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Anti-van der Waerden numbers of graph products of cycles

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Abstract

A k-term arithmetic progression (k-AP) of a graph G is a list of k distinct vertices such that the distance between consecutive pairs is constant. Given a coloring of the vertices of G, a k-AP is rainbow if each vertex in the AP is colored distinctly. This allows for the definition of the anti-van der Waerden number of a graph G, which is the least positive integer r such that every surjective r-coloring of the vertices of G contains a rainbow k-AP. This paper focuses on 3-term arithmetic progressions for graph products that involve cycles. Specifically, the anti-van der Waerden numbers of $P_m \square C_n$, $C_m \square C_n$ and $G \square C_{2n+1}$ are determined precisely.

1 Introduction

The study of van der Waerden numbers started with Bartel van der Waerden showing in 1927 that given a fixed number of colors r, and a fixed integer k there is some N (a van der Waerden number) such that if $n \geq N$, then no matter how you color $[n] = \{1, 2, \ldots, n\}$ with r-colors, there will always be a monochromatic k-term arithmetic progression (see [16]). Around this time, in 1917, it is interesting to note that Schur proved that given r colors, you can find an N (a Schur number) such that if $n \geq N$, then no matter how you color $[n]$ there must be a monochromatic solution to $x+y = z$ (see [15]). In addition, in 1928, Ramsey showed (here graph theory language is used but was not in Ramsey's original formulation) that given r colors and some constant

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k you can find an N (a Ramsey number) such that if $n \geq N$, then no matter how you color the edges of a complete graph K_n you can always find a complete subgraph K_k that is monochromatic (see [13]).

These types of problems that look for monochromatic structures have been categorized as Ramsey-type problems and each of them has a dual version. For example, an anti-van der Waerden number is when given integers n and k, find the minimum number of colors such that coloring $\{1, \ldots, n\}$ ensures a rainbow k-term arithmetic progression. It was not until 1973 when Erdős, Simonovits and Sós, in [7], started looking at the dual versions of these problems which are now well-studied (see [9] for a survey).

Results on colorings and balanced colorings of $[n]$ that avoid rainbow arithmetic progressions have been studied in [1] and [2]. Rainbow free colorings of $[n]$ and $\mathbb{Z}_n$ were studied in [6] and [11]. Although Butler et al., in [6], considered arithmetic progressions of all lengths, many results on 3-APs were produced. In particular, the authors of [6] determined $\mathrm{aw}(\mathbb{Z}_n, 3)$ (see Theorem 1.1 with additional cycle notation). Further, the authors of [6] determined that $3 \leq \mathrm{aw}(\mathbb{Z}_p, 3) \leq 4$ for every prime number p and that $\mathrm{aw}(\mathbb{Z}_n, 3)$ can be determined by the prime factorization of n. This result was then generalized by Young in [17].

Theorem 1.1. *[6] Let n be a positive integer with prime decomposition $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ for $e_i \geq 0$, $i = 0, \ldots, s$, where primes are ordered so that $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 3$ for $1 \leq i \leq \ell$ and $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 4$ for $\ell + 1 \leq i \leq s$. Then,*

$$\mathrm{aw}(\mathbb{Z}_n, 3) = \mathrm{aw}(C_n, 3) = \begin{cases} 2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j & \text{if } n \text{ is odd,} \\ 3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j & \text{if } n \text{ is even.} \end{cases}$$

As mentioned, Butler et al. also studied arithmetic progressions on $[n]$ and obtained bounds on $\mathrm{aw}([n], 3)$ and conjectured the exact value that was later proven in [4]. This result on $[n]$ is presented as Theorem 1.2 and includes path notation.

Theorem 1.2. *[4] If $n \geq 3$ and $7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}$, then*

$$\mathrm{aw}([n], 3) = \mathrm{aw}(P_n, 3) = \begin{cases} m + 2 & \text{if } n = 3^m, \\ m + 3 & \text{otherwise.} \end{cases}$$

It is also interesting to note that 3-APs in $[n]$ or $\mathbb{Z}_n$ satisfy the equation $x_1 + x_3 = 2x_2$. Thus, rainbow numbers for other linear equations have also been considered (see [5], [8], [10] and [12]).

Studying the anti-van der Waerden numbers of graphs is a natural extension of determining the anti-van der Waerden numbers of $[n] = \{1, 2, \ldots, n\}$, which behave like paths, and $\mathbb{Z}_n$, which behave like cycles. In particular, the set of arithmetic progressions on $[n]$ is isomorphic to the set of arithmetic progressions on P_n and the set of arithmetic progressions on $\mathbb{Z}_n$ is isomorphic to the set of arithmetic progressions on C_n. This relationship was first introduced and explored in [3] where the anti-van der Waerden number was bounded by the radius and diameter of a graph, the anti-van der Waerden number of trees and hypercubes were investigated and an upper bound of four was conjectured for the anti-van der Waerden number of graph products. Then, in [14], the authors confirmed the upper bound of four for any graph product (see Theorem 1.3). This paper continues in this vein.

Theorem 1.3. *[14] If G and H are connected graphs and $|G|, |H| \geq 2$, then*

$$\text{aw}(G\square H, 3) \leq 4.$$

Something that makes anti-van der Waerden numbers challenging is that they are not subgraph monotone. A particular example,

$$4 = \text{aw}([9], 3) = \text{aw}(P_9, 3) < \text{aw}(P_8, 3) = \text{aw}([8], 3) = 5,$$

even though P_8 is a subgraph of P_9, and a general statement,

$$\text{aw}(C_n, 3) = \text{aw}(\mathbb{Z}_n, 3) \leq \text{aw}([n], 3) = \text{aw}(P_n, 3),$$

were both given, without the graph theory interpretation, in [6]. One tool that does allow a kind of monotonicity when studying the anti-van der Waerden numbers of graphs is when a subgraph is isometric, that is, the subgraph preserves distances. This insight was used extensively in [14] to get an upper bound on the anti-van der Waerden number of graph products and will also be leveraged in this paper. First, some definitions and background inspired by [6] and used in [3] and [14] are provided.

Graphs in this paper are undirected so edge $\{u, v\}$ will be shortened to $uv \in E(G)$. If $uv \in E(G)$, then u and v are *neighbors* of each other. The *distance* between vertex u and v in graph G is denoted $\text{d}_G(u, v)$, or just $\text{d}(u, v)$ when context is clear, and is the smallest length of any $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*.

A *k-term arithmetic progression in graph G* (k-AP) is a set of vertices $\{v_1, \ldots, v_k\}$ such that $d(v_i, v_{i+1}) = d$ for all $1 \leq i \leq k - 1$. A k-term arithmetic progression is *degenerate* if $v_i = v_j$ for any $i \neq j$. Note that technically, since a k-AP is a set, the order of the elements does not matter. However, oftentimes k-APs will be presented in the order that provides the most intuition.

An *exact r-coloring of a graph G* is a surjective function $c : V(G) \to [r]$. A set of vertices S is *rainbow* under coloring c if for every $v_i, v_j \in V(G)$, $c(v_i) \neq c(v_j)$ when $i \neq j$. Given a set $S \subset V(G)$, define $c(S) = \{c(s) \mid s \in S\}$.

The *anti-van der Waerden number of graph G with respect to k*, denoted $\text{aw}(G, k)$, is the least positive number r such that every exact r-coloring of G contains a rainbow k-term arithmetic progression. If $|V(G)| = n$ and no coloring of the vertices yields a rainbow k-AP, then $\text{aw}(G, k) = n + 1$.

Graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and for any $uv \in E(G')$, we have that $u, v \in V(G')$ and $uv \in E(G)$. A subgraph G' of G is an *induced subgraph* if whenever u and v are vertices of G' and uv is an edge of G, then uv is an edge of G'. If S is a nonempty set of vertices of G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S and is denoted $G[S]$. An *isometric subgraph G'* of G is a subgraph such that $d_{G'}(u,v) = d_G(u,v)$ for all $u, v \in V(G')$.

If $G = (V, E)$ and $H = (V', E')$ the *Cartesian product*, written $G\Box H$, has vertex set $\{(x, y) : x \in V \text{ and } y \in V'\}$ and (x, y) and (x', y') are adjacent in $G\Box H$ if either $x = x'$ and $yy' \in E'$ or $y = y'$ and $xx' \in E$.

This paper will use the convention that if

$$V(G) = \{u_1, \ldots, u_{n_1}\} \quad \text{and} \quad V(H) = \{w_1, \ldots, w_{n_2}\},$$

then $V(G\Box H) = \{v_{1,1}, \ldots, v_{n_1,n_2}\}$ where $v_{i,j}$ corresponds to the vertices $u_i \in V(G)$ and $w_j \in V(H)$.

Also, if $1 \le i \le n_2$, then G_i denotes the ith labeled copy of G in $G\Box H$. Likewise, if $1 \le j \le n_1$, then H_j denotes the jth labeled copy of H in $G\Box H$. In other words, G_i is the induced subgraph $G_i = G\Box H[\{v_{1,i}, \ldots, v_{n_1,i}\}]$, and H_j is the induced subgraph $H_j = G\Box H[\{v_{j,1}, \ldots, v_{j,n_2}\}]$. Notice that the i subscript in G_i corresponds to the ith vertex of H and the j in the subscript in H_j corresponds to the jth vertex of G. See Example 1.4 below.

Example 1.4. Consider the graph $P_3\Box C_5$ where $V(P_3) = \{u_1, u_2, u_3\}$ and $V(C_5) = \{w_1, w_2, w_3, w_4, w_4\}$. Let $G = P_3$ and $H = C_5$ as in the definition. Now, G_4 is a subgraph of $P_3\Box C_5$ that is isomorphic to P_3 and corresponds to vertex w_4 of C_5. Similarly, H_2 is a subgraph of $P_3\Box C_5$ that is isomorphic to C_5 and corresponds to vertex u_2 of P_3. See Figure 1 below.

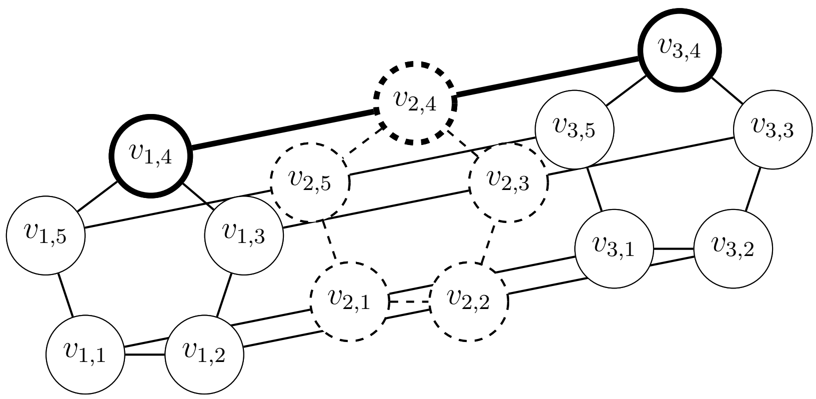

Figure 1: Image for Example 1.4. The subgraph G_4 is in bold and H_2 is dashed.

The paper continues with Section 2 recapping and expanding many fundamental results from [14]. Section 3 establishes $\text{aw}(P_m\Box C_n, 3)$ for all m and n. Section 4 is an investigation of $\text{aw}(G\Box C_n, 3)$. In particular, $\text{aw}(C_m\Box C_n, 3)$ is determined for all m and n. Further, Section 4 determines $\text{aw}(G\Box C_n, 3)$ for any G when n is odd. Finally, Section 5 provides the reader with some conjectures and open questions.

2 Background and Fundamental Tools

Distance preservation in subgraphs can be leveraged to guarantee the existence of rainbow 3-APs. Thus, this section starts with some basic distance and isometry results.

Proposition 2.1. *If $v_{i,j}, v_{h,k} \in V(G\Box H)$, then*

$$\mathrm{d}_{G\Box H}(v_{i,j}, v_{h,k}) = \mathrm{d}_G(u_i, u_h) + \mathrm{d}_H(w_j, w_k).$$

Proof. Note that $\mathrm{d}_{G\Box H}(v_{i,j}, v_{h,k}) \leq \mathrm{d}_G(u_i, u_h) + \mathrm{d}_H(w_j, w_k)$ because a path of length $\mathrm{d}_G(u_i, u_h)+\mathrm{d}_H(w_j, w_k)$ can be constructed using a $u_i - u_h$ geodesic in G and combining it with a $w_j - w_k$ geodesic in H.

To show the other inequality, let P be a $v_{i,j} - v_{h,k}$ geodesic, say

$$P = \{v_{i,j} = x_1, x_2, \ldots, x_y = v_{h,k}\}.$$

Note that for every edge $v_{j_1,j_2}v_{\beta_1,\beta_2} \in E(P)$, either $j_1 = \beta_1$ and $w_{j_2}w_{\beta_2} \in E(H)$, or $j_2 = \beta_2$ and $u_{j_1}u_{\beta_1} \in E(G)$. Then, $x_\ell x_{\ell+1}$ must correspond either to an edge from a $u_i - u_h$ walk or from a $w_j - w_k$ walk and P must correspond to a walk in G and also a walk in H. In other words, the length of P is the sum of the length of the corresponding walks in G and H. Thus, the length of P is at least the sum of the lengths of a $u_i - u_h$ geodesic in G and a $w_j - w_k$ geodesic in H. So,

$$\mathrm{d}_G(u_i, u_h) + \mathrm{d}_H(w_j, w_k) \leq \mathrm{d}_{G\Box H}(v_{i,j}, v_{h,k}).$$

□

Corollary 2.2. *If G' is an isometric subgraph of G and H' is an isometric subgraph of H, then $G'\Box H'$ is an isometric subgraph of $G\Box H$.*

Proof. Let $V(G) = \{u_1, \ldots, u_{n_1}\}$ and $V(H) = \{w_1, \ldots, w_{n_2}\}$. Then let $v_{i,j}, v_{h,k} \in V(G'\Box H')$. Observe,

$$\begin{aligned}\mathrm{d}_{G'\Box H'}(v_{i,j}, v_{h,k}) &= \mathrm{d}_{G'}(u_i, u_h) + \mathrm{d}_{H'}(w_j, w_k)\\ &= \mathrm{d}_G(u_i, u_h) + \mathrm{d}_H(w_j, w_k)\\ &= \mathrm{d}_{G\Box H}(v_{i,j}, v_{h,k}).\end{aligned}$$

□

Lemma 2.3 is powerful since it guarantees isometric subgraphs. Isometric subgraphs are important when investigating anti-van der Waerden numbers because distance preservation implies k-AP preservation.

Lemma 2.3. *[14] If G is a connected graph on at least three vertices with an exact r-coloring c where $r \geq 3$, then there exists a subgraph G' in G with at least three colors where G' is either an isometric path or $G' = C_3$.*

Theorem 2.4 is used when isometric $P_m \square P_n$ subgraphs are found within $G \square H$.

Theorem 2.4. *[14] For $m, n \geq 2$,*

$$\text{aw}(P_m \square P_n, 3) = \begin{cases} 3 & \text{if } m = 2 \text{ and } n \text{ is even, or } m = 3 \text{ and } n \text{ is odd,} \\ 4 & \text{otherwise.} \end{cases}$$

Lemma 2.5 helps restrict the number of colors each copy of G or H can have within $G \square H$.

Lemma 2.5. *[14] Assume G and H are connected with $|V(H)| \geq 3$. Suppose c is an exact, rainbow-free r-coloring of $G \square H$, such that $r \geq 3$ and $|c(V(G_i))| \leq 2$ for $1 \leq i \leq n$. If $w_i w_j \in E(H)$, then $|c(V(G_i) \cup V(G_j))| \leq 2$.*

To prove Lemmas 3.1 and 3.4 requires the use of Lemma 2.6.

Lemma 2.6. *If G and H are connected, $|G|, |H| \geq 2$ and c is an exact r-coloring of $G \square H$, $3 \leq r$, that avoids rainbow 3-APs, then $|c(V(G_i))| \leq 2$ for $1 \leq i \leq |H|$.*

Proof. If $|G| = 2$ the result is immediate, so let $3 \leq |G|$. For the sake of contradiction, assume *red*, *blue*, *green* $\in |c(V(G_i))|$ for some $1 \leq i \leq |H|$. By Lemma 2.3, there must exist an isometric path or a C_3 in G_i containing *red*, *blue*, and *green*. If there is such a C_3, then there is a rainbow 3-AP which is a contradiction. So, assume P_ℓ is a shortest isometric path in G_i containing *red*, *blue*, and *green*, for some positive integer $3 \leq \ell$.

Case 1. ℓ is odd.

Without loss of generality, suppose the two leaves of P_ℓ are colored *red* and *blue*. Since P_ℓ is shortest the rest of the vertices are colored *green*. Since ℓ is odd there exists a *green* vertex equidistant from the *red* and *blue* vertices which creates a rainbow 3-AP, a contradiction.

Case 2. ℓ is even.

Let $u_i \in V(H)$ be the vertex that corresponds to G_i and note that u_i has a neighbor since H is connected. Let P_2 be a path on two vertices in H containing u_i and ρ be the isometric subgraph in G that corresponds to P_ℓ. Thus, the subgraph $P_2 \square \rho$ of $G \square H$ is isometric and, by Theorem 2.4, contains a rainbow 3-AP, a contradiction.

All cases give a contradiction, thus $|c(V(G_i))| \leq 2$. □

Corollary 2.7 is a strengthening of Lemma 2.5 and follows from Lemmas 2.5 and 2.6. It is used to help analyze $\text{aw}(P_m \square C_{2k+1})$.

Corollary 2.7. *If G and H are connected graphs, $|G| \geq 2$, $|H| \geq 3$, c is an exact, rainbow-free r-coloring of $G \square H$ with $r \geq 3$, and $v_i v_j \in E(H)$, then*

$$|c(V(G_i) \cup V(G_j))| \leq 2.$$

Lemma 2.8. *[3] Let G be a connected graph on m vertices and H be a connected graph on n vertices. Let c be an exact r-coloring of $G\square H$ with no rainbow 3-APs. If $G_1, G_2, \dots, G_n$ are the labeled copies of G in $G\square H$, then $|c(V(G_j)) \setminus c(V(G_i))| \leq 1$ for all $1 \leq i, j \leq n$.*

Proposition 2.9. *If G and H are connected graphs, $|G| \geq 2$, $|H| \geq 3$, c is an exact, rainbow-free r-coloring of $G\square H$ with $r \geq 3$, then there is a color in $c(G\square H)$ that appears in every copy of G.*

Proof. Suppose $c(G\square H) = \{c_1, \dots, c_r\}$. First, for the sake of contradiction, assume $|c(V(G_i))| = 1$ for every $1 \leq i \leq |H|$. Then define a coloring $c' : V(H) \to c(G\square H)$ such that $c'(w_i) \in c(V(G_i))$. Then Lemma 2.3 implies that there is either an isometric path or C_3 in H with 3 colors. If there is an isometric C_3, say (w_1, w_2, w_3), then $\{v_{1,1}, v_{1,2}, v_{1,3}\}$ is a rainbow 3-AP in $G\square H$ with respect to c, a contradiction. So, there must be an isometric path in H with 3 colors. Suppose $P = (w_1, \dots, w_n)$ is a shortest such path. Without loss of generality, $c'(w_1) = c_2$, $c'(w_n) = c_3$ and $c'(w_i) = c_1$ for all $1 < i < n$. Then there exist $u_1, u_2 \in V(G)$ such that $u_1u_2 \in E(G)$. Thus, $\{v_{1,1}, v_{1,n}, v_{2,2}\}$ is a rainbow 3-AP in $G\square H$ with respect to c, a contradiction.

Thus, there exists some G_i such that $|c(V(G_i))| \geq 2$, without loss of generality, say $c_1, c_2 \in c(V(G_i))$. Then Lemma 2.6 implies $c(V(G_i)) = \{c_1, c_2\}$. Note that $c_3 \in c(V(G_j))$ for some $j \neq i$. Lemma 2.8 implies that $c_1 \in c(V(G_j))$ or $c_2 \in c(V(G_j))$. Without loss of generality, suppose $c_1 \in c(V(G_j))$ implying $c(V(G_j)) = \{c_1, c_3\}$ by Lemma 2.6. It will be shown that c_1 appears in every copy of G.

Now, for $k \notin \{i, j\}$, Lemma 2.8 implies that $|c(V(G_i)) \setminus c(V(G_k))| \leq 1$ and $|c(V(G_j)) \setminus c(V(G_k))| \leq 1$. Thus, for all $k \notin \{i, j\}$, either $c_1 \in c(V(G_k))$ or $c_2, c_3 \in c(V(G_k))$ implying $c(V(G_k)) = \{c_2, c_3\}$ by Lemma 2.6. Now, define $c' : V(H) \to \{red, blue\}$ by

$$c'(w_k) = \begin{cases} red & \text{if } c_1 \in c(V(G_k)), \\ blue & \text{if } c(V(G_k)) = \{c_2, c_3\}. \end{cases}$$

For the sake of contradiction, assume $blue \in c'(V(H))$. Then there must exist *red* and *blue* neighbors in H, call them w_{ℓ_1}, w_{ℓ_2}. Without loss of generality, say $c'(w_{\ell_1}) = red$ and $c'(w_{\ell_2}) = blue$ so that $c_1 \in c(V(G_{\ell_1}))$ and $c(V(G_{\ell_2})) = \{c_2, c_3\}$. Then $c_1, c_2, c_3 \in c(V(G_{\ell_1})) \cup c(V(G_{\ell_2}))$ and $3 \leq |c(V(G_{\ell_1})) \cup c(V(G_{\ell_2}))|$, contradicting Corollary 2.7. Thus, $c'(V(H)) = \{red\}$, the desired result. □

3 Graph Products of Paths and Cycles

As a reminder, the conventions for $G\square H$ will be used to label the vertices of $P_m\square C_n$. In particular, letting $G = P_m$ and $H = C_n$ gives the following:

- $V(P_m) = \{u_1, u_2, \dots, u_m\}$ with edges u_iu_{i+1} for $1 \leq i \leq m - 1$,
- $V(C_n) = \{w_1, w_2, \dots, w_n\}$ with edges w_iw_{i+1} for $1 \leq i \leq n - 1$ and w_nw_1,

- G_i is the ith copy of P_m in $P_m \square C_n$ and has vertex set $\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\}$, and
- H_i is the ith copy of C_n in $P_m \square C_n$ and has vertex set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$.

A fact about $P_m \square C_n$ is that

$$\mathrm{d}_{P_m \square C_n}(v_{i,j}, v_{k,\ell}) = |i-k| + \min\{(j-\ell) \bmod n, (\ell - j) \bmod n\}.$$

Note that the standard representative of the equivalence class of $\mathbb{Z}_n$ is chosen, i.e. $(j-\ell) \bmod n, (\ell - j) \bmod n \in \{0, 1, \ldots, n-1\}$.

Lemma 3.1. *For any positive integer k,* $\mathrm{aw}(P_2 \square C_{2k+1}, 3) = 3$.

Proof. For the sake of contradiction, let c be an exact, rainbow-free 3-coloring of $P_2 \square C_{2k+1}$. Swapping the roles of G and H in Lemma 2.6 gives

$$|c(V(H_1))|, |c(V(H_2))| \leq 2.$$

Without loss of generality, suppose $c(V(H_1)) = \{red, blue\}$, $green \in c(V(H_2))$ with $c(v_{2,1}) = green$. Note that $c(v_{1,1}) \in \{red, blue\}$ and define P_ℓ to be a shortest path in H_1 containing $v_{1,1}$ that contains colors *red* and *blue*. Without loss of generality, let $P_\ell = (v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,\ell})$ and let ρ be the isometric subgraph of C_{2k+1} that corresponds to P_ℓ. Note that $P_2 \square \rho$ is an isometric subgraph in $P_2 \square C_{2k+1}$ that contains three colors and $\ell \leq k+1$.

If ℓ is even, then $P_2 \square \rho$ has a rainbow 3-AP by Theorem 2.4 so $P_2 \square C_{2k+1}$ has a rainbow 3-AP, a contradiction.

If ℓ is odd and $\ell \leq k$, extending P_ℓ by one additional vertex (and likewise extending ρ to be ρ') maintains isometry. That is, there is an isometric path $P_{\ell+1}$ in H_1 that contains $v_{1,1}$ and has colors *red* and *blue*. Thus, $P_2 \square \rho'$ is an isometric subgraph of $P_2 \square C_{2k+1}$ that contains three colors and it contains a rainbow 3-AP by Theorem 2.4, another contradiction.

Finally, consider the case when ℓ is odd and $\ell = k+1$. Note that $c(v_{1,i}) = red$ for $k+3 \leq i \leq 2k+1$, else the minimality of P_ℓ would be contradicted. Also, $j = \frac{3k+4}{2}$ is an integer and $k+3 \leq j \leq 2k+1$ when $k \geq 2$. Thus, $\{v_{2,1}, v_{1,j}, v_{1,\ell}\}$ is a rainbow 3-AP, a contradiction.

Therefore, no such c exists and $\mathrm{aw}(P_2 \square C_{2k+1}, 3) = 3$. □

Lemma 3.2. *For integers m and k with $2 \leq m$ and $1 \leq k$,*

$$\mathrm{aw}(P_m \square C_{2k+1}, 3) = 3.$$

Proof. For a base case, note that Lemma 3.1 implies $\mathrm{aw}(P_2 \square C_{2k+1}, 3) = 3$ for all $1 \leq k$. As the inductive hypothesis, suppose that $\mathrm{aw}(P_\ell \square C_{2k+1}, 3) = 3$ for some $2 \leq \ell$. Let c be a rainbow-free, exact 3-coloring of $P_{\ell+1} \square C_{2k+1}$ and let H_i denote the ith copy of C_{2k+1}. By hypothesis and the fact that c is rainbow-free,

$$\left| c\left(\bigcup_{i=1}^{\ell} V(H_i) \right) \right| \leq 2 \text{ and } \left| c\left(\bigcup_{i=2}^{\ell+1} V(H_i) \right) \right| \leq 2.$$

Thus, the inclusion-exclusion principle gives $\left|c\left(\bigcup_{i=2}^{\ell} V(H_i)\right)\right| = 1$. Without loss of generality, assume

$$c\left(\bigcup_{i=2}^{\ell} V(H_i)\right) = \{red\}, \quad blue \in c(V(H_1)), \quad and \quad green \in c\left(V(H_{\ell+1})\right).$$

In particular, assume $c(v_{1,1}) = blue$ and $c(v_{\ell+1,j}) = green$ for some $j \leq k+1$.

Suppose ℓ is even. Then $\{v_{1,1}, v_{\frac{\ell+2}{2},i}, v_{\ell+1,j}\}$ is a rainbow 3-AP for $i = \frac{j+1}{2}$ if j is odd, and $i = \frac{2k+j+2}{2}$ if j is even. On the other hand, suppose ℓ is odd. Then, $\{v_{1,1}, v_{\frac{\ell+1}{2},i}, v_{\ell+1,j}\}$ is a rainbow 3-AP for $i = \frac{j+2}{2}$ if j is even, and $i = \frac{2k+j+1}{2}$ if j is odd.

In any case, there is a rainbow 3-AP, a contradiction, so $\mathrm{aw}(P_{\ell+1}\square C_{2k+1}, 3) = 3$. Thus, by induction, $\mathrm{aw}(P_m\square C_{2k+1}, 3) = 3$ for any $2 \leq m$. $\square$

Determination of $\mathrm{aw}(P_2\square C_{2k})$ requires two strategies since there are k values for which $\mathrm{aw}(P_2\square C_{2k}) = 3$ and k values for which $\mathrm{aw}(P_2\square C_{2k}) = 4$. Essentially, $\mathrm{aw}(P_2\square C_n) = 4$ when $n = 4\ell$ and is determined by providing a coloring where one pair of vertices that are diametrically opposed are colored distinctly and everything else is a third color. This avoids rainbow 3-APs since the diameter of $P_2\square C_{4\ell}$ is odd and because each vertex $v \in V(C_{4\ell})$ has exactly one vertex whose distance from v realizes the diameter of $C_{4\ell}$. Note that this is different than what happens in $P_2\square C_{2k+1}$ since each vertex $v \in V(C_{2k+1})$ has two vertices whose distance from v realizes the diameter of C_{2k+1}. When the diameter of $P_2\square C_{2k}$ is even, this coloring, and every other coloring, ends up creating an isometric $P_2\square P_{2j}$ with 3-colors. Then, it is only a matter of applying Theorem 2.4 to find the rainbow 3-AP.

Lemma 3.3. *For integers m and k with $2 \leq m, k$, $\mathrm{aw}(P_m\square C_{2k}, 3) = 4$ if $\mathrm{diam}(P_m\square C_{2k})$ is odd.*

Proof. Define $c : V(P_m\square C_{2k}) \to \{red, blue, green\}$ by

$$c(v_{i,j}) = \begin{cases} blue & \text{if } i = j = 1, \\ green & \text{if } i = m \text{ and } j = k+1, \\ red & \text{otherwise.} \end{cases}$$

Note that any rainbow 3-AP must contain $v_{1,1}$ and $v_{m,k+1}$ since they are the only *blue* and *green* vertices, respectively. This will be shown by proving $v_{1,1}$ and $v_{m,k+1}$ are not part of any nondegenerate 3-AP. For the sake of contradiction, assume there exists $v_{i,j} \in V(P_m\square C_n)$ such that $\{v_{1,1}, v_{i,j}, v_{m,k+1}\}$ is a nondegenerate 3-AP.

One way this can happen is if $\mathrm{d}(v_{1,1}, v_{i,j}) = \mathrm{d}(v_{i,j}, v_{m,k+1})$. Without loss of generality, suppose $1 \leq j \leq k+1$. Then

$$\mathrm{d}(v_{1,1}, v_{i,j}) = (i-1) + (j-1) = i + j - 2,$$

and

$$\mathrm{d}(v_{i,j}, v_{m,k+1}) = (m-i) + (k+1-j) = m + k + 1 - i - j.$$

By assumption, $i+j-2 = m+k+1-i-j$ which implies that $m+k+1 = 2i+2j-2$. However, $\mathrm{diam}(P_m \Box C_{2k}) = m+k-1$ is odd, a contradiction.

The only other possible way that $\{v_{1,1}, v_{i,j}, v_{m,k+1}\}$ is a 3-AP is if $\mathrm{d}(v_{i,j}, v_{1,1}) = \mathrm{diam}(P_m \Box C_{2k})$ or $\mathrm{d}(v_{i,j}, v_{m,k+1}) = \mathrm{diam}(P_m \Box C_{2k})$. However, this implies $v_{i,j} \in \{v_{1,1}, v_{m,k+1}\}$ which gives a degenerate 3-AP.

Thus, the exact 3-coloring c of $P_m \Box C_{2k}$ is rainbow free so $4 \leq \mathrm{aw}(P_m \Box C_{2k}, 3)$. Theorem 1.3 gives an upper bound of 4 which implies $\mathrm{aw}(P_m \Box C_{2k}, 3) = 4$. □

Lemma 3.4. *For any integer k with $2 \leq k$,*

$$\mathrm{aw}(P_2 \Box C_{2k}, 3) = \begin{cases} 3 & \text{if } k \text{ is odd,} \\ 4 & \text{if } k \text{ is even.} \end{cases}$$

Proof. If k is even, then $\mathrm{diam}(P_2 \Box C_{2k}) = 1 + k$ is odd, and x so by i Lemma 3.3 $\mathrm{aw}(P_2 \Box C_{2k}) = 4$.

Now assume k is odd and let c be an exact 3-coloring of $P_2 \Box C_{2k}$. For the sake of contradiction, assume c is rainbow-free. By Lemma 2.6, $|c(V(H_1))|, |c(V(H_2))| \leq 2$. Without loss of generality, suppose $c(V(H_1)) = \{red, blue\}$, and $green \in c(V(H_2))$ with $c(v_{2,1}) = green$. Now, define P_ℓ as a shortest path in H_1 containing $v_{1,1}$ that contains colors *red* and *blue*, and let ρ be the isometric subgraph of C_{2k} that corresponds to P_ℓ. Note that $P_2 \Box \rho$ is an isometric subgraph in $P_2 \Box C_{2k}$ that contains three colors. If ℓ is even, then Theorem 2.4 gives a rainbow 3-AP, a contradiction.

Suppose ℓ is odd. Since $\mathrm{diam}(H_1) = \mathrm{diam}(C_{2k}) = k$ is odd, the length of P_ℓ is even and P_ℓ is isometric, it follows that P_ℓ can be extended by one vertex in either direction while maintaining isometry. In other words, there is an isometric path $P_{\ell+1}$ in H_1 that contains $v_{1,j}$ and the colors *red* and *blue*. Thus, $P_2 \Box P_{\ell+1}$ is an isometric subgraph of $P_2 \Box C_{2k}$ that contains three colors which means it has a rainbow 3-AP by Theorem 2.4, a contradiction.

Therefore, when k is odd, every exact 3-coloring of $P_2 \Box C_{2k}$ has a rainbow 3-AP and $\mathrm{aw}(P_2 \Box C_{2k}, 3) = 3$. □

Before getting to more general results an analysis of $\mathrm{aw}(P_3 \Box C_n)$ needs to happen. Similar to the $\mathrm{aw}(P_2 \Box C_n)$ situation, there are very subtle and important differences when n is odd versus when n is even.

Lemma 3.5. *For any integer k with $2 \leq k$,*

$$\mathrm{aw}(P_3 \Box C_{2k}, 3) = \begin{cases} 3 & \text{if } k \text{ is even,} \\ 4 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. If k is odd, then $\mathrm{diam}(P_3 \Box C_{2k}) = 2 + k$ is odd, and so by Lemma 3.3 $\mathrm{aw}(P_2 \Box C_{2k}) = 4$.

Suppose k is even and c is an exact, rainbow-free 3-coloring of $P_3 \Box C_{2k}$. Then an argument similar to the argument in the proof of Lemma 3.2 can be used to establish, without loss of generality, that $c(V(H_1)) = \{red, blue\}$, $c(V(H_2)) = \{red\}$, $c(V(H_3)) = \{red, green\}$, $c(v_{1,1}) = blue$ and $c(v_{3,j}) = green$ for some $1 \le j \le k+1$.

If j is odd, then $\{v_{1,1}, v_{2,\frac{j+1}{2}}, v_{3,j}\}$ is a rainbow 3-AP, contradicting that $P_3 \square C_{2k}$ is rainbow free. So, suppose j is even. Then $j+1 \le k+1$ implying that the path $P_{j+1} = (w_1, \ldots, w_{j+1})$ is an isometric subgraph of C_{2k}. So, $P_3 \square P_{j+1}$ is an isometric subgraph of $P_3 \square C_{2k}$. Since $c(P_3 \square P_{j+1}) = \{red, blue, green\}$, Theorem 2.4 implies that $P_3 \square C_{2k}$ contains a rainbow 3-AP. □

Lemma 3.6. *If $m \ge 2$ is even and $k \ge 1$, then*

$$\mathrm{aw}(P_m \square C_{4k+2}, 3) = 3.$$

Proof. Lemma 3.4 implies $\mathrm{aw}(P_2 \square C_{4k+2}, 3) = 3$. Suppose $\mathrm{aw}(P_\ell \square C_{4k+2}, 3) = 3$ for some even $\ell \ge 2$. Then, let c be an exact 3-coloring of $P_{\ell+2} \square C_{4k+2}$ that avoids rainbow 3-APs, and let H_i denote the ith copy of C_{4k+2}. By hypothesis,

$$\left| c\left(\bigcup_{i=1}^{\ell} V(H_i) \right) \right| \le 2 \quad \text{and} \quad \left| c\left(\bigcup_{i=3}^{\ell+2} V(H_i) \right) \right| \le 2.$$

By the inclusion-exclusion principle, $\left| c\left(\bigcup_{i=3}^{\ell} V(H_i) \right) \right| = 1$. Without loss of generality, suppose $c\left(\bigcup_{i=3}^{\ell} V(H_i) \right) = \{red\}$, so that Proposition 2.9 implies $red \in c(H_i)$ for $1 \le i \le \ell+2$. Further, without loss of generality, suppose $blue \in c(V(H_1) \cup V(H_2))$ and $green \in c(V(H_{\ell+1}) \cup V(H_{\ell+2}))$. Say, $c(v_{i,1}) = blue$ and $c(v_{h,j}) = green$ for $i \in \{1,2\}$, $h \in \{\ell+1, \ell+2\}$ and $1 \le j \le 2k+1$ such that i is maximal and h is minimal. If $i = 2$ and $h = 3$, then $|c(H_2) \cup c(H_3)| \ge 3$ which contradicts Corollary 2.7. So assume $h - i \ge 2$. Thus, $c(V(H_{i+1})) = \{red\}$ and $c(V(H_{h-1})) = \{red\}$.

Case 1. Suppose $\mathrm{d}(v_{i,1}, v_{h,j})$ is even.

Then either $\mathrm{d}_{P_{\ell+2}}(u_i, u_h) = h - i$ and $\mathrm{d}_{C_{4k+2}}(w_1, w_j) = j - 1$ are both odd or both even. If they are both even, then $\{v_{i,1}, v_{\frac{i+h}{2}, \frac{j+1}{2}}, v_{h,j}\}$ is a rainbow 3-AP. If they are both odd, then $\{v_{i,1}, v_{\frac{i+h+1}{2}, \frac{j}{2}}, v_{h,j}\}$ is a rainbow 3-AP.

Case 2. Suppose $\mathrm{d}(v_{i,1}, v_{h,j})$ is odd.

If $j < 2k+2$, then $\{v_{h,j}, v_{i,1}, v_{h-1,j+1}\}$ is a rainbow 3-AP. So, suppose $j = 2k+2$. Then $\mathrm{d}_{C_{4k+2}}(w_1, w_j) = 2k+1$ is odd implying that $\mathrm{d}_{P_{\ell+2}}(u_i, u_h)$ is even. Thus, either $i = 1$ and $h = \ell+1$, or $i = 2$ and $h = \ell+2$. First, suppose $i = 1$ and $h = \ell+1$. Then the 3-AP $\{v_{\ell+1,j}, v_{1,1}, v_{\ell+2,j+1}\}$ implies $c(v_{\ell+2,j+1}) = green$. Since i is maximal, $c(V(H_2)) = \{red\}$. Thus, $\{v_{1,1}, v_{\ell+2,j+1}, v_{2,2}\}$ is a rainbow 3-AP since $j+1 = 2k+3$. For $i = 2$ and $j = \ell+2$, the 3-APs $\{v_{2,1}, v_{\ell+2,j}, v_{1,2}\}$ and $\{v_{\ell+2,j}, v_{1,2}, v_{\ell+1,j+1}\}$ yield a rainbow 3-AP.

Thus, $\mathrm{aw}(P_{\ell+2} \square C_{4k+2}, 3) = 3$ and by induction, $\mathrm{aw}(P_m \square C_{4k+2}, 3) = 3$ for any even $m \ge 2$. □

Replacing $4k + 2$ with $4k$ and $2k + 2$ with $2k + 1$ gives the proof of Lemma 3.7, thus the proof has been omitted.

Lemma 3.7. *If $m \geq 3$ is odd and $k \geq 1$, then*

$$\mathrm{aw}(P_m \square C_{4k}, 3) = 3.$$

Lemmas 3.2, 3.3, 3.6, and 3.7 yield the following theorem.

Theorem 3.8. *If $m \geq 2, n \geq 3$ then*

$$\mathrm{aw}(P_m \square C_n, 3) = \begin{cases} 4 & \text{if } n \text{ is even and } \mathrm{diam}(P_m \square C_n) \text{ is odd,} \\ 3 & \text{otherwise.} \end{cases}$$

4 Graph Products of Cycles with Other Graphs

This section starts with a general result, Theorem 4.1, and then uses the general result to establish $\mathrm{aw}(C_m \square C_n, 3)$.

Theorem 4.1. *For any integer k with $1 \leq k$, $\mathrm{aw}(G \square C_{2k+1}, 3) = 3$ for any connected graph G with $|G| \geq 2$.*

Proof. Let $V(G) = \{u_1, \ldots, u_n\}$ and H_i denote the ith labeled copy of C_{2k+1}. Lemma 3.1 implies that $\mathrm{aw}(P_2 \square C_{2k+1}, 3) = 3$, so suppose $|G| \geq 3$. Let $c : V(G \square C_{2k+1}) \to \{red, blue, green\}$ be an exact 3-coloring, and, for the sake of contradiction, assume c is rainbow-free. Since $|G| \geq 3$, Proposition 2.9 implies that, without loss of generality, *red* is in every copy of C_{2k+1}. So, define $c' : V(G) \to \{red, blue, green\}$ by

$$c'(u_i) = \begin{cases} red & \text{if } c(V(H_i)) = \{red\}, \\ \mathcal{C} & \text{if } \mathcal{C} \in c(V(H_i)) \setminus \{red\}. \end{cases}$$

Since Lemma 2.6 implies that $|c(V(H_i))| \leq 2$ for all $1 \leq i \leq n$, it follows that c' is well-defined. By Lemma 2.3, there either exists a C_3 in G containing *red*, *blue*, and *green* or an isometric path in G containing *red*, *blue*, and *green*.

First, suppose $C_3 \cong G[\{u_{i_1}, u_{i_2}, u_{i_3}\}]$ contains *red*, *blue*, and *green*. Then, without loss of generality, there exists neighboring copies H_{i_1} and H_{i_2} of H, in $G \square C_{2k+1}$, such that $c(V(H_{i_1})) = \{red, blue\}$ and $c(V(H_{i_2})) = \{red, green\}$, contradicting Corollary 2.7.

Finally, suppose there exists an isometric path P in G such that $c'(V(P)) = \{red, blue, green\}$. Now, by Lemma 3.2, there exists a rainbow 3-AP in the isometric subgraph $P \square C_{2k+1}$, a contradiction. □

Just as Lemma 3.2 was generalized into Theorem 4.1 which showed that

$$\mathrm{aw}(G \square C_{2k+1}, 3) = 3$$

for all connected G with at least 2 vertices, significant time was spent on the conjecture that a similar generalization would help show $\mathrm{aw}(G\Box C_{4k+2}, 3) = 3$ when $\mathrm{diam}(G)$ is odd and $\mathrm{aw}(G\Box C_{4k}, 3) = 3$ when $\mathrm{diam}(G)$ is even. However, these conjectures do not hold because it cannot be guaranteed that an isometric $P_{2j}\Box C_{4k+2}$ subgraph of $G\Box C_{4k+2}$ or $P_{2j+1}\Box C_{4k}$ subgraph of $G\Box C_{4k}$ exists that contains three colors. The following example provides such a G.

Example 4.2. Consider the graph in Figure 2 which is $G\Box C_4$, where G is a C_{10} with a leaf. That is $V(G) = \{w_1, \ldots, w_{11}\}$ with edges w_iw_{i+1} for $1 \le i \le 9$ and the additional edges w_1w_{10} and $w_{10}w_{11}$. Define $c : V(G\Box C_4) \to \{red, blue, green\}$ by $c(v_{2,1}) = blue$, $c(v_{7,3}) = green$, and $c(v) = red$ for all $v \in V(G\Box C_4) \setminus \{v_{2,1}, v_{7,3}\}$. In order for $G\Box C_4$ to contain a rainbow 3-AP, there must exist a red $v \in V(G\Box C_4)$ such that

$$\mathrm{d}(v_{2,1}, v) = \mathrm{d}(v, v_{7,3}), \quad \mathrm{d}(v, v_{2,1}) = \mathrm{d}(v_{2,1}, v_{7,3}), \quad \text{or} \quad \mathrm{d}(v, v_{7,3}) = \mathrm{d}(v_{7,3}, v_{2,1}).$$

By construction, every vertex v of $G\Box C_4$ is such that $\mathrm{d}(v, v_{2,1})$ and $\mathrm{d}(v, v_{7,3})$ have different parity, thus $\mathrm{d}(v_{2,1}, v) \neq \mathrm{d}(v, v_{7,3})$ for all $v \in V(G)$. To show that there are no vertices v of G distinct from $v_{2,1}, v_{7,3}$ such that $\mathrm{d}(v, v_{2,1}) = \mathrm{d}(v_{2,1}, v_{7,3})$ or $\mathrm{d}(v, v_{7,3}) = \mathrm{d}(v_{7,3}, v_{2,1})$, a discussion about *eccentricity* is needed. For a vertex v of a graph G, the *eccentricity* of v, denoted $\epsilon(v)$, is the distance between v and a vertex furthest from v in G. In other words,

$$\epsilon(v) = \max_{u \in V(G)} \mathrm{d}(u, v).$$

In this example, $\epsilon(v_{2,1}) = \epsilon(v_{7,3}) = \mathrm{d}(v_{2,1}, v_{7,3}) = 7$ and both eccentricities are uniquely realized. So, there are no non-degenerate 3-APs in $G\Box C_4$ containing $v_{2,1}$ and $v_{7,3}$. Thus, $\mathrm{aw}(G\Box C_4, 3) = 4$.

Note that the graph in Figure 2 is the only example presented in this paper of a graph product with even diameter and anti-van der Waerden number (with respect to 3) equal to 4. This is discussed more in Section 5.

Theorem 4.1 gives the following result.

Corollary 4.3. *If m or n is odd with $m, n \ge 3$, then* $\mathrm{aw}(C_m\Box C_n, 3) = 3$.

Lemmas 3.6 and 3.7 are used to prove Lemma 4.4.

Lemma 4.4. *If m and n are even with $m \equiv n \pmod 4$, then* $\mathrm{aw}(C_m\Box C_n, 3) = 3$.

Proof. Let c be an exact 3-coloring of $C_m\Box C_n$. Lemma 2.3 implies that $C_m\Box C_n$ either contains an isometric path or a C_3 with three colors. Since there are no C_3 subgraphs in $C_m\Box C_n$, it follows that $C_m\Box C_n$ must contain an isometric path with three colors. Call a shortest such path P. Suppose P intersects k copies of C_n, and, without loss of generality, suppose these copies are $H_1, \ldots, H_k$.

Notice that there are vertices v and v' of P in $V(H_1)$ and $V(H_k)$, respectively. If $k > \frac{m}{2} + 1$, then any shortest path from v to v' would be contained in the subgraph

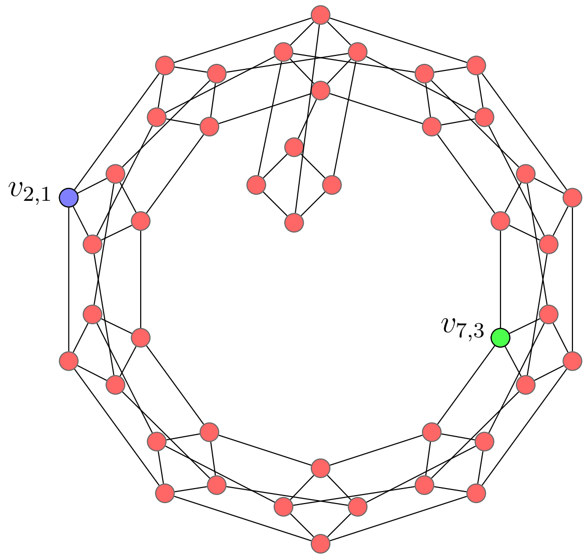

Figure 2: Image for Example 4.2: Graph $G\square C_4$, counterexample of generalizing Lemma 3.7.

induced by the vertices of $H_k, H_{k+1}, \dots, H_n, H_1$. So, no shortest path between v and v' would be contained in P, implying that P is not isometric, a contradiction.

Thus, $k \le \frac{m}{2}+1$, and P is a subgraph of $P_{\frac{m}{2}+1}\square C_n$ where $P_{\frac{m}{2}+1}$ is the subgraph of C_m induced by $\{u_1, \dots, u_{\frac{m}{2}+1}\}$. Thus, P is an isometric subgraph of $C_m\square C_n$ because $P_{\frac{m}{2}+1}$ is isometric in C_m. Since there are three colors in P, there are three colors in $P_{\frac{m}{2}+1}\square C_n$. Furthermore, since $m \equiv n \pmod 4$, $\frac{m}{2}$ and $\frac{n}{2}+1$ have different parity. So, Lemma 3.6 or Lemma 3.7 implies that $P_{\frac{m}{2}+1}\square C_n$ contains a rainbow 3-AP. Thus, $C_m\square C_n$ contains a rainbow 3-AP. □

In the proof of Lemma 4.5, the fact that each vertex in an even cycle realizes the diameter with exactly one other vertex will be used.

Lemma 4.5. *If m and n are even with $m \not\equiv n \pmod 4$, then* $\mathrm{aw}(C_m\square C_n, 3) = 4$.

Proof. Define $k = \frac{m}{2}+1$ and $\ell = \frac{n}{2}+1$ and the coloring $c : V(C_m\square C_n) \to \{red, blue, green\}$ by

$$c(v_{i,j}) = \begin{cases} blue & \text{if } i = j = 1, \\ green & \text{if } i = k, j = \ell, \\ red & \text{otherwise.} \end{cases}$$

Since $v_{1,1}$ and $v_{k,\ell}$ are the only *blue* and *green* vertices, any rainbow 3-AP must contain them. This result will be proved by showing $v_{1,1}$ and $v_{k,\ell}$ are not part of any nondegenerate 3-AP. For the sake of contradiction, assume there exists $v_{i,j} \in V(C_m\square C_n)$ such that $\{v_{1,1}, v_{i,j}, v_{k,\ell}\}$ is a nondegenerate 3-AP.

One way this can happen is if $\mathrm{d}(v_{1,1}, v_{i,j}) = \mathrm{d}(v_{i,j}, v_{k,\ell})$. Without loss of generality, up to a relabelling of the vertices, suppose $1 \le i \le k$ and $1 \le j \le \ell$. Then

$$\mathrm{d}(v_{1,1}, v_{i,j}) = (i-1) + (j-1) = i + j - 2,$$

and

$$\mathrm{d}(v_{i,j}, v_{k,\ell}) = (k - i) + (\ell - j) = k + \ell - i - j.$$

By assumption, $i + j - 2 = k + \ell - i - j$, which implies that

$$2i + 2j - 2 = k + \ell = \frac{m}{2} + \frac{n}{2} + 2. \tag{1}$$

However, $m \not\equiv n \pmod 4$ implies $\frac{m}{2} + \frac{n}{2}$ is odd, which contradicts equation (1).

The only other possible way that $\{v_{1,1}, v_{i,j}, v_{k,\ell}\}$ is a 3-AP is if $\mathrm{d}(v_{i,j}, v_{1,1}) = \mathrm{d}(v_{1,1}, v_{k,\ell})$ or $\mathrm{d}(v_{i,j}, v_{k,\ell}) = \mathrm{d}(v_{k,\ell}, v_{1,1})$. However,

$$\epsilon(v_{1,1}) = \epsilon(v_{k,\ell}) = \mathrm{diam}(C_m \square C_n)$$

is uniquely realized. This implies $v_{i,j} \in \{v_{1,1}, v_{k,\ell}\}$ yielding a degenerate 3-AP.

Thus, the exact 3-coloring c of $C_m \square C_n$ is rainbow free so $4 \leq \mathrm{aw}(C_m \square C_n, 3)$. Theorem 1.3 gives an upper bound of 4 which implies $\mathrm{aw}(C_m \square C_n, 3) = 4$. □

Conglomerating Corollary 4.3, Lemma 4.4 and Lemma 4.5 yields Theorem 4.6.

Theorem 4.6. *If $m, n \geq 3$, then*

$$\mathrm{aw}(C_m \square C_n, 3) = \begin{cases} 4 & \text{if } m \text{ and } n \text{ are even and } \mathrm{diam}(C_m \square C_n) \text{ is odd,} \\ 3 & \text{otherwise.} \end{cases}$$

5 Future Work

Recall that Example 4.2 was the only example presented in this paper of a graph product with even diameter and anti-van der Waerden number (with respect to 3) equal to 4. One of the key factors in allowing this to happen was a pair of vertices u and v such that $\epsilon(u) = \epsilon(v) = \mathrm{d}(u, v) < \mathrm{diam}(u, v)$. Such vertices will be called *almost peripheral vertices* whose name comes from *peripheral vertices* which are vertices that realize the diameter.

Conjecture 5.1. *If $G \square H$ has no almost peripheral vertices and* $\mathrm{diam}(G \square H)$ *is even, then* $\mathrm{aw}(G \square H, 3) = 3$.

In particular, the authors believe that trees do not contain any almost peripheral vertices. For this reason, it is believed that Conjecture 5.2 holds if Conjecture 5.1 holds.

Conjecture 5.2. *If T is a tree, n is even, and* $\mathrm{diam}(T \square C_n)$ *is even, then* $\mathrm{aw}(T \square C_n, 3) = 3$.

This result would provide a more specific case of when the even cycle analog of Theorem 4.1 holds.

Another way to extend Theorem 4.1 would be considering $\mathrm{aw}(G \square C_n, k)$ for some $k > 3$. For $k = 3$, Theorem 4.1 showed that when n is odd, $\mathrm{aw}(G \square C_n, k) = k$ for any connected G of order at least 2. However, there may be other properties of n that guarantee $\mathrm{aw}(G\Box C_n, k) = k$ for $k > 3$. Some preliminary work analyzing $\mathrm{aw}(P_m\Box C_n, 4)$ suggests that for any n, there exists an m such that $\mathrm{aw}(P_m\Box C_n, 4) \geq 5$.

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