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Chorded pancyclicity with distance two degree condition and doubly chorded pancyclicity

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Abstract

A graph G of order $n \geq 3$ is pancyclic if G contains a cycle of each length from 3 to n, and vertex pancyclic (edge pancyclic) if every vertex (edge) is contained on a cycle of each length from 3 to n. A chord of a cycle is an edge between two nonadjacent vertices of the cycle, and a chorded cycle is a cycle containing at least one chord. We define a graph G of order $n \geq 4$ to be chorded pancyclic if G contains a chorded cycle of each length from 4 to n. In this paper, we improve some known results on chorded pancyclic, chorded vertex pancyclic, and chorded edge pancyclic graphs.

1 Introduction

We consider only simple graphs in this paper. Let G be a graph and let H be a subgraph of G. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$, $N_H(u) = N_G(u) \cap V(H)$, and $d_H(u) = |N_H(u)|$. Let

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$S \subseteq V(G)$. For $u \in V(G) - S$, $N_S(u) = N_G(u) \cap S$. The subgraph of G induced by S is denoted by $\langle S \rangle$, $G - S = \langle V(G) - S \rangle$, and $G - H = \langle V(G) - V(H) \rangle$. If $S = \{u\}$, then we write $G - u$ for $G - S$. If C is a cycle with a given orientation and $x \in V(C)$, then x^+ (x^-) denotes the first successor (predecessor) of x on C. If $x, y \in V(C)$, then $C[x, y]$ denotes the subpath of C from x to y (including x and y) in the given direction. The reverse sequence of $C[x, y]$ is denoted by $C^-[y, x]$. We also write $C(x, y] = C[x^+, y]$, $C[x, y) = C[x, y^-]$, and $C(x, y) = C[x^+, y^-]$, and consider them as both paths and vertex sets. If C is a cycle, say $C = x_1, x_2, \ldots, x_t, x_1$, then we assume that an orientation of C is given from x_1 to x_t clockwise. For two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, $G_1 + G_2$, and $G_1 \times G_2$ denote the *union*, the *join*, and the *cartesian product* of G_1 and G_2, respectively. A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbors. For an integer $k \geq 3$, a cycle of length k is called a *k-cycle*. Let P_t be a path on t vertices for an integer $t \geq 1$. We denote the distance between two vertices x and y in G by $\mathrm{dist}_G(x, y)$. For a graph G, we let

$$\sigma_2(G) = \min\{d_G(u) + d_G(v) \mid u, v \in V(G), uv \notin E(G)\},$$
$$\mu_2(G) = \min\{d_G(u) + d_G(v) \mid u, v \in V(G), \mathrm{dist}_G(u, v) = 2\}, \text{ and}$$

$\sigma_2(G)$ and $\mu_2(G)$ are both equal to ∞ when G is complete. For terminology and notation not defined here, see [10].

In 1960, Ore proved the following theorem which is one of the most fundamental results on Hamiltonian graphs.

Theorem 1.1 (Ore [12]). *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

A graph G of order $n \geq 3$ is said to be *pancyclic* if G contains a cycle of each length from 3 to n. In 1971, Bondy [2] proposed the following famous meta-conjecture.

Bondy's Meta-Conjecture. Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic. There may be a simple family of exceptional graphs.

The following extension of Ore's theorem (Theorem 1.1) by Bondy supports the meta-conjecture.

Theorem 1.2 (Bondy [3]). *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is pancyclic or $G = K_{n/2, n/2}$* (n is even).

Let G be a graph of order $n \geq 3$, and let $r \geq 3$ be an integer. A graph G is called *vertex pancyclic* (*edge pancyclic*) if every vertex (edge) is contained on a k-cycle for each $3 \leq k \leq n$ in G. A graph G is *r-pancyclic* if G contains a k-cycle for each $r \leq k \leq n$, and G is also called *vertex r-pancyclic* (*edge r-pancyclic*) if every vertex (edge) is contained on a k-cycle for each $r \leq k \leq n$ in G.

Recently, chorded pancyclic properties have been well-studied (see [1, 4–7, 9]). A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. We say a cycle is *chorded* (*doubly chorded*) if the cycle has at least one chord (at least two chords), and we call such a cycle a *chorded cycle* (*doubly chorded cycle*). Further, we say a graph G of order $n \geq 4$ is *chorded pancyclic* (*doubly chorded pancyclic*) if G contains a chorded cycle (doubly chorded cycle) of each length from 4 to n. In this paper, we improve some known results on chorded pancyclic graphs.

A survey of results and problems on chorded cycles can be found in [11].

2 Chorded Pancyclic Graphs

Bondy's meta-conjecture was extended in [6] to almost any nontrivial condition that implies a graph is Hamiltonian will imply it is chorded pancyclic, possibly with some class of well-defined exceptional graphs, and some small order exceptional graphs. As support for this extension, the following theorem which is the extension of Theorems 1.1 and 1.2 was proved.

Theorem 2.1 (Cream et al. [6]). *Let G be a graph of order $n \geq 4$. If $\sigma_2(G) \geq n$, then G is chorded pancyclic, $G = K_{n/2,n/2}$* (n is even), *or $G = K_2 \times K_3$.*

We improve Theorem 2.1 by considering the distance two degree condition. Since $\mu_2(G) \geq \sigma_2(G)$, the following theorem is stronger than Theorem 2.1.

Theorem 2.2. *Let G be a graph of order $n \geq 4$. If $\mu_2(G) \geq n$, then G is chorded pancyclic, $G = K_{n/2,n/2}$* (n is even), *or $G = K_2 \times K_3$.*

Remark 2.1. The degree condition of Theorem 2.2 is sharp. Let a, b be integers such that $a \geq 1$, $b \geq 1$, and $a + b \geq 3$. Consider the graph $G = (K_a \cup K_b) + K_1$ of order n. Then $\mu_2(G) = a + b = n - 1$, and G does not contain a Hamiltonian cycle. Thus $\mu_2(G) \geq n$ is necessary.

The following theorem by Zhang and Song will be used in the proof of Theorem 2.2.

Theorem 2.3 (Zhang and Song [13]). *Let G be a graph of order $n \geq 4$. If $\mu_2(G) \geq n$, then G is vertex 4-pancyclic or $G = K_{n/2,n/2}$* (n is even).

Proof of Theorem 2.2. Let G, n be as described in Theorem 2.2. By Theorem 2.3, G is Hamiltonian. If $n = 4$, then either $G = K_{2,2}$ or G is a 4-cycle with chords and is then chorded pancyclic. Suppose $n = 5$. Let $C = v_1, v_2, v_3, v_4, v_5, v_1$ be a Hamiltonian cycle in G. If $d_G(v_i) = 4$ for some $1 \leq i \leq 5$, then G is chorded pancyclic. Thus, we may now assume that $d_G(v_i) \leq 3$ for each $1 \leq i \leq 5$. Without loss of generality, we may assume that $v_1v_3 \notin E(G)$. Then $\mathrm{dist}_G(v_1, v_3) = 2$. Since $\mu_2(G) \geq n = 5$, without loss of generality, we may assume that $d_G(v_1) \geq 3$. Then we have $d_G(v_1) = 3$, $v_1v_4 \in E(G)$, and $d_G(v_4) = 3$. If $v_2v_5 \in E(G)$, then there exists a chorded 4-cycle, and G is chorded pancyclic. Thus we may assume that $v_2v_5 \notin E(G)$.

By symmetry, $v_3v_5 \notin E(G)$. Thus $d_G(v_2) = d_G(v_3) = 2$. Therefore, $d_G(v_5) = 2$. Since $\mathrm{dist}_G(v_2, v_5) = 2$ and $\mu_2(G) \geq n = 5$, this is a contradiction. Thus we suppose $n \geq 6$. It follows from Theorem 2.3 that G is 4-pancyclic or $G = K_{n/2,\, n/2}$ (n is even). Suppose that $G \neq K_{n/2,\, n/2}$ (n is even) and $G \neq K_2 \times K_3$. If G is complete, then Theorem 2.2 holds. Thus we may assume that G is not complete. Then there exist two distinct vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$. We now choose two such distinct vertices x, y with the smallest number of common neighbors. Partition $V(G) - \{x, y\}$ as follows:

$$\begin{aligned}
M &= N_G(x) \cap N_G(y),\\
X &= N_G(x) - M,\\
Y &= N_G(y) - M, \text{ and}\\
D &= V(G) - (\{x, y\} \cup M \cup X \cup Y).
\end{aligned}$$

Suppose $|M| \leq 1$. Since $\mathrm{dist}_G(x, y) = 2$ and $\mu_2(G) \geq n$, we have

$$\begin{aligned}
n \leq \mu_2(G) \leq d_G(x) + d_G(y) &\leq |V(G) - \{x, y\}| + |M|\\
&\leq (n-2) + 1 = n - 1.
\end{aligned}$$

This is a contradiction. Thus $|M| \geq 2$. Set $|M| = 2 + r$, $r \geq 0$.

Claim 2.1. *We have* $|D| \leq r$.

Proof. Suppose not, and let $|D| = r + t$, $t \geq 1$. Then we have

$$\begin{aligned}
n \leq \mu_2(G) \leq d_G(x) + d_G(y) &\leq |V(G) - \{x, y\}| - |D| + |M|\\
&= (n-2) - (r+t) + (2+r) = n - t.
\end{aligned}$$

Since $t \geq 1$, this is a contradiction. Thus $|D| \leq r$. □

Claim 2.2. *There exists a chorded n-cycle in G.*

Proof. Since $n \geq 6$ and G contains a Hamiltonian cycle, say C, it is easy to see that C is a chorded n-cycle by the distance two degree condition. □

Claim 2.3. *There exists a chorded* 4*-cycle in G.*

Proof. Suppose that the claim does not hold. Since $|M| \geq 2$ by the above fact, let a and b be any two distinct vertices in M. If $ab \in E(G)$, then a, y, b, x, a is a 4-cycle with chord ab, a contradiction. Thus we may assume that $ab \notin E(G)$. This implies M is an independent set. Note $\mathrm{dist}_G(a, b) = 2$. By the choice of x and y, $|N_G(a) \cap N_G(b)| \geq 2 + r$, $r \geq 0$. Let $w \in N_G(a) \cap N_G(b)$. Since M is independent, $w \notin M$. If $w \in X$, then a, w, b, x, a is a 4-cycle with chord xw, a contradiction. Thus $w \notin X$. Similarly, $w \notin Y$. Therefore, $w \in \{x, y\} \cup D$ and then $|D| \geq r$. By Claim 2.1, we obtain $|D| = r$. Since a and b are any two distinct vertices in M, we have $N_D(v) = D$ for any $v \in M$ if $D \neq \emptyset$. If $D \neq \emptyset$, then D is an independent set, otherwise, when $|D| \geq 2$, there exists a chorded 4-cycle in $\langle M \cup D \rangle$. Thus

$\langle M \cup D\rangle = K_{2+r,r}$ if $D \neq \emptyset$. If $X = \emptyset = Y$, then $G = K_{2+r,2+r} = K_{n/2,n/2}$, a contradiction. Thus we may assume that $X \cup Y \neq \emptyset$, and then, without loss of generality, we may also assume that $|X| \geq |Y|$. Since $\mathrm{dist}_G(a, b) = 2$, we have

$$\begin{aligned}|M| + |\{x, y\}| + |X| + |Y| + |D| = n \leq \mu_2(G) &\leq d_G(a) + d_G(b)\\ &\leq 2(|\{x, y\}| + |D|) + |N_{X\cup Y}(a)| + |N_{X\cup Y}(b)|.\end{aligned}$$

Since $|M| = 2 + r$ and $|D| = r$,

$$(2 + r) + 2 + |X| + |Y| + r \leq 2(2 + r) + |N_{X\cup Y}(a)| + |N_{X\cup Y}(b)|,$$

and therefore,

$$|X| + |Y| \leq |N_{X\cup Y}(a)| + |N_{X\cup Y}(b)|. \tag{1}$$

Since $N_{X\cup Y}(a) \cap N_{X\cup Y}(b) = \emptyset$, it follows from (1) that

$$N_{X\cup Y}(a) \cup N_{X\cup Y}(b) = X \cup Y. \tag{2}$$

Let $w_1 \in X$, and without loss of generality, we may assume that $aw_1 \in E(G)$. Note that $w_1v \notin E(G)$ for any $v \in X - \{w_1\}$, otherwise, say $w_1v' \in E(G)$ for some $v' \in X - \{w_1\}$, then w_1, v', x, a, w_1 is a 4-cycle with chord xw_1, a contradiction. Note that $w_1y \notin E(G)$ by the definition of X. Also $w_1t \notin E(G)$ for any $t \in M - \{a\}$, otherwise, again a chorded 4-cycle exists. Thus $N_G(w_1) \subseteq \{a, x\} \cup Y \cup D$. If $Y = \emptyset$, then since $\mathrm{dist}_G(w_1, y) = 2$,

$$|M| + |\{x, y\}| + |X| + |D| = n \leq \mu_2(G) \leq d_G(w_1) + d_G(y) \leq (|\{a, x\}| + |D|) + |M|,$$

and thus $|X| \leq 0$, a contradiction. Therefore, $Y \neq \emptyset$. If $N_Y(w_1) = \emptyset$, then similarly, $|X| \leq 0$, again a contradiction. Therefore, $N_Y(w_1) \neq \emptyset$. Let $w_1z_1 \in E(G)$ for $z_1 \in Y$. By (2), we have $bz_1 \in E(G)$, as otherwise, if $az_1 \in E(G)$, then a, y, z_1, w_1, a is a 4-cycle with chord az_1, a contradiction.

We now claim that $|M| = 2$. Suppose that this claim does not hold. Then $|M| \geq 3$, and let $v \in M - \{a, b\}$. Since M is independent, $av \notin E(G)$. By the same argument as (2), $N_{X\cup Y}(a) \cup N_{X\cup Y}(v) = X \cup Y$. Since $az_1 \notin E(G)$, $vz_1 \in E(G)$. Then v, y, b, z_1, v is a 4-cycle with chord yz_1, a contradiction. Thus $|M| = 2$ and so $r = 0$. By Claim 2.1, $D = \emptyset$.

We note that $|N_X(u)| \leq 1$ and $|N_Y(u)| \leq 1$ for any $u \in \{a, b\}$, otherwise, there exists a chorded 4-cycle, a contradiction. If $|X| \geq 3$, then by (2), one of a and b has at least two adjacencies in X, a contradiction. Thus $|X| \leq 2$, and similarly, $|Y| \leq 2$.

If $|X \cup Y| = 2$, then $G = K_2 \times K_3$, a contradiction. Thus we may assume that $|X \cup Y| \geq 3$. Then by $|X| \geq |Y|$ which is our previous assumption, we have $|X| = 2$. Let $w_2 \in X - \{w_1\}$. Then note $bw_2 \in E(G)$ since $aw_2 \notin E(G)$. Suppose $|Y| = 1$. Then $n = 7$. Since $\mathrm{dist}_G(y, w_1) = 2$, $d_G(y) + d_G(w_1) \geq n = 7$. On the other hand, since $d_G(y) = 3$ and $d_G(w_1) = 3$, we have $d_G(y) + d_G(w_1) = 6$, a contradiction. Thus $|Y| = 2$. Now $n = 8$. Let $z_2 \in Y - \{z_1\}$. Then $az_2 \in E(G)$ since $bz_2 \notin E(G)$. If

$w_1z_2 \in E(G)$, then a, z_2, w_1, x, a is a 4-cycle with chord aw_1, a contradiction. Hence $w_1z_2 \notin E(G)$. Since $\mathrm{dist}_G(y, w_1) = 2$, $d_G(y) + d_G(w_1) \geq n = 8$. On the other hand, since $d_G(y) = 4$ and $d_G(w_1) = 3$, we have $d_G(y) + d_G(w_1) = 7$, a contradiction. This completes the proof of Claim 2.3. □

Claim 2.4. *If G contains a chorded 4-cycle, then there exists a chorded 5-cycle in G.*

Proof. Suppose that $C = v_1, v_2, v_3, v_4, v_1$ is a 4-cycle in G with chord v_2v_4. Recall that G is connected. Since $n \geq 6$, there exists some $x \in V(G - C)$ such that $xv \in E(G)$ for some $v \in V(C)$. By symmetry, we may assume that $v = v_1$ or $v = v_2$.

Case 1. Suppose $v = v_1$, that is, $xv_1 \in E(G)$.

If $xv' \in E(G)$ for some $v' \in V(C) - \{v_1\}$, then there exists a chorded 5-cycle. Thus $xv \notin E(G)$ for any $v \in V(C) - \{v_1\}$. Note $\mathrm{dist}_G(x, v_2) = 2$. By the distance two degree condition, x and v_2 share at least two common neighbors, and the common neighbors except v_1 must be off of C. Let $y \in V(G - C) - \{x\}$ be such a common neighbor. Then v_1, x, y, v_2, v_4, v_1 is a 5-cycle with chord v_1v_2.

Case 2. Suppose $v = v_2$, that is, $xv_2 \in E(G)$.

Considering Case 1, we may assume that $xv_1, xv_3 \notin E(G)$. Note $\mathrm{dist}_G(x, v_1) = 2$. By the distance two degree condition, x and v_1 share at least two common neighbors, and let y be such a common neighbor except v_2. If $y \in V(G - C) - \{x\}$, then y, x, v_2, v_4, v_1, y is a 5-cycle with chord v_1v_2. This implies $y = v_4$, and then $xv_4 \in E(G)$. If $v_1v_3 \in E(G)$, then x, v_2, v_3, v_1, v_4, x is a 5-cycle with chord v_1v_2. Thus we may assume that $v_1v_3 \notin E(G)$. Then $\mathrm{dist}_G(v_1, v_3) = 2$. Since $d_G(v_1) + d_G(v_3) \geq n \geq 6$, there exists some $z \in V(G-C) - \{x\}$ such that $zv' \in E(G)$ for some $v' \in \{v_1, v_3\}$. By symmetry, we may assume that $zv_1 \in E(G)$. Then we are now in a case analogous to Case 1.

This completes the proof of Claim 2.4. □

If $n = 6$, $G \neq K_{3,3}$ and $G \neq K_2 \times K_3$, then G is chorded pancyclic by Claims 2.2–2.4. Thus we may assume that $n \geq 7$.

Claim 2.5. *There exists a chorded k-cycle for each $6 \leq k \leq n - 1$ in G.*

Proof. Recall that G is 4-pancyclic since $G \neq K_{n/2,n/2}$. Let $6 \leq k \leq n - 1$ and consider a chordless k-cycle $C = v_1, v_2, \ldots, v_k, v_1$ in G. Let $L = \{v_1, v_3, v_4, v_6\}$. Now we claim that $|N_L(x)| \leq 2$ for any $x \in V(G-C)$. Suppose that $|N_L(x')| \geq 3$ for some $x' \in V(G-C)$. By symmetry, it is sufficient to consider the cases when $\{v_1, v_3, v_4\} \subseteq N_L(x')$ and $\{v_1, v_3, v_6\} \subseteq N_L(x')$. If $\{v_1, v_3, v_4\} \subseteq N_L(x')$, then $x', v_3, v_4, \ldots, v_1, x'$ is a k-cycle with chord $x'v_4$. If $\{v_1, v_3, v_6\} \subseteq N_L(x')$, then $x', v_3, v_4, \ldots, v_1, x'$ is a k-cycle with chord $x'v_6$. Thus the claim holds. Since $|N_L(x)| \leq 2$ for any $x \in V(G - C)$,

$\mathrm{dist}_G(v_1, v_3) = 2$, and $\mathrm{dist}_G(v_4, v_6) = 2$, we have

$$\begin{aligned} 2n \le 2\mu_2(G) &\le (d_G(v_1) + d_G(v_3)) + (d_G(v_4) + d_G(v_6)) \\ &= \sum_{v\in L} d_{G-C}(v) + \sum_{v \in L} d_C(v) \\ &\le 2(n-k) + 2\cdot 4 \\ &= 2n - 2k + 8, \end{aligned}$$

and then $k \le 4$. Since $k \ge 6$, this is a contradiction. Thus Claim 2.5 holds. □

Claims 2.2–2.5 imply that G is chorded pancyclic. This completes the proof of Theorem 2.2. ■

Finally, we consider an improvement of Bondy's theorem (Theorem 1.2). If a graph G contains a chorded 4-cycle, then G contains a 3-cycle. Thus if G is chorded pancyclic, then G is pancyclic. Note that $K_2 \times K_3$ in Theorem 2.2 is pancyclic. Thus the following corollary holds by Theorem 2.2.

Corollary 2.1. *Let G be a graph of order $n \ge 3$. If $\mu_2(G) \ge n$, then G is pancyclic or $G = K_{n/2,\,n/2}$ (n is even).*

3 Doubly Chorded Edge (Vertex) Pancyclic Graphs

In this section, we first consider an extension of edge pancyclicity. Let $r \ge 4$ be an integer. A graph G of order $n \ge 4$ is *chorded edge r-pancyclic* (*doubly chorded edge r-pancyclic*) if every edge is contained on a chorded cycle (doubly chorded cycle) of each length from r to n in G.

The following result is a consequence of a theorem in [8].

Theorem 3.1. (Faudree et al. [8, Theorem 2]) *Let G be a graph of order $n \ge 3$. If $\sigma_2(G) \ge n + 1$, then G is edge pancyclic.*

In 2018, Cream et al. extended Theorem 3.1 as follows.

Theorem 3.2. (Cream et al. [7, Theorem 20]) *Let G be a graph of order $n \ge 5$. If $\sigma_2(G) \ge n + 1$, then G is chorded edge 5-pancyclic.*

In this section we extend Theorems 3.1 and 3.2.

Theorem 3.3. *Let G be a graph of order $n \ge 5$. If $\sigma_2(G) \ge n+1$, then G is doubly chorded edge 5-pancyclic.*

Remark 3.1. The graph $K_{n/2,\,n/2}$ of even order n verifies that the $\sigma_2(G)$ condition in Theorem 3.3 is sharp. Theorem 3.3 is sharp in terms of 5-pancyclicity. We consider the graph G of order n, $n \equiv 3 \pmod 4$, obtained from $K_{(n-1)/2,\,(n+1)/2}$ along with a perfect matching in the larger partite set. Then G is $(n+1)/2$-regular, and $\sigma_2(G) = n + 1$. However, G does not contain K_4. Hence there exist no doubly chorded 4-cycles containing any specified edge in G.

Proof of Theorem 3.3. Let G, n be as described in Theorem 3.3, and e be any specified edge in G. If G is complete, then the theorem holds. Thus we may assume that G is not complete.

Claim 3.1. *Let $a, b \in V(G)$ with $ab \notin E(G)$, $M = N_G(a) \cap N_G(b)$, $A = N_G(a) - M$, $B = N_G(b) - M$, and $D = V(G) - (\{a, b\} \cup M \cup A \cup B)$. Then $|M| \geq |D| + 3 \geq 3$.*

Proof. Let a, b, M, A, B, D be as described in Claim 3.1. Since $ab \notin E(G)$, by the $\sigma_2(G)$ condition,

$$n + 1 \leq d_G(a) + d_G(b) = 2|M| + |A| + |B|.$$

Since $n = |M| + |\{a, b\}| + |A| + |B| + |D|$, we have $|M| \geq |D| + 3 \geq 3$. □

Claim 3.2. *There exists a doubly chorded n-cycle containing e in G.*

Proof. By Theorem 3.2, there exists a chorded n-cycle (say C) containing e in G. Since $n \geq 5$ and $\sigma_2(G) \geq n + 1 \geq 6$, it is easy to see that C has at least two chords. Thus the claim holds. □

Claim 3.3. *There exists a doubly chorded 5-cycle containing e in G.*

Proof. Let $e = x_1x_2$. We claim that there exists some $y \in V(G) - \{x_1, x_2\}$ such that $x_1y \notin E(G)$. Suppose not. Since $\sigma_2(G - x_1) \geq (n+1) - 2 = n - 1$, $G - x_1$ contains a Hamiltonian cycle (say C) by Ore's theorem (Theorem 1.1). Now it is trivial to find a path of length 3 on C starting at x_2, and then there exists a doubly chorded 5-cycle containing e. Thus the claim holds. We partition $V(G)$ as follows: $M = N_G(x_1) \cap N_G(y)$, $X = N_G(x_1) - M$, $Y = N_G(y) - M$, and $D = V(G) - (\{x_1, y\} \cup M \cup X \cup Y)$.

Case 1. Suppose $x_2 \in M$.

If $x_2m' \in E(G)$ for some $m' \in M - \{x_2\}$, then for any $m \in M - \{x_2, m'\}$, x_1, x_2, m', y, m, x_1 is a 5-cycle with chords x_1m' and x_2y containing e. If $m_1m_2 \in E(G)$ for any $m_1, m_2 \in M - \{x_2\}$, then $x_1, x_2, y, m_2, m_1, x_1$ is a 5-cycle with chords x_1m_2 and m_1y containing e. Thus we may assume that M is an independent set. If $z \in N_X(x_2) \cap N_X(m')$ for some $m' \in M - \{x_2\}$, then x_2, x_1, z, m', y, x_2 is a 5-cycle with chords x_1m' and x_2z containing e. Thus $N_X(x_2) \cap N_X(m) = \emptyset$ for any $m \in M - \{x_2\}$. Similarly, $N_Y(x_2) \cap N_Y(m) = \emptyset$ for any $m \in M - \{x_2\}$. Since $x_2m \notin E(G)$ for any $m \in M - \{x_2\}$, by the $\sigma_2(G)$ condition,

$$n + 1 \leq d_G(x_2) + d_G(m) \leq 2|\{x_1, y\}| + |X| + |Y| + 2|D|.$$

Since $n = |M| + |\{x_1, y\}| + |X| + |Y| + |D|$, we have $|M| \leq |D| + 1$. This contradicts Claim 3.1.

Case 2. Suppose $x_2 \notin M$.

In this case, note $x_2 \in X$. If $|N_M(x_2)| \geq 2$, then there exists a doubly chorded 5-cycle containing e. Thus we may assume that $|N_M(x_2)| \leq 1$. Since $|M| \geq 3$ by Claim 3.1, there exists $m_1 \in M$ such that $x_2m_1 \notin E(G)$. Now we consider

x_2 (respectively, m_1) as x_1 (respectively, y) in Case 1. Note that x_1 is a common neighbor of x_2 and m_1. Then we are in a case analogous to Case 1.

Hence Claim 3.3 holds. □

Claim 3.4. *There exists a doubly chorded k-cycle for each $6 \le k \le n-1$ containing e in G.*

Proof. By Theorem 3.2, G is chorded edge 5-pancyclic. Let $C = v_1, v_2, \dots, v_k, v_1$, $6 \le k \le n-1$ be a chorded cycle of length k containing e in G. Without loss of generality, we may assume that v_1v_i, $3 \le i \le k-1$ is a chord of C. Then note that $|C(v_1, v_i)| \ge 1$ and $|C(v_i, v_1)| \ge 1$. Also we may assume that $C[v_1, v_i]$ contains e. Let $e = v_jv_{j+1}$ for $1 \le j \le i-1$.

Case 1. Suppose $|C(v_1, v_i)| \ge 5$.

In this case, $i \ge 7$. By symmetry, we may assume that $1 \le j \le i-4$. First suppose $j \ge 2$. If $v_2v_i \in E(G)$, then C is the desired cycle. Thus we may assume that $v_2v_i \notin E(G)$. Similarly, $v_{i-3}v_{i+1} \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_i)$ and $y \in N_{G-C}(v_{i-3}) \cap N_{G-C}(v_{i+1})$ with $x \ne y$. Then $C[v_2, v_{i-3}], y, C[v_{i+1}, v_1], v_i, x, v_2$ is a k-cycle with chords v_1v_2 and v_iv_{i+1} containing e.

Next suppose $j = 1$. We have $v_2v_5, v_{i-1}v_k \notin E(G)$, otherwise, C is the desired cycle. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_5)$ and $y \in N_{G-C}(v_{i-1}) \cap N_{G-C}(v_k)$ with $x \ne y$. Then $v_1, v_2, x, C[v_5, v_{i-1}], y, C^-[v_k, v_i], v_1$ is a k-cycle with chords $v_{i-1}v_i$ and v_kv_1 containing e.

Case 2. Suppose $|C(v_1, v_i)| = 4$.

In this case, $i = 6$. By symmetry, we may assume that $1 \le j \le 3$. First suppose $j = 1$, that is, $e = v_1v_2$. We have $v_2v_5, v_5v_k \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_5)$ and $y \in N_{G-C}(v_5) \cap N_{G-C}(v_k)$ with $x \ne y$. Then $v_1, v_2, x, v_5, y, C^-[v_k, v_6], v_1$ is a k-cycle with chords v_5v_6 and v_kv_1 containing e.

Next suppose $j = 2$, that is, $e = v_2v_3$. We have $v_2v_6, v_3v_7 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_6)$ and $y \in N_{G-C}(v_3) \cap N_{G-C}(v_7)$ with $x \ne y$. Then $v_2, v_3, y, C[v_7, v_1], v_6, x, v_2$ is a k-cycle with chords v_1v_2 and v_6v_7 containing e.

Finally, suppose $j = 3$, that is, $e = v_3v_4$. Assume $|C(v_6, v_1)| \ge 2$. We have $v_2v_6, v_5v_9 \notin E(G)$ (if $k = 8$, then $v_9 = v_1$). By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_6)$ and $y \in N_{G-C}(v_5) \cap N_{G-C}(v_9)$ with $x \ne y$. Then $C[v_2, v_5], y, C[v_9, v_1], v_6, x, v_2$ is a k-cycle with chords v_1v_2 and v_5v_6 containing e. Next assume that $|C(v_6, v_1)| = 1$. We have $v_2v_5, v_4v_6 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_5)$ and $y \in N_{G-C}(v_4) \cap N_{G-C}(v_6)$ with $x \ne y$. Now we make a new cycle $C' = v_2, v_3, v_4, y, v_6, v_5, x, v_2$. Then C' is a k-cycle with chord v_4v_5 containing $e = v_3v_4$, and $yv_5, v_3v_6 \notin E(G)$. Since $|N_{G-C'}(y) \cap N_{G-C'}(v_5)| \ge 1$, we first take $z \in N_{G-C'}(y) \cap N_{G-C'}(v_5)$. Next we take $w \in N_{G-C'}(v_3) \cap N_{G-C'}(v_6)$ with $w \ne z$. Then $v_3, w, v_6, y, z, v_5, v_4, v_3$ is a k-cycle with chords v_4y and v_5v_6 containing e.

Case 3. Suppose $|C(v_1, v_i)| = 3$.

In this case, $i = 5$. By symmetry, we may assume that $j \in \{1, 2\}$. First suppose $j = 1$, that is, $e = v_1v_2$. Assume $|C(v_5, v_1)| \geq 2$. We have $v_5v_k, v_2v_6 \notin E(G)$. Since $|N_{G-C}(v_5) \cap N_{G-C}(v_7)| \geq 1$ when $|C(v_5, v_1)| = 2$, i.e., $k = 7$, we first take $x \in N_{G-C}(v_5) \cap N_{G-C}(v_k)$. Next we take $y \in N_{G-C}(v_2) \cap N_{G-C}(v_6)$ with $x \neq y$. Then $v_1, v_2, y, C[v_6, v_k], x, v_5, v_1$ is a k-cycle with chords v_5v_6 and v_kv_1 containing e. Next assume $|C(v_5, v_1)| = 1$. We have $v_1v_3, v_2v_4 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_1) \cap N_{G-C}(v_3)$ and $y \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$ with $x \neq y$. Now we make a new cycle $C' = v_1, v_2, y, v_4, v_3, x, v_1$. Then C' is a k-cycle with chord v_2v_3 containing $e = v_1v_2$, and $v_1v_3, v_2x \notin E(G)$. By Claim 3.1, there exist $z \in N_{G-C}(v_1) \cap N_{G-C}(v_3)$ and $w \in N_{G-C}(v_2) \cap N_{G-C}(x)$. If $z \neq w$, then $v_1, v_2, w, x, v_3, z, v_1$ is a k-cycle with chords v_1x and v_2v_3 containing e. If $z = w$, then $v_1, v_2, y, v_4, v_3, z, v_1$ is a k-cycle with chords v_2v_3 and zv_2 containing e.

Next suppose $j = 2$, that is, $e = v_2v_3$. Assume $|C(v_5, v_1)| \geq 2$. We have $v_2v_4, v_3v_8 \notin E(G)$ (if $k = 7$, then $v_8 = v_1$). By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$ and $y \in N_{G-C}(v_3) \cap N_{G-C}(v_8)$ with $x \neq y$. Then

$$v_2, v_3, y, C[v_8, v_1], v_5, v_4, x, v_2$$

is a k-cycle with chords v_1v_2 and v_3v_4 containing e. Next assume $|C(v_5, v_1)| = 1$. We have $v_2v_4, v_3v_5 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$ and $y \in N_{G-C}(v_3) \cap N_{G-C}(v_5)$ with $x \neq y$. Now we make a new cycle $C'' = v_2, v_3, y, v_5, v_4, x, v_2$. Then C'' is a k-cycle with chord v_3v_4 containing $e = v_2v_3$, and this case is the same as C' above.

Case 4. Suppose $|C(v_1, v_i)| = 2$.

In this case, $i = 4$. Since $k \geq 6$, $|C(v_4, v_1)| \geq 2$. By symmetry, we may assume that $j \in \{1, 2\}$. First suppose $j = 1$, that is, $e = v_1v_2$. Assume $|C(v_4, v_1)| \geq 3$. We have $v_2v_4, v_3v_7 \notin E(G)$. By Claim 3.1, we first take $x \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$. Next we take $y \in N_{G-C}(v_3) \cap N_{G-C}(v_7)$ with $x \neq y$. Then $v_1, v_2, x, v_4, v_3, y, C[v_7, v_1]$ is a k-cycle with chords v_1v_4 and v_2v_3 containing e. Next assume $|C(v_4, v_1)| = 2$. We have $v_1v_3, v_2v_4 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_1) \cap N_{G-C}(v_3)$ and $y \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$. If $x \neq y$, then $v_1, v_2, y, v_4, v_3, x, v_1$ is a k-cycle with chords v_1v_4 and v_2v_3 containing e. If $x = y$, then $v_1, v_2, x, v_4, v_5, v_6, v_1$ is a k-cycle with chords v_1x and v_1v_4 containing e.

Next suppose $j = 2$, that is, $e = v_2v_3$. Assume $|C(v_4, v_1)| \geq 3$. We have $v_2v_4, v_3v_7 \notin E(G)$. Since $|N_{G-C}(v_2) \cap N_{G-C}(v_4)| \geq 1$, we first take $x \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$. Next we take $y \in N_{G-C}(v_3) \cap N_{G-C}(v_7)$ with $x \neq y$. Then $v_2, v_3, y, C[v_7, v_1], v_4, x, v_2$ is a k-cycle with chords v_1v_2 and v_3v_4 containing e. Next assume $|C(v_4, v_1)| = 2$. We have $v_1v_3, v_2v_4 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_1) \cap N_{G-C}(v_3)$ and $y \in N_{G-C}(v_2) \cap N_{G-C}(v_4)$. If $x \neq y$, then $v_1, x, v_3, v_2, y, v_4, v_1$ is a k-cycle with chords v_1v_2 and v_3v_4 containing e. Thus we may assume that $x = y$. Noting $v_2v_6 \notin E(G)$, by Claim 3.1, there exists $z \in N_{G-C}(v_2) \cap N_{G-C}(v_6)$ with $z \neq x$. Then $v_1, x, v_3, v_2, z, v_6, v_1$ is a k-cycle with chords v_1v_2 and xv_2 containing e.

Case 5. Suppose $|C(v_1, v_i)| = 1$.

In this case, $i = 3$. By symmetry, we may assume that $j = 1$, that is, $e = v_1v_2$. Since $k \geq 6$, $|C(v_3, v_1)| \geq 3$. We have $v_2v_k, v_3v_6 \notin E(G)$. By Claim 3.1, there exist $x \in N_{G-C}(v_2) \cap N_{G-C}(v_k)$ and $y \in N_{G-C}(v_3) \cap N_{G-C}(v_6)$ with $x \neq y$. Then $v_1, v_3, y, C[v_6, v_k], x, v_2, v_1$ is a k-cycle with chords v_1v_k and v_2v_3 containing e.

Thus Claim 3.4 holds. □

By Claims 3.2–3.4, Theorem 3.3 holds. ■

Let $m \geq 4$ and $k \geq 1$ be integers, and let G be a graph of order $n \geq m$. We say G is *doubly chorded* (P_k, m)*-pancyclic* if any path P_k is contained on a doubly chorded cycle of each length from m to n in G.

Corollary 3.1. *Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq k + 3$. If $\sigma_2(G) \geq n + 2k - 3$, then G is doubly chorded $(P_k, k + 3)$-pancyclic.*

Proof. Contract P_k to an edge e to obtain a new graph G' of order $n - (k - 2) \geq 5$. Then $\sigma_2(G') \geq n + 2k - 3 - 2(k - 2) = n + 1$, and $n + 1 \geq (n - k + 2) + 1 = |G'| + 1$ since $k \geq 2$. By Theorem 3.3, G' is doubly chorded edge 5-pancyclic. Thus e is contained on a doubly chorded cycle of each length from 5 to $n - (k - 2)$. Now we expand e back to P_k. Then each doubly chorded cycle in G' containing e expands to a doubly chorded cycle in G containing P_k. These cycles have each length from $5 + (k - 2) = k + 3$ to n. Thus the corollary holds. ■

Finally, we consider an extension of vertex pancyclicity. Similar to the definitions of (doubly) chorded edge r-pancyclic graphs ($r \geq 4$), we define (doubly) chorded vertex r-pancyclic graphs.

Theorem 3.4 (Cream et al. [7, Theorem 6])**.** *Let G be a graph of order $n \geq 5$. If $\sigma_2(G) \geq n + 1$, then G is chorded vertex 5-pancyclic.*

On the assumption that G in Theorem 3.4 is claw-free, Beck et al. proved the following theorem.

Theorem 3.5 (Beck et al. [1, Theorem 4.4])**.** *Let G be a claw-free graph of order $n \geq 5$. If $\sigma_2(G) \geq n + 1$, then G is doubly chorded vertex 5-pancyclic.*

We show that the same result as Theorem 3.5 holds, even if G is not claw-free.

Corollary 3.2. *Let G be a graph of order $n \geq 5$. If $\sigma_2(G) \geq n + 1$, then G is doubly chorded vertex 5-pancyclic.*

Remark 3.2. The two graphs in Remark 3.1 show that the $\sigma_2(G)$ condition and 5-pancyclicity in Corollary 3.2 are sharp.

Proof of Corollary 3.2. We consider any edge e in G such that an endvertex of e is any specified vertex. Then we can prove Corollary 3.2 by Theorem 3.3. ■

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