - 2$, then we construct I_{i+2} by replacing each vertex of G^6 with a copy of I_i.

8. If $r \leq i$ and $i = s + r - 1$, then we construct I_{i+1} by connecting two copies of I_i with all new edges in color $s + r$.

9. If $s + r \leq i < k$, then we construct I_k by replacing each vertex of I_{s+r} with a copy of G^7.

One can obtain the desired construction I_k for case b_2 from procedures 5,7 and 9, case b_4 from procedures 5 and 7–9, case b_6 from procedures 5–7 and 9, and case b_8 from procedures 5–9.

Now, it suffices to show that $GR_k((k-r)K_3, rH) \leq w(k,r)+1$ with $H \in \{G_{20}, G_{21}\}$ and $GR_k((k-s-r)P_3, sK_3, rG_{22}) \leq f(k,s,r)+1$ for all $k \geq 1$ and s, r with $0 \leq s \leq k$ and $0 \leq r \leq k$. For the simplicity of the notations, we use C1.4 and C1.5 to represent the cases $GR_k((k-r)K_3, rH)$ with $H \in \{G_{20}, G_{21}\}$ and $GR_k((k-s-r)P_3, sK_3, rG_{22})$, respectively. We proceed the proofs by induction on $k + r$ for C1.4 and $k + s + 2r$ for C1.5. The case for $k = 1$ is trivial. By Theorems 2.2–2.6, $GR_2(H) = R_2(H) = w(2,2)+1$, $GR_2(K_3, H) = R(K_3, H) = w(2,1)+1$ and $GR_2(K_3) = R_2(K_3) = w(2,0)+1$ for C1.4; $GR_2(G_{22}) = R_2(G_{22}) = f(2,0,2)+1$, $GR_2(K_3) = R_2(K_3) = f(2,2,0)+1$, $GR_2(K_3, G_{22}) = R(K_3, G_{22}) = f(2,1,1)+1$, $GR_2(P_3, G_{22}) = R(P_3, G_{22}) = f(2,0,1)+1$ and $GR_2(P_3, K_3) = R(P_3, K_3) = f(2,1,0)+1$ for C1.5. The case $r = 0$ for C1.4 is Theorem 2.1, and the cases $s = k$ and $s + r = 0$ for C1.5 are Theorems 2.1 and 2.3, respectively. Therefore, we may assume that $k \geq 3$ and $1 \leq r \leq k$ for C1.4; $k \geq 3$, $1 \leq s + r \leq k$ and $0 \leq s < k$ for C1.5. Suppose that Theorem 1.4 holds for all $k' + r' < k + r$ and Theorem 1.5 holds for all $k' + s' + 2r' < k + s + 2r$, where k' is the total number of colors, s' is the number of colors assigned to K_3 and r' is the number of colors assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$. Now we consider a complete graph G with $w(k,r)+1$ and $f(k,s,r)+1$ vertices for $C1.4$ and $C1.5$, respectively. Let $c : E(G) \to [k]$ be any Gallai k-coloring of G. Suppose that (G, c) contains neither a monochromatic copy of $H \in \{G_{20}, G_{21}\}$ in any of the first r colors nor a monochromatic copy of K_3 in any of the last $k-r$ colors for C1.4 and (G, c) contains no a monochromatic copy of G_{22} in any of the first r colors, K_3 in any of the middle s colors and P_3 in any of the last $k-s-r$ colors for C1.5. Choose (G, c) with k minimum.

Let X_1 and X_2 be disjoint sets of $V(G)$ such that X_1 is color i-adjacent to X_2, $i \in [k]$. It is easily seen that $G[X_1 \cup X_2]$ contains a monochromatic copy of $H \in \{G_{20}, G_{21}\}$ in color i if condition 1 holds or $H \in \{G_{20}, G_{21}, G_{22}\}$ in color i if any one of conditions 2–4 holds.

1. Both $G[X_1]$ and $G[X_2]$ have an edge in color i and $|X_1 \cup X_2| \geq 5$.
2. $|X_1| \geq 2$ and $G[X_2]$ contains a K_3 in color i.
3. $|X_1| \geq 1$ and $G[X_2]$ contains a $K_4 - e$ in color i.
4. $G[X_1]$ has an edge in color i and $G[X_2]$ has a P_3 in color i.

Let $t_1, t_2, \ldots, t_m \in V(G)$ be a maximum sequence of vertices chosen as follows: for each $j \in [m]$, all edges between t_j and $V(G) \setminus \{t_1, t_2, \ldots, t_j\}$ are colored the same color under c. Let $T = \{t_1, t_2, \ldots, t_m\}$. Notice that T is possibly empty. For each $t_j \in T$, let $c(t_j)$ be the unique color on the edges between t_j and $V(G) \setminus \{t_1, t_2, \ldots, t_j\}$.

Claim 1. *$c(t_i) \neq c(t_j)$ for all $i, j \in [m]$ with $i \neq j$. Thus all colors in $\{c(t_1), \ldots, c(t_m)\}$ are assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$.*

Proof. Suppose that $c(t_i) = c(t_j)$ for some $i, j \in [m]$ with $i \neq j$. We may assume that t_j is the first vertex in the sequence $t_1, \ldots, t_m$ such that $c(t_i) = c(t_j)$ for some $i \in [m]$ with $i < j$. We may further assume that the color $c(t_i)$ is red. Thus the edge $t_i t_j$ is colored with red under c. Let $A = V(G) \setminus \{t_1, t_2, \ldots, t_j\}$. Then all the edges between $\{t_i, t_j\}$ and A are colored with red under c. For C1.4, as t_j is the first vertex in the sequence $t_1, \ldots, t_m$ such that $c(t_i) = c(t_j)$ for some $i \in [m]$ with $i < j$, by the pigeonhole principle, we see that $j \leq k+1$. Since $A = V(G) \setminus \{t_1, t_2, \ldots, t_j\}$, we have $|A| \geq |G| - (k+1) > k+1 \geq 4$ by (a). Therefore, red cannot be assigned to K_3 as $t_i t_j$ is a red edge. Furthermore, to avoid condition 1, there are no red edges in $(G[A], c)$. By induction, $|A| \leq w(k-1, r-1)$. By (a), we have $|G| \leq w(k-1, r-1) + k + 1 < w(k, r) + 1$, contrary to the fact that $|G| = w(k, r) + 1$. For C1.5, by (h) and (i), $|G| - (s+r) \geq 3$ as $s + r \geq 1$ and $f(k-1, s-1, r) \geq 2$. So no color in $\{c(t_1), \ldots, c(t_j)\}$ can be assigned to P_3. By the pigeonhole principle again, we see that $j \leq s + r + 1$ as t_j is the first vertex in the sequence $t_1, \ldots, t_m$ such that $c(t_i) = c(t_j)$ for some $i \in [m]$ with $i < j$. Since $A = V(G) \setminus \{t_1, t_2, \ldots, t_j\}$ and $|G| - (s+r) \geq 3$, we have $|A| \geq |G| - (s+r+1) \geq 2$. Therefore, red cannot be assigned to K_3 as $t_i t_j$ is a red edge. To avoid condition 4, there does not exist a red P_3 in $(G[A], c)$. By induction, $|A| \leq f(k, s, r-1)$. By (h), we have $|G| \leq f(k, s, r-1) + s + r + 1 < f(k, s, r) + 1$, which is a contradiction. Thus $c(t_i) \neq c(t_j)$ for all $i, j \in [m]$ with $i \neq j$.

Since $|G| - (s+r) \geq 3$, all colors in $\{c(t_1), \ldots, c(t_m)\}$ will be assigned to K_3 or $H \in \{G_{20}, G_{21}, G_{22}\}$. This implies that $|T| \leq k$ for C1.4 and $|T| \leq s + r$ for

C1.5. If there exists a color in $\{c(t_1), \ldots, c(t_m)\}$ which is assigned to K_3, say green, then $(G \setminus T, c)$ contains no green edges. By induction, $|G \setminus T| \leq w(k-1, r)$ and $|G \setminus T| \leq f(k-1, s-1, r)$. By (b) and (i), we have $|G| \leq w(k-1, r)+k < w(k, r)+1$ and $|G| \leq f(k-1, s-1, r) + s + r < f(k, s, r) + 1$, which are impossible and thus the statement follows. $\square$

By Claim 1, we see that $|T| \leq r$. Consider a Gallai-partition of $G \setminus T$ with parts $V_1, V_2, \ldots, V_\ell$ such that $\ell \geq 2$ is as small as possible. Assume that $|V_1| \geq |V_2| \geq \ldots \geq |V_\ell|$. Let $\mathcal{G}$ be the reduced graph of $G \setminus T$ with vertices $v_1, \ldots, v_\ell$. By Theorem 1.1, we may further assume that the edges of $\mathcal{G}$ are colored with red or blue. Clearly, any monochromatic copy of $H \in \{G_{20}, G_{21}, G_{22}\}$ or K_3 or P_3 in $\mathcal{G}$ would yield a monochromatic copy of $H \in \{G_{20}, G_{21}, G_{22}\}$ or K_3 or P_3 in $G \setminus T$. Let

$$\mathcal{V}_r = \{V_i \mid V_i \text{ is red-adjacent to } V_1 \text{ under } c,\ i \in \{2, \ldots, \ell\}\} \text{ and}$$
$$\mathcal{V}_b = \{V_i \mid V_i \text{ is blue-adjacent to } V_1 \text{ under } c,\ i \in \{2, \ldots, \ell\}\}.$$

Let $R = \bigcup_{V_i \in \mathcal{V}_r} V_i$ and $B = \bigcup_{V_i \in \mathcal{V}_b} V_i$. Then we see that $|G| = |V_1 \cup R \cup B \cup T| = |V_1| + |R| + |B| + |T|$. Without loss of generality, we may assume that $|B| \leq |R|$. Obviously, $|R| \geq 2$, otherwise the vertex in R or B can be added to T, contrary to the maximality of m in T. Therefore, red cannot be assigned to P_3 for C1.5.

Claim 2. $|V_1| \geq 2$.

Proof. For C1.4, by Theorem 2.5, $\ell \leq R_2(H) - 1 = 17$ with $H \in \{G_{20}, G_{21}\}$. As $|G \setminus T| \geq w(3, 1) = 20$ for all $k \geq 3$ and $1 \leq r \leq k$, it follows that $|V_1| \geq 2$. For C1.5, if $|V_1| = 1$, then $(G \setminus T, c)$ is only colored with red and blue. By Claim 1, $r \geq 1$ as $k \geq 3$. Moreover, since red cannot be assigned to P_3, $|G \setminus T| \geq f(3, 1, 1) = 16$. If there is at most one of red and blue which is assigned to G_{22}, then by Theorems 2.2, 2.4 and 2.6, $|G \setminus T| = \ell \leq R(K_3, G_{22}) - 1 = 10$, a contradiction. If both red and blue are assigned to G_{22}, then by Theorem 2.5, $\ell \leq R_2(G_{22}) - 1 = 21$. Thus by Claim 1, $r \geq 3$, and hence $|G \setminus T| \geq f(3, 0, 3) - 2 > 21$, which is a contradiction. $\square$

Claim 3. *No vertex in T is red-adjacent to $V(G) \setminus T$ under c.*

Proof. Suppose not. Then red must be assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$ by Claim 1. As $|R| \geq 2$ and $|V_1| \geq 2$, to avoid condition 3, there are no red edges in either $(G[V_1], c)$ or $(G[R], c)$. For C1.4, by induction, $|V_1| \leq w(k-1, r-1)$ and $|B| \leq |R| \leq w(k-1, r-1)$. Then by (a), we have $|G| \leq 3w(k-1, r-1) + r < w(k, r) + 1$, a contradiction. For C1.5, by induction, $|B| \leq |R| \leq f(k-1, s, r-1)$. If $|B| \leq 1$, then $(G[V_1 \cup B], c)$ contains no red edges and thus $|V_1 \cup B| \leq f(k-1, s, r-1)$. By (j), we have $|G| \leq 2f(k-1, s, r-1) + r < f(k, s, r) + 1$, a contradiction again. Therefore, $|B| \geq 2$ and hence blue cannot be assigned to P_3. If blue is assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$, then, to avoid condition 2, there is not a blue K_3 in $(G[V_1], c)$. By induction, $|V_1| \leq f(k-1, s+1, r-2) \leq 3f(k-2, s, r-2)$. Note that $r \geq 2$ as both red and blue are assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$. Then $|B| \leq |R| \leq f(k-1, s, r-1) \leq 6f(k-2, s, r-2)$. By (q), we have $|G| \leq 15f(k-2, s, r-2) + r < f(k, s, r) + 1$, which is impossible. So blue must be assigned to K_3 and there are no blue edges in $(G[V_1], c)$. Now $(G[V_1], c)$ contains neither a red nor a blue edge. By induction, $|V_1| \leq f(k-2, s-1, r-1)$. As $(G[R], c)$ contains no a red edge, to avoid a blue K_3, R has at most two parts of $\{V_2, \ldots, V_\ell\}$. It follows that $|B| \leq |R| \leq 2|V_1|$. By (k), we have $|G| \leq 5f(k-2, s-1, r-1) + r < f(k, s, r) + 1$, yielding a contradiction. □

Claim 4. $|B| \geq 2$.

Proof. Suppose not. Assume that red is assigned to K_3. Then by Claims 1–3, there are no red edges in either $(G[V_1 \cup B \cup T], c)$ or $(G[R], c)$. By induction, $|V_1 \cup B \cup T| \leq w(k-1, r)$ and $|R| \leq w(k-1, r)$; $|V_1 \cup B \cup T| \leq f(k-1, s-1, r)$ and $|R| \leq f(k-1, s-1, r)$. By (b) and (i), we have $|G| \leq 2w(k-1, r) < w(k, r) + 1$ and $|G| \leq 2f(k-1, s-1, r) < f(k, s, r) + 1$, which are impossible. Therefore, red must be assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$. If $(G[V_1], c)$ does not contain red edge, and neither does $(G[V_1 \cup B \cup T], c)$ by Claim 3. Moreover, to avoid condition 2, there is not a red K_3 in $(G[R], c)$ as $|V_1| \geq 2$. By induction, $|V_1 \cup B \cup T| \leq w(k-1, r-1)$ and $|R| \leq w(k, r-1)$; $|V_1 \cup B \cup T| \leq f(k-1, s, r-1)$ and $|R| \leq f(k, s+1, r-1)$. By (c) and (l), we have $|G| \leq w(k-1, r-1) + w(k, r-1) < w(k, r) + 1$ and $|G| \leq f(k-1, s, r-1) + f(k, s+1, r-1) < f(k, s, r) + 1$, which are impossible. Thus there are red edges in $(G[V_1], c)$. For C 1.4, we see that $|V_1 \cup R| \geq 5$ as $|G| > r + 5$ by (d). To avoid conditions 1 and 2, there does not exist a red edge in $(G[R], c)$ and a red K_3 in $(G[V_1], c)$. Similar to the above arguments, $|G| < w(k, r) + 1$. For C1.5, to avoid condition 4, there is not a red P_3 in $(G[R], c)$. By the above arguments, we only need to consider the case that $(G[R], c)$ contains red edges. Clearly, $(G[V_1], c)$ does not contain a red P_3, and neither does $(G[V_1 \cup B \cup T], c)$. By induction, $|V_1 \cup B \cup T| \leq f(k, s, r-1)$ and $|R| \leq f(k, s, r-1)$. By (h), we have $|G| \leq 2f(k, s, r-1) < f(k, s, r) + 1$, this yields a contradiction. □

By Claims 2 and 4, blue is assigned to K_3 or $H \in \{G_{20}, G_{21}, G_{22}\}$. Define $Y_1 = \{V_i : |V_i| = 1, i \in \{2, \ldots, \ell\}\}$ and $Y_2 = \{V_i : |V_i| \geq 2, i \in \{2, \ldots, \ell\}\}$. Then $|Y_1 \cup Y_2| = |R \cup B|$. Let $|R \cap Y_t|$ and $|B \cap Y_t|$ be the number of the common parts in $\{V_2, \ldots, V_\ell\}$, where $t = 1, 2$.

Claim 5. *Suppose that blue is assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$ and $(G[B], c)$ contains no blue edges (respectively, blue P_3). Then $|B \cap Y_2| \leq 3$. In particular,*

(1) if $|B \cap Y_2| = 3$, then $|B \cap Y_1| = 0$;

(2) if $|B \cap Y_2| = 2$, then $|B \cap Y_1| \leq 1$ (respectively, $|B \cap Y_1| \leq 2$);

(3) if $|B \cap Y_2| = 1$, then $|B \cap Y_1| \leq 2$ (respectively, $|B \cap Y_1| \leq 4$);

(4) if $|B \cap Y_2| = 0$, then $|B \cap Y_1| \leq 4$ (respectively, $|B \cap Y_1| \leq 6$);

(5) $|B| \leq 3|V_1|$.

Moreover, if red is assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$ and $(G[R], c)$ contains no red edges (respectively, a red P_3), then similar properties hold for R.

Proof. We only give the proof for B. The proof for R is similar. It is easily seen that $|B \cap Y_2| \leq 3$. If $(G[B], c)$ contains no blue edges, then the edges between any two parts of $\{V_2, \ldots, V_\ell\}$ in B are colored with red. In order to avoid a red $H \in \{G_{20}, G_{21}, G_{22}\}$, B has at most three parts unless $|B \cap Y_2| = 0$. However, if $|B \cap Y_2| = 0$, then $|B \cap Y_1| \leq 4$, otherwise we can get a red K_5 that contains all red $H \in \{G_{20}, G_{21}, G_{22}\}$. If $(G[B], c)$ contains no a blue P_3, then Y_1 is red-adjacent to Y_2 in $(G[B], c)$. Also, all the parts of Y_2 in B are red-adjacent to each other. For (1), there exists a red $K_4 - e$ in $(G[B \cap Y_2], c)$. Thus, to avoid condition 3, we have $|B \cap Y_1| = 0$. Suppose that $|B \cap Y_1| \geq 3$ for (2). Let $V_i, V_j, V_k \in Y_1$. Since $(G[B \cap Y_2], c)$ contains red edges, to avoid condition 4, there exists a blue P_3 in $(G[V_i \cup V_j \cup V_k], c)$, which is a contradiction. Suppose that $|B \cap Y_1| \geq 5$ for (3). Then there will be a red K_3 in $(G[B \cap Y_1], c)$ as $R(P_3, K_3) = 5$. To avoid condition 2, we have $|B \cap Y_2| = 0$, which is also a contradiction. For (4), to avoid a red G_{22}, $(G[B], c)$ must have at most $R(P_3, G_{22}) - 1 = 6$ parts. Since $|B \cap Y_2| = 0$, it follows that $|B \cap Y_1| \leq 6$. Recall that $|V_1| \geq 2$. So (5) holds based on (1)–(4). $\square$

Claim 6. *No vertex in T is blue-adjacent to $V(G) \setminus T$ under c.*

Proof. Suppose not. Then blue will be assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$ by Claim 1. As $|B| \geq 2$ and $|V_1| \geq 2$, to avoid condition 3, there does not exist blue edges in either $(G[V_1], c)$ or $(G[B], c)$. If red is assigned to K_3, then there are no red edges in either $(G[V_1], c)$ or $(G[R], c)$. Now $(G[V_1], c)$ contains neither a red nor a blue edge. By induction, $|V_1| \leq w(k-2, r-1)$ and $|V_1| \leq f(k-2, s-1, r-1)$. Since $(G[R], c)$ contains no a red edge, the edges between any two parts in R are colored with blue. Thus, to avoid a blue $H \in \{G_{20}, G_{21}, G_{22}\}$, R contains at most four parts of $\{V_2, \ldots, V_\ell\}$. In particular, if R has four parts, then $|R| = 4$. Hence $|B| \leq |R| \leq 3|V_1|$. By (d) and (k), we have $|G| \leq 7w(k-2, r-1) + r < w(k, r) + 1$ and $|G| \leq 7f(k-2, s-1, r-1) + r < f(k, s, r) + 1$, which are impossible. Recall that red cannot be assigned to P_3. Therefore, red must be assigned to $H \in \{G_{20}, G_{21}, G_{22}\}$ and thus $r \geq 2$.

Assume that $(G[V_1], c)$ contains red edges. For C1.4, we see that $|V_1 \cup R| \geq 5$ as $|G| > r + 6$ by (d). To avoid condition 1, there are no red edges in $(G[R], c)$. By induction, $|V_1| \leq w(k-1, r-1)$ and $|B| \leq |R| \leq w(k-1, r-1)$. By (a), we have $|G| \leq 3w(k-1, r-1) + r < w(k, r) + 1$, a contradiction. For C1.5, if $(G[R], c)$ contains no a red edge, then by the similar arguments as in the proof of Claim 3, we can show that $|G| \leq 16f(k-2, s, r-2) < f(k, s, r) + 1$. Thus $(G[R], c)$ contains red edges. To avoid condition 4, there is not a red P_3 in either $(G[V_1], c)$ or $(G[R], c)$. So by induction and (5), $|V_1| \leq f(k-1, s, r-2) \leq 2f(k-2, s, r-2)$ and $|B| \leq |R| \leq 3|V_1|$. By (q), we have $|G| \leq 14f(k-2, s, r-2) + r < f(k, s, r) + 1$, a contradiction. Therefore, we conclude that $(G[V_1], c)$ contains no red edges. Now there is neither a red nor a blue edge in $(G[V_1], c)$. By induction, $|V_1| \leq w(k-2, r-2)$ and $|V_1| \leq f(k-2, s, r-2)$. To avoid condition 2, there is not a red K_3 in $(G[R], c)$. Thus R has at most $R(K_3, H) - 1 = 8$ parts with $H \in \{G_{20}, G_{21}\}$ for C1.4 and $R(K_3, G_{22}) - 1 = 10$ parts for C1.5. Recall that $(G[B], c)$ contains no blue edges. So $|B| \leq 3|V_1|$ by (5). By (e) and (q), we have $|G| \leq 12w(k-2, r-2) + r < w(k, r) + 1$

and $|G| \leq 14f(k-2, s, r-2) + r < f(k, s, r) + 1$, which are impossible. $\square$

Obviously, if $(G[V_1], c)$ contains red or blue edges, then so does $(G[V_1 \cup T], c)$. By Claims 3 and 6, if $(G[V_1], c)$ does not contain any red or blue edges, then neither does $(G[V_1 \cup T], c)$. We next consider the following three cases.

Case 1. *Blue and red are the colors assigned to* K_3*.*

Recall that $|R| \geq |B| \geq 2$. So $(G[V_1 \cup T], c)$ contains neither red nor blue edges. By induction, $|V_1 \cup T| \leq w(k-2, r)$ and $|V_1 \cup T| \leq f(k-2, s-2, r)$. Furthermore, by Theorem 2.4, we see that $\ell \leq R_2(K_3) - 1 = 5$. By (b) and (m), $|G| \leq 5w(k-2, r) < w(k, s) + 1$ and $|G| \leq 5f(k-2, s-2, r) < f(k, s, r) + 1$, which are impossible. Thus this completes the proof of Case 1.

Case 2. *Blue* (*respectively, red*) *is assigned to* K_3 *and red* (*respectively, blue*) *is assigned to* $H \in \{G_{20}, G_{21}, G_{22}\}$*.*

Clearly, $r \geq 1$ and $s \geq 1$ in this case. As $|R| \geq |B| \geq 2$, $(G[V_1 \cup T], c)$ contains no blue (respectively, red) edges. For C1.4, if $(G[V_1], c)$ does not contain red (respectively, blue) edge, then neither does $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq w(k-2, r-1)$. Since $\ell \leq R(K_3, H) - 1 = 8$ with $H \in \{G_{20}, G_{21}\}$, by (d), we have $|G| \leq 8w(k-2, r-1) < w(k, r)+1$, contrary to the fact that $|G| = w(k, r)+1$, and hence $(G[V_1], c)$ contains red (respectively, blue) edges. For the former case, as $|G| > r + 6$ and $|T| \leq r$, we see that $|V_1 \cup R| \geq 5$. To avoid conditions 1 and 2, there does not exist red edges in $(G[R], c)$ and a red K_3 in $(G[V_1 \cup T], c)$. By induction, $|B| \leq |R| \leq w(k-1, r-1)$ and $|V_1 \cup T| \leq w(k-1, r-1)$. By (a), we have $|G| \leq 3w(k-1, r-1) < w(k, r) + 1$, yielding a contradiction. For the latter case, obviously, $(G[R], c)$ has no red edges. To avoid conditions 1 and 2, there does not exist blue edges in $(G[B], c)$ unless $|V_1 \cup B| \leq 4$, and a blue K_3 in $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq w(k-1, r-1)$, $|R| \leq w(k-1, r)$ and $|B| \leq w(k-1, r-1)$. By (f), we have $|G| \leq w(k-1, r) + 2w(k-1, r-1) < w(k, r) + 1$, which is impossible.

For C1.5, we may first assume that $(G[V_1], c)$ contains no red (respectively, blue) edges. By induction, $|V_1 \cup T| \leq f(k-2, s-1, r-1)$. Obviously, $\ell \leq R(K_3, G_{22}) - 1 = 10$. In particular, if $9 \leq \ell \leq 10$, then there exists a red K_4 in the reduced graph $\mathcal{G}$ as $R(K_3, K_4) = 9$. Note that a monochromatic copy of K_4 in $\mathcal{G}$ only has four vertices in $G \setminus T$, otherwise we can get a monochromatic copy of G_{22}. This means that $|G| \leq 6|V_1 \cup T| + 4 \leq 8|V_1 \cup T|$. By (k), we have $|G| \leq 8f(k-2, s-1, r-1) < f(k, s, r)+1$, a contradiction. Therefore, there exist red (respectively, blue) edges in $(G[V_1], c)$. For the former case, if $(G[R], c)$ contains no red edges, then, to avoid condition 2, there is not a red K_3 in $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq f(k-1, s, r-1)$ and $|B| \leq |R| \leq f(k-1, s, r-1)$. By (j), we have $|G| \leq 3f(k-1, s, r-1) < f(k, s, r)+1$, which is a contradiction. So $(G[R], c)$ contains red edges. To avoid condition 4, there is not a red P_3 in either $(G[R], c)$ or $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq f(k-1, s-1, r-1)$. Moreover, by Theorem 2.6, R has at most $R(P_3, K_3) - 1 = 4$ parts of $\{V_2, \ldots, V_\ell\}$ and thus there are at most two independent red edges between the parts in R. It is easy to check that $|R| \leq 4$ when R has two such edges, $|R| \leq |V_1| + 2$ when R has only one such edge and $|R| \leq 2|V_1|$ when R does not contain such an edge. It follows that $|B| \leq |R| \leq 2|V_1|$. By (n), we have $|G| \leq 5f(k-1, s-1, r-1) < f(k, s, r)+1$, this yields a contradiction.

For the latter case, it is obvious that $(G[R], c)$ contains no a red edge. To avoid a blue G_{22}, R has at most four parts of $\{V_2, \ldots, V_\ell\}$ and the edges between any two parts are colored with blue. Assume that $(G[B], c)$ contains blue edges. Then there is not a blue P_3 in $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq f(k-1, s-1, r-1) \leq 2f(k-2, s-1, r-1)$. If R has four parts, then $|R| \leq 4$, and if R has three parts, then, to avoid a blue G_{22}, none of these parts contain blue edges unless $|R \cap Y_2| \leq 1$. By induction, $|B| \leq |R| \leq 3f(k-2, s-1, r-1)$. By (k), we have $|G| \leq 8f(k-2, s-1, r-1) < f(k, s, r) + 1$, which is a contradiction. Finally, if $|R \cap Y_2| \leq 1$ or R has at most two pars, then $|B| \leq |R| \leq 2|V_1|$ and thus, by (n), we have $|G| \leq 5f(k-1, s-1, r-1) < f(k, s, r) + 1$, a contradiction again. So $(G[B], c)$ has no a blue edge and thus there is not a blue K_3 in $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq f(k-1, s, r-1) \leq 3f(k-2, s-1, r-1)$. Note that $\ell \geq 3$ as $|B| \geq 2$. This implies that each part of $\{V_1, \ldots, V_\ell\}$ cannot be only red or blue-adjacent to the rest of parts, otherwise we can get a Gallai-partition of $G \setminus T$ with only two parts. Hence there are no red edges in each part. To avoid red K_3, B has at most two parts. So by induction, $|B| \leq 2f(k-2, s-1, r-1)$. Similar to above arguments, if R has at least three parts, then $|R| \leq 3f(k-2, s-1, r-1)$ and thus $|G| < f(k, s, r) + 1$. Now we consider the case that R has two parts. According to above calculations, at least one part in R contains blue edges. If R only has one part with blue edges, then, by induction, $|R| \leq |V_1| + f(k-2, s-1, r-1)$. By (o), we have $|G| \leq 2f(k-1, s, r-1) + 3f(k-2, s-1, r-1) < f(k, s, r) + 1$, a contradiction. If both parts contain blue edges, then, to avoid condition 4, there does not exist a blue P_3 in both parts. By induction, $|R| \leq 2f(k-1, s-1, r-1)$. By (p), we have $|G| \leq f(k-1, s, r-1) + 2f(k-1, s-1, r-1) + 2f(k-2, s-1, r-1) < f(k, s, r) + 1$, a contradiction again. Finally, if R only has one part, then $|R| \leq |V_1|$. Similar to the above calculations, there is a contradiction. Thus this completes the proof of Case 2.

Case 3. *Blue and red are assigned to* $H \in \{G_{20}, G_{21}, G_{22}\}$.

In this case, we see that $r \geq 2$. Suppose that $(G[V_1 \cup T], c)$ contains no blue edges. If $(G[V_1 \cup T], c)$ contains no red edges, then by induction, $|V_1 \cup T| \leq w(k-2, r-2)$ and $|V_1 \cup T| \leq f(k-2, s, r-2)$. For C1.4, we see that $\ell \leq R_2(H) - 1 = 17$ with $H \in \{G_{20}, G_{21}\}$. For C1.5, to avoid condition 2, there is not a red K_3 in $(G[R], c)$. So R has at most $R(K_3, G_{22}) - 1 = 10$ parts of $\{V_2, \ldots, v_\ell\}$. Since a monochromatic copy of K_4 in the reduced graph $\mathcal{G}$ only contains four vertices in $G \setminus T$, $|B| \leq |R| \leq 6|V_1| + 4 \leq 8|V_1|$ as $R(K_3, K_4) = 9$. By (e) and (q), we have $|G| \leq 17w(k-2, r-2) < w(k, r) + 1$ and $|G| \leq 17f(k-2, s, r-2) < f(k, s, r) + 1$, which are impossible. Hence $(G[V_1 \cup T], c)$ contains red edges. Note that $|V_1 \cup R| \geq 5$ for C1.4 as $|G| > r+6$ by (d). To avoid conditions 1 and 4, there are no red edges for C1.4 and a red P_3 for C1.5 in $(G[R], c)$. If $(G[R], c)$ contains no red edges for both C1.4 and C1.5, then there does not exist a red K_3 in $(G[V_1 \cup T], c)$. By induction, $|V_1 \cup T| \leq w(k-1, r-2) \leq w(k-1, r)$ and $|B| \leq |R| \leq w(k-1, r-1)$; $|V_1 \cup T| \leq f(k-1, s+1, r-2) \leq 3f(k-2, s, r-2)$

and $|B| \leq |R| \leq f(k-1,s,r-1) \leq 6f(k-2,s,r-2)$. By (f) and (q), we have $|G| \leq w(k-1,r)+2w(k-1,r-1) < w(k,r)+1$ and $|G| \leq 15f(k-2,s,r-2) < f(k,s,r)+1$, which are also impossible. Thus $(G[R],c)$ contains red edges but does not contain red P_3 for C1.5, and hence there is not a red P_3 in $(G[V_1 \cup T],c)$. By induction, $|V_1 \cup T| \leq f(k-1,s,r-2) \leq 2f(k-2,s,r-2)$. Furthermore, by (5), $|B| \leq |R| \leq 3|V_1|$. By (q), we have $|G| \leq 14f(k-2,s,r-2) < f(k,s,r)+1$, yielding a contradiction. Therefore, there exist blue edges in $(G[V_1 \cup T],c)$.

Claim 7. *Suppose that $(G[B],c)$ contains no blue edges.*

(6) If $|B \cap Y_2| = 3$, then $|B| \leq 3w(k-2,r-2)$ for C1.4 and $|B| \leq 3f(k-2,s,r-2)$ for C1.5;

(7) if $|B \cap Y_2| \leq 2$, then $|B| \leq |V_1| + w(k-2,r-2)$ for C1.4 and $|B| \leq 2f(k-1,s,r-2)$ or $|B| \leq |V_1| + f(k-2,s,r-2)$ for C1.5.

Proof. Let $|B \cap Y_2| = 3$. To avoid condition 3, there are no red edges in each part of $B \cap Y_2$. By induction and (1), (6) holds. Let $|B \cap Y_2| \leq 1$. By (3) and (4), we see that $|B| \leq |V_1| + 2$. Let $|B \cap Y_2| = 2$. By (2), $|B \cap Y_1| \leq 1$. So we first assume that $|B \cap Y_1| = 1$. By the same reason as above, $|B| \leq 2w(k-2,r-2)+1$ and $|B| \leq 2f(k-2,s,r-2)+1$. We next assume that $|B \cap Y_1| = 0$. Now B only has two parts. To avoid condition 1, there is at least one part without a red edge for C1.4 unless $|B| = 4$. By induction, $|B| \leq |V_1| + w(k-2,r-2)$. For C1.5, if both two parts have red edges, then, to avoid condition 4, there is not a red P_3 in either of them. By induction, $|B| \leq 2f(k-1,s,r-2)$. If there is at least one part without a red edge, then by induction, $|B| \leq |V_1| + f(k-2,s,r-2)$. □

For C1.4, if there exist red edges in $(G[V_1 \cup T],c)$, then $(G[V_1 \cup T],c)$ contains red and blue edges and so does $(G[V_1],c)$ by Claim 6, and hence $|V_1| \geq 3$. Recall that $|R| \geq 2$ and $|B| \geq 2$. To avoid conditions 1 and 2, $(G[R],c)$ contains no red edges and $(G[V_1 \cup T],c)$ contains neither a red nor a blue K_3. By induction, $|V_1 \cup T| \leq w(k,r-2) \leq w(k-1,r)$ and $|B| \leq |R| \leq w(k-1,r-1)$. By (f), we have $|G| \leq w(k-1,r)+2w(k-1,r-1) < w(k,r)+1$, yielding a contradiction. Thus $(G[V_1 \cup T],c)$ contains no a red edge. Similar to above arguments, there does not exist a blue K_3 in $(G[V_1 \cup T],c)$ and a red K_3 in $(G[R],c)$. By induction, $|V_1 \cup T| \leq w(k-1,r-2)$ and $|R| \leq w(k,r-1)$. Note that $|V_1 \cup B| \geq 5$ as $|G| > w(k,r-1)+r+4$ by (c). Since $(G[V_1],c)$ contains blue edges, there are no blue edges in $(G[B],c)$. By Claim 5, we see that $|B \cap Y_2| \leq 3$. Furthermore, by Claim 7, $|B| \leq 3w(k-2,r-2) \leq w(k-1,r-2)+w(k-2,r-2)$. Then by (g), we have $|G| \leq 2w(k-1,r-2)+w(k,r-1)+w(k-2,r-2) < w(k,r)+1$, contrary to the fact that $|G| = w(k,r)+1$. Thus this completes the proof of C1.4.

For C1.5, as $(G[V_1 \cup T],c)$ contains blue edges, there is not a blue P_3 in $(G[B],c)$. Assume that $(G[B],c)$ contains blue edges. Then there is not a blue P_3 in $(G[V_1 \cup T],c)$. We may further assume that $(G[V_1 \cup T],c)$ contains red edges. If $(G[R],c)$ also has red edges, then there is not a red P_3 in either $(G[V_1 \cup T],c)$ or $(G[R],c)$. By induction, $|V_1 \cup T| \leq f(k,s,r-2) \leq 2f(k-2,s,r-2)$. Moreover, by (5),

$|B| \leq |R| \leq 3|V_1|$. Then by (q), we have $|G| \leq 14f(k-2,s,r-2) < f(k,s,r)+1$, a contradiction. So $(G[R],c)$ contains no red edges. Thus there is not a red K_3 in $(G[V_1 \cup T],c)$. By induction, $|V_1 \cup T| \leq f(k,s+1,r-2) \leq f(k,s+2,r-2)$ and $|B| \leq |R| \leq f(k-1,s,r-1)$. By (r), we have $|G| \leq f(k,s+2,r-2)+2f(k-1,s,r-1) < f(k,s,r)+1$, a contradiction again. So $(G[V_1 \cup T],c)$ contains no a red edge. Hence there does not exist a red K_3 in $(G[R],c)$. So by induction, $|V_1 \cup T| \leq f(k-1,s,r-2)$ and $|R| \leq f(k,s+1,r-1)$. As $(G[B],c)$ contains no a blue P_3, by (5), $|B| \leq 3|V_1|$. Then by (s), we have $|G| \leq 4f(k-1,s,r-2)+f(k,s+1,r-1) < f(k,s,r)+1$, which is impossible. Therefore, there are no blue edges in $(G[B],c)$.

Clearly, $(G[V_1 \cup T],c)$ contains no a blue K_3. Assume that $(G[V_1 \cup T],c)$ contains red edges. If $(G[R],c)$ has no red edges, then there does not exist a red K_3 in $(G[V_1 \cup T],c)$. By induction, $|V_1 \cup T| \leq f(k,s+2,r-2)$ and $|B| \leq |R| \leq f(k-1,s,r-1)$. By (r), we have $|G| \leq f(k,s+2,r-2)+2f(k-1,s,r-1) < f(k,s,r)+1$, a contradiction. Thus there exist red edges in $(G[R],c)$, and hence there is not a red P_3 in either $G[V_1 \cup T],c)$ or $(G[R],c)$. By induction, $|V_1 \cup T| \leq f(k,s+1,r-2)$ and $|R| \leq f(k,s,r-1)$. Since $(G[B],c)$ contains no blue edges, by Claim 5, $|B \cap Y_2| \leq 3$. By (t), (6) and (7), we have $|G| \leq f(k,s+1,r-2)+f(k,s,r-1)+\{3f(k-2,s,r-2);2f(k-1,s,r-2)\} < f(k,s,r)+1$, which is a contradiction. By (7) again, what is left is to consider the case that $|V_1 \cup T| \leq f(k,s+1,r-2)$ and $|B| \leq |V_1|+f(k-2,s,r-2)$. As $(G[R],c)$ contains no a red P_3, we have the following claim.

Claim 8. $|R| \leq f(k,s+1,r-2)+2f(k-1,s,r-2)$.

Proof. By Claim 5, $|R \cap Y_2| \leq 3$. Similar to the proofs of (6) and (7), we see that $|R| \leq 3f(k-1,s,r-2)$ when $|R \cap Y_2| = 3$, $|R| \leq |V_1|+4$ when $|R \cap Y_2| \leq 1$ and $|R| \leq 2f(k-1,s,r-2)+2$ when $|R \cap Y_2| = 2$ and $|R \cap Y_1| \geq 1$. Furthermore, if $|R \cap Y_2| = 2$ and $|R \cap Y_1| = 0$, then we have $|R| \leq 2f(k,s,r-2) = 2f(k-1,s,r-2)$ or $|R| \leq |V_1|+f(k-1,s,r-2) \leq |V_1|+2f(k-1,s,r-2)$. Recall that $|V_1 \cup T| \leq f(k,s+1,r-2)$. Also, by Table 2, $2 \leq f(k-1,s,r-2) \leq f(k,s+1,r-2)$. Therefore, $|R| \leq 3f(k-1,s,r-2) \leq f(k,s+1,r-2)+2f(k-1,s,r-2)$. □

By Claim 8 and (u), we have $|G| \leq 3f(k,s+1,r-2)+f(k-2,s,r-2)+2f(k-1,s,r-2) < f(k,s,r)+1$, contrary to the fact that $|G| = f(k,s,r)+1$. So $(G[V_1 \cup T],c)$ contains no red edges.

Obviously, there does not exist a red K_3 in $(G[R],c)$. As $(G[V_1 \cup T],c)$ contains no a blue K_3, by induction, $|V_1 \cup T| \leq f(k-1,s+1,r-2)$ and $|R| \leq f(k,s+1,r-1)$. Then by (t), (6) and (7), we have $|G| \leq f(k-1,s+1,r-2)+f(k,s+1,r-1)+\{3f(k-2,s,r-2);2f(k-1,s,r-2)\} < f(k,s,r)+1$, yielding a contradiction. Therefore, by (7) again, it remains to consider the case that $|V_1 \cup T| \leq f(k-1,s+1,r-2) \leq 3f(k-2,s,r-2)$ and $|B| \leq |V_1|+f(k-2,s,r-2) \leq 4f(k-2,s,r-2)$. As $(G[R],c)$ contains no a red K_3, we have the following claim.

Claim 9. $|R| \leq 10f(k-2,s,r-2)$.

Proof. Suppose that R contains a part of $\{V_2,\dots,V_\ell\}$ with red edges, say V^r. In order to avoid the red K_3, V^r must be blue-adjacent to the rest of parts in R. Furthermore, there is neither a red nor a blue K_3 in $(G[R\backslash V^r],c)$. So R contains at most $R_2(K_3)=6$ parts of $\{V_2,\dots,V_\ell\}$. Note that R has at least 5 parts of $\{V_2,\dots,V_\ell\}$, otherwise we have $|R|\leq 4|V_1|$ and then, by (v), $|G|\leq 6f(k-1,s+1,r-2)+f(k-2,s,r-2)<f(k,s,r)+1$, a contradiction. Moreover, each part of $R\setminus V^r$ cannot be only red or blue-adjacent to the rest of parts in $R\setminus V^r$, otherwise we can obtain a blue or a red K_3 as $R(K_2,K_3)=3$. This means that all parts of $R\setminus V^r$ contain neither red nor blue edges. By induction, $|R|\leq |V_1|+5f(k-2,s,r-2)\leq 8f(k-2,s,r-2)$. Then by (q), we have $|G|\leq 15f(k-2,s,r-2)<f(k,s,r)+1$, which is a contradiction.

Suppose that R has a part of $\{V_2,\dots,V_\ell\}$ with blue edges, say V^b. Let $N_b(V^b)$ and $N_r(V^b)$ be the vertex sets of R such that $R=V^b\cup N_r(V^b)\cup N_b(V^b)$ and all the vertices of $N_b(V^b)$ and $N_r(V^b)$ are blue and red-adjacent to V^b, respectively. Then there does not exist a red K_3 and a blue P_3 in $(G[N_b(V^b)],c)$. By induction, $|N_b(V^b)|\leq f(k,s+1,r-2)$. Moreover, there are no red edges in $(G[N_r(V^b)],c)$ and thus $|N_r(V^b)\cap Y_2|\leq 3$. Similar to the proofs of (6) and (7), we have $|N_r(V^b)|\leq 3f(k-2,s,r-2)$ when $|N_r(V^b)\cap Y_2|=3$ and $|N_r(V^b)|\leq |V_1|+f(k-2,s,r-2)$ or $|N_r(V^b)|\leq 2f(k-1,s,r-2)$ when $|N_r(V^b)\cap Y_2|\leq 2$. It follows that $|R|\leq |V_1|+3f(k-2,s,r-2)+f(k,s+1,r-2)$ when $|N_r(V^b)\cap Y_2|=3$ and $|R|\leq 2|V_1|+f(k-2,s,r-2)+f(k,s+1,r-2)$ or $|R|\leq |V_1|+2f(k-1,s,r-2)+f(k,s+1,r-2)$ when $|N_r(V^b)\cap Y_2|\leq 2$. By (w), we have $|G|\leq 3f(k-1,s+1,r-2)+f(k,s+1,r-2)+\{4f(k-2,s,r-2);f(k-1,s+1,r-2)+2f(k-2,s,r-2);2f(k-1,s,r-2)+f(k-2,s,r-2)\}<f(k,s,r)+1$, which are impossible.

By the above discussions, we can see that all the parts of $\{V_2,\dots,V_\ell\}$ in R contain neither red nor blue edges. Since R contains at most $R(K_3,G_{22})-1=10$ parts of $\{V_2,\dots,V_\ell\}$, by induction, $|R|\leq 10f(k-2,s,r-2)$. □

By Claim 9 and (q), we have $|G|\leq 17f(k-2,s,r-2)<f(k,s,r)+1$, yielding a contradiction. Thus this completes the proof of C1.5 and the proof of Case 3.

This completes the proofs of Theorems 1.4 and 1.5.

Acknowledgements

The authors are grateful to both referees for their helpful comments and suggestions which improved the presentation of this paper.

Appendix

Using the expressions of $w(k,r)$ and $f(k,s,r)$, we obtain the following functions. Note that the sequence for all cases in each of the following functions is consistent with $w(k,r)$ or $f(k,s,r)$.

$$w(k-1,r)=\begin{cases}\frac{2}{5}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ 5^{(k-r-1)/2}\cdot 17^{r/2},\\ 4\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{8}{5}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}\quad w(k-2,r)=\begin{cases}\frac{1}{5}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{2}{5}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ \frac{8}{5}\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{4}{5}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}\quad w(k,r-1)=\begin{cases}\frac{8}{17}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{20}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ 5\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ 2\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}$$

$$w(k-1,r-1)=\begin{cases}\frac{4}{17}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{8}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ 2\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}\quad w(k-2,r-1)=\begin{cases}\frac{8}{85}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{4}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{2}{5}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}\quad w(k,r-2)=\begin{cases}\frac{5}{17}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{10}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ \frac{40}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{20}{17}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}$$

$$w(k-1,r-2)=\begin{cases}\frac{2}{17}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{5}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ \frac{20}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{8}{17}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}\quad w(k-2,r-2)=\begin{cases}\frac{1}{17}\cdot 5^{(k-r)/2}\cdot 17^{r/2},\\ \frac{2}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{r/2},\\ \frac{8}{17}\cdot 5^{(k-r-1)/2}\cdot 17^{(r-1)/2},\\ \frac{4}{17}\cdot 5^{(k-r)/2}\cdot 17^{(r-1)/2},\end{cases}$$

$$f(k-1,s,r-1)=\begin{cases}4\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{1}{17}\cdot\lfloor\frac{32}{5}\cdot 5^{s/2}\rfloor\cdot 17^{r/2},\\ 10\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{16}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ 5^{(s-1)/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ 5^{s/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 10\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\quad f(k,s,r-1)=\begin{cases}\lfloor\frac{32}{5}\cdot 5^{s/2}\rfloor\cdot 17^{(r-2)/2},\\ \frac{1}{17}\cdot\lfloor\frac{32}{5}\cdot 5^{s/2}\rfloor\cdot 17^{r/2},\\ 16\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{16}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ 2\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ 10\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\quad f(k-1,s-1,r-1)=\begin{cases}\frac{16}{5}\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{16}{85}\cdot 5^{s/2}\cdot 17^{r/2},\\ \lfloor\frac{32}{5}\cdot 5^{(s-1)/2}\rfloor\cdot 17^{(r-2)/2},\\ \frac{1}{17}\cdot\lfloor\frac{32}{5}\cdot 5^{(s-1)/2}\rfloor\cdot 17^{r/2},\\ 2\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ 2\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{4}{5}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ 4\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}$$

$$f(k-2,s-1,r-1)=\begin{cases}2\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{16}{85}\cdot 5^{s/2}\cdot 17^{r/2},\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{1}{17}\cdot\lfloor\frac{32}{5}\cdot 5^{(s-1)/2}\rfloor\cdot 17^{r/2},\\ 5^{(s-1)/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 2\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{1}{5}\cdot 5^{s/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 4\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\quad f(k,s+2,r-2)=\begin{cases}5\cdot 5^{s/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{10}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 5\cdot 5^{(s-1)/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{20}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ \frac{50}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{80}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{20}{17}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ \frac{160}{17}\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\quad f(k,s+1,r-2)=\begin{cases}4\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{4}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 10\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{10}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ \frac{32}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{32}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{16}{17}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ \frac{80}{17}\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}$$

$$f(k-1,s-1,r)=\begin{cases}\frac{2}{5}\cdot 5^{s/2}\cdot\lfloor 21\cdot 17^{(r-2)/2}\rfloor,\\ \frac{4}{5}\cdot 5^{s/2}\cdot 17^{r/2},\\ 5^{(s-1)/2}\cdot\lfloor 21\cdot 17^{(r-2)/2}\rfloor,\\ 2\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \lfloor\frac{32}{5}\cdot 5^{(s-1)/2}\rfloor\cdot 17^{(r-1)/2},\\ 2\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ 16\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\qquad f(k-2,s-2,r)=\begin{cases}\frac{1}{5}\cdot 5^{s/2}\cdot\lfloor 21\cdot 17^{(r-2)/2}\rfloor,\\ \frac{2}{5}\cdot 5^{s/2}\cdot 17^{r/2},\\ \frac{1}{5}\cdot 5^{(s-1)/2}\cdot\lfloor 42\cdot 17^{(r-2)/2}\rfloor,\\ \frac{4}{5}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ 2\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{16}{5}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{4}{5}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ \lfloor\frac{32}{5}\cdot 5^{(s-2)/2}\rfloor\cdot 17^{(r-1)/2},\end{cases}$$

$$f(k-1,s+1,r-2)=\begin{cases}5^{s/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{4}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 5\cdot 5^{(s-1)/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{10}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ \frac{20}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{32}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{10}{17}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ \frac{80}{17}\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2},\end{cases}\qquad f(k-2,s,r-2)=\begin{cases}5^{s/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{2}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 5^{(s-1)/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-2)/2}\rfloor,\\ \frac{4}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ \frac{10}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{16}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{4}{17}\cdot 5^{s/2}\cdot 17^{(r-1)/2},\\ \frac{1}{17}\cdot\lfloor 32\cdot 5^{(s-2)/2}\rfloor\cdot 17^{(r-1)/2},\end{cases}$$

$$f(k,s,r-2)=f(k-1,s,r-2)=\begin{cases}2\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{2}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 4\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{4}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ \frac{16}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{16}{17}\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ \frac{1}{17}\cdot\lfloor\frac{32}{5}\cdot 5^{s/2}\rfloor\cdot 17^{(r-1)/2},\\ \frac{1}{17}\cdot\lfloor 32\cdot 5^{(s-2)/2}\rfloor\cdot 17^{(r-1)/2},\end{cases}\qquad f(k,s+1,r-1)=\begin{cases}10\cdot 5^{s/2}\cdot 17^{(r-2)/2},\\ \frac{16}{17}\cdot 5^{s/2}\cdot 17^{r/2},\\ 20\cdot 5^{(s-1)/2}\cdot 17^{(r-2)/2},\\ \frac{32}{17}\cdot 5^{(s-1)/2}\cdot 17^{r/2},\\ 5\cdot 5^{(s-1)/2}\cdot\lfloor\frac{21}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 10\cdot 5^{(s-1)/2}\cdot 17^{(r-1)/2},\\ 5^{s/2}\cdot\lfloor\frac{42}{17}\cdot 17^{(r-1)/2}\rfloor,\\ 20\cdot 5^{(s-2)/2}\cdot 17^{(r-1)/2}.\end{cases}$$

We create the following Table 3 by taking the maximum values from each of cases a_1–a_4 and b_1–b_8 in Tables 1 and 2.

Case	a_1	a_2	a_3	a_4				
$(f): \frac{w(k-1,r)+2w(k-1,r-1)}{w(k,r)}$	$\frac{74}{85}$	$\frac{33}{34}$	1	$\frac{9}{10}$				
$(g): \frac{2w(k-1,r-2)+w(k,r-1)+w(k-2,r-2)}{w(k,r)}$	$\frac{13}{17}$	$\frac{16}{17}$	$\frac{133}{136}$	$\frac{27}{34}$				
Case	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$(o): \frac{2f(k-1,s,r-1)+3f(k-2,s-1,r-1)}{f(k,s,r)}$	$\frac{2}{3}$	$\frac{56}{85}$	$\frac{16}{21}$	$\frac{64}{85}$	$\frac{147}{170}$	$\frac{7}{8}$	$\frac{84}{85}$	1
$(p): \frac{f(k-1,s,r-1)+2f(k-1,s-1,r-1)+2f(k-2,s-1,r-1)}{f(k,s,r)}$	$\frac{24}{35}$	$\frac{48}{85}$	$\frac{77}{105}$	$\frac{52}{85}$	$\frac{76}{85}$	$\frac{3}{4}$	$\frac{65}{68}$	$\frac{13}{16}$
$(r): \frac{f(k,s+2,r-2)+2f(k-1,s,r-1)}{f(k,s,r)}$	$\frac{241}{357}$	$\frac{57}{85}$	$\frac{275}{357}$	$\frac{13}{17}$	$\frac{67}{85}$	$\frac{27}{34}$	$\frac{31}{34}$	$\frac{50}{51}$
$(s): \frac{4f(k-1,s,r-2)+f(k,s+1,r-1)}{f(k,s,r)}$	$\frac{6}{7}$	$\frac{12}{17}$	$\frac{6}{7}$	$\frac{12}{17}$	$\frac{169}{170}$	$\frac{117}{136}$	$\frac{169}{170}$	$\frac{46}{51}$
$(t): \frac{f(k,s+1,r-2)+f(k,s,r-1)+3f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{1199}{1785}$	$\frac{41}{85}$	$\frac{284}{357}$	$\frac{19}{34}$	$\frac{13}{17}$	$\frac{37}{68}$	$\frac{31}{34}$	$\frac{2}{3}$
$(t): \frac{f(k,s+1,r-2)+f(k,s,r-1)+2f(k-1,s,r-2)}{f(k,s,r)}$	$\frac{24}{35}$	$\frac{36}{85}$	$\frac{17}{21}$	$\frac{1}{2}$	$\frac{66}{85}$	$\frac{33}{68}$	$\frac{157}{170}$	$\frac{31}{51}$
$(t): \frac{f(k-1,s+1,r-2)+f(k,s+1,r-1)+3f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{275}{357}$	$\frac{13}{17}$	$\frac{571}{714}$	$\frac{27}{34}$	$\frac{31}{34}$	$\frac{125}{136}$	$\frac{16}{17}$	1
$(t): \frac{f(k-1,s+1,r-2)+f(k,s+1,r-1)+2f(k-1,s,r-2)}{f(k,s,r)}$	$\frac{40}{51}$	$\frac{12}{17}$	$\frac{83}{102}$	$\frac{25}{34}$	$\frac{157}{170}$	$\frac{117}{136}$	$\frac{81}{85}$	$\frac{16}{17}$
$(u): \frac{3f(k,s+1,r-2)+f(k-2,s,r-2)+2f(k-1,s,r-2)}{f(k,s,r)}$	$\frac{293}{357}$	$\frac{9}{17}$	$\frac{344}{357}$	$\frac{21}{34}$	$\frac{69}{85}$	$\frac{9}{17}$	$\frac{81}{85}$	$\frac{11}{17}$
$(v): \frac{6f(k-1,s+1,r-2)+f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{13}{17}$	$\frac{13}{17}$	$\frac{16}{17}$	$\frac{16}{17}$	$\frac{13}{17}$	$\frac{13}{17}$	$\frac{16}{17}$	1
$(w): \frac{3f(k-1,s+1,r-2)+f(k,s+1,r-2)+4f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{278}{357}$	$\frac{12}{17}$	$\frac{653}{714}$	$\frac{14}{17}$	$\frac{66}{85}$	$\frac{12}{17}$	$\frac{31}{34}$	$\frac{44}{51}$
$(w): \frac{4f(k-1,s+1,r-2)+f(k,s+1,r-2)+2f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{278}{357}$	$\frac{12}{17}$	$\frac{337}{357}$	$\frac{29}{34}$	$\frac{66}{85}$	$\frac{12}{17}$	$\frac{16}{17}$	$\frac{46}{51}$
$(w): \frac{3f(k-1,s+1,r-2)+f(k,s+1,r-2)+2f(k-1,s,r-2)+f(k-2,s,r-2)}{f(k,s,r)}$	$\frac{283}{357}$	$\frac{11}{17}$	$\frac{13}{14}$	$\frac{13}{17}$	$\frac{67}{85}$	$\frac{11}{17}$	$\frac{157}{170}$	$\frac{41}{51}$

Table 3: The ratios of the sum of functions to $w(k,r)$ and $f(k,s,r)$.

References

[1] F. R. K. Chung and R. Graham, Edge-colored complete graphs with precisely colored subgraphs, *Combinatorica* 3 (1983), 315–324.

[2] M. Clancy, Some small Ramsey numbers, *J. Graph Theory* 1 (1977), 89–91.

[3] R. J. Faudree, R. Gould, M. Jacobson and C. Magnant, Ramsey numbers in rainbow triangle free colorings, *Australas. J. Combin.* 46 (2010), 269–284.

[4] J. Fox, A. Grinshpun and J. Pach, The Erdős-Hajnal conjecture for rainbow triangles, *J. Combin. Theory Ser. B* 111 (2015), 75–125.

[5] S. Fujita and C. Magnant, Gallai-Rasmey numbers for cycles, *Discrete math.* 311 (2011), 1247–1254.

[6] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* 26 (2010), 1–30.

[7] T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hung.* 18 (1967), 25–66.

[8] R. E. Greenwood and A. M. Gleason, Combinatorial Relations and Chromatic Graphs, *Canad. J. Math.* 7 (1955), 1–7.

[9] A. Gyárfás, G. Sárközy, A. Sebő and S. Selkow, Ramsey-type results for Gallai colorings, *J. Graph Theory* 64 (2010), 233–243.

[10] A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* 46 (2004), 211–216.

[11] G. R. T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory* 13 (1989), 245–248.

[12] X.-H. Li and L.-G. Wang, Gallai-Ramsey numbers for a class of graphs with five vertices, *Graphs Combin.* 36 (6) (2020), 1603–1618.

[13] H. Liu, C. Magnant, A. Saito, I. Schiermeyer and Y. Shi, Gallai-Ramsey number for K_4, *J. Graph Theory* 94 (2020), 192–205.

[14] C. Magnant and I. Schiermeyer, Gallai-Ramsey number for K_5, 2019, arXiv:1901.03622v1.

[15] Y. Mao, Z. Wang, C. Magnant and I. Schiermeyer, Gallai-Ramsey numbers for fans, 2019, arXiv:1902.10706v1.

[16] Y. Mao, Z. Wang, C. Magnant and I. Schiermeyer, Ramsey and Gallai-Ramsey number for wheels, *Graphs Combin.* 38 (2022).

[17] S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* Dynamic Survey 1 (2017).

[18] Z.-X. Song, B. Wei, F. Zhang and Q. Zhao, A note on Gallai-Ramsey number of even wheels, *Discrete Math.* 343 (2020), 111725.

[19] Z. Wang, Y. Mao, C. Magnant and J. Zou, Ramsey and Gallai-Ramsey numbers for two classes of unicyclic graphs, *Graphs Combin.* 37 (1) (2021), 337–354.

[20] H. Wu, C. Magnant, P. S. Nowbandegani and S. Xia, All partitions have small parts Gallai-Ramsey numbers of bipartite graphs, *Discrete Appl. Math.* 254 (2019), 196–203.

[21] F. Zhang, Z.-X. Song and Y. Chen, Gallai-Ramsey number of odd cycles with chords, arXiv:1809.00227v2.

[22] F. Zhang, Z.-X. Song and Y. Chen, Gallai-Ramsey number of even cycles with chords, *Discrete Math.* 345 (2022), 112738.

[23] Q. Zhao and B. Wei, Gallai-Ramsey numbers for graphs with chromatic number three, *Discrete Appl. Math.* 304 (2021), 110–118.

[24] J. Zou, Y. Mao, C. Magnant, Z. Wang and C. Ye, Gallai-Ramsey numbers for books, *Discrete Appl. Math.* 268 (2019), 164–177.

(Received 14 Aug 2021; revised 17 Apr 2022)