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On deformed dodecahedron tilings

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Abstract

We prove the conjecture of Gao, Shi and Yan, that there is only one two-parametric family of edge-to-edge tilings of the sphere by 12 congruent pentagons. We also prove that all tilings from this family are isohedral.

1 Introduction

A *deformed dodecahedron* is an edge-to-edge tiling of the sphere by 12 congruent pentagons. Gao, Shi and Yan [6] showed that 12 is the minimal number of tiles for edge-to-edge tilings of the sphere by congruent pentagons, and they completely classified this minimal case. The work was the beginning of the whole program of the complete classification of edge-to-edge tilings of the sphere by congruent pentagons [3, 9, 10].

Gao, Shi and Yan concluded that there are five types of deformed dodecahedra, given by Figures 1, 2, 4 and 8. In each picture, the thin, thick and dashed edges have respective lengths a, b, c. Moreover, by [6, Lemma 1], the pentagon must be *simple*, in the sense that the boundary is a simple closed curve. They further conjectured that the first four types must be the central projection of the *regular dodecahedron* to the sphere, where all the inner angles are $2\pi/3$, and all the edges have the spherical length $\arccos(\sqrt{5}/3)$. Here and subsequently, the sphere always has the unit radius.

In this paper, we prove this conjecture, which means we prove the following theorem.

Theorem. *The set of edge-to-edge tilings of the sphere by* 12 *congruent pentagons is a two-parametric family such that the angles and the lengths of edges are illustrated in Figure 1.*

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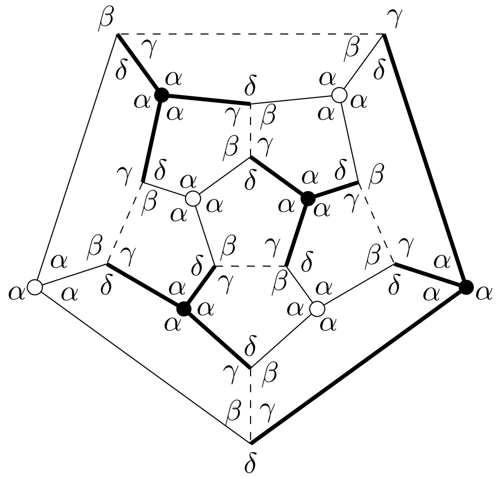

Figure 1: Type 5 tiling, where the thin, thick and dashed edges have respective lengths a, b, c, and $\alpha = 2\pi/3$, $\beta + \gamma + \delta = 2\pi$.

In Section 2, we use two short technical lemmas (Lemmas 2.1 and 2.2) to prove a tiling of type 1 or 4 is the regular dodecahedron. It is harder to prove that a tiling of type 2 or 3 is also the regular dodecahedron. In Sections 3 and 4, we use symmetry of the tiling and spherical trigonometry calculations to carry out the proof. Therefore the only deformed dodecahedron is the type 5 tiling in Figure 1. Such a tiling allows two free parameters, and Wang and Yan [11] give a detailed description of the two dimensional moduli. In Section 5, we determine the symmetry group of the tiling and show that it acts transitively on the tiles. We also prove that the regular dodecahedron is the only equilateral tiling (Proposition 5.1). This justifies a claim implicit in the main theorem of [3]. Moreover, we discuss the symmetries of other related deformed tilings.

2 Tilings of types 1 and 4

Tilings of types 1 and 4 of Gao, Shi and Yan [6] are given by Figure 2. The thin and thick edges have respective lengths a and b.

We recall the following technical results.

Lemma 2.1 ([10, Lemma 2]). *For the simple spherical pentagon on the left of Figure 3, if $x = y$ and $w = z$, then $\beta > \gamma$ if and only if $\delta < \epsilon$.*

Lemma 2.2 ([6, Lemma 21]). *For the spherical pentagon on the left of Figure 3, consider four equalities*

$$x = y, \quad w = z, \quad \beta = \gamma, \quad \delta = \epsilon.$$

If any three equalities hold, then the fourth equality also holds.

We remark that Lemma 2.1 requires the pentagon to be simple, and Lemma 2.2 has no such requirement. The right of Figure 3 is a non-simple pentagon. The

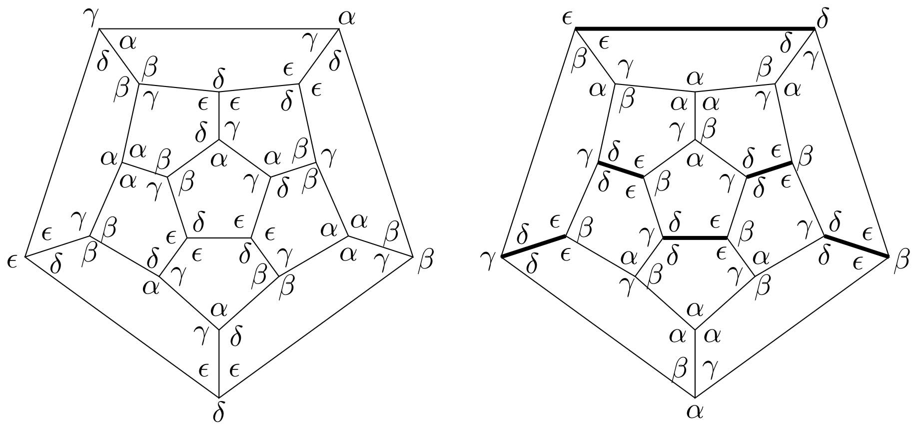

Figure 2: Tilings of types 1 and 4.

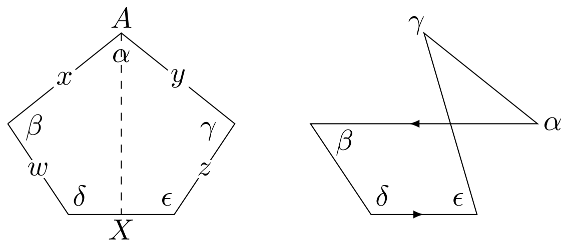

Figure 3: Geometrical constraint for pentagon.

boundary curve is still closed. We may pick one side of the boundary curve, and only pick angles $\alpha, \beta, \gamma, \delta, \epsilon$ on the chosen side. Then Lemma 2.2 still holds for this choice of angles.

For the case $\beta, \gamma, \delta, \epsilon, x, y, z, w < \pi$, Lemma 2.1 can be proved as follows. Let A be the top vertex, and let X be the middle point of the bottom edge. Let $2u$ be the length of the bottom edge. Then A, X are connected on the left by edges x, w, u at angles β, δ, and connected on the right by edges y, z, u at angles γ, ϵ. By Cauchy's arm lemma (see [1, Chapter 14], or [4, 7]), if $x = y$ and $w = z$, then $\beta > \gamma$ and $\delta > \epsilon$ (or $\beta < \gamma$ and $\delta < \epsilon$) imply AX as calculated from the left is strictly longer (or shorter) than AX as calculated from the right. The contradiction proves Lemma 2.1.

It is not obvious that the pentagons in our tiling satisfy the less than π condition assumed in the argument above. Although it is conceivable to modify the proof above in case the condition fails, we choose to rely on [10] for the full proof.

In a type 1 tiling (the left of Figure 2), the angle sums at vertices give

$$3\alpha = \alpha + \gamma + \delta = 2\beta + \gamma = \delta + 2\epsilon = 2\pi.$$

If $\beta > \gamma$, then by Lemma 2.1, we get $\delta < \epsilon$. Then from $2\beta + \gamma = \delta + 2\epsilon = 3\alpha$, we get $\gamma < \alpha$ and $\delta < \alpha$. Therefore $\alpha + \gamma + \delta < 3\alpha$, and we get a contradiction. Similarly, if $\beta < \gamma$, then we get $\alpha + \gamma + \delta > 3\alpha$, which is again a contradiction. This proves $\beta = \gamma$. Then by the angle sum equalities above, all angles $\alpha, \beta, \gamma, \delta, \epsilon$ are equal to $2\pi/3$. As all the edges have the same length, the tiling is the regular dodecahedron.

In a type 4 tiling (the right of Figure 2), the angle sums at vertices give

$$3\alpha = \alpha + \beta + \gamma = \beta + 2\epsilon = \gamma + 2\delta = 2\pi.$$

The equalities imply that $\beta < \gamma$ if and only if $\delta < \epsilon$, and $\beta > \gamma$ if and only if $\delta > \epsilon$. Since this violates Lemma 2.1, we must have $\beta = \gamma$ and $\delta = \epsilon$. Then by the angle sums above, this further implies all angles are equal to $2\pi/3$. Then by Lemma 2.2, we get $a = b$, and the tiling is again the regular dodecahedron.

3 Tilings of type 2

A type 2 tiling is given by the left of Figure 4. The thin and thick edges have respective lengths a and b. The angle sums at vertices give

$$3\alpha = \alpha + \beta + \gamma = \beta + 2\epsilon = \gamma + 2\delta = 2\pi. \tag{1}$$

This implies $\alpha = 2\pi/3$ and $\beta + \gamma = 4\pi/3$. From $\beta, \gamma > 0$, we get $\beta, \gamma < 4\pi/3$. We also note that $a < \pi$ because two arcs of lengths $\geq \pi$ connected at one end must intersect again, which violates the simple pentagon requirement.

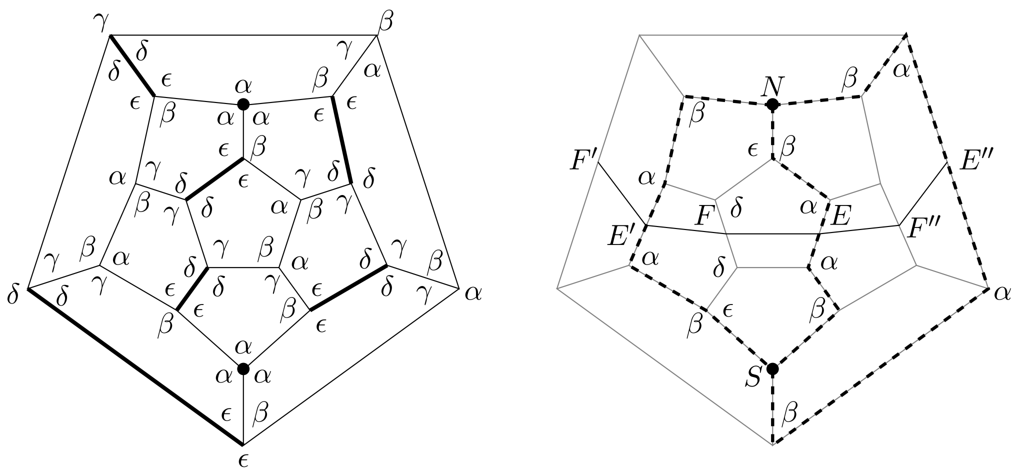

Figure 4: Type 2 tiling.

To show that a type 2 tiling is the regular dodecahedron, we take advantage of the symmetry of the tiling. Let N and S be the two vertices where the three angles are α, α, α. On the right of Figure 4, we connect N and S by three dashed paths, each consisting of five edges of the same length a connected at alternate angles $\beta, \alpha, \alpha, \beta$. The dashed paths divide the sphere into three "timezones" consisting of four tiles each. By an *axis*, we mean a line through the center of the sphere. The three timezones are congruent by the rotations around the axis through N, by angles $\alpha = 2\pi/3$ and $2\alpha = 4\pi/3$. Since the rotations also fix S, we find that S is the antipodal point of N. We may regard N and S as the north and south poles.

The dashed paths consist of five edges of the same length a, connected at alternate angles $\beta, \alpha, \alpha, \beta$. Such description implies that the rotation of the sphere around the axis through the middle point E of a dashed path (i.e., the middle point of the middle edge in the path), by angle π is an isometry of the path to itself that exchanges the two ends N and S. Since this rotation exchanges N and S, we find that E lies in the equator with respect to the two poles. Therefore the length of NE is $\pi/2$, and we get the left of Figure 5.

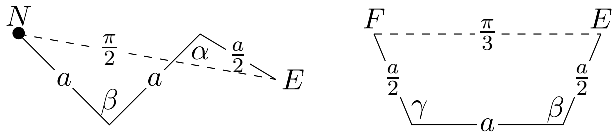

Figure 5: Paths between NE and EF.

We also have the similar symmetries around the middle points E', E'' of the other two dashed paths. Moreover, the rotations around the axis through N, by α and 2α, exchange the three dashed paths, and in particular exchange E, E', E''.

Next we consider another path connecting N to S. The path consists of five edges of lengths a, b, a, b, a at alternate angles $\epsilon, \delta, \delta, \epsilon$. The path has the similar symmetry as the dashed path, and the same argument shows that the middle point F of the middle edge lies in the equator with respect to the two poles. Moreover, we have three such middle points F, F', F'', and the rotations around the axis through N, by α and 2α, exchange the three points.

All the middle points E, F, E', F', E'', F'' lie in the equator with respect to the two poles. Moreover, the arcs EF and $E'F$ have the same length, because they are mapped to each other by the congruence between the two pentagons containing the two edges. This implies that the six arcs have equal length of $2\pi/6 = \pi/3$, and we get the right of Figure 5.

Next we try to derive equalities from the two pictures in Figure 5. We need the formula that calculates the fourth edge in a spherical quadrilateral.

Lemma 3.1. *In the spherical quadrilateral on the left of Figure 6, we have*

$$\begin{aligned}\cos x = {} & \cos u \cos v \cos w + \cos u \sin v \sin w \cos \psi - \sin u \cos v \sin w \cos \phi \cos \psi \\ & + \sin v \sin u \cos w \cos \phi + \sin u \sin w \sin \phi \sin \psi.\end{aligned}$$

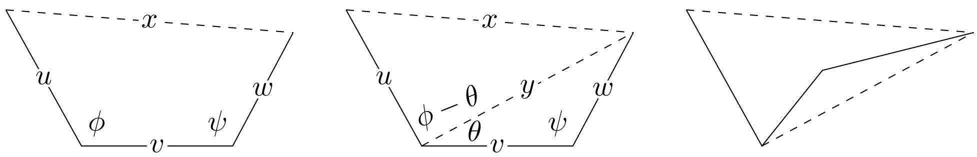

Figure 6: The length of the fourth edge in a quadrilateral.

We remark that the proof below relies on the middle picture in Figure 6, which implicitly assumes the quadrilateral is simple, and y lies inside the quadrilateral. In fact, the proof can be adapted to all the cases, such as y lying outside the quadrilateral, or if the quadrilateral is not simple. For example, in case of the right picture, we may choose θ to have the negative value. Then the whole proof is still valid.

For the non-simple case, the key requirement for the lemma is that the angles ϕ, ψ should be on the "same side", as explained after Lemmas 2.1 and 2.2 (also see the right of Figure 3).

Proof. In the middle of Figure 6, we divide the quadrilateral into two triangles. By applying the cosine law twice, we get

$$\begin{aligned}\cos x &= \cos u \cos y + \sin u \sin y \cos(\phi - \theta)\\ &= \cos u(\cos v \cos w + \sin v \sin w \cos \psi)\\ &\quad + \sin u \sin y(\cos \phi \cos \theta + \sin \phi \sin \theta).\end{aligned}$$

By the cosine law, we also have

$$\begin{aligned}\cos w &= \cos v \cos y + \sin v \sin y \cos \theta\\ &= \cos v(\cos v \cos w + \sin v \sin w \cos \psi) + \sin v \sin y \cos \theta.\end{aligned}$$

This is the same as

$$\sin^2 v \cos w = \cos v \sin v \sin w \cos \psi + \sin v \sin y \cos \theta.$$

If v is not a multiple of π, then we may divide by $\sin v$ and get a formula for $\sin y \cos \theta$. We also have the sine law $\sin y \sin \theta = \sin w \sin \phi$. Substituting the two formulae to the formula for $\cos x$, we get the formula in the lemma.

If $v = \pi$, then the two ends of v are antipodal points, and we get the left of Figure 7. This implies that u and y form half of the great circle, and $y = \pi - u$. We get a triangle with edges x, y, w. Moreover, the angle between y, w is $\psi - \phi$. Then the formula in the lemma is the cosine law for the triangle. The discussion applies to the general case that v is any odd multiple of π.

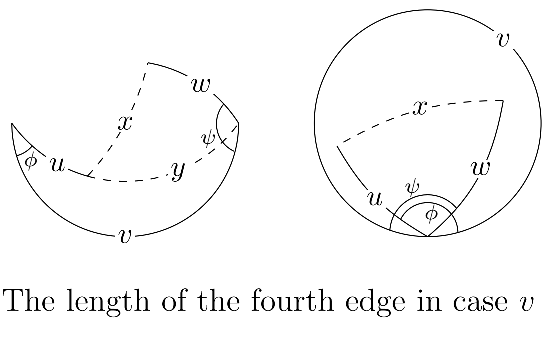

Figure 7: The length of the fourth edge in case $v = \pi$ or 2π.

If $v = 2\pi$, then v is one great circle, and we get the right of Figure 7. In particular, we get a triangle with edges x, u, w. Moreover, the angle between u, w is $\phi + \psi - \pi$. Then the formula in the lemma is the cosine law for the triangle. The discussion applies to the general case that v is any even multiple of π. ∎

We apply Lemma 3.1 to the first of Figure 5, with

$$u = v = a,\ w = a/2,\ x = \pi/2,\ \phi = \beta,\ \psi = 2\pi - \alpha = 4\pi/3.$$

We further divide the resulting equation by $\cos(a/2)$ (which is nonzero by $a < \pi$) to get a quadratic equation for $t = \cos a$

$$3(1 - \cos\beta)t^2 + (\cos\beta + \sqrt{3}\sin\beta - 1)t + 2\cos\beta - \sqrt{3}\sin\beta = 0. \tag{2}$$

Similarly, we apply Lemma 3.1 to the second of Figure 5 with

$$u = w = a/2,\ v = a,\ x = \pi/3,\ \phi = \gamma = 4\pi/3 - \beta,\ \psi = \beta,$$

and get another quadratic equation

$$\begin{aligned}(\cos\beta + \sqrt{3}\sin\beta + 2)(1 - \cos\beta)t^2 + (2\cos^2\beta + 2\sqrt{3}\cos\beta\sin\beta + 1)t\\ - \cos^2\beta - \sqrt{3}\cos\beta\sin\beta + \cos\beta - \sqrt{3}\sin\beta - 1 = 0.\end{aligned} \tag{3}$$

Since the two quadratic equations (2) and (3) share the same root $t = \cos a$, the resultant between the two quadratic polynomials must vanish. This gives

$$(1 - \cos\beta)(2\cos\beta + 1)(2\sqrt{3}(4\cos\beta - 1)\sin\beta + (8\cos^2\beta - 4\cos\beta + 5)) = 0.$$

By $\beta \neq 0$, we know $1 - \cos\beta \neq 0$. If $2\cos\beta + 1 = 0$, then by $\beta + \gamma = 4\pi/3$, we get $\beta = \gamma = 2\pi/3$. Further by the angle sums (1), we find all angles equal to $2\pi/3$. By Lemma 2.2, this implies $a = b$, and the tiling is the regular dodecahedron.

It remains to consider the case $2\cos\beta + 1 \neq 0$. The equation becomes

$$2\sqrt{3}(4\cos\beta - 1)\sin\beta + (8\cos^2\beta - 4\cos\beta + 5) = 0. \tag{4}$$

Using $\cos^2\beta + \sin^2\beta = 1$, this implies a polynomial equation for $\cos\beta$

$$(2\cos\beta + 1)(128\cos^3\beta - 144\cos^2\beta + 30\cos\beta + 13) = 0.$$

By $2\cos\beta+1 \neq 0$, the cubic factor $128\cos^3\beta - 144\cos^2\beta + 30\cos\beta + 13$ vanishes. The factor has only one real root, and the unique real root satisfies $-1/4 < \cos\beta < -1/5$. By $\beta < 4\pi/3$, this implies $\pi/2 < \beta < 2\pi/3$. Then by the angle sums (1), we find all angles are less than π. Therefore the tile is convex, and the isosceles triangle T with two sides a and the top angle $\alpha = 2\pi/3$ is contained in the pentagon of area $4\pi/12 = \pi/3$. This implies the area of T is less than $\pi/3$. By the top angle $\alpha = 2\pi/3$, this further implies $a < \pi/2$. Therefore we get $\cos a > 0$.

On the other hand, we may derive a formula of $t = \cos a$ in terms of $\cos\beta$. First, by (4), we have

$$\sin\beta = -\frac{8\cos^2\beta - 4\cos\beta + 5}{2\sqrt{3}(4\cos\beta - 1)}.$$

Second, we may cancel t^2 in (2) and (3) to obtain a linear equation in t, and then substitute the equality above so that the coefficients are polynomials of $\cos\beta$. The result is

$$(2\cos\beta+1)\big((4\cos\beta-1)^2 t-(16\cos^2\beta-8\cos\beta-5)\big)=0.$$

By $2\cos\beta+1\neq 0$, we get

$$\cos a = t = \frac{16\cos^2\beta-8\cos\beta-5}{(4\cos\beta-1)^2}.$$

By $-1/4<\cos\beta<-1/5$, the numerator is

$$16\cos^2\beta-8\cos\beta-5=16(\cos\beta-\tfrac{1}{4})^2-6\le 16(-\tfrac{1}{4}-\tfrac{1}{4})^2-6=-2<0.$$

This implies $\cos a<0$, a contradiction.

4 Tilings of type 3

A type 3 tiling is given by the left of Figure 8. Again the thin and thick edges have respective lengths a and b. The angle sums at vertices give

$$3\alpha=\alpha+\beta+\gamma=\beta+2\delta=\gamma+2\epsilon=2\pi. \tag{5}$$

This implies $\alpha=2\pi/3$ and $\beta+\gamma=4\pi/3$. Again we have $a<\pi$.

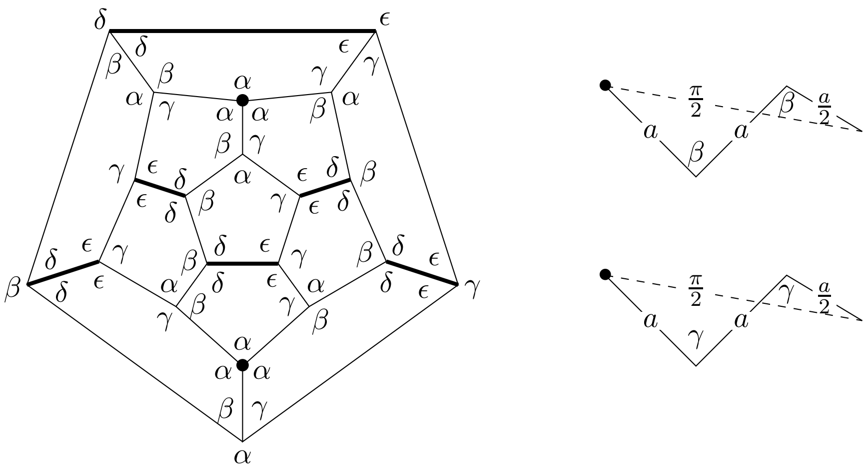

Figure 8: Type 3 tiling.

Similar to the type 2 tiling, we have two sets of three congruent paths connecting the two antipodal poles (vertices • where the three angles are α,α,α). Each path consists of five edges of the same length a, connected at alternate angles β,β,β,β or $\gamma,\gamma,\gamma,\gamma$. Using a picture similar to the right of Figure 4, we may derive two pictures on the right of Figure 8, similar to the left of Figure 5. Applying Lemma 3.1 to the upper right picture, and dividing by $\cos(a/2)$, we get a quadratic equation for $u = \cos\beta$:

$$(\cos a - 1)^2 u^2 - (2\cos a + 1)(\cos a - 1)u + \cos^2 a + \cos a - 1 = 0. \tag{6}$$

If we apply Lemma 3.1 to the lower right picture, we get the same equation for $u = \cos\gamma$. Therefore both $\cos\beta$ and $\cos\gamma = \cos(4\pi/3 - \beta)$ are roots of (6).

If $\cos\beta = \cos\gamma$, then by the strictly monotone property of cosine on $[0, 4\pi/3]$, we get $\beta = \gamma$. By the angle sums (5), this implies all angles equal to $2\pi/3$. As explained for type 2 tiling, this implies the tiling is the regular dodecahedron.

So we assume $\cos\beta$ and $\cos\gamma$ are two distinct roots of the quadratic equation (6). The sum and product of the two roots give

$$\cos(\beta + \pi/3) = \frac{2\cos a + 1}{\cos a - 1}, \quad \frac{1}{2}\sin(2\beta + \pi/6) = \frac{\cos^2 a + \cos a + 1}{(\cos a - 1)^2}.$$

Then the double angle formula gives a quadratic equation for $\cos a$:

$$(3\cos a + 5)(3\cos a + 1) = 0.$$

We get $a = \arccos(-1/3)$.

In Figure 9, let A, B, C, D, E be the vertices of the pentagon where the angles $\alpha, \beta, \gamma, \delta, \epsilon$ are located. By $AB = AC = a = \arccos(-1/3)$ and $\angle BAC = \alpha = 2\pi/3$, we use the cosine law to get $BC = a$. Therefore $\triangle ABC$ is an equilateral triangle, and we have $\angle ABC = \angle ACB = 2\pi/3$. Then by $\beta + \gamma = 4\pi/3$ and $BD = CE$, we find BC and DE intersect at the middle point X of both arcs, and $\triangle BDX$ and $\triangle CEX$ are congruent. Therefore the area of the pentagon is the same as the area π of $\triangle ABC$. This contradicts the fact that twelve such pentagons tile the sphere of total area 4π.

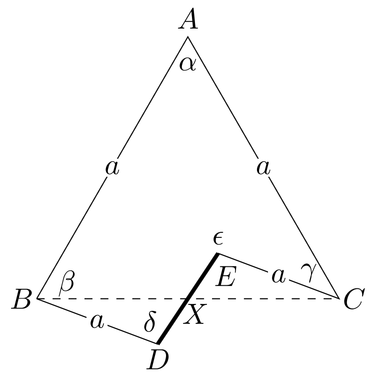

Figure 9: The pentagon for type 3 tiling.

5 The symmetry

The type 5 tiling is given by Figure 1. There are eight vertices where α, α, α meet. Four of these vertices have thin edges, which we denote by $\circ$. The other four vertices have thick edges, which we denote by $\bullet$.

We note that two nearby $\circ$ are connected by edges a, c, a at alternate angles β, β, and two nearby $\bullet$ are connected by edges b, c, b at alternate angles γ, γ. See the left of Figure 10. This implies that $\circ$ are at equal distance from each other. Therefore the four $\circ$ are the vertices of a regular tetrahedron $T_\circ$. Similarly, the four $\bullet$ are the vertices of another regular tetrahedron $T_\bullet$.

We also note that two nearby $\circ$ and $\bullet$ are connected by edges a, b at angle δ. Therefore the distances between pairs of nearby $\circ$ and $\bullet$ are the same. This implies that any $\circ$ is the center of a face of $T_\bullet$, and any $\bullet$ is the center of a face of $T_\circ$. In other words, the two tetrahedra are dual of each other.

The dual regular tetrahedra structure underlies the symmetry of the deformed dodecahedron. The tiling is symmetric with respect to the 3-fold rotations around axes through any of $\circ$ or $\bullet$. By combining these rotations, we can move any tile to any other tile. Therefore the tiling is *isohedral*, in the sense that the full symmetry group G of the tiling acts transitively on the tiles.

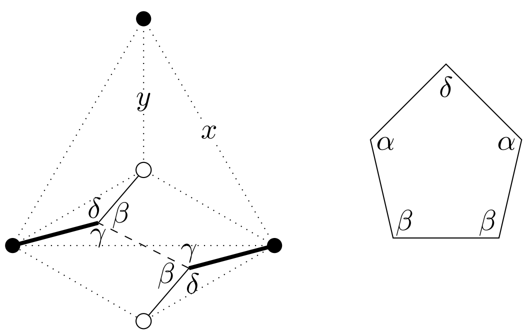

Figure 10: Regular tetrahedron $T_\bullet$, and equilateral tile.

Let H be the subgroup of symmetries that preserve one tile (say the center tile in Figure 1). Then the transitivity implies that G/H corresponds bijectively to $\{12 \text{ tiles}\}$. We may use this to determine G.

Suppose a, b, c are distinct. Then the only symmetry of the center tile is the identity, and fixing the center tile implies fixing all the tiles. Therefore H is the trivial group, and the order of G is 12. It turns out that G is the chiral tetrahedral group T.

Suppose $a = c \neq b$. If $\alpha = \beta$, then by Lemma 2.2, we have $\delta = \gamma$. By $3\alpha = \beta + \gamma + \delta = 2\pi$, we find all angles are equal to $2\pi/3$. Then by Lemma 2.2, we get $b = c$, contradicting the assumption. Therefore we have $\alpha \neq \beta$. This implies H is still the trivial group, and G is still the chiral tetrahedral group T. The same happens to the case $b = c \neq a$.

Suppose $a = b \neq c$. Then by Lemma 2.2, we have $\beta = \gamma$. Therefore the tile is symmetric with respect to the flipping that preserves the δ angle. Moreover, the flipping of the center tile determines the action on all the other tiles. Therefore H has order 2, and G has order 24. In fact, G is the pyritohedral group T_h.

Suppose $a = b = c$. Then the pentagon is equilateral. In the proposition below, we prove that the tiling is the regular dodecahedron. Then the subgroup H is the symmetry group of the regular pentagon, which is the dihedral group of order 10. Moreover, G is the icosahedral group I_h.

The symmetry of deformed dodecahedron tiling is summarized in Table 1.

edges	symmetry	order
$a \neq b$	T	12
$a = b \neq c$	T_h	24
$a = b = c$	I_h	120

Table 1: Symmetry of the deformed dodecahedron tiling.

Proposition 5.1. *An edge-to-edge tiling of the sphere by* 12 *congruent equilateral pentagons is the regular dodecahedron.*

The main theorem of [3] implicitly assumed that there is only one tiling of the sphere by 12 congruent equilateral pentagons. However, this was not justified because the paper was only concerned with more than 12 tiles. The proposition fills the gap.

Proof. We only need to prove the proposition for the type 5 tiling. By Lemma 2.2, we get $\beta = \gamma$ in the equilateral pentagon. Therefore the pentagon is the equilateral one on the right of Figure 10. Moreover, we have $2\beta + \delta = 2\pi$, which implies $\beta < \pi$.

The triangle with three $\bullet$ vertices in the left of Figure 10 (now all edges have length a) is a face of the regular tetrahedron, and the center of the face is a $\circ$ vertex. Therefore we know $\cos x = -1/3$ and $\cos y = 1/3$. Applying the cosine law to $\cos y$, and using $\delta = 2\pi - 2\beta$, we get

$$\cos^2 a + \sin^2 a \cos 2\beta = 1/3. \tag{7}$$

By applying Lemma 3.1 to $\cos x$, and using $\beta = \gamma$, we get

$$\begin{aligned}&(1-\cos\beta)^2\cos^3 a + (1-\cos^2\beta)\cos^2 a\\ &+(-\cos^2\beta + 2\cos\beta)\cos a + (\cos^2\beta - 1) = -1/3.\end{aligned} \tag{8}$$

By (7), we get

$$\cos^2 a = \frac{3\cos^2\beta - 2}{2(\cos^2\beta - 1)}.$$

Then we replace $\cos^2 a$ in (8), including the $\cos^2 a$ part of $\cos^3 a$, and get

$$\frac{2(2\cos\beta + 1)}{3(\cos\beta + 1)}\cos a = 0.$$

This implies $2\cos\beta + 1 = 0$ or $\cos a = 0$.

If $2\cos\beta + 1 = 0$, then by $\beta < \pi$, we get $\beta = 2\pi/3$. This implies all angles are equal to $2\pi/3$. Therefore the pentagon is regular, and the tiling is the regular dodecahedron.

If $\cos a = 0$, then $a = \pi/2$. By (7), this implies $\cos 2\beta = 1/3$, and $2\beta < \pi/2$ or $3\pi/2 < 2\beta < 2\pi$. If $2\beta < \pi/2$, then by that fact that the equilateral triangle with edge $a = \pi/2$ has interior angles $\pi/2$, the two lower edges (connecting α and β) on the right of Figure 10 intersect. This violates the simple pentagon requirement. If $3\pi/2 < 2\beta < 2\pi$, then $\delta = 2\pi - 2\beta < \pi/2$. This implies the pentagon is convex. Therefore the area of the pentagon is bigger than the area of the isosceles triangle with top angle β and side length a. By $a = \pi/2$, the area of the isosceles triangle is β, and 12 pentagons has area $> 12\beta$. Since $12\beta > 9\pi$ is bigger than the area of the sphere, we get a contradiction. □

The connection between the deformed dodecahedron and the regular tetrahedron is the *pentagonal subdivision* construction introduced in [9, Section 3.1]. The pentagonal subdivision of the tetrahedron is the deformed dodecahedron in Figure 1, also given by the left of Figure 11. The pentagonal subdivision of the regular octahedron (with six $\bullet$ vertices) or the regular cube (with eight $\circ$ vertices) is an edge-to-edge tiling of the sphere by 24 congruent pentagons, given by the middle of Figure 11. The pentagonal subdivision of the regular icosahedron (with twelve $\bullet$ vertices) or the regular dodecahedron (with twenty $\circ$ vertices) is a tiling by 60 congruent pentagons, given by the right of Figure 11.

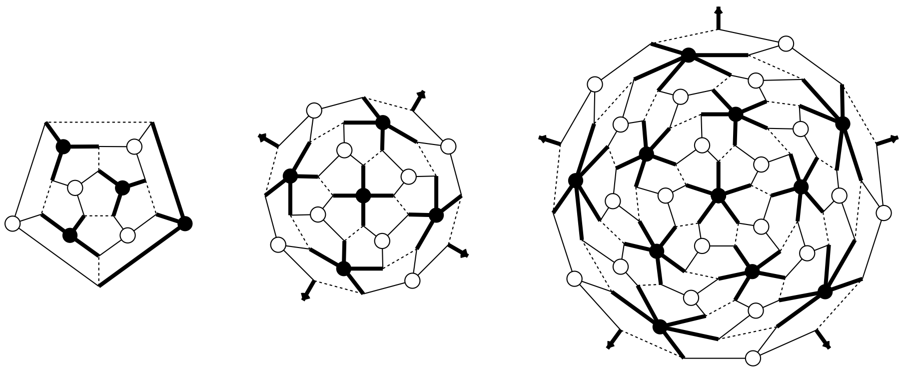

Figure 11: Pentagonal subdivision tilings.

The symmetries of the underlying regular Platonic solids give the symmetries of the pentagonal subdivisions. These are generated by 3 fold rotations around $\circ$, and 3, 4 or 5 fold rotations around $\bullet$. Then it is easy to see that the action is transitive on all the tiles.

Unlike the deformed dodecahedron, in the pentagonal subdivision tiling with 24 or 60 tiles, $\bullet$ vertices are the only vertices of degree greater than 3. Therefore if a symmetry preserves a tile, then it must fix the $\bullet$ vertex of the tile. Therefore the only way for the symmetry not to fix the tile is $a = c$, and the pentagon is symmetric with respect to the flipping that fixes $\bullet$. Such tilings are exactly Case 1.2, $H = \alpha^4$ or α^5 of [10, Proposition 26] (we also note that, by the calculation in [3], the pentagons in equilateral tilings are not symmetric). The tilings are unique, with the angles at $\bullet$ being $\pi/2$ for 24 tiles and $2\pi/5$ for 60 tiles, and all other angles being $2\pi/3$. For these two cases, the order of the symmetry group is respectively 48 and 120. For all the other cases, the order of the symmetry group is respectively 24 and 60.

Finally, we compare the symmetry of the deformed dodecahedron with the other *deformed Platonic solids*, which are edge-to-edge tilings of the sphere by congruent polygons, that are combinatorially Platonic solids.

By the classification in [8, Theorem 1] (also see [5]), the *deformed tetrahedron* and the *deformed octahedron* (respectively denoted $\bullet F_4$ and $\bullet G_8$ in [8]) are given by the central projections of the left and right of Figure 12 to the sphere. Moreover, the regular icosahedron (denoted H_{20} in [8]) is not deformable, i.e., must be the regular one.

By [2, Theorem 2] (also see [5]), the *deformed cube* is given by the central projection of the middle of Figure 12. Of course, the main theorem of this paper gives the deformed dodecahedron.

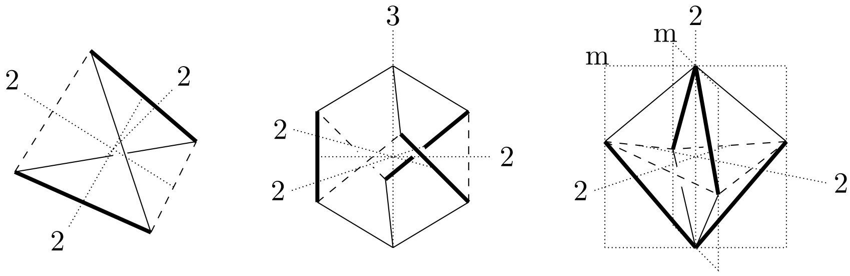

Figure 12: Deformed tetrahedron, cube, octahedron and their symmetries. The number n attached to an axis means n-fold rotation. A plane with “m” means mirror reflection.

In general, when thin, thick and dashed edges have distinct lengths, we have the following symmetry groups:

- Deformed tetrahedron: dihedral group D_2.
- Deformed cube: dihedral group D_3.
- Deformed octahedron: dihedral group D_{2d}.
- Deformed dodecahedron: chiral tetrahedral group T.
- Icosahedron: icosahedral group I_h.

The symmetry group may become bigger when some edges of different types have the same length. In all cases, the symmetry groups of deformed Platonic solids are always isohedral. On the other hand, we note that the trapezohedron of 12 faces, which is dual to an antiprism of 12 vertices, is isohedral when the tiles are convex, and may not be isohedral with concave tiles [2, Theorem 5].

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