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AN ASYMMETRIC CONTEST FOR PROPERTIES OF ARBITRARY VALUE

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Abstract

In this paper, we study the following game: We consider two players whom we shall call the defense and the offense respectively. There are n properties $T_1, \ldots, T_n$ with values $v_1, \ldots, v_n$ on which values both players agree. The offense has total resources normalized to 1 while the defense has total resources h; the interesting case is $h > 1$. The process of the game is that the defense first apportions its total resource among the properties T_i and then, with full knowledge of this assignment, the offense divides its unit resource among the same properties. If the defense assigns h_i and the offense assigns a_i at T_i, the offense wins v_i if and only if $a_i > h_i$. The problem is to spread the defense's resource h in such a way that the total value of the properties taken by the offense under the offense's best strategy is minimized, i.e., $(h_1, h_2, \ldots, h_n)$ must be such that

$$\max_{a_1+a_2+\ldots+a_n=1} \sum_{\{i \mid a_i>h_i\}} v_i$$

is minimal under the constraints $h_1 + h_2 + \ldots + h_n \leqslant h$, each $h_i \geqslant 0$. We shall prove that the problem is equivalent to a problem in the theory of linear equations. From this it will follow that for every n there is a finite set of defenses $(h_1, h_2, \ldots, h_n)$ such that whatever h is, and whatever $v_1, v_2, \ldots, v_n$ are, an optimal strategy is in the set. We shall also show how such a set may be constructed.

1. Introduction

In 1965–66 the authors studied an asymmetric game which had an interesting combinatorial structure; except for an internal Bell Laboratories memorandum the work never reached publication. However, it turns out to be closely related to Shapley's balanced sets of coalitions for a multi-person game [7]) and thus to the notion of the core. We therefore present our earlier results, fortified by an important counterexample due to N. G. de Bruijn which has clarified the structure considerably. We are particularly grateful also to L. S. Shapley [8]) for a number of references; we have not, however, attempted to connect our problem in detail with the flourishing theory of n-person games.

We consider two players, whom we shall call defense and offense rather than first and second player in order to emphasize the asymmetry. There are n properties $T_1, \ldots, T_n$ with values $v_1, \ldots, v_n$ on which values both players agree. The offense has total resources normalized to 1 while the defense has total resources h; the interesting case is $h > 1$.

The process of the game is that the defense first apportions its total resource h among the properties T_i and then, with full knowledge of this assignment, the offense divides its unit resource among the same properties. If the defense assigns h_i and the offense assigns a_i at T_i, the offense wins v_i if and only if $a_i > h_i$. The problem is to spread the defense's resource h in such a way that the total value of the properties taken by the offense under the offense's best strategy is minimized, i.e., $(h_1, h_2, \ldots, h_n)$ must be such that

$$\max_{a_1+a_2+\ldots+a_n=1} \sum_{\{i \mid a_i > h_i\}} v_i$$

is minimal under the constraints $h_1 + h_2 + \ldots + h_n \leqslant h$, each $h_i \geqslant 0$. Remember that we assume the offense to be designed with full knowledge of the defense, that is, of $(h_1, \ldots, h_n)$. We shall prove that the problem is equivalent to a problem in the theory of linear equations. From this it will follow that for every n there is a finite set of defenses $(h_1, h_2, \ldots, h_n)$ such that whatever h is, and whatever $v_1, v_2, \ldots, v_n$ are, an optimal strategy is in the set. We shall also show how such a set may be constructed.

In the present mathematical model, no T_i needs to be defended with strength > 1 to protect it from the offense. More generally, among all defenses for which the total property value which is lost is the same, it is sufficient to consider those with minimal h. If for a T_i the defense is 0 then it is sure to be lost. We shall call a defense "*essential for n*" if $0 < h_i < 1$ for $i = 1, 2, \ldots, n$. In general, a defense for n properties will have the form $(1, 1, \ldots, 1, h_1, h_2, \ldots, h_k, 0, 0, \ldots, 0)$ where $(h_1, h_2, \ldots, h_k)$ is essential for k properties, $k \geqslant 0$.

2. The main theorem

We consider a set $T := \{T_1, \ldots, T_n\}$ of n properties. Let $(h_1, \ldots, h_n)$ be any defense,

$$h := \sum_{j=1}^{n} h_j.$$

With each nonempty subset $S \subset T$ we shall associate an equation or an inequality and we shall order these in a special way described below. For the given defense $(h_1, \ldots, h_n)$ and for any $S \subset T$ there are three possibilities, namely

$$\sum_{T_j \in S} h_j \gtreqless 1.$$

First we consider all subsets S for which

$$\sum_{T_j \in S} h_j = 1$$

and with each we associate the equation

$$\sum_{T_j \in S} x_j = 1.$$

In this way we obtain a system of linear equations of which $(h_1, \ldots, h_n)$ apparently is a solution. Let l be the number of these equations. From this system of equations we pick a maximal collection of linearly independent equations. In the following we shall use k $(k \leqslant l)$ to denote the number of these equations, number the corresponding sets $S \subset T$ as $S_1, S_2, \ldots, S_k$ and write the k equations as

$$\sum_{j=1}^{n} \varepsilon_{ij} x_j = 1, \qquad (i = 1, \ldots, k), \tag{1}$$

where $\varepsilon_{ij} = 1$ if $T_j \in S_i$ and $\varepsilon_{ij} = 0$ if $T_j \notin S_i$. The remaining equations and the corresponding subsets of T are numbered with indices $k+1, k+2, \ldots, l$. Hence each of the equations

$$\sum_{j=1}^{n} \varepsilon_{ij} x_j = 1, \qquad (i = k+1, \ldots, l) \tag{2}$$

is linearly dependent on the set of equations (1). Next we consider subsets $S \subset T$ for which

$$\sum_{T_j \in S} h_j > 1.$$

With each of these we associate an inequality

$$\sum_{T_j \in S} x_j > 1. \tag{3}$$

Again $(h_1, \ldots, h_n)$ satisfies all the inequalities obtained in this way. The subsets S now under consideration are then numbered $S_{l+1}, \ldots, S_m$ and the inequalities are written as

$$\sum_{j=1}^{n} \varepsilon_{ij} x_j > 1, \qquad (i = l+1, \ldots, m). \tag{4}$$

Finally we look at the subsets $S \subset T$ for which

$$\sum_{T_j \in S} h_j < 1.$$

Continuing the process started above we number these subsets $S_{m+1}, \ldots, S_r$, where $r = 2^n - 1$ and write the inequalities as

$$\sum_{j=1}^{n} \varepsilon_{ij} x_j < 1, \qquad (i = m+1, \ldots, r). \tag{5}$$

In the following, when describing a defense, we refer to (1) to (5) and use the symbols k, l and m as defined above.

Definitions

A defense $(h_1, \ldots, h_n)$ is said to be *saturated* if, in (1), $k = n$; that is $(h_1, \ldots, h_n)$ is the solution of a set of n linearly independent equations of the form (1).

A defense $(h_1', \ldots, h_n')$ is said to *dominate* $(h_1, \ldots, h_n)$ if

(i)
$$\forall_{S \subset T} \left[\sum_{T_j \in S} h_j \geqslant 1 \Rightarrow \sum_{T_j \in S} h_j' \geqslant 1 \right]$$

and

(ii)
$$\sum_{j=1}^{n} h_j' \leqslant \sum_{j=1}^{n} h_j$$

both hold. If a defense is dominated only by itself we shall call it *undominated*.

If, for two defenses $(h_1, \ldots, h_n)$ and $(h_1', \ldots, h_n')$,

$$\forall_{S \subset T} \left[\operatorname{sgn}\left(\sum_{T_j \in S} h_j - 1 \right) = \operatorname{sgn}\left(\sum_{T_j \in S} h_j' - 1 \right) \right]$$

the two defenses are called *equivalent* *). Notice that two equivalent defenses need not have the same total h.

*) $\operatorname{sgn} 0 = 0$.

Our first main result is the following.

Theorem 1

For every defense $(h_1, \ldots, h_n)$ *there exists a saturated defense* $(h_1', \ldots, h_n')$ *which dominates* $(h_1, \ldots, h_n)$.

Proof. Let us assume that there exists a defense $(h_1, \ldots, h_n)$ for which the conclusion is false, i.e., there is no saturated defense which dominates $(h_1, \ldots, h_n)$. Then there exists such a defense with minimal n, $n \geqslant 2$. First we shall show that such a defense is essential for n. For if, say, h_n were 0 or 1, we could then consider the set $T \backslash \{T_n\}$ with the defense $(h_1, \ldots, h_{n-1})$. Since n was minimal there is a dominating saturated defense $(h_1', \ldots, h_{n-1}')$ for $T \backslash \{T_n\}$. If we adjoin to this defense the previously removed h_n for property T_n, we have a dominating saturated defense on T. This last assertion is not trivial and the reader should take the time to convince himself, thus familiarizing himself with our way of using (1) to (5) to describe defenses. We may now assume that the defense $(h_1, \ldots, h_n)$ is essential for n. If it is itself saturated, we are finished. If not, then, for this defense, $k < n$, and the system (1) determines an $(n-k)$-dimensional subspace D of Euclidean n-space. If we impose the additional constraints (4) and (5), and the further conditions

$$0 < x_i < 1, \qquad i = 1, 2, \ldots, n, \tag{6}$$

we define a convex subset H of D. H is not empty, since $(h_1, \ldots, h_n) \in H$; since the inequalities (4), (5) and (6) are strict, we know by continuity that H contains other points as well. In some nonempty portion of the boundary of H,

$$\sum_{i=1}^{n} x_i \leqslant h.$$

For if the equation

$$\sum_{i=1}^{n} x_i = h$$

is linearly dependent on (1), then

$$\sum_{i=1}^{n} x_i = h$$

everywhere in H and on its boundary. If

$$\sum_{i=1}^{n} x_i = h$$

is linearly *independent* of (1), then it is possible to proceed from $(h_1, \ldots, h_n)$ towards the boundary of H so that

$$\sum_{i=1}^{n} x_i$$

decreases. At any such boundary point, at least one of the inequalities in (4), (5) and (6) must be an equality. If one of (4) or (5) became an equality, then this equality would have to be linearly independent of the set (1), for otherwise $\Sigma\, \varepsilon_{ij}\, x_j$ in that inequality could not change in value. Thus, at such a boundary point, k would increase. At a boundary point at which (6) were to become an equality, we would have an equivalent defense which is not essential for n, and this case has already been covered in the previous paragraph.

Thus in all cases, there is either a dominating saturated defense, or a dominating defense with larger k. This process can be continued until $k = n$, and the theorem is proved. □

We have thus proved that we can restrict ourselves to saturated defenses. But every saturated defense $(h_1, h_2, \ldots, h_n)$ is the solution of n linearly independent equations

$$\sum_{j=1}^{n} \varepsilon_{ij}\, x_j = 1, \qquad i = 1, 2, \ldots, n,$$

where each ε_{ij} is 0 or 1. There are, moreover, obviously at most $\binom{2^n-1}{n}$ such systems, and hence only a finite number of saturated defenses for n properties.

The exact number of such defenses as a function of n is unknown. The number for $n = 1$ to 5 respectively has been found to be 1, 2, 4, 9, and 26, where two defenses obtainable from each other by permutation are counted only once.

Let

$$\sum_{j=1}^{n} \varepsilon_{ij}\, x_j = 1, \qquad i = 1, 2, \ldots, k,$$

be a set of equations of form (1). Let R_j $(j = 1, 2, \ldots, n)$ be the subset of these equations in which $\varepsilon_{ij} = 1$ rather than 0.

The set of R_j is a balanced set in the sense of Shapley, e.g. ref. 7, and the x_j are their weights. The collection of saturated defenses corresponds in this way to the collection of minimal balanced sets [7]). This correspondence was, to our knowledge, first utilized by Graver [4]). Complementation of balanced sets has its analog in a transformation of saturated defenses described in the next section.

3. (0–1) determinants

Consider a matrix $\mathsf{D} = (d_{ij})$ where all d_{ij} are 0 or 1. We shall use the symbol D to denote the determinant of this matrix. Let $x_1, x_2, \ldots, x_n$ be the solution of the system

$$\sum_{j=1}^{n} d_{ij}\, \xi_j = D \qquad (i = 1, \ldots, n). \tag{7}$$

In this paper we are interested only in those (0, 1)-matrices for which $x_1, \ldots, x_n$ are all $\geqslant 0$ but in the following two lemmas we do not use this. We define a *complementation* of a matrix as replacing all zeros by ones and all ones by zeros in a specified collection of columns. Let $\mathsf{D}(j_1, j_2, \ldots, j_k)$ denote the result of complementing the columns $j_1, j_2, \ldots, j_k$ and let $D(j_1, j_2, \ldots, j_k)$ be the determinant of this matrix.

Lemma 1. $D(j_1, j_2, \ldots, j_k) = (-1)^{k-1}(x_{j_1} + x_{j_2} + \ldots + x_{j_k} - D)$.
Proof. By Cramer's rule

$$D + D(j) = x_j.$$

Furthermore

$$D(j_1, j_2, \ldots, j_{k-1}, j_k) + D(j_1, j_2, \ldots, j_{k-2}, j_k) +$$
$$+ D(j_1, j_2, \ldots, j_{k-1}) + D(j_1, j_2, \ldots, j_{k-2}) = 0$$

because it is the determinant of a matrix with two columns consisting only of ones. The lemma now immediately follows by induction on k.

Lemma 2. *If* $\mathsf{D}^* = (d_{ij}^*) = \mathsf{D}(j_1, j_2, \ldots, j_k)$ *then the solution of*

$$\sum_{j=1}^{n} d_{ij}^*\, \xi_j = D^* \qquad (i = 1, 2, \ldots, n)$$

is

$$\begin{aligned} \xi_j &= (-1)^k x_j &&\text{if} \quad j \notin (j_1, j_2, \ldots, j_k), \\ &= (-1)^{k-1} x_j &&\text{if} \quad j \in (j_1, j_2, \ldots, j_k). \end{aligned}$$

Proof. Apply Cramer's rule and then lemma 1. □

Consequence. We can now find saturated defenses by starting with (0, 1)-matrices with nonzero determinant, solving the corresponding set of equations

$$\sum_{j=1}^{n} d_{ij}\, \xi_j = D \qquad (i = 1, 2, \ldots, n),$$

and complementing the columns corresponding to the negative x_j (or alternatively those corresponding to the positive x_j) in the solution. If $(x_1^*, x_2^*, \ldots, x_n^*)$ is the solution of the equations corresponding to the matrix D^* thus obtained, then

$$\left(\left|\frac{x_1^*}{D^*}\right|, \left|\frac{x_2^*}{D^*}\right|, \ldots, \left|\frac{x_n^*}{D^*}\right|\right)$$

is a saturated defense.

Remark. If $(h_1, h_2, \ldots, h_n)$ is a saturated defense and

$$s = \sum_{i=1}^{n} h_i > 1$$

then

$$\left(\frac{h_1}{s-1}, \frac{h_2}{s-1}, \ldots, \frac{h_n}{s-1}\right)$$

is also a saturated defense. It is found by complementing all columns of D (cf. lemmas 1 and 2). This means that in constructing all defenses one could make the restriction that

$$\sum_{i=1}^{n} h_i \geqslant 2$$

(because either s or $s/(s-1) \geqslant 2$).

For numerical purposes this is not an efficient way to list all saturated defenses, given n. A more efficient way is described below.

Definition. Consider the set of (0, 1)-matrices $D = (d_{ij})$ for which the system (7) has a solution $(x_1, x_2, \ldots, x_n)$ with all $x_i \geqslant 0$. For each of these D we define $\hat{D} := D/\gcd(x_1, x_2, \ldots, x_n)$. This is obviously an integer. Let M_n be the maximum of $\hat{D}$ over the set. Then any saturated defense for n properties consists of rationals with a common denominator $\leqslant M_n$.

As it is sufficient to list all essential defenses in compiling a list of defenses one can proceed as follows. Let m be any integer, $n \leqslant m \leqslant N$ (we shall bound N in a moment) and partition m into n positive integers. If

$$m = n + k = p_1 + p_2 + \ldots + p_n$$

is one of these partitionings, and $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n$ then

$$(p_1/d, p_2/d, \ldots, p_n/d)$$

with $p_1 \leqslant d \leqslant M_n$ may be a saturated defense. This is the case if there are n linearly independent equations (1) for which $(p_1/d, p_2/d, \ldots, p_n/d)$ is the solution. This is often not so and therefore quite a lot of these potential defenses are excluded.

Example. $m = 5 = 2 + 1 + 1 + 1$, $d = 3$. Since $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the solution of the 4 independent equations

$$x_1 + x_2 = x_1 + x_3 = x_1 + x_4 = x_2 + x_3 + x_4 = 1$$

we have found a saturated defense. For a total defensive strength $\geqslant n/2$ there is no essential defense except $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ so we can stop the above construction at $N = [\frac{1}{2}\, n\, M_n]$, where $[x]$ denotes the greatest integer $\leqslant x$.

4. The value of M_n

The number M_n defined in sec. 3 is interesting in itself. Little is known about the growth of M_n as n increases. Clearly, by Hadamard's inequality, $M_n \leqslant n^{n/2}$, but this is undoubtedly a very loose estimate. In the set of (0, 1)-matrices used to define M_n there are elements with a very large determinant namely of the order of magnitude $n^{(n+1)/2}\, 2^{-n}$ (cf. ref. 6, p. 107 and other estimates in refs 1, 2 and 3), but for these gcd $(x_1, x_2, \ldots, x_n)$ is also large and hence the corresponding $\hat{D}$ small. For $n = 2, 3, 4, 5, 6$, we found $M_n = 1, 2, 3, 5, 9$, i.e., for $n \leqslant 6$ we have $M_n = 1 + [2^{n-3}]$. We now show that this is a lower bound for all n. To do this we restrict ourselves to (0, 1)-matrices $\mathsf{D} = (d_{ij})$ for which the solution $(x_1, x_2, \ldots, x_n)$ of (7) has the property that all x_i are $\geqslant 0$ and gcd $(x_1, x_2, \ldots, x_n) = 1$. In this subset $\hat{D} = D$.

Theorem 2

$M_n \geqslant 1 + 2^{n-3}$ *for* $n \geqslant 3$.

Proof. For n even ($\geqslant 4$) let $\mathsf{D}_n = (d_{ij}^{(n)})$ be a (0, 1)-matrix with determinant D_n with the following properties:

(a) $D_n = 2^{n-3} + 1$;

(b) if $(x_1, x_2, \ldots, x_n)$ is the solution of the system

$$\sum_{j=1}^{n} d_{ij}^{(n)}\, \xi_j = D_n \qquad (i = 1, \ldots, n),$$

then $x_1 + x_2 + \ldots + x_{n-2} = 2^{n-3}$, $x_{n-1} = 2^{n-4}$, $x_n = 2^{n-4} + 1$ (note that gcd $(x_1, x_2, \ldots, x_n) = 1$);

(c) D_n has the form

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & 1 & 1 \\ & & & & 0 & 1 \\ & & & & \vdots & \vdots \\ & & & & 0 & 1 \\ & & & & 1 & 0 \\ & & & & \vdots & \vdots \\ & & & & 1 & 0 \end{vmatrix}.$$

Define D_{n+1} as follows:

$$D_{n+1} := \left(\begin{array}{c|c} 1\ 1\ \dots\ 1\ 0\ 0 & 1 \\ \hline & 1 \\ D_n & 0 \\ & \vdots \\ & 0 \end{array}\right).$$

From the last column subtract the preceding two columns and then expand D_{n+1} by the first row. This gives

$$D_{n+1} = \sum_{i=1}^{n-2} (-1)^{i-1} (-1)^{n-i+1} x_i + D_n = 2^{n-2} + 1$$

and the solution of

$$\sum_{j=1}^{n+1} d_{ij}^{(n+1)} \xi_j = 2^{n-2} + 1$$

is

$$(2x_1, 2x_2, \dots, 2x_{n-2}, 2x_{n-1}-1, 2x_n - 1, 1),$$

which is easily checked by substitution. At the end of this proof we will have shown that a D_n with properties (a), (b), (c) exists for all even $n \geqslant 4$. The preceding step of the proof then implies that $M_n \geqslant 1 + 2^{n-3}$ for odd $n \geqslant 5$. Next define D_{n+2}' as follows:

$$D_{n+2}' := \left(\begin{array}{c|c} 0\ 0\ \dots\ 0\ 1 & 1 \\ \hline & 0 \\ & 0 \\ D_{n+1} & 1 \\ & \vdots \\ & 1 \end{array}\right).$$

Expanding by the first row we have

$$D_{n+2}' = -D_{n+1} + D_{n+1}^*$$

where D_{n+1} and D_{n+1}^* have complementary last columns, i.e., by lemma 1,

$$D_{n+2}' = -(2^{n-2} + 1) + (1 - (2^{n-2} + 1)) = -(2^{n-1} + 1).$$

The solution of

$$\sum_{j=1}^{n+2} d_{ij}^{(n+2)} \xi_j = 2^{n-1} + 1$$

is

$$(2x_1, 2x_2, \ldots, 2x_{n-2}, 2x_{n-1} - 1, 2x_n - 1, 2^{n-3} + 1, 2^{n-3}),$$

which is again easily checked by substitution. If we define D_{n+2} by interchanging the last two columns of D_{n+2}' then D_{n+2} satisfies conditions (a), (b) and (c). Theorem 2 is now proved by induction if we give an example for $n = 4$ and $n = 3$. For $n = 4$ the example is

$$D_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

For $n = 3$ we have

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2. \qquad \square$$

Remark. The inequality in theorem 2 cannot be replaced by equality. This is shown by the following example:

$$D_7 = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{vmatrix} = 18.$$

The set of equations associated with the preceding D_7 is

$$\begin{aligned}
x_1 + x_2 + x_3 \qquad\qquad\qquad\qquad &= 18, \\
x_1 + x_2 \qquad + x_4 \qquad\qquad + x_7 &= 18, \\
x_1 + x_2 \qquad\qquad + x_5 + x_6 \qquad &= 18, \\
x_1 \qquad + x_3 + x_4 \qquad + x_6 \qquad &= 18, \\
x_1 \qquad + x_3 \qquad + x_5 + x_6 + x_7 &= 18, \\
x_2 + x_3 + x_4 + x_5 \qquad\qquad &= 18, \\
x_2 + x_3 + x_4 \qquad + x_6 + x_7 &= 18.
\end{aligned}$$

The solution is $(x_1, \ldots, x_7) = (7, 6, 5, 4, 3, 2, 1)$. The balanced-set form of this example is due to Jacqueline Shalhevet in 1968. $D_8 = 38$ can now be constructed by an ingenious idea due to Peleg [5]). We observe each equation contains either x_2, x_6, or both. Let us now add $x_8 = l$ to each equation which contains only *one* of x_2 and x_6, and let x_2 and x_6 also be augmented by l. Then 18 is increased by $2l$; an additional equation $x_2 + x_6 + x_8 = 18 + 2l$ implies $8 + 3l = 18 + 2l$, $l = 10$, so that 18 is increased to $18 + 2l = 38$.

Shapley [8]) has also obtained the relation

$$D_{n+1} \geqslant 2\, D_n \left(1 - \frac{1}{n}\right).$$

5. A complete analysis for $n = 5$

We begin with table I.

TABLE I

Essential saturated defenses for $n \leqslant 5$

n	partition	defenses
3	$1 + 1 + 1 = 3$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
4	$1 + 1 + 1 + 1 = 4$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
	$2 + 1 + 1 + 1 = 5$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
5	$1 + 1 + 1 + 1 + 1 = 5$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
	$2 + 1 + 1 + 1 + 1 = 6$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
	$3 + 1 + 1 + 1 + 1 = 7$	$(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
	$2 + 2 + 1 + 1 + 1 = 7$	$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$
	$3 + 2 + 1 + 1 + 1 = 8$	$(\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$
	$3 + 2 + 2 + 1 + 1 = 9$	$(\frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$

We may now study the case of 5 properties in detail, and obtain a complete list of possible defenses and the range of h for which each defense must be considered. Assume $v_1 \geqslant v_2 \geqslant v_3 \geqslant v_4 \geqslant v_5$, and that the offense always takes the highest value it can obtain. A list of all defenses for $n = 5$ may be derived from all essential defenses for $n \leqslant 5$ by adding 1's and 0's. Each defense needs to be considered from the lowest h for which it is possible up to the lowest h for a better defense, that is, one that is guaranteed to protect more property value. For instance, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ has $h = 2$. Therefore a defense that protects T_1 to T_4 equally in such a way that the offense can get only T_1, and has not defended T_5 at all, is possible for $h \geqslant 2$, and guarantees that the offense gets no more than $v_1 + v_5$. For $2 \leqslant h < 2{\cdot}5$ there is no better way to defend the properties. At $h = 2{\cdot}5$ the defenses $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ become possible, each of which guarantees the defense better success (loss of only $v_2 + v_5$ and v_1, respectively). Notice that in the preceding examples, the order of the v_i uniquely determines which T_i the offense will attack. For some saturated defenses, this is not the case. For instance with $h = 1{\cdot}4$, $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ allows the offense his choice of $v_1 + v_2$ or $v_1 + v_3 + v_4$, and either total value may be higher. Only saturated defenses of the form

$$(\underbrace{1, \ldots, 1}_{k_1}, \underbrace{\alpha, \ldots, \alpha}_{k_2}, \underbrace{0, \ldots, 0}_{k_3}),$$

where some of the k_i may vanish, are "pure" in the sense that they lead to unique determination of the T_i which the offense will attack. Figure 1 (see the

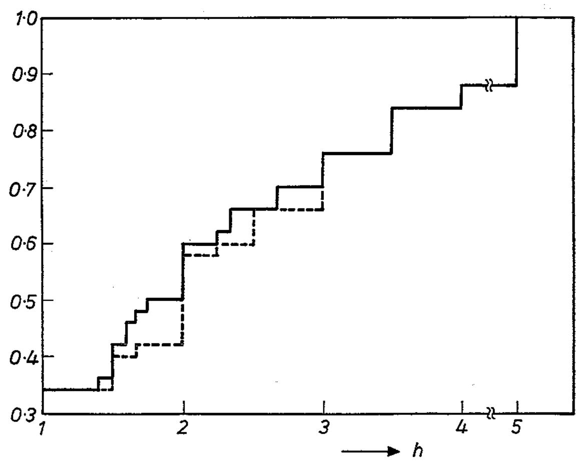

Fig. 1. Fraction of total property value saved by the defense. Drawn curve: all saturated defenses; dashed curve: only pure defenses. Property values: (17, 12, 8, 7, 6).

end of this section for details) shows, for a particular set of v_i, the difference between considering all strategies and only "pure" strategies (for which the theory is easy).

From table I we may now compile the corresponding complete list of undominated strategies for $n = 5$, together with the total h required, and the properties which will be taken. An entry such as "12 or 134" means that the defense, although undominated, is not pure: the offense will obtain either $v_1 + v_2$ or $v_1 + v_3 + v_4$ at its discretion.

TABLE II

Saturated defenses for $n = 5$

defense	necessary $h = \Sigma h_i$	properties lost ($v_1 \geqslant v_2 \geqslant v_3 \geqslant v_4 \geqslant v_5$)
(1, 0, 0, 0, 0)	1	2345
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1·25	123
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$1{\cdot}\dot{3}$	125
$(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	1·4	12 or 134
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$	1·5	145
$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1·5	12 or 234
$(\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	1·6	13 or 234
$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$1{\cdot}\dot{6}$	15 or 235
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$1{\cdot}\dot{6}$	12
$(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1·75	1 or 234
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1·75	13 or 345
$(\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$	1·8	14 or 23 or 245
(1, 1, 0, 0, 0)	2	345
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$	2	15
$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	2	1 or 23
$(\frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	2·25	1 or 24
$(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$2{\cdot}\dot{3}$	23
$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$2{\cdot}\dot{3}$	1 or 34
$(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$	2·5	25
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	2·5	1
$(1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$2{\cdot}\dot{6}$	2 or 34
(1, 1, 1, 0, 0)	3	45
$(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	3	2
$(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	3·5	3
(1, 1, 1, 1, 0)	4	5
(1, 1, 1, 1, 1)	5	none

Using table II we may now make a plot of all the saturated defenses which must be considered for any given value of h. For instance, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, which requires $h \geqslant 1{\cdot}\dot{3}$ and gives the offense 125, is poorer than $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$, which requires $h \geqslant 1{\cdot}5$ and gives the offense 145. However, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ is not poorer than any strategy *between* these two in table II, and hence must be considered for $1{\cdot}\dot{3} < h < 1{\cdot}5$.

The plot allows us to find all defenses that must be considered for a given range of h. For instance, if $1{\cdot}4 \leqslant h < 1{\cdot}5$, then the *defense* has the choice of giving the offense either 2345, or 125, or the *offense's* choice of 12 or 134. The corresponding strategies, namely (1, 0, 0, 0, 0), or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, or $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, are found in table II.

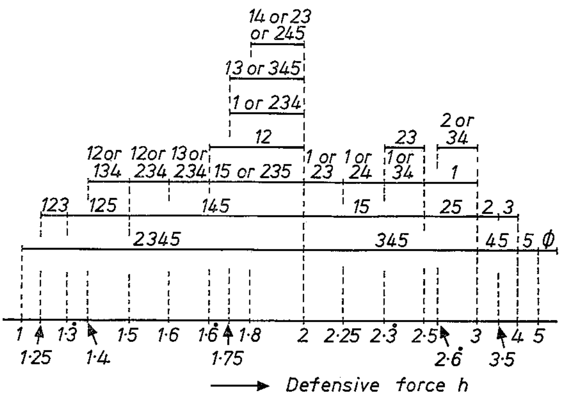

Fig. 2. Properties lost by saturated defenses; $n = 5$.

Using the plot of fig. 2 the effectiveness of various total defensive strengths was studied for a particular set of 5 properties. In the chosen example, the v_i are given by (17, 12, 8, 7, 6), so that the total value of all properties is 50. Figure 1 shows the proportion of total value *saved* by the best appropriate defense as a function of total defensive strength h. The dashed curve gives the best that can be done by pure defenses alone, and shows the value of the more complicated defenses.

6. Undominated defenses

In the case of 5 properties, each of the saturated defenses is actually the unique best defense for some set of values $(v_1, \ldots, v_n)$ and some h, in the sense that no other defense could equal its effectiveness. It is an immediate consequence of theorem 1 that a defense which has this property, i.e. it is undominated, is a saturated defense. On the other hand a saturated defense is not necessarily an undominated defense as is shown by the following examples, which are alternate versions of an example due to N. G. de Bruijn:

(a) $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are both saturated (both with total strength $h = 3$). The first one is dominated by the second one.

(b) $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with $h = \frac{11}{3}$ is saturated but dominated by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ with $h = \frac{7}{2}$.

We remark that in any saturated defense $(h_1, h_2, \ldots, h_n)$, any element h_k can be repeated an arbitrary number of times (say s) thus leading to a saturated defense on $n + s - 1$ properties. If h_k is not of the form m^{-1} $(m \in N)$ or 0, then this saturated defense will in fact be dominated if s is sufficiently large. This is our understanding of De Bruijn's example.

7. Concluding remarks

For the benefit of any readers who might develop a further interest in the problem, it is perhaps worth recording two other conjectures suggested by our work that turn out to be false:

(a) Any saturated defense contains repeated values of h_i.
Counterexample: $(\frac{7}{10}, \frac{6}{10}, \frac{5}{10}, \frac{4}{10}, \frac{3}{10}, \frac{2}{10}, \frac{1}{10})$, or the example for D_7 at the end of sec. 4.

(b) Any saturated defense written with lowest common denominator D must contain the element $1/D$.
Counterexample: $(\frac{6}{13}, \frac{5}{13}, \frac{5}{13}, \frac{4}{13}, \frac{4}{13}, \frac{3}{13}, \frac{3}{13}, \frac{2}{13}, \frac{2}{13})$.

At present the authors are studying the similar problem for the situation that the offense does *not* have knowledge of the assignment of the defense. The difference in values of the two games will give some indication of the value of "inside information". For example, it is not hard to show that if $n = 2$ and $\frac{3}{2} < h < 2$ the expected value of properties taken by the offense is

$$v_1 v_2 (v_1 + v_2)^{-1},$$

whereas in the game we have treated the offense gets v_2. Thus the value of the offense's advance knowledge of the defense is $v_2^2 (v_1 + v_2)^{-1}$.

Finally, we note that, mutatis mutandis, our results also apply to the following game with *defense*-last-move: Defense has total strength $h = 1$, offense total strength $b \geqslant 1$, defense divides forces after seeing how offense has chosen to divide forces, properties lost if $b_i > h_i$. In this game, the saturated offenses correspond to the saturated defenses we have studied, and sets of properties lost are the complements of the sets we have found.

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