ANNALES
UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

SECTIO MATHEMATICA

TOMUS XXX.

1987

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UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

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DIE DICHTESTE PACKUNG VON 25 KREISEN IN EINEM QUADRAT

Von

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(Eingegangen am 28. Juni 1983)

Bekanntlich ist das Problem der dichtesten Packung von k kongruenten nicht übereinandergreifenden Kreisen in einem abgeschlossenen Quadrat äquivalent zu dem Problem der Verteilung von k Punkten in einem abgeschlossenen Quadrat in der Weise, daß der Mindestabstand m_k irgend zweier Punkte möglichst groß wird. Dieses Packungs- bzw. Verteilungsproblem wurde bisher für $k = 2, \ldots, 9$ [2], [3], [5] und $k = 16$ [6] gelöst. Von J. SCHAER [4] und M. GOLDBERG [1] wurden darüber hinaus zahlreiche Vermutungen über dichteste Kreispackungen in einem Quadrat für k Kreise mit $k > 9$ angegeben. U. a. wurde von Goldberg die in Fig. 1 dargestellte Kreispackung für 25 Kreise als die dichteste vermutet. Als Verteilungsproblem interpretiert bedeutet dies, daß die 25 Punkte quadratgitterförmig angeordnet sind (Fig. 2). Falls ein Einheitsquadrat zugrundegelegt wird, ist der vermutete maximale Mindestabstand je zweier Punkte $m_{25} = \frac{1}{4}$.

Diese Vermutung soll durch den Beweis des folgenden Satzes bestätigt werden.

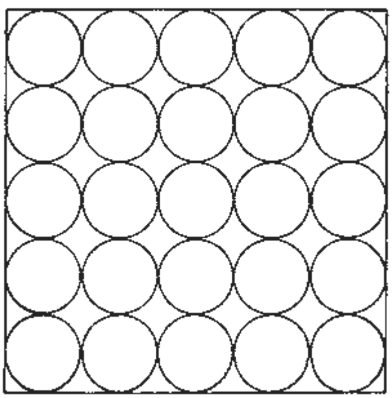

Fig. 1

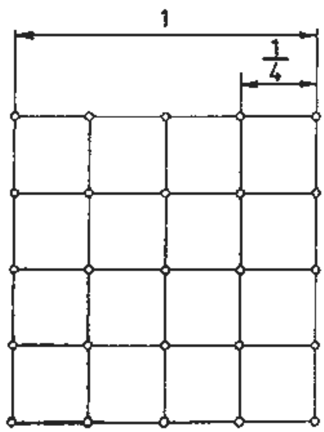

Fig. 2

Satz. *Es bezeichne $d(P_i, P_j)$ den Abstand zweier Punkte P_i und P_j. Dann gilt für irgend 25 Punkte P_i $(1 \leqq i \leqq 25)$ eines abgeschlossenen Einheitsquadrats*

$$\min_{1 \leqq i < j \leqq 25} d(P_i, P_j) \leqq m = \frac{1}{4},$$

und Gleichheit gilt nur für die vermutete Konfiguration.

Zum Beweis betrachten wir irgendeine Menge S von 25 Punkten P_i $(1 \leqq i \leqq 25)$ eines abgeschlossenen Einheitsquadrats mit

$$(1) \qquad \min_{1 \leqq i < j \leqq 25} d(P_i, P_j) \geqq \frac{1}{4}.$$

Wir zeigen in mehreren Schritten, daß es genau eine solche Menge S gibt, nämlich gerade die vermutete, für die in (1) offensichtlich das Gleichheitszeichen gilt. Um die Sprechweise zu vereinfachen, bezeichnen wir nach Schaer eine Konfiguration von k Punkten mit gegenseitigen Abständen $\geqq d$ in einem Quadrat als (k, d)-Konfiguration.

1. Im ersten Teil des Beweises überdecken wir das vorgegebene Einheitsquadrat, das wir mit Q bezeichnen, durch 25 Teilquadrate Q_i $(1 \leqq i \leqq 25)$ mit der Seitenlänge $\frac{1}{5}$ und beginnen mit der Bezeichnung $Q_1, Q_2, \ldots$ links oben (Fig. 3). Dann zeigen wir, daß in jedem Teilquadrat Q_i genau ein Punkt $P_i \in S$ liegt. Zur Verkürzung der Beweisführung teilen wir die 25 Teilqua-

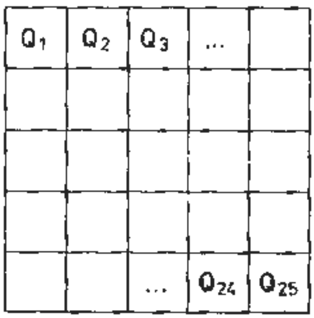

Fig. 3

drate Q_i bezüglich ihrer Lage im Quadrat Q in sechs Klassen ein. Jede Klasse werde durch das Element mit dem kleinsten Index repräsentiert. Dann enthält die Klasse $\{Q_1\}$ die vier "Eckenquadrate", die Klasse $\{Q_2\}$ acht an die Elemente von $\{Q_1\}$ anliegende "Seitenquadrate", die Klasse $\{Q_3\}$ die vier "Seitenmittenquadrate", entsprechend enthalten die Klassen $\{Q_7\}$ und $\{Q_8\}$ je vier Quadrate, und die Klasse $\{Q_{13}\}$ enthält das Quadrat Q_{13}.

Durch diese Klasseneinteilung reduziert sich der Nachweis, daß in jedem Teilquadrat Q_i genau ein Punkt $P_i \in S$ liegt, von 25 auf 6 Fälle. Nehmen wir nun an, in einem Teilquadrat Q_i lägen zwei Punkte aus S. Da $m = \frac{1}{4}$ kleiner als eine Diagonale, jedoch größer als eine Seite eines Q_i ist, können die beiden angenommenen Punkte nur in einer Umgebung gegenüberliegender Ecken liegen, d.h., entweder "linksoben-rechtsunten" oder "linksunten-rechtsoben". Damit erhöht sich zwar die Zahl der zu untersuchenden Fälle, aber sie verdoppelt sich nicht, wie wir sehen werden. Um zu beweisen, daß in jedem Teilquadrat Q_i genau einer der 25 Punkte $P_i \in S$ liegt, zeigen wir:

Wenn in einem Teilquadrat Q_i mehr als ein Punkt $P_i \in S$ liegt, so ist im Quadrat Q eine $\left(25, \frac{1}{4}\right)$-Konfiguration nicht möglich.

Zum Beweis verwenden wir das in [6] beschriebene Reduzierungsverfahren: Wenn ein Punkt $P_i \in S$ eine vorgegebene Lage einnimmt, dann kön-

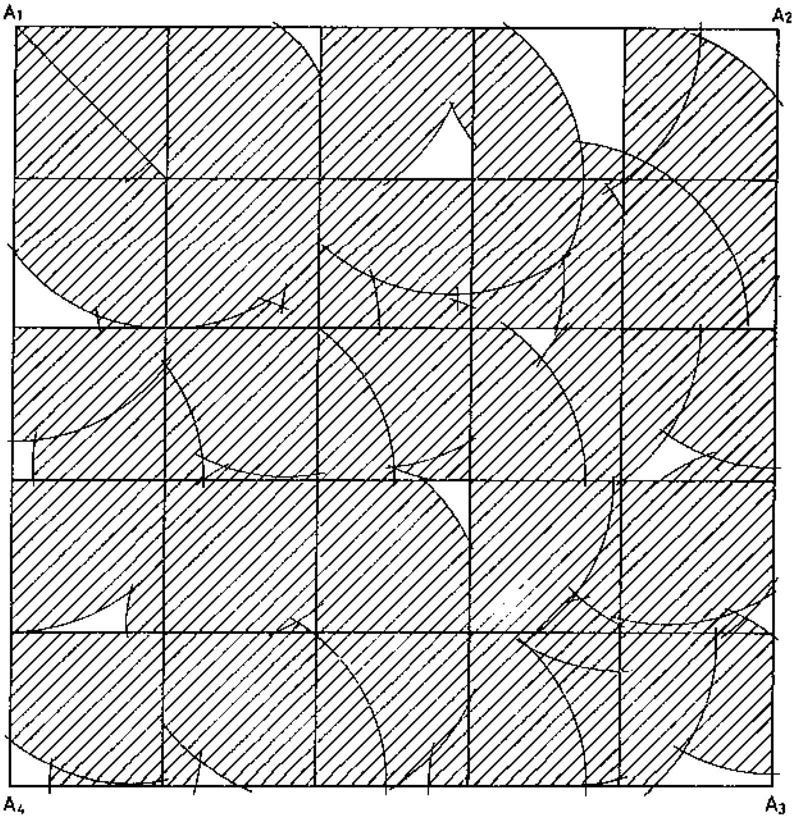

Fig. 4

nen weitere Punkte aus S wegen der Bedingung (1) nicht innerhalb eines Kreises um P_i mit dem Radius $m = \frac{1}{4}$ liegen. Das hat zur Folge, daß die mögliche Lage der weiteren Punkte aus S auf gewisse, nicht innerhalb solcher Kreise liegende Restgebiete der Teilquadrate Q_i eingeschränkt wird. Wir vereinbaren, daß bei allen Verkleinerungen von Bereichen, die sich durch Reduzierungsschritte ergeben, entstehende Randpunkte stets zu den verbleibenden Restgebieten gehören sollen.

1.1. Wir nehmen an, zwei Punkte aus S mit gegenseitigem Abstand $\geqq \frac{1}{4}$ liegen im Teilquadrat Q_1 „linksoben-rechtsunten" (Fig. 4). Zur Verkürzung der Konstruktion liege der zweite Punkt rechtsunten in dem Bogendreieck oberhalb der Diagonale. Dann bestimmen wir mit Hilfe der Bedingung (1) in den benachbarten Teilquadraten die Gebiete, in denen weitere Punkte aus S liegen können. Diese Gebiete sind in Fig. 4 nicht schraffiert. Durch ständige Fortsetzung dieses Reduzierungsverfahrens, das bezüglich der Reihenfolge der beteiligten Teilquadrate natürlich nicht eindeutig ist, ergibt sich, daß in Q_8 und in Q_{18} kein Punkt aus S liegen kann, während in Q_1 nach Voraussetzung genau zwei Punkte aus S liegen. In den Restgebieten aller übrigen Q_i kann wegen der Bedingung (1) höchstens je ein Punkt aus S liegen. Daher können im Quadrat Q höchstens 24 Punkte aus S liegen, womit eine $\left(25, \frac{1}{4}\right)$-Konfiguration in Q nicht möglich ist.

Zur analytischen Bestätigung der vorstehenden Konstruktion führen wir ein kartesisches Koordinatensystem ein, so daß die untere Seite des Quadrats Q auf die x-Achse und die linke Seite auf die y-Achse fällt. Weiter bezeichnen wir die im folgenden mehrfach gebrauchten Symmetrieachsen von Q wie folgt:

$$d_1{:}x+y = 1, \quad d_2{:}x-y = 0,$$

$$d_3{:}x = \frac{1}{2}, \quad d_4{:}y = \frac{1}{2}.$$

Lassen wir die Teilquadrate Q_1, Q_8 und Q_{18} außer Betracht, so ergibt die Rechnung, daß in allen Teilquadraten Q_i die nichtschraffierten Restgebiete einen Durchmesser $< m$ besitzen. Daher können diese Gebiete nur höchstens je einen Punkt aus S enthalten. Jeder Punkt aus dem Restgebiet von Q_8 besitzt zu jedem Punkt aus dem Restgebiet von Q_{14} einen Abstand $< m$. Entsprechend besitzt jeder Punkt aus dem Restgebiet von Q_{18} zu jedem Punkt aus dem Restgebiet von Q_{13} einen Abstand $< m$. Daher kann kein Punkt aus S z.B. in Q_8 und Q_{18} liegen.

Nach Spiegelung der gesamten Konstruktion an der Diagonale d_1 von Q folgt: Wenn der zweite Punkt aus S in Q_1 rechtsunten im Bogendreieck unterhalb von d_1 liegt, enthalten auch in diesem Falle zwei Teilquadrate Q_i keinen Punkt aus S.

Setzen wir nun voraus, daß die beiden Punkte aus S in Q_1 „linksuntenrechtsoben" liegen, dann führt eine analoge Verfahrensweise zu dem Ergebnis, daß in Q_3 und Q_{19} kein Punkt aus S liegt, in Q_1 nach Annahme zwei Punkte liegen und in den Restgebieten der übrigen Q_i wegen der Bedingung (1) höchstens je ein Punkt aus S liegen kann. Somit können in Q höchstens 24 Punkte aus S liegen, und damit ist eine $\left(25, \frac{1}{4}\right)$-Konfiguration in Q nicht möglich. (Für diesen Fall wie auch für die restlichen Fälle wird auf die Wiedergabe entsprechender Figuren verzichtet.)

1.2. Analog zu dem in 1.1. ausführlich beschriebenen Reduzierungsverfahren wird nun der Reihe nach vorausgesetzt, die beiden Punkte aus S liegen jeweils in den Teilquadraten Q_2, Q_3, Q_7, Q_8 und Q_{13}. Dann führt das Verfahren in allen Fällen dazu, daß unter der genannten Voraussetzung eine $\left(25, \frac{1}{4}\right)$-Konfiguration in Q nicht möglich ist.

1.3. Damit haben wir gezeigt, daß eine $\left(25, \frac{1}{4}\right)$-Konfiguration in Q zur Folge hat, daß in $Q_1, Q_2, Q_3, Q_7, Q_8, Q_{13}$ höchstens je ein Punkt aus S liegt. Spiegeln wir nun die o.g. sechs Teilquadrate mit den zugehörigen Konstruktionen an allen Symmetrieachsen d_1, d_2, d_3, d_4 des Quadrats Q, so leuchtet ein, daß die geführten Nachweise für alle 25 Teilquadrate Q_i gelten. Somit ist beweisen: Unter der Voraussetzung, daß in einem beliebigen der 25 Teilquadrate Q_i zwei Punkte aus S mit Mindestabstand $m = \frac{1}{4}$ liegen, ist in Q eine $\left(25, \frac{1}{4}\right)$-Konfiguration nicht möglich. Daher kann in jedem Teilquadrat Q_i höchstens ein Punkt aus S liegen. Da die Anzahl der Punkte aus S mit der Anzahl der Teilquadrate Q_i übereinstimmt, ist mit geeigneter Bezeichnung

$$P_i \in Q_i \; (1 \leqq i \leqq 25). \tag{2}$$

2. Durch (2) wird bereits eine gewisse Verteilung der Punkte $P_i \in S$ im Quadrat Q festgelegt. Wir werden jetzt das Gebiet für eine mögliche Lage der Punkte $P_i \in Q_i$ $(1 \leqq i \leqq 25)$ schrittweise verkleinern.

2.1. Im ersten Schritt verkleinern wir die Teilquadrate Q_i durch Wegnahme von Parallelstreifen der Breite $\frac{1}{40}$ und zwar verkleinern wir, wie in Fig. 5 dargestellt, die Quadrate aus $\{Q_1\}$, $\{Q_2\}$ und $\{Q_7\}$ um je zwei Streifen, die Quadrate aus $\{Q_3\}$ und $\{Q_8\}$ um je drei Streifen und Q_{13} um vier Streifen. Als Ergebnis entstehen unterschiedlich große Teilgebiete der Q_i (Fig. 5).

Zum Beweis haben wir für jeden einzelnen Streifen zu zeigen, daß in ihm der betreffende Punkt $P_i \in Q_i$ nicht liegt, und zwar beweisen wir: Wenn in

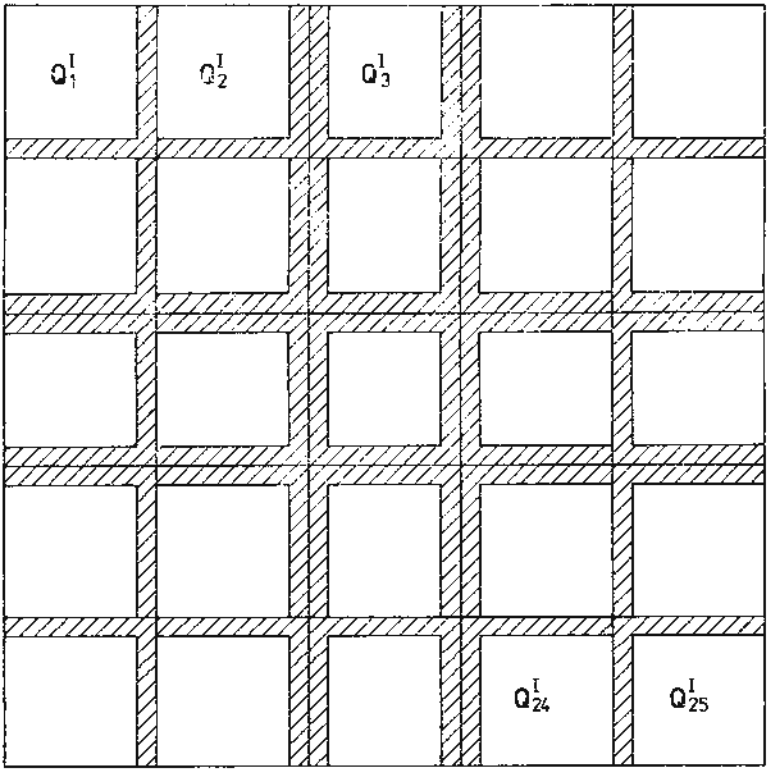

Fig. 5

einem solchen Streifen ein Punkt $P_i \in S$ liegt, dann ist in Q eine $\left(25. \frac{1}{4}\right)$-Konfiguration nicht möglich. Der Beweis verläuft analog zu **1.** wieder in der Weise, daß von einem angenommenen, in einem Streifen liegenden Punkt P_i ausgehend, die Quadrate Q_i mit Hilfe der Bedingung (1) in einer Reihe von Schritten reduziert werden, so daß schließlich in zwei benachbarten Quadraten Q_i, Q_j für die mögliche Lage von Punkten aus S Restgebiete D_i, D_j übrigbleiben, für die zwei Punkte P_i, P_j mit $P_i \in D_i$ und $P_j \in D_j$ der Ungleichung $max\ d(P_i, P_j) < m$ genügen. Das heißt aber, in einem der beiden Restgebiete und damit auch in dem dieses Restgebiet enthaltenden Teilquadrat kann kein Punkt aus S liegen.

Die Konstruktionen, die für je einen Repräsentanten der sechs Klassen $\{Q_i\}$ ($i = 1, 2, 3, 7, 8, 13$) ausgeführt und durch Rechnung nachgeprüft wurden, ergeben die Richtigkeit der Behauptung. Die entstehenden unterschiedlich großen rechteckigen Teilgebiete der Q_i für die mögliche Lage der Punkte P_i bezeichnen wir mit Q_i^I ($1 \leq i \leq 25$) (Fig. 5).

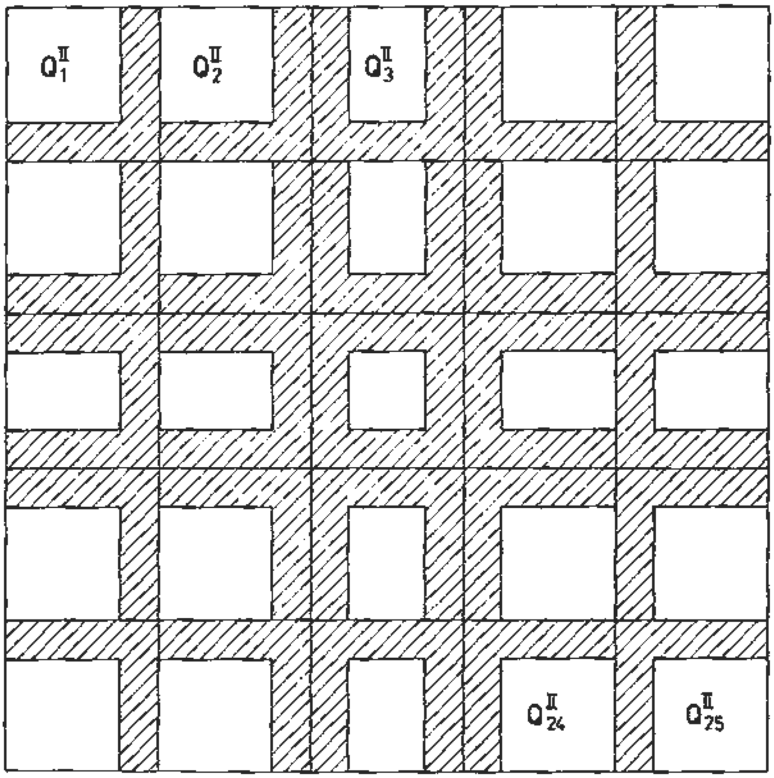

Fig. 6

2.2. In einem zweiten Schritt werden die Teilgebiete Q_i^I erneut durch Wegnahme von Streifen der Breite $\frac{1}{40}$ verkleinert, so daß wieder kleinere Teilgebiete entstehen, wie sie in Figur 6 dargestellt sind. Grundsätzlich verläuft jede Reduzierung nach dem in **1.** verwendeten Verfahren. Im einzelnen werden die Teilgebiete Q_i^I aus $\{Q_1\}$, $\{Q_2\}$ und $\{Q_7\}$ um zwei Streifen, die Teilgebiete Q_i^I aus $\{Q_3\}$ und $\{Q_8\}$ um drei Streifen und Q_{13}^I um vier Streifen reduziert. Wir haben damit die mögliche Lage der Punkte $P_i \in S$ auf Teilgebiete der Q_i^I eingeschränkt, die wir mit Q_i^{II} $(1 \leq i \leq 25)$ bezeichnen.

Im Verlaufe dieses zweiten Reduzierungsschrittes stellt sich zwangsläufig die Frage, ob es nicht möglich wäre, die beiden vorgenommenen Reduzierungen zu vereinigen und jedes Quadrat Q_i von vornherein um Streifen der Breite $\frac{1}{20}$ zu verkleinern. Ein solches Vorgehen muß natürlich möglich sein, denn in den Streifen der Breite $\frac{1}{20}$ kann ja, wie sich gezeigt hat, kein Punkt aus S liegen. Jedoch hat sich bei der Bearbeitung des Problems ergeben, daß in diesem Falle das Reduzierungsverfahren unverhältnismäßig umfangreich wird. Aus diesem Grunde ist einer allmählichen Verkleinerung der Teilquadrate der Vorzug gegeben worden. Dies bezieht sich auch auf die folgenden Verkleinerungen der Q_i^{II}.

2.3. Im nächsten Schritt werden die Teilgebiete Q_i^{II} durch Wegnahme von Streifen der Breite $\frac{1}{20}$ erneut eingeschrumpft, und zwar verkleinern wir wie folgt: Von allen Q_i^{II} aus $\{Q_1\}$, $\{Q_2\}$ und $\{Q_7\}$ werden zwei Streifen und von allen Q_i^{II} aus $\{Q_3\}$ und $\{Q_8\}$ wird ein Streifen weggenommen. Q_{13}^{II} bleibt unverändert, Dadurch entstehen die in Figur 7 dargestellten nichtschraffierten rechteckigen Teilgebiete der Q_i^{II}, die wir mit Q_i^{III} $(1 \leqq i \leqq 25)$ bezeichnen.

2.4. Mit einem vierten und zugleich letzten Reduzierungsschritt verkleinern wir die Teilgebiete Q_i^{III} durch Wegnahme von Streifen unterschiedlicher Breite. Jede Verkleinerung wird nach dem in **1.** verwendeten Verfahren durchgeführt. Im einzelnen verkleinern wir, wie dies in Figur 8 dargestellt ist: Von allen Q_i^{III} aus $\{Q_1\}$ werden zwei Streifen der Breite $\frac{1}{40}$ weg-

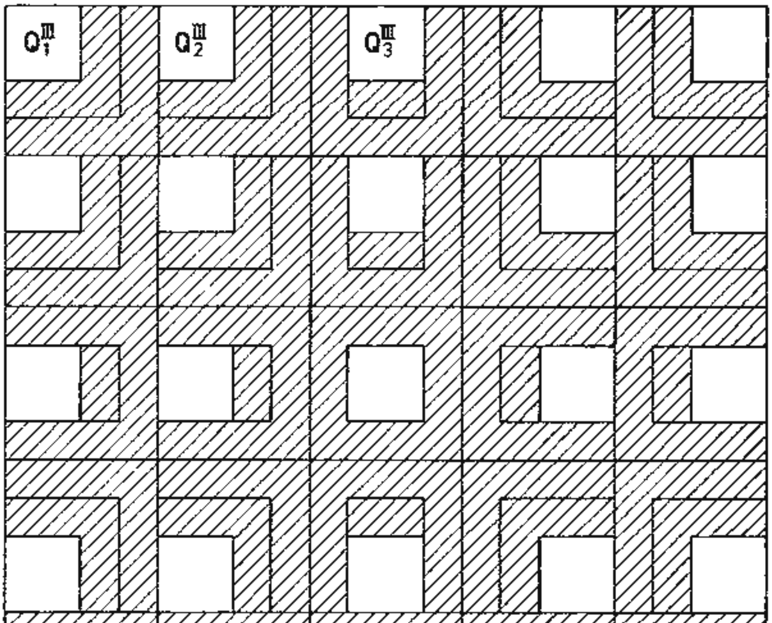

Fig. 7

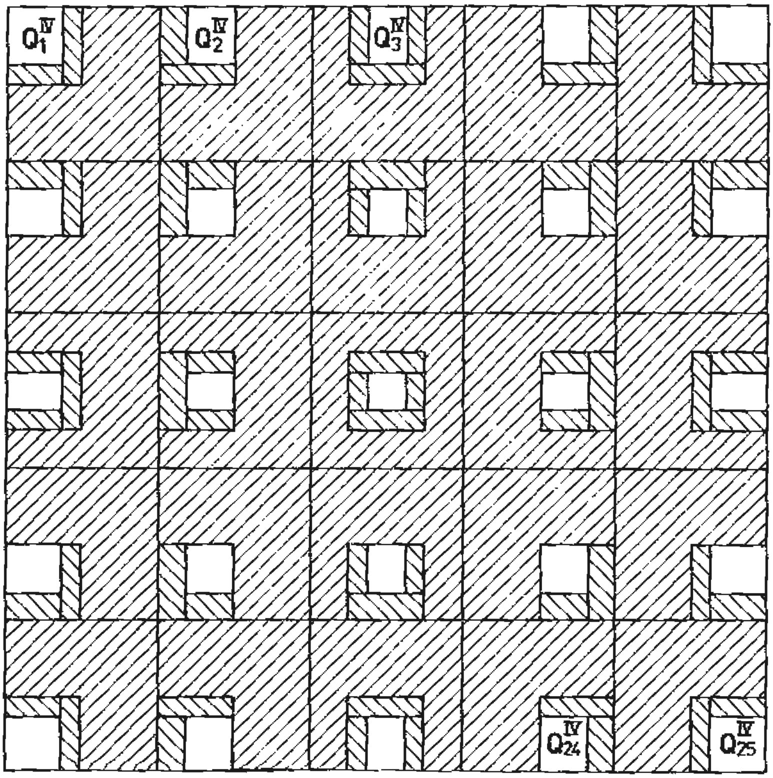

Fig. 8

genommen, entsprechend von allen Q_i^3 aus $\{Q_2\}$ je ein Streifen der Breit $\frac{1}{40}$ und $\frac{3}{80}$, von den Q_i^{III} aus $\{Q_3\}$ drei Streifen der Breite $\frac{1}{40}$, von den Q_i^{III} aus $\{Q_7\}$ zwei Streifen der Breite $\frac{3}{80}$, von den Q_i^{III} aus $\{Q_8\}$ je zwei Streifen der Breite $\frac{1}{40}$ und ein Streifen der Breite $\frac{3}{80}$ und von Q_{13}^{III} vier Streifen der Breite $\frac{1}{40}$. Die mögliche Lage der Punkte $P_i \in S$ ist damit erneut eingeschränkt worden, und zwar auf Teilgebiete der Q_i^{III}, die wir mit Q_i^{IV} $(1 \leqq i \leqq 25)$ (Fig. 8) bezeichnen.

3. Im nächsten Beweisschritt denken wir im Quadrat Q ein Quadratgitter, dessen Gitterlinien parallel zu den Seiten von Q verlaufen und dessen Gitterpunkte R_i auf den Gitterlinien jeweils den Abstand $\frac{1}{4}$ haben (Fig. 9).

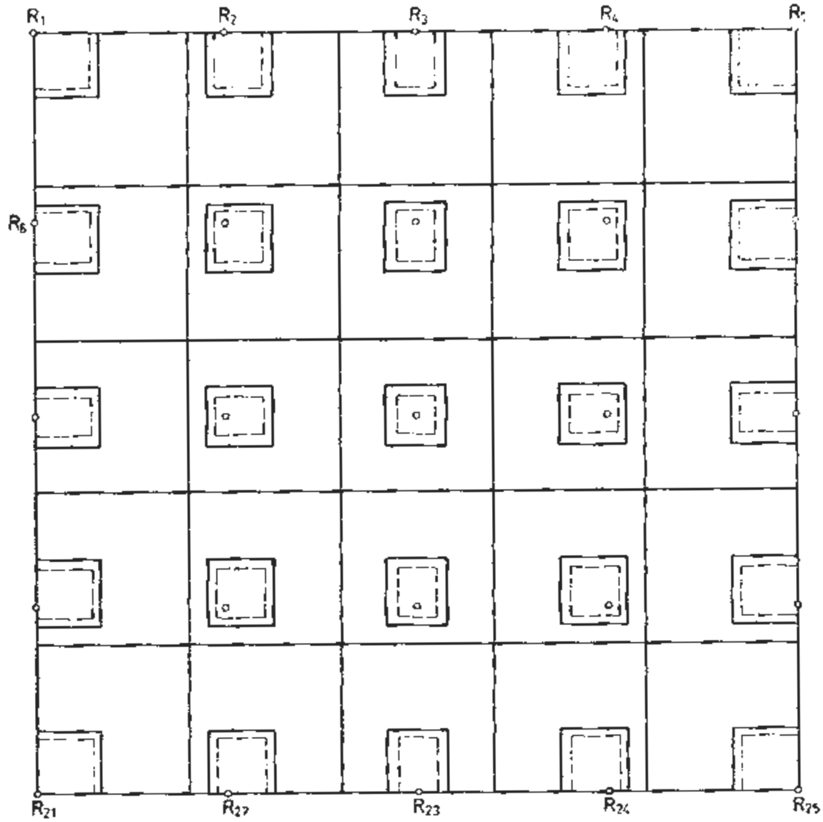

Fig. 9

Die Gitterpunkte seien so numeriert, daß $R_i \in Q_i$ $(1 \leqq i \leqq 25)$ gilt. Um die Voraussetzung für das in **4.** verwendete Verfahren von J. SCHAER zu schaffen, konstruieren wir zu jedem Punkt R_i als Zentrum das homothetische Bild des Quadrats Q mit der Seitenlänge $\frac{1}{12}$ und bezeichnen diese 25 neuen Quadrate mit Q_i^1. Wegen $Q_i^1 \supset Q_i^{1\prime}$, was durch Rechnung leicht nachzuprüfen ist, gilt dann auch $P_i \in Q_i^1$ $(1 \leqq i \leqq 25)$.

4. Das nun folgende Verfahren, das J. SCHAER zum Beweis der besten Verteilung von 9 Punkten in einem Quadrat verwendet hat [3], schliest unseren Beweis ab. Es ermöglicht nämlich, die Lage der Punkte $P_i \in Q_i^1$ mit Seitenlängen $s_1 = \frac{1}{12}$ auf Quadrate $Q_i^2 \subset Q_i^1$ mit Seitenlängen s_2 und $s_2 < s_1$ zu reduzieren. Durch ständige Wiederholung dieses Prozesses kann jeder Punkt P_i schrittweise auf immer kleinere Quadrate Q_i^n der Seitenlänge s_n $(n = 1, 2, \ldots; s_1 > s_2 > \ldots > s_n > \ldots)$ eingeschränkt werden.

Um dies zu sehen, betrachten wir drei benachbarte Quadrate, z.B. Q_i^1, Q_j^1, Q_k^1 (Fig. 10) und wollen Q_i^1 verkleinern. (Wir verwenden hier den Begriff „Nachbarquadrat" in einem erweiterten Sinne, nämlich in der Weise, daß zwei Teilquadrate Q_i^1, Q_j^1 auch Nachbarquadrate heißen sollen, wenn die sie enthaltenden Quadrate Q_i, Q_j Nachbarquadrate im üblichen Sinne sind.) Dazu konstruieren wir, wie in Figur 10 ausgeführt, ein Rechteck der Seitenlänge s_1 und der Diagonalenlänge $\frac{1}{4}$, welches das Nachbarquadrat Q_j^1 ganz und einen Teil des Quadrats Q_i^1 enthält. Dieses Rechteck enthält den Punkt $P_j \in Q_j$ gewiß, daher kann das schraffierte Teilrechteck des Rechtecks als Teil von Q_i^1 als Gebiet, in dem P_i möglicherweise liegt, ausgeschlossen werden. Von Q_i^1 bleibt demnach ein Restrechteck übrig. Dieselbe Überlegung stellen wir nun mit den anderen Nachbarquadrat Q_k^1 an. Dadurch reduziert sich das Restrechteck von Q_i^1 entsprechend, und es entsteht ein kleineres Quadrat Q_i^2 mit der Seitenlänge s_2, in dem der Punkt P_i liegt.

Diese Verkleinerungsmethode wird von allen unmittelbaren Nachbarquadraten aus auf jedes Quadrat Q_i^1 $(1 \leq i \leq 25)$ angewendet. Die Eckenquadrate besitzen jeweils nur zwei unmittelbare Nachbarquadrate, werden also am wenigsten reduziert. Alle anderen 21 Quadrate Q_i^1 haben drei oder vier Nachbarquadrate, daher werden sie stärker verkleinert. Danach konstruieren wir bezüglich der in diesen 21 Restgebieten liegenden Punkte R_i wieder zu Q homothetische Quadrate mit der Seitenlänge s_2. Es läßt sich

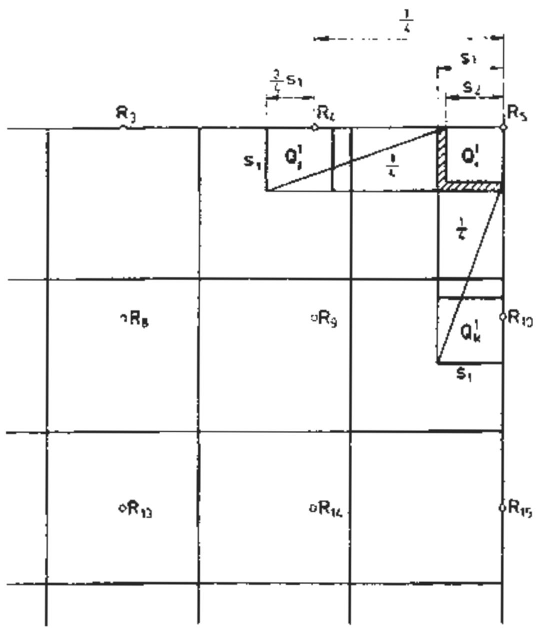

Fig. 10

leicht nachweisen, daß diese konstruierten Quadrate Q_i^2 die reduzierten Restgebiete ganz enthalten. Damit haben wir zweierlei erreicht: Erstens sind alle reduzierten Gebiete wieder Quadrate und zweitens liegen sie bezüglich der Punkte R_i wieder homothetisch zum Quadrat Q, was ganz entscheidend für das Verfahren ist. Danach wird die beschriebene Methode auf die Quadrate Q_i^2 angewendet. Da alle Quadrate nach einem Verkleinerungsschritt die gleiche Größe haben, genügt es, den Effekt des Verkleinerungsprozesses an einem Eckenquadrat zu verfolgen.

Nach Figur 10 und mit $s_1 = \frac{1}{12}$ ist

$$s_1^2 + \left(\frac{1}{4} + \frac{3}{4} s_1 - s_2\right)^2 = \left(\frac{1}{4}\right)^2.$$

Durch Fortsetzung des Verfahrens erhalten wir allgemein

$$s_n^2 + \left(\frac{1}{4} + \frac{3}{4} s_n - s_{n+1}\right)^2 = \left(\frac{1}{4}\right)^2. \tag{3}$$

Wir betrachten die Folge der Längen s_n der Quadratseiten von Q_i^n mit dem ersten Glied $s_1 = \frac{1}{12}$ und zeigen, daß $\{s_n\}$ eine Nullfolge ist. Aus (3) folgt

$$s_{n+1} = \frac{1}{4}\left(1 + 3s_n - \sqrt{1 - 16s_n^2}\right) =$$

$$= \frac{1}{4}\,\frac{(1+3s_n)^2 - (1-16s_n^2)}{(1+3s_n) + \sqrt{1-16s_n^2}} = \frac{1}{4}\,\frac{6s_n + 25s_n^2}{1 + 3s_n + \sqrt{1-16s_n^2}}.$$

Nach Division durch s_n erhalten wir

$$\frac{s_{n+1}}{s_n} = \frac{6 + 12s_n + 13s_n}{4 + 12s_n + 4\sqrt{1-16s_n^2}}.$$

Mit $s_n \leqq s_1 = \frac{1}{12}$ folgt

$$\frac{s_{n+1}}{s_n} \leqq \frac{6 + 12s_n + \frac{13}{12}}{4 + 12s_n + 4\sqrt{1 - \frac{16}{144}}} < \frac{85 + 144s_n}{92 + 144s_n} = 1 - \frac{7}{92 + 144s_n}.$$

Wegen $s_n \leqq \frac{1}{12}$ folgt daraus schließlich

$$\frac{s_{n+1}}{s_n} < 1 - \frac{7}{104} < 1.$$

Damit ist die Reihe $\sum_{n=1}^{\infty} s_n$ konvergent und daher die Folge $\{s_n\}$ ihrer Glieder eine Nullfolge. Wenn aber die Längen der Quadratseiten gegen Null streben und die Quadrate Q_i^n stets homothetisch bezüglich R_i zum Quadrat Q liegen, müssen als Folge davon die Punkte P_i an den vermuteten Stellen R_i liegen.

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A LOCALIZATION OF SOME CONGRUENCE CONDITIONS IN VARIETIES WITH NULLARY OPERATIONS

By

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(Received June 4, 1985)

Some of congruence conditions in an algebra with a nullary operation can be localized in the "neighbourhood of the nullary operation". The best known such example is e.g. the weak regularity. Recall that an algebra A is *regular* if every two congruences on A coincide whenever they have a congruence class in common. The local version is: an algebra A with a nullary operation O is *weakly regular* if every two congruences on A coincide whenever they have a congruence class containing O in common. Although lattices or semilattices are regular only rarely, there exist large classes of weakly regular lattices or semilattices (e.g. implication semilattices). The foregoing case has motivated other similar investigations. H. P. GUMM and A. URSINI [10] use the concept of "permutability at O". Other such attempt was done in [5] to give examples of broad classes of algebras (semilattices) which are not distributive but they are "distributive at O". We are also informed on the work of J. DUDA [9] devoted to some problems of "aritmeticity at O". The aim of this paper is to continue this study for permutability and n-permutability at O.

We say that an algebra A is with O if the O is a nullary operation of A. A variety $\mathcal{V}$ is with O if this O is a nullary operation of the type of $\mathcal{V}$.

DEFINITION 1. An algebra A with O is *permutable at O* if

$$[O]_{\Theta\cdot\Phi} = [O]_{\Phi\cdot\Theta}$$

for each two congruences $\Theta, \Phi\in\mathrm{Con}\ A$. A is *n-permutable* ($n\geq 2$) *at O* if $[O]_\Gamma = [O]_\Psi$ where $\Gamma = \Theta\cdot\Phi\cdot\Theta\cdot\ldots$, $\Psi = \Phi\cdot\Theta\cdot\Phi\cdot\ldots$ (both with n factors) for each $\Theta, \Phi\in\mathrm{Con}\ A$. A variety $\mathcal{V}$ with O is *permutable at O* or *n-permutable at O* if each $A\in\mathcal{V}$ has this property.

THEOREM 1. *For a variety $\mathcal{V}$ with O, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is permutable at O;*

(2) *there exists a binary polynomial $b(x, y)$ such that*

$$b(x, x) = 0 \text{ and } b(x, 0) = x.$$

Proof. (1)⇒(2): Let $\mathcal{V}$ be a variety with O which is permutable at O and $F_2(x, y)$ be a free algebra of $\mathcal{V}$ with free generators x, y. Let $\Theta = \Theta(x, y)$, $\Phi = \Theta(y, 0)$ be congruences on $F_2(x, y)$. Then $x\Theta y\Phi O$ implies $x \in [O]_{\Theta \cdot \Phi}$ i.e. also $x \in [O]_{\Phi \cdot \Theta}$ It implies the existence of $v \in F_2(x, y)$ with $x\Phi v\Theta O$. Hence $v = b(x, y)$ for some binary polynomial b and

$$\langle x, b(x, y)\rangle \in \Theta(y, 0) \text{ gives } b(x, 0) = x$$

$$\langle b(x, y), 0\rangle \in \Theta(x, y) \text{ gives } b(x, x) = 0.$$

(2)⇒(1): If $\mathcal{V}$ satisfies (2), $A \in \mathcal{V}$, $\Theta, \Phi \in \operatorname{Con} A$, $a \in A$ and $a \in [O]_{\Theta \cdot \Phi}$, then there exists an element $c \in A$ such that $a\Theta c\Phi O$. Put $v = b(a, c)$. Clearly

$$0 = b(a, a)\Theta b(a, c)\,\Phi b(a, 0) = a, \text{ thus } a \in [O]_{\Phi \cdot \Theta}.$$

Example 1. Any *variety of pseudocomplemented semilattices* (or lattices) is permutable at O (but not permutable in a general case). If the asterik denotes the pseudocomplementation, then the polynomial $b(x, y) = x \wedge y^*$ satisfies (2) of Theorem 1.

Let A be an algebra. A binary relation R on A is *compatible* if it has the Substitution property with respect to all operations of A, i.e. if R is a subalgebra of the direct product $A \times A$. A binary compatible relation on A is called *diagonal relation* or *tolerance* or *quasiorder* if it is reflexive or reflexive and symmetric or reflexive and trasitive, respectively. It is known that the set of all diagonal relations (tolerances, quasiorders) on A forms an algebraic lattice, see [2]. Hence, there exists the least diagonal relation (tolerance, quasiorder) containing the given pair $\langle a, b\rangle$ of elements a, b of A; denote it by $R(a, b)$ (or $T(a, b)$ or $Q(a, b)$, respectively).

H. Werner [12] gives wery useful relational characterization of permutability:

For a variety $\mathcal{V}$, the following conditions are equivalent:

(a) $\mathcal{V}$ is permutable;
(b) for each $A \in \mathcal{V}$, every diagonal relation on A is a congruence on A;
(c) for each $A \in \mathcal{V}$, every tolerance on A is a congruence on A.

The next two theorems will localize the foregoing conditions (b) and (c) "at O".

Definition 2. Let A be an algebra with O. A binary relation R on A is *reflexive at O*, if $\langle a, 0\rangle \in R$ implies $\langle a, a\rangle \in R$ for each $a \in A$;
symmetric at O, if $\langle a, 0\rangle \in R$ implies $\langle 0, a\rangle \in R$ for each $a \in A$.

Lemma. Let A be an algebra and a, b, x, y be elements of A.

(a) $\langle x, y\rangle \in R(a, b)$ if and only if there exist an $(n+1)$-ary polynomial p and elements $c_1, \ldots, c_n \in A$ such that $x = p(a, c_1, \ldots, c_n)$, $y = p(b, c_1, \ldots, c_n)$;
(b) $\langle x, y\rangle \in T(a, b)$ if and only if there exist an $(n+2)$-ary polynomial p and elements $c_1, \ldots, c_n \in A$ such that $x = p(a, b, c_1, \ldots c_n)$, $y = p(b, a, c_1 \ldots, c_n)$;

(c) $\langle x, y\rangle \in Q(a, b)$ if and only if there exist unary algebraic functions $\tau_1, \ldots, \tau_n$ over A such that $\tau_1(a) = x$, $\tau_n(b) = y$, and $\tau_i(b) = \tau_{i+1}(a)$ for $i = 1, \ldots, n-1$.

The proof is clear, see e.g. [2], [6], [7], [8].

THEOREM 2. *Let $\mathcal{V}$ be a variety with O. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is permutable at O;*
(2) *every reflexive at O compatible binary relation on each $A \in \mathcal{V}$ is symmetric at O.*

PROOF. (1)⇒(2): Let $\mathcal{V}$ be a variety with O which is permutable at O. Let $A \in \mathcal{V}$ and R be a reflexive at O compatible relation on A. Suppose $\langle a, 0\rangle \in R$ for $a \in A$. Then $\langle a, a\rangle \in R$ and, for the polynomial $b(x, y)$ of Theorem 1,

$$\langle 0, a\rangle = \langle b(a, a), b(a, 0)\rangle \in R.$$

(2)⇒(1): Let $F_1(x)$ be a free algebra of $\mathcal{V}$ with one free generator x and let $R = R(x, 0)$. By (2), $\langle 0, x\rangle \in R$ and, by the Lemma, there exists a binary polynomial b such that $0 = b(x, x)$, $x = b(x, 0)$. By Theorem 1, (1) is evident.

The next theorem characterizes the local version of the condition (c) of the quoted Werner's result:

THEOREM 3. *Let $\mathcal{V}$ be a variety with O. The following conditions are equivalent:*

(1) *$T(0, x) = \Theta(0, x)$ for each $x \in A$ and every A of $\mathcal{V}$;*
(2) *$T(0, a)\cdot T(0, b)\cdot T(0, a) = T(0, b)\cdot T(0, a)\cdot T(0, b)$ for each a, b of A end every A of $\mathcal{V}$.*

PROOF. (1)⇒(2): Let $A \in \mathcal{V}$, $a, b, x, y \in A$ and (1) holds. Suppose $\langle x, y\rangle \in T(0, a)\cdot T(0, b)\cdot T(0, a)$. By (1), it implies

$$(*) \qquad \langle x, y\rangle \in \Theta(0, a)\cdot\Theta(0, b)\cdot\Theta(0, a).$$

By Theorem 3.5. in [1], (1) implies

$$h(\Theta(c, d)) = \Theta(h(c), h(d))$$

for each $c, d \in A$ and every homomorphism h of A (see also [4] or [11]). Thus $(*)$ gives us for the canonical homomorphism $h\colon A \to A/\Theta(0, b)$

$\langle h(x), h(y)\rangle \in \Theta(h(0), h(a))\cdot\Theta(h(0), h(a)) = \Theta(h(0), h(a))$, i.e.

$\langle x, y\rangle \in \Theta(0, b)\cdot\Theta(0, a)\cdot\Theta(0, b) = T(0, b)\cdot T(0, a)\cdot T(0, b)$.

Hence $T(0, a)\cdot T(0, b)\cdot T(0, a) \subseteq T(0, b)\cdot T(0, a)\cdot T(0, b)$, thus (2) is evident.

(2)⇒(1): Let $A \in \mathcal{V}$ and $x \in A$. Clearly $\omega = T(0, 0)$, thus

$T(0, x) = \omega\cdot T(0, x)\cdot\omega = T(0, x)\cdot\omega\cdot T(0, x) = T(0, x)\cdot T(0, x)$

proving the transitivity of $T(0, x)$. Hence $T(0, x) = \Theta(0, x)$.

The foregoing result can be compared with [3]. It was proven in [7] that the following conditions are equivalent in every variety $\mathcal{V}$:

(i) $\mathcal{V}$ is n-permutable;
(ii) every quasiorder on each $A \in \mathcal{V}$ is a congruence on A.

The aim of the last part of the paper is to characterize n-permutable at O varieties, give a local version of the foregoing conditon (ii) and compare these local conditions.

THEOREM 4. *Let $\mathcal{V}$ be a variety with O, the following conditions are equivalent:*

(1) $Q(0, x) = \Theta(0, x)$ *for each* $x \in A$ *and every* A *of* $\mathcal{V}$;
(2) *there exist an integer n and binary polynomials $d_1, \ldots, d_{n-1}$ such that* $d_1(x, 0) = x$, $d_{n-1}(x, x) = 0$,
$d_i(x, x) = d_{i+1}(x, 0)$ for $i = 1, \ldots, n-2$.

PROOF. (1)$\Rightarrow$(2): Let $F_1(x) \in \mathcal{V}$ be a free algebra with one free generator x and $Q(0, x)$ be a quasiorder on $F_1(x)$ generated by $\langle 0, x\rangle$. Since $Q(0, x) = \Theta(0, x)$, clearly $\langle x, 0\rangle \in Q(0, x)$. By the Lemma, there exist unary algebraic functions $\tau_1, \ldots, \tau_{n-1}$ over $F_1(x)$ such that

$x = \tau_1(0)$, $\tau_{n-1}(x) = 0$, and $\tau_i(x) = \tau_{i+1}(0)$ for $i = 1, \ldots, n-2$. Since $F_1(x)$ has one free generator, there exist binary polynomials $d_1, \ldots, d_{n-1}$ such that $\tau_i(\xi) = d_i(x, \xi)$ for $i = 1, \ldots, n-1$, whence (2) is evident.

(2)$\Rightarrow$(1): Let $\mathcal{V}$ be a variety with O satisfying (2) and $A \in \mathcal{V}$. Clearly $Q(0, x) \subseteq \Theta(0, x)$ for each $x \in A$. To prove the converse inclusion, we need only to show $\langle x, 0\rangle \in Q(0, x)$. By (1) and the reflexivity and compatibility of $Q(0, x)$, we have

$x = d_1(x, 0) Q(0, x) d_1(x, x) = d_2(x, 0) Q(0, x) d_2(x, x) = \ldots = d_{n-1}(x, x) = 0$.
Since $Q(0, x)$ is transitive, we conclude $\langle x, 0\rangle \in Q(0, x)$.

EXAMPLE 2. The following varieties of groupoids with O satisfies (2) of Theorem 4:

(a) the variety given by $x \cdot x = 0$, $x \cdot (0 \cdot x) = x$, then $n = 3$ and $d_1(x, y) = x \cdot y$, $d_2(x, y) = x \cdot (y \cdot x)$;

(b) the variety given by $x \cdot x = 0$, $x \cdot (0 \cdot 0) = x$, then $n = 4$ and $d_1(x, y) = x \cdot y$, $d_2(x, y) = x \cdot (y \cdot x)$, $d_3(x, y) = x \cdot (0 \cdot y)$;

(c) more generally, any variety given either by

$$x \cdot x = 0,\ x \cdot (\underbrace{0 \cdot (0 \cdot (\ldots (0}_{\text{k times}} \cdot 0)) \ldots) = x$$

or by

$$x \cdot x = 0,\ x \cdot (\underbrace{0 \cdot (0 \cdot (\ldots (0}_{\text{k times}} \cdot x)) \ldots) = x.$$

In the first case, we can put $n = 2k$, in the second one $n = 2k+1$, and

$$d_1(x, y) = x \cdot y,$$
$$d_2(x, y) = x \cdot (y \cdot x),$$
$$d_3(x, y) = x \cdot (0 \cdot y),$$
$$d_4(x, y) = x \cdot (0 \cdot (x \cdot y)),$$
$$\vdots$$

THEOREM 5. *Let $\mathcal{V}$ be a variety with 0 and $n \geq 3$ an integer. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is n-permutable at 0;*

(2) *there exist a binary polynomial d and ternary polynomials $q_1, \ldots, q_{n-2}$ such that*

$d(x, x) = 0,\ q_1(x, z, z) = x,$
$q_{i-1}(x, x, z) = q_i(x, z, z)$ *for* $i = 2, \ldots, n-2,$
$q_{n-2}(x, x, 0) = d(x, 0).$

PROOF. (1)⇒(2): Let $F_n(x_1, \ldots, x_n)$ be a free algebra with free generators $x_1, \ldots, x_n$. Put $\Phi = \Theta(x_n, 0)$ and $\Psi = \omega$ if n is odd and $\Phi = \omega$, $\Psi = \Theta(x_n, 0)$ if n is even. Let

$\Theta_1 = \Theta(x_1, x_2) \vee \Theta(x_3, x_4) \vee \ldots \vee \Phi,$
$\Theta_2 = \Theta(x_2, x_3) \vee \Theta(x_4, x_5) \vee \ldots \vee \Psi.$

Then

$x_1 \Theta_1 x_2 \Theta_2 x_3 \Theta_1 \ldots 0,$

.e. $x_1 \in [O]_\Gamma$ for $\Gamma = \Theta_1 \cdot \Theta_2 \cdot \Theta_1 \cdot \ldots$. By (1), there exist elements $a_0, a_1, \ldots, a_n$ of $F_n(x_1, \ldots, x_n)$ such that

(**) $$x_1 = a_0 \Theta_2 a_1 \Theta_1 a_2 \Theta_2 a_3 \ldots a_n = 0.$$

Therefore, there exist *n-ary polynomials* $p_0, \ldots, p_n$ such that $a_i = p_i(x_1, \ldots, x_n)$ and (**) implies in the free algebras $F_n(x_1, \ldots, x_n)/\Theta_1$, $F_n(x_1, \ldots, x_n)/\Theta_2$ the following identities:

$$x_1 = p_0(x_1, \ldots, x_n),\ p_n(x_1, \ldots, x_n) = 0,$$

for n even:

$$p_{i-1}(x_1, x_1, x_3, x_3, \ldots, x_{n-1}, x_{n-1}) = p_i(x_1, x_1, x_3, x_3, \ldots, x_{n-1}, x_{n-1})$$

if i is even,

$p_{i-1}(x_1, x_3, x_3, \ldots, x_{n-1}, x_{n-1}, 0) = p_i(x_1, x_3, x_3, \ldots, x_{n-1}, x_{n-1}, 0)$ if i is odd, and for n odd:

$$p_{i-1}(x_1, x_3, x_3, \ldots, x_n, x_n) = p_i(x_1, x_3, x_3, \ldots, x_n, x_n) \text{ if } i \text{ is odd},$$

$$p_{i-1}(x_1, x_1, x_3, x_3, \ldots, x_{n-2}, x_{n-2}, 0) = p_i(x_1, x_1, \ldots, x_{n-2}, x_{n-2}, 0)$$

if i is even.

Now, we can put $d(x, y) = p_{n-1}(x, \ldots, x, y)$ and for $i = 1, \ldots, n-2$.

$$q_i(x, y, z) = p_i(x, \ldots, x, y, z, \ldots, z, 0) \quad \text{if } i \not\equiv n \bmod 2$$

$$q_i(x, y, z) = p_i(\underbrace{x, \ldots, x}_{i \text{ times}}, y, z, \ldots, z) \quad \text{if } i \equiv n \bmod 2.$$

Then $q_1(x, z, z) = p_1(x, z, \ldots, z, 0) = p_0(x, z, \ldots, z, 0) = x$ for n even $q_1(x, z, z) = p_1(x, z, \ldots, z) = p_0(x, z, \ldots, z) = x$ for n odd, and for $i = 2, \ldots$ $\ldots, n-2$ we obtain

$$q_{i-1}(x, x, z) = p_{i-1}(\underbrace{x, \ldots, x}_{i \text{ times}}, z, \ldots, z, 0) = p_i(\underbrace{x, \ldots, x}_{i \text{ times}}, z, \ldots, z, 0) =$$

$$= q_i(x, z, z) \text{ if } i \not\equiv n \bmod 2$$

$$q_{i-1}(x, x, z) = p_{i-1}(\underbrace{x, \ldots, x}_{i \text{ times}}, z, \ldots, z) = p_i(\underbrace{x_1, \ldots, x}_{i \text{ times}}, z, \ldots, z) = q_i(x, z, z)$$

$$\text{if } i \equiv n \bmod 2.$$

Finaly, we have

$$q_{n-2}(x, x, 0) = p_{n-2}(x, \ldots, x, 0) = p_{n-1}(x, \ldots, x, 0) = d(x, 0)$$

$d(x, x) = p_{n-1}(x, \ldots, x) = p_n(x, \ldots, x) = 0$, proving (2).

(2)$\Rightarrow$(1): Let $\mathcal{V}$ be a variety with O satisfying (2), let $A \in \mathcal{V}$, Θ, $\Phi \in \operatorname{Con} A$ and $a \in [0]_\Gamma$ where $\Gamma = \Theta \cdot \Phi \cdot \Theta \ldots$ (n factors). Then there exist $a_1, \ldots a_n \in A$ with $a = a_1$ and

$$a_1 \Theta a_2 \Phi a_3 \Theta a_4 \ldots 0.$$

Put $v_i = q_i(a_i, a_{i+1}, a_{i+2})$ for $i = 1, \ldots, n-2$ and $v_{n-1} = d(a_{n-1}, a_n)$. It is easy to show that

$$a \Phi v_1 \Theta v_2 \Phi v_3 \ldots 0,$$

.e. $a \in [O]_{\Phi \cdot \Theta \cdot \Phi \cdot \ldots}$ proving (1).

Example 3. The variety of groupoids with O satisfying

$$x+0 = x, \; x+(y+y) = x, \; (x+x)+(x+x) = 0$$

is 3-permutable at O. We can put $q_1(x, y, z) = x+(y+z)$, $d(x, y) = (x+x)+$ $+(y+y)$.

Theorem 6. *Let $\mathcal{V}$ be a variety with O. If $\mathcal{V}$ is n-permutable at O (for some $n \geq 2$), then $Q(0, x) = \Theta(0, x)$ for each $x \in A$ and every A of $\mathcal{V}$.*

Proof. If $n = 2$, it follows directly from Theorem 1 and Theorem 4. Suppose $n \geq 3$. For the polynomials $d, q_1, \ldots, q_{n-1}$ from Theorem 5, we can construct

$d_{n-1}(x, y) = d(x, y)$ and $d_i(x, y) = q_i(x, y, 0)$ for $i = 1, \ldots, n-2$.
By Theorem 4, the assertion is clear.

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DIE ÜBERDECKUNG DER EUKLIDISCHEN EBENE DURCH ZWEI KREISPACKUNGEN

Von

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(Eingegangen am 10. April 1985)

Unter einem Kreis $K(M, r)$ in der euklidischen Ebene E wollen wir im weiteren stets die abgeschlossene Kreisscheibe mit dem Mittelpunkt M und dem Radius r verstehen. Entsprechend [1] definieren wir eine Menge $\{K(M_i, r)\}$ von kongruenten Kreisen als eine Kreispackung in E genau dann, wenn jeder Punkt von E in das Innere von höchstens einem Kreis aus $\{K(M_i, r)\}$ gehört. Weiterhin bezeichnen wir eine Menge $\{K(M_i, r_i)\}$ von abzählbar unendlich vielen Kreisen als eine Kreisüberdeckung von E genau dann, wenn jeder Punkt von E zu wenigstens einem Kreis aus $\{K(M_i, r_i)\}$ gehört. Wir bezeichnen mit $\mathfrak{P}(r)$ bzw. $\mathfrak{Q}(r)$, $r \in \mathbf{R}$ und $r > 0$, eine Kreispackung, deren Kreise alle den Radius r haben. Mit $\mathfrak{U}(r_1, r_2)$, $r_1, r_2 \in \mathbf{R}$ und $r_1, r_2 > 0$, bezeichnen wir eine Kreisüberdeckung der Ebene, bei der Kreise mit den Radien r_1 und r_2 vorkommen, aber keine anderen. Ebenfalls bezeichnen wir nach [1] eine Kreispackung $\mathfrak{P}(r)$ in E genau dann als gesättigt, wenn wir in dem von den Kreisen aus $\mathfrak{P}(r)$ freigelassenen Teil der Ebene E keinen weiteren Kreis K mit dem Radius r einlagern können, so daß $\mathfrak{P}(r) \cup K$ ebenfalls eine Kreispackung bildet.

Nach [2] bestimmen die Stützkreise in der Menge der Mittelpunkte der Kreise einer gesättigten Kreispackung $\mathfrak{P}(r)$ eindeutig eine normale Zerlegung der Ebene in Stützpolygone. Diese Zerlegung bezeichnen wir im weiteren mit $\mathfrak{Z}$. Unter der freien Fläche eines Polygons der Zerlegung $\mathfrak{Z}$, die zur Kreispackung $\mathfrak{P}(r)$ gehört, wollen wir die Punktmenge verstehen, die aus denjenigen Punkten des betrachteten Polygons gebildet wird, die nicht im Innern eines Kreises aus $\mathfrak{P}(r)$ liegen.

Wir betrachten nun das folgende Problem:

Gibt es zwei Kreispackungen $\mathfrak{P}(r_1)$ und $\mathfrak{Q}(r_2)$ in der Ebene E, die zusammen eine Kreisüberdeckung $\mathfrak{U}(r_1, r_2)$ bilden?

Diese Frage kann positiv beantwortet werden, wie ein Beispiel zeigt. Folglich sollen alle Kreispackungen $\mathfrak{P}(r_1)$ und $\mathfrak{Q}(r_2)$ in E bestimmt werden, die eine Kreisüberdeckung $\mathfrak{U}(r_1, r_2)$ bilden. Zur Lösung dieses Problems können wir o.E.d.A. $0 < r_2 \leq r_1 = 1$ setzen. Der Fall $r_1 = r_2 = 1$ wurde, ebenso wie das hier behandelte Problem, in [3] ausführlich dargestellt und ist in [4] kurz nachzulesen. Die Ergebnisse, die für diesen Fall entstanden, sind hier als Satz 1 und Satz 2 angegeben, da sie für die weiteren Betrachtungen notwendig sind.

Satz 1. *Dafür, daß es zu einer Einheitskreispackung $\mathfrak{P}(1)$ der Ebene E eine weitere Einheitskreispackung $\mathfrak{Q}(1)$ derselben Ebene gibt, die mit $\mathfrak{P}(1)$ die Ebene überdeckt, ist es notwendig und hinreichend, daß $\mathfrak{P}(1)$ eine gitterförmige Kreispackung ist, in der es zwei erzeugende Gittervektoren* $\mathbf{a}$ *und* $\mathbf{b}$ *mit* $|\mathbf{a}| = |\mathbf{b}| = 2$ *und* $60° \leqq |\sphericalangle(\mathbf{a}, \mathbf{b})| \leqq 90°$ *gibt. Die Kreispackung* $\mathfrak{Q}(1)$ *erhält man aus* $\mathfrak{P}(1)$, *indem man* $\mathfrak{P}(1)$ *mit* $\mathbf{v} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ *verschiebt.*

Bedenken wir nun, daß sich jede Überdeckung der Ebene, die sich durch zwei Einheitskreispackungen erzeugen läßt, in zwei Einheitskreispackungen zerlegbar ist, und umgekehrt, daß sich jede Überdeckung, die in zwei Einheitskreispackungen zerlegbar ist, auch wieder aus diesen zwei Einheitskreispackungen zusammensetzen läßt, so können wir Satz 1 auch wie folgt formulieren.

Satz 2. *Eine Einheitskreisüberdeckung* $\mathfrak{U}(1)$ *der Ebene läßt sich genau dann in zwei Einheitskreispackungen* $\mathfrak{P}(1)$ *und* $\mathfrak{Q}(1)$ *zerlegen, wenn es sich um eine gitterförmige Überdeckung handelt, in der es zwei erzeugende Gittervektoren* $\mathbf{u}$ *und* $\mathbf{v}$ *mit* $\mathbf{u} = \mathbf{a}$ *und* $\mathbf{v} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ *gibt, wobei* $|\mathbf{a}| = |\mathbf{b}| = 2$ *und* $60° \leqq |\sphericalangle(\mathbf{a}, \mathbf{b})| \leqq 90°$ *ist.*

Nun betrachten wir das am Anfang genannte Problem für den Fall, daß $0 < r_2 = r < r_1 = 1$ gilt.

Zur Lösung dieses Problems reicht es aus, alle diejenigen Kreispackungen $\mathfrak{P}(1)$ von Einheitskreisen zu bestimmen, zu denen es jeweils eine weitere Kreispackung $\mathfrak{Q}(r)$ von kongruenten Kreisen mit dem Radius r, $0 < r < 1$, gibt, die zusammen mit $\mathfrak{P}(1)$ eine Überdeckung der Ebene bilden. Dazu wollen wir unter einem zusammenhängenden Bereich, kurz z-Bereich, in E eine Punktmenge G von E verstehen, die die folgenden beiden Bedingungen erfüllt:

1. G ist abgeschlossen und hat innere Punkte.

2. Zu zwei verschiedenen Punkten A und B aus G gibt es eine Kurve, die die beiden Punkte so verbindet, daß jeder Kurvenpunkt $P, P \neq A$ und $P \neq B$, ein innerer Punkt von G ist.

Dann gilt der

Hilfssatz 1. Ist G ein z-Bereich und $K(M_1, r)$ ein Kreis, der mit G gemeinsame innere Punkte hat, jedoch G nicht vollständig überdeckt, so gibt es keine Packung $\mathfrak{Q}(r)$ von kongruenten Kreisen des Radius r mit $K(M_1, r) \in \mathfrak{Q}(r)$, die G überdeckt.

Aus diesem Hilfssatz erhalten wir sofort die

Folgerung 1. Ist G ein z-Bereich, in dem es zwei Punkte gibt, deren Abstand größer als $2r$ ist, so gibt es keine Packung $\mathfrak{Q}(r)$, die G überdeckt.

Die Kontraposition von Hilfssatz 1 bezeichnen wir als

FOLGERUNG 2. Wird ein z-Bereich G durch eine Packung $\mathfrak{Q}(r)$ überdeckt, so muß G durch einen einzigen Kreis $K(M, r)$ aus $\mathfrak{Q}(r)$ überdeckt werden.

Der nächste Hilfssatz wird die Menge der zu untersuchenden Einheitskreispackungen bereits einschränken.

HILFSSATZ 2. Ist $\mathfrak{P}(1)$ eine nicht gesättigte Einheitskreispackung in der Ebene E, so gibt es keine weitere Kreispackung $\mathfrak{Q}(r)$ in E, die zusammen mit $\mathfrak{P}(1)$ die Ebene überdeckt.

Zum Beweis betrachten wir eine nicht gesättigte Einheitskreispackung $\mathfrak{P}(1)$. Dann muß es in der Ebene E einen Punkt M geben, so daß der Kreis $K(M, 1)$ mit den Kreisen von $\mathfrak{P}(1)$ keine gemeinsame inneren Punkte hat. Weiterhin betrachten wir alle die Punkte von $K\left(M, \frac{3}{2}\right)$, die nicht im Innern eines Kreises aus $\mathfrak{P}(1)$ liegen. Diese Punktmenge bezeichnen wir mit G. Im Bild 1 ist G schraffiert. Es ist nun leicht einzusehen, daß G ein z-Bereich ist, in dem es stets zwei Punkte gibt, deren Abstand größer als 2 ist. Folglich gibt es in G auch zwei Punkte, deren Abstand größer als $2r$ ist.

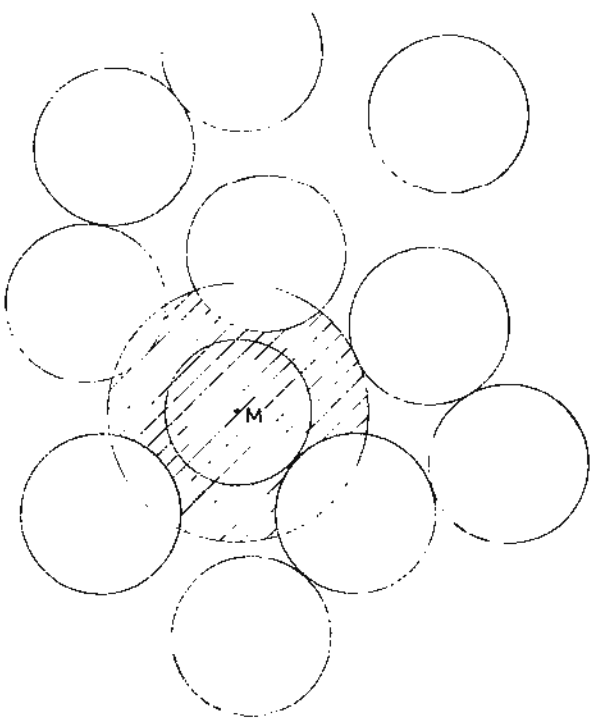

Bild 1.

Wegen Folgerung 1 gibt es nun keine Kreispackung $\mathfrak{Q}(r)$, die G überdeckt. Da G aber ein Teil der von $\mathfrak{P}(1)$ unbedeckten Fläche von E ist, gibt es zu $\mathfrak{P}(1)$ auch keine Kreispackung $\mathfrak{Q}(r)$, die mit $\mathfrak{P}(1)$ die Ebene überdeckt. Damit ist der Hilfssatz bewiesen.

Hilfssatz 3. Gibt es zu einer Einheitskreispackung $\mathfrak{P}(1)$ keine weitere Einheitskreispackung $\mathfrak{Q}(1)$, die mit $\mathfrak{P}(1)$ die Ebene überdeckt, dann gibt es zu $\mathfrak{P}(1)$ auch keine Kreispackung $\mathfrak{Q}(r)$, die mit $\mathfrak{P}(1)$ die Ebene überdeckt.

Der Beweis von Hilfssatz 3 ergibt sich als eine Folgerung aus dem Beweis von Satz 1 zusammen mit dem Hilfssatz 2.

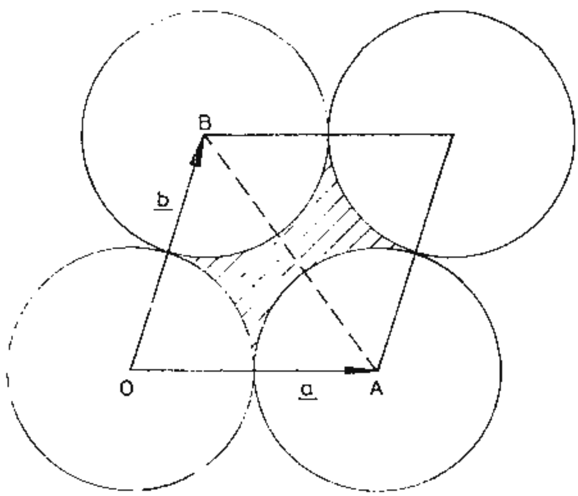

Bild 2.

Unter Verwendung der angegebenen Hilfssätze und Satz 1 gelingt es uns nun das am Anfang gestellte Problem zu lösen. Wegen Hilfssatz 3 und Satz 1 brauchen wir nur noch diejenigen Einheitskreispackungen $\mathfrak{P}(1)$ zu betrachten, die gitterförmig sind und die sich durch zwei Vektoren $\mathbf{a}$ und $\mathbf{b}$ mit $|\mathbf{a}| = |\mathbf{b}| = 2$ und $60° \leq |\sphericalangle(\mathbf{a}, \mathbf{b})| \leq 90°$ erzeugen lassen.

Nun ist aber in allen denjenigen Einheitskreispackungen, die sich durch $\mathbf{a}$ und $\mathbf{b}$ mit $|\mathbf{a}| = |\mathbf{b}| = 2$ und $60° < |\sphericalangle(\mathbf{a}, \mathbf{b})| \leq 90°$ erzeugen lassen, die freie Fläche des durch $\mathbf{a}$ und $\mathbf{b}$ erzeugten Grundparallelogramms ein z-Bereich (Bild 2).

Damit gibt es aber in diesem Bereich zwei Punkte, nämlich die Mittelpunkte gegenüberliegender Seiten, deren Abstand gleich 2 und damit größer als $2r$ ist.

Wegen der Folgerung 1 gibt es zu einer solchen Einheitskreispackung keine Kreispackung $\mathfrak{Q}(r)$, die mit dieser Einheitskreispackung die Ebene überdeckt.

Es bleibt für das in dieser Arbeit zu behandelnde Problem nur noch die dichteste gitterförmige Einheitskreispackung der Ebene übrig, zu der es eventuell eine Kreispackung $\mathfrak{Q}(r)$ gibt, die mit dieser Einheitskreispackung die Ebene überdeckt.

Betrachten wir also die dichteste gitterförmige Einheitskreispackung der Ebene, die wir mit $\mathfrak{P}_{60°}$ bezeichnen (Bild 3).

Die dazugehörige Zerlegung $\mathfrak{Z}$ der Ebene ist eine Zerlegung der Ebene in gleichseitige Dreiecke der Seitenlänge 2.

Betrachten wir ein solches Dreieck und bezeichnen es mit ABC. G sei die freie Fläche dieses Dreiecks, die ein z-Bereich ist. Nun nehmen wir an,

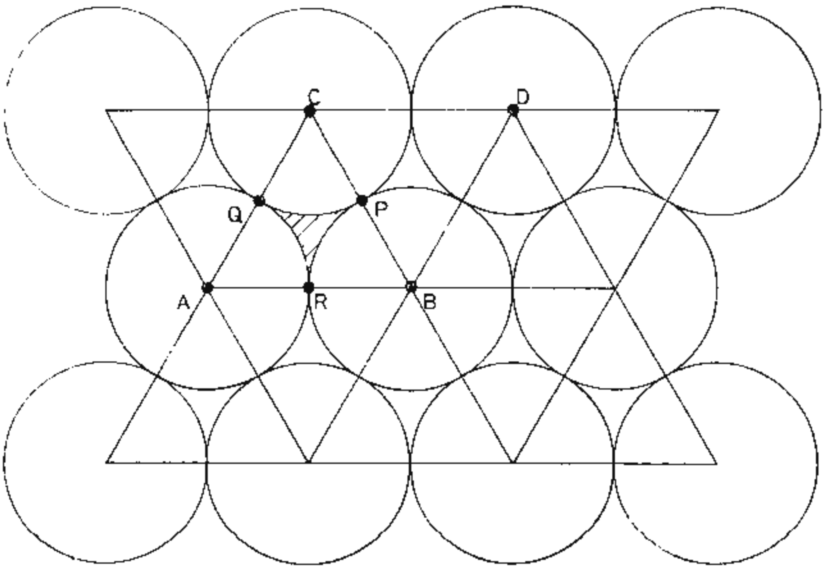

Bild 3.

daß es zu $\mathfrak{P}_{60°}$ eine Kreispackung $\mathfrak{Q}(r)$, $0 < r < 1$, gibt, die mit $\mathfrak{P}_{60°}$ die Ebene überdeckt. Dann muß es einen Kreis $K(M, r)$ aus $\mathfrak{Q}(r)$ geben, der mit G gemeinsame innere Punkte hat. Wegen Folgerung 2 muß $K(M, r)$ den Bereich G ganz überdecken.

Bezeichnen P, Q und R die Mittelpunkte der Seiten des Dreiecks ABC, so sind P, Q und R Punkte von G. Da $K(M, r)$ den z-Bereich G überdecken muß, müssen P, Q und R zu $K(M, r)$ gehören. P, Q und R bilden ein gleichseitiges, also spitzwinkliges Dreieck, und folglich ist der kleinste Kreis $K(M, r_{min})$, der G überdeckt, durch die Punkte P, Q und R eindeutig bestimmt.

Diese Punkte sind Randpunkte von $K(M, r_{min})$ und $K(M, r_{min})$ ist Inkreis des Dreiecks ABC (Bild 4). Folglich ist $r_{min} = \frac{1}{3}\sqrt{3}$, also $r \geqq \frac{1}{3}\sqrt{3}$. Damit gibt es zu $\mathfrak{P}_{60°}$ und damit zu einer beliebigen Einheitskreispackung $\mathfrak{P}(1)$ keine weitere Kreispackung $\mathfrak{Q}(r)$, mit $0 < r < \frac{1}{3}\sqrt{3}$, die mit $\mathfrak{P}(1)$ die Ebene überdeckt.

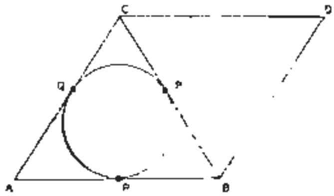

Bild 4.

Zunächst beschäftigen wir uns mit dem Fall, daß $r = \frac{1}{3}\sqrt{3}$ ist. Wir zeigen, daß es zu $\mathfrak{P}_{60°}$ eine Kreispackung $\mathfrak{Q}(r)$ mit $r = \frac{1}{3}\sqrt{3}$ gibt, die mit $\mathfrak{P}_{60°}$ die Ebene überdeckt.

Dazu bedenken wir, daß kein Punkt des Inkreises eines Dreiecks außerhalb des Dreiecks liegt. Dann betrachten wir alle Inkreise der Dreiecke aus $\mathfrak{Z}$ (Bild 5a).

Diese Menge bildet eine Kreispackung $\mathfrak{Q}_1\left(\frac{1}{3}\sqrt{3}\right)$, da keine zwei solche Inkreise innere Punkte gemeinsam haben. Dies ist aber nicht die einzige Kreispackung $\mathfrak{Q}\left(\frac{1}{3}\sqrt{3}\right)$, die mit $\mathfrak{P}_{60°}$ eine Überdeckung der Ebene bildet. Wir können nämlich um jeden Mittelpunkt eines Einheitskreises aus $\mathfrak{P}_{60°}$ einen weiteren Kreis mit dem Radius $\frac{1}{3}\sqrt{3}$ legen (Bild 5b). Jeder solche

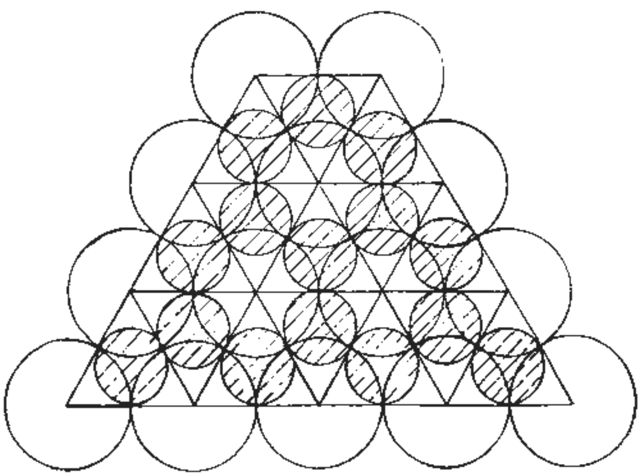

Bild 5a.

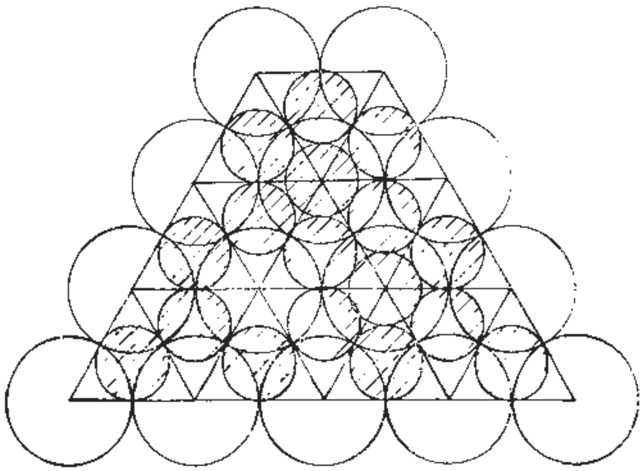

Bild 5b.

Kreis hat mit keinem Kreis aus $\mathfrak{Q}_1\left(\frac{1}{3}\sqrt{3}\right)$ gemeinsame innere Punkte. Damit gibt es also zu $\mathfrak{P}_{60°}$ beliebig viele Kreispackungen $\mathfrak{Q}\left(\frac{1}{3}\sqrt{3}\right)$, die mit $\mathfrak{P}_{60°}$ die Ebene überdecken. Jedoch muß jede dieser Kreispackungen $\mathfrak{Q}\left(\frac{1}{3}\sqrt{3}\right)$ eine Teilmenge $\left\{K\left(M_i, \frac{1}{3}\sqrt{3}\right)\right\}$ enthalten, die zu $\mathfrak{Q}_1\left(\frac{1}{3}\sqrt{3}\right)$ kongruent ist. Alle die Kreise, die man noch zu $\mathfrak{Q}_1\left(\frac{1}{3}\sqrt{3}\right)$ hinzufügen kann, sind für die Überdeckung der Ebene nicht notwendig, da diese ganz im Innern von Einheitskreisen aus $\mathfrak{P}_{60°}$ liegen. In der Überdeckung der Ebene gibt es also „überflüssige" Kreise vom Radius $\frac{1}{3}\sqrt{3}$.

Nun nehmen wir an, daß es zur Einheitskreispackung $\mathfrak{P}_{60°}$ eine weitere Kreispackung $\mathfrak{Q}(r)$ mit $\frac{1}{3}\sqrt{3} < r < 1$ gibt, die mit $\mathfrak{P}_{60°}$ die Ebene überdeckt.

ABC sei wieder ein Dreieck aus der zu $\mathfrak{P}_{60°}$ eindeutig bestimmten Zerlegung $\mathfrak{Z}$, und *P*, *Q* und *R* sind die Mittelpunkte der Seiten des Dreiecks *ABC* (Bild 6). *G* bezeichnet die freie Fläche des Dreiecks *ABC*, die ein *z*-Bereich ist.

Da $\mathfrak{Q}(r)$ mit $\mathfrak{P}_{60°}$ die Ebene überdeckt, muß es einen Kreis $K(M, r)$ aus $\mathfrak{Q}(r)$ geben, der mit *G* gemeinsame innere Punkte hat. Wegen Folgerung 2 muß $K(M, r)$ den *z*-Bereich *G* ganz überdecken. Da aber $r > \frac{1}{3}\sqrt{3}$ ist, gehört wenigstens einer der Punkte *P*, *Q*, *R* in das Innere von $K(M, r)$.

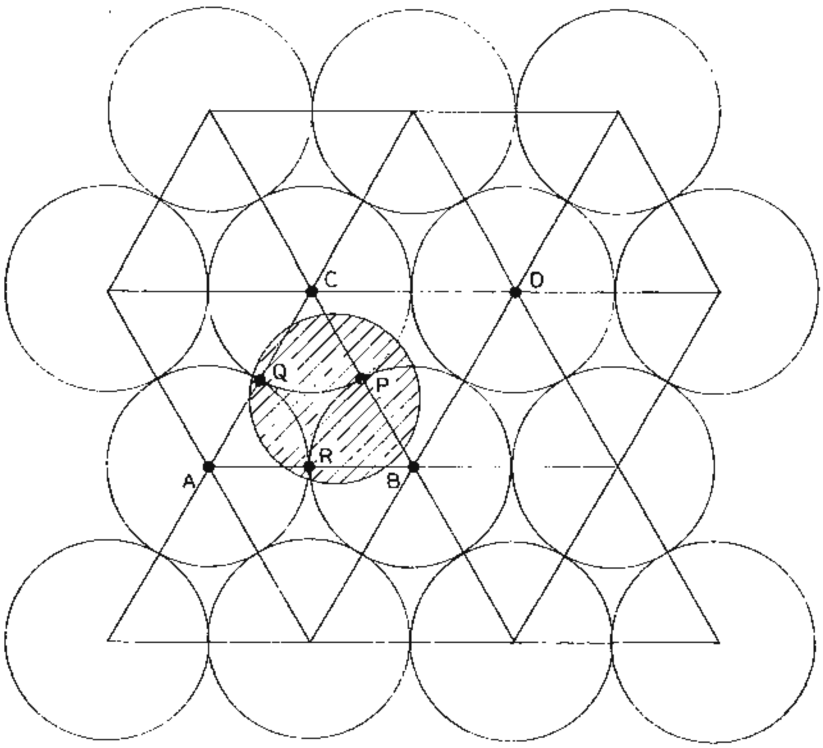

Bild 6.

Nehmen wir o.E.d.A. an, daß P innerer Punkt von $K(M, r)$ ist. D bezeichnet den dritten Eckpunkt des Dreiecks, das mit dem Dreieck ABC die Seite BC gemeinsam hat, für die P Mittelpunkt ist. G' bezeichnet die freie Fläche dieses Dreiecks. G' ist natürlich auch ein z-Bereich, von dem P ein Randpunkt ist. Da P aber innerer Punkt von $K(M, r)$ ist, hat $K(M, r)$ mit G' gemeinsame innere Punkte. Wegen Folgerung 2 muß $K(M, r)$ außer G auch noch den Bereich G' überdecken. Das heißt, $K(M, r)$ muß G und G' gleichzeitig überdecken. Da aber das Viereck $ABCD$ ein Rhombus der Seitenlänge 2 ist (in den $|BC| = 2$ gilt), gibt es zwei Punkte von $G \cup G'$, nämlich die Mittelpunkte von gegenüberliegenden Seiten, deren Abstand die Größe 2 hat. Folglich läßt sich $G \cup G'$ nicht mit einem Kreis, dessen Radius kleiner als 1 ist, überdecken.

Damit ist klar, daß es zu $\mathfrak{P}_{60°}$ und damit zu einer beliebigen Einheitskreispackung $\mathfrak{P}(1)$ keine weitere Kreispackung $\mathfrak{Q}(r)$ mit $\frac{1}{3}\sqrt{3} < r < 1$ gibt, die mit $\mathfrak{P}(1)$ die Ebene überdeckt. Dieses Ergebnis fassen wir zusammen im

SATZ 3. *Zu einer Einheitskreispackung* $\mathfrak{P}(1)$ *der Ebene E gibt es genau dann eine weitere Kreispackung* $\mathfrak{Q}(r)$ *mit* $0<r<1$, *die mit* $\mathfrak{P}(1)$ *die Ebene überdeckt, wenn* $\mathfrak{P}(1)$ *die dichteste gitterförmige Einheitskreispackung in E ist. Für den Radius der Kreise aus* $\mathfrak{Q}(r)$ *muß dabei* $r=\frac{1}{3}\sqrt{3}$ *gelten.*

Analog zum Satz 2 können wir dieses Ergebnis auch anders ausdrücken.

SATZ 4. *Eine Überdeckung* $\mathfrak{U}(r_1, r_2)$, $r_1>r_2>0$, *der Ebene, an der zwei Sorten von Kreisen beteiligt sind, läßt sich genau dann in eine Kreispackung* $\mathfrak{P}(r_1)$ *und in eine Kreispackung* $\mathfrak{Q}(r_2)$ *zerlegen, wenn* $\mathfrak{P}(r_1)$ *die dichteste gitterförmige Packung dieser Kreise ist,* $r_2=\frac{1}{3}\sqrt{3}\,r_1$ *gilt und* $\mathfrak{Q}(r_2)$ *eine Packung ist.*

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ON COMMUTING MAPPINGS AND FIXED POINTS

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(Received July 17, 1985)

Introduction. The notion of a 2-metric space was first introduced by S. GÄHLER [1]. Recently, KHAN and FISHER [2] have established a necessary and sufficient condition which guarantees the existence of a common fixed point for certain pairs of continuous mappings in 2-metric as well as in metric spaces.

More recently T. KUBIAK [3], using the same technique, proved theorems similar to those of [2] under weaker conditions. In this note, following the idea of KUBIAK, we shall give a necessary and sufficient condition for the existence of a common fixed point for certain commuting mappings. The following definitions are taken from [2] and given here for the sake of completeness.

DEFINITION 1. A 2-metric space is a set X with a real-valued function d defined on $X\times X\times X$, such that

(i) to each pair of distinct points x, y in X, there exists a point z in X such that $d(x, y, z)\neq 0$,

(ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal,

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$,

(iv) $d(x, y, z)\leq d(x, y, w)+d(x, w, z)+d(w, y, z)$, for all w in X.

It is clear that d is non-negative.

DEFINITION 2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent with limit x in X if $\lim\limits_{n\to\infty} d(x_n, x, a) = 0$ for all a in X.

It follows from (iv) that if the sequence $\{x_n\}$ converges to x then

$$\lim_{n\to\infty} d(x_n, a, b) = d(x, a, b)$$

for all a, b in X.

DEFINITION 3. A 2-metric d on a set X is said to be continuous on X if it is sequentially continuous in two of its three arguments.

DEFINITION 4. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a Cauchy sequence if $\lim_{m, n\to\infty} d(x_m, x_n, a) = 0$ for all a in X.

DEFINITION 5. A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Main Results. In this section we present two fixed point theorems for mappings in 2-metric and metric spaces. The method of proof is the one employed in [2] and [3].

THEOREM 1. *Let S and T be continuous self-mappings of a complete 2-metric space (X, d) and let d be continuous. Then S and T have a common fixed point in X if and only if there exist mappings A and B of X into $SX\cap TX$ such that $AS = SA$, $BT = TB$ and satisfying condition*

$$\text{(1)}\quad \{d(Ax, By, a)\}^2 \le d(Sx, Ty, a)\{\alpha_1 d(Ax, Sx, a)+ \\ +\alpha_2 d(By, Ty, a)+\alpha_3 d(Ax, Ty, a)+\alpha_4 d(By, Sx, a)\}+ \\ +\alpha_5 d(Ax, Sx, a)d(Ax, Ty, a)+\alpha_6 d(By, Sx, a)d(By, Ty, a)+ \\ +\alpha_7\{d(Ax, Sx, a)\}^2+\alpha_8\{d(By, Ty, a)\}^2+\alpha_9 d(Ax, Sx, a)d(By, Sx, a)+ \\ +\alpha_{10}d(Ax, Ty, a)d(By, Ty, a)+\alpha_{11}d(Ax, Sx, a)d(By, Ty, a)+ \\ +\alpha_{12}d(By, Sx, a)d(Ax, Ty, a)$$

for all x, y, a in X where α_i, $i = 1, 2, \ldots, 12$ are some non-negative constants such that

$$\text{(E}_1\text{)}\quad \left(b+(b^2+4ac)^{1/2}\right)\left(b'+(b'^2+4a'c')^{1/2}\right) < 4aa'$$

in which

$$\begin{cases} a = 1-\alpha_5-\alpha_7 > 0, \\ b = \alpha_1+\alpha_3+\alpha_5+\alpha_{10}+\alpha_{11}, \\ c = \alpha_2+\alpha_3+\alpha_8+\alpha_{10}, \end{cases} \qquad \begin{cases} a' = 1-\alpha_6-\alpha_8 > 0, \\ b' = \alpha_2+\alpha_4+\alpha_6+\alpha_9+\alpha_{11}, \\ c' = \alpha_1+\alpha_4+\alpha_7+\alpha_9, \end{cases}$$

and

$$\text{(E}_2\text{)}\quad \alpha_3+\alpha_4+\alpha_{12} < 1.$$

Indeed, A, B, S and T then have a unique common fixed point.

PROOF. To prove the necessity we observe that the mappings $A = B$, $Ax = z$ for all x in X, where z is a common fixed point of S and T, satisfy the condition (1) for every $\alpha_i \ge 0$, $i = 1, 2, \ldots, 12$ satisfying (E_1) and (E_2).

To prove the sufficiency, suppose that two mappings A and B with the properties stated in the theorem exist. Let x_0 be an arbitrary point in X. Since AX and BX are contained in $SX\cap TX$ we may define the sequence $\{x_n\}$ in X such that $Sx_{2n-1} = Bx_{2n-2}$, $Tx_{2n} = Ax_{2n-1}$ for $n = 1, 2, \ldots$.

Then

$$\{d(Sx_{2n-1}, Tx_{2n}, a)\}^2 = \{d(Ax_{2n-1}, Bx_{2n-2}, a)\}^2 \leq$$
$$\leq d(Sx_{2n-1}, Tx_{2n-2}, a)\{\alpha_1 d(Ax_{2n-1}, Sx_{2n-1}, a) + \alpha_2 d(Bx_{2n-2}, Tx_{2n-2}, a) +$$
$$+ \alpha_3 d(Ax_{2n-1}, Tx_{2n-2}, a) + \alpha_4 d(Bx_{2n-2}, Sx_{2n-1}, a)\} +$$
$$+ \alpha_5 d(Ax_{2n-1}, Sx_{2n-1}, a) d(Ax_{2n-1}, Tx_{2n-2}, a) +$$
$$+ \alpha_6 d(Bx_{2n-2}, Sx_{2n-1}, a) d(Bx_{2n-2}, Tx_{2n-2}, a) +$$
$$+ \alpha_7 \{d(Ax_{2n-1}, Sx_{2n-1}, a)\}^2 + \alpha_8 \{d(Bx_{2n-2}, Tx_{2n-2}, a)\}^2 +$$
$$+ \alpha_9 d(Ax_{2n-1}, Sx_{2n-1}, a) d(Bx_{2n-2}, Sx_{2n-1}, a) +$$
$$+ \alpha_{10} d(Ax_{2n-1}, Tx_{2n-2}, a) d(Bx_{2n-2}, Tx_{2n-2}, a) +$$
$$+ \alpha_{11} d(Ax_{2n-1}, Sx_{2n-1}, a) d(Bx_{2n-2}, Tx_{2n-2}, a) +$$
$$+ \alpha_{12} d(Bx_{2n-2}, Sx_{2n-1}, a) d(Ax_{2n-1}, Tx_{2n-2}, a).$$

Letting

$$U = d(Sx_{2n-1}, Tx_{2n}, a), \quad V = d(Sx_{2n-1}, Tx_{2n-2}, a)$$

we obtain

$$U^2 \leq \alpha_7 U^2 + (\alpha_1 + \alpha_{11}) UV + (\alpha_2 + \alpha_8) V^2 + + \{\alpha_5 U + (\alpha_3 + \alpha_{10}) V\} d(Tx_{2n}, Tx_{2n-2}, a). \tag{2}$$

But

$$d(Tx_{2n}, Tx_{2n-2}, a) \leq d(Tx_{2n}, Sx_{2n-1}, a) + d(Sx_{2n-1}, Tx_{2n-2}, a) + + d(Tx_{2n}, Tx_{2n-2}, Sx_{2n-1})$$

in which $d(Tx_{2n}, Tx_{2n-2}, Sx_{2n-1}) = 0$ since from (1) we have

$$\{d(Sx_{2n-1}, Tx_{2n}, Tx_{2n-2})\}^2 \leq \alpha_7 \{d(Sx_{2n-1}, Tx_{2n}, Tx_{2n-2})\}^2.$$

Thus

$$d(Tx_{2n}, Tx_{2n-2}, a) \leq U + V$$

and from (2) we obtain

$$(1 - \alpha_5 - \alpha_7) U^2 - (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_{10} + \alpha_{11}) VU - (\alpha_2 + \alpha_3 + \alpha_8 + \alpha_{10}) V^2 \leq 0. \tag{3}$$

It now follows that

$$d(Sx_{2n-1}, Tx_{2n}, a) \leq \beta d(Sx_{2n-1}, Tx_{2n-2}, a) \tag{4}$$

for $n = 1, 2, \ldots$ and all a in X, where

$$\beta = \frac{b + (b^2 + 4ac)^{1/2}}{2a}.$$

Similarly, from (1) it follows that

$$d(Sx_{2n+1}, Tx_{2n}, a) \leq \gamma d(Sx_{2n-1}, Tx_{2n}, a) \tag{5}$$

for $n = 1, 2, \ldots$ and all a in X, where

$$\gamma = \frac{b' + (b'^2 + 4a'c')^{1/2}}{2a'}.$$

By induction we obtain

$$d(Sx_{2n-1}, Tx_{2n}, a) \leq \beta(\beta\gamma)^{n-1} d(Sx_1, Tx_0, a), \tag{6}$$

$$d(Sx_{2n+1}, Tx_{2n}, a) \leq (\beta\gamma)^n d(Sx_1, Tx_0, a), \tag{7}$$

for $n = 1, 2, \ldots$ and all a in X.

We now have

$$d(Sx_{2n-1}, Sx_{2n+1}, a) \leq d(Sx_{2n-1}, Sx_{2n+1}, Tx_{2n}) + \\ + d(Sx_{2n-1}, Tx_{2n}, a) + d(Sx_{2n+1}, Tx_{2n}, a),$$

and from (1) (or now from (5)) $d(Sx_{2n-1}, Sx_{2n+1}, Tx_{2n}) = 0$. Thus

$$d(Sx_{2n-1}, Sx_{2n+1}, a) \leq \beta(\beta\gamma)^{n-1}(1+\gamma) d(Sx_1, Tx_0, a). \tag{8}$$

We shall now show that $\{Sx_{2n-1}\}$ is a Cauchy sequence.

For $n < m$, it follows from (iv), (8) and (E_1) that

$$d(Sx_{2n-1}, Sx_{2m-1}, a) \leq \sum_{i=n}^{m-1} d(Sx_{2i-1}, Sx_{2i+1}, a) + \\ + \sum_{i=n}^{m-2} d(Sx_{2i-1}, S_{2i+1}, Sx_{2m-1}) \leq \\ \leq \frac{\beta(\beta\gamma)^{n-1}(1+\gamma)}{1-\beta\gamma} [d(Sx_1, Tx_0, a) + + d(Sx_1, Tx_0, Sx_{2m-1})].$$

It also follows from (iv) and (8) that

$$d(Sx_1 Tx_0, Sx_{2m-1}) \leq \sum_{i=1}^{m-1} d(Sx_{2i-1}, Sx_{2i+1}, Tx_0) + \\ + \sum_{i=1}^{m-2} d(Sx_{2i-1}, Sx_{2i+1}, Sx_1) = 0.$$

Hence

$$d(Sx_{2n-1}, Sx_{2m-1}, a) \leq \frac{\beta(\beta\gamma)^{n-1}(1+\gamma)}{1-\beta\gamma} d(Sx_1, Tx_0, a)$$

for $n = 1, 2, \ldots, m > n$ and all a in X, which implies that $\{Sx_{2n-1}\}$ is a Cauchy sequence with a limit u in X since X is complete. Since

$$d(Tx_{2n}, u, a) \leq d(Tx_{2n}, u, Sx_{2n-1}) + d(Tx_{2n}, Sx_{2n-1}, a) + d(Sx_{2n-1}, u, a),$$

it follows from (6) that $\lim\limits_{n\to\infty} Tx_{2n} = u$, thus

$$\lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Bx_{2n-2} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n-1} = u.$$

Since A and B commute with S and T, respectively, we have

$$\{d(SAx_{2n-1}, TBx_{2n}, a)\}^2 = \{d(ASx_{2n-1}, BTx_{2n}, a)\}^2 \leq$$
$$\leq d(S^2x_{2n-1}, T^2x_{2n}, a)\{\alpha_1 d(SAx_{2n-1}, S^2x_{2n-1}, a) + \alpha_2 d(TBx_{2n}, T^2x_{2n}, a) +$$
$$+ \alpha_3 d(SAx_{2n-1}, T^2x_{2n}, a) + \alpha_4 d(TBx_{2n}, S^2x_{2n-1}, a)\} +$$
$$+ \alpha_5 d(SAx_{2n-1}, S^2x_{2n-1}, a) d(SAx_{2n-1}, T^2x_{2n}, a) +$$
$$+ \alpha_6 d(TBx_{2n}, S^2x_{2n-1}, a) d(TBx_{2n}, T^2x_{2n}, a) + \alpha_7\{d(SAx_{2n-1}, S^2x_{2n-1}, a)\}^2 +$$
$$+ \alpha_8\{d(TBx_{2n}, T^2x_{2n}, a)\}^2 + \alpha_9 d(SAx_{2n-1}, S^2x_{2n-1}, a) d(TBx_{2n}, S^2x_{2n-1}, a) +$$
$$+ \alpha_{10} d(SAx_{2n-1}, T^2x_{2n}, a) d(TBx_{2n}, T^2x_{2n}, a) +$$
$$+ \alpha_{11} d(SAx_{2n-1}, S^2x_{2n-1}, a) d(TBx_{2n}, T^2x_{2n}, a) +$$
$$+ \alpha_{12} d(TBx_{2n}, S^2x_{2n-1}, a) d(SAx_{2n-1}, T^2x_{2n}, a),$$

and since d, S and T are continuous, we have letting $n \to \infty$

$$\{d(Su, Tu, a)\}^2 \leq (\alpha_3 + \alpha_4 + \alpha_{12})\{d(Su, Tu, a)\}^2$$

for all a in X. From (E_2) it follows that $d(Su, Tu, a) = 0$ for all a in X, which implies that $Su = Tu$.

Now

$$\{d(Au, TBx_{2n}, a)\}^2 = \{d(Au, BTx_{2n}, a)\}^2 \leq d(Su, T^2x_{2n}, a)\{\alpha_1 d(Au, Su, a) +$$
$$+ \alpha_2 d(TBx_{2n}, T^2x_{2n}, a) + \alpha_3 d(Au, T^2x_{2n}, a) + \alpha_4 d(TBx_{2n}, Su, a)\} +$$
$$+ \alpha_5 d(Au, Su, a) d(Au, T^2x_{2n}, a) + \alpha_6 d(TBx_{2n}, Su, a) d(TBx_{2n}, T^2x_{2n}, a) +$$
$$+ \alpha_7\{d(Au, Su, a)\}^2 + \alpha_8\{d(TBx_{2n}, T^2x_{2n}, a)\}^2 +$$
$$+ \alpha_9 d(Au, Su, a) d(TBx_{2n}, Su, a) + \alpha_{10} d(Au, T^2x_{2n}, a) d(TBx_{2n}, T^2x_{2n}, a) +$$
$$+ \alpha_{11} d(Au, Su, a) d(TBx_{2n}, T^2x_{2n}, a) + \alpha_{12} d(TBx_{2n}, Su, a) d(Au, T^2x_{2n}, a)$$

and on letting $n \to \infty$ we have

$$\{d(Au, Tu, a)\}^2 \leq \alpha_5 d(Au, Su, a) d(Au, Tu, a) + \alpha_7\{d(Au, Su, a)\}^2 =$$
$$= (\alpha_5 + \alpha_7)\{d(Au, Tu, a)\}^2,$$

for all a in X. Thus

$$Au = Tu.$$

Similarly,

$$Bu = Su,$$

so that

$$Au = Bu = Su = Tu.$$

Now using the fact that $Su = Au$, $TAu = TBu = BTu = BAu$ we obtain from (1)

$$\{d(Au, BAu, a)\}^2 \le (\alpha_3 + \alpha_4 + \alpha_{12})\{d(Au, BAu, a)\}^2$$

for all a in X, which in wiew of (E_2) implies that

$$Au = BAu.$$

Similarly,

$$Bu = ABu.$$

Letting $Au = z$ we have

$$Sz = Az = Bz = Tz = z.$$

Thus z is a common fixed point of A, B, S and T.

To show that z is the unique common fixed point of A, B, S and T, suppose that w is a second common fixed point of them. Then from (1) it follows that

$$\{d(z, w, a)\}^2 = \{d(Az, Bw, a)\}^2 \le (\alpha_3 + \alpha_4 + \alpha_{12})\{d(z, w, a)\}^2,$$

for all a in X, which in wiew of (E_2) results in $z = w$ completing the proof of the theorem.

We also have the following theorem for metric spaces.

THEOREM 2. *Let S and T be continuous self-mappings of a complete metric space (X, d). Then S and T have a common fixed point in X if and only if there exist mappings A and B of X into $SX \cap TX$ such that $AS = SA$, $BT = TB$ and satisfying condition*

$$\begin{aligned}\{d(Ax, By)\}^2 \le d(Sx, Ty)\{\alpha_1 d(Ax, Sx) + \alpha_2 d(By, Ty) + \\ + \alpha_3 d(Ax, Ty) + \alpha_4 d(By, Sx)\} + \alpha_5 d(Ax, Sx) d(Ax, Ty) + \\ + \alpha_6 d(By, Sx) d(By, Ty) + \alpha_7 \{d(Ax, Sx)\}^2 + \alpha_8 \{d(By, Ty)\}^2 + \\ + \alpha_9 d(Ax, Sx) d(By, Sx) + \alpha_{10} d(Ax, Ty) d(By, Ty) + \\ + \alpha_{11} d(Ax, Sx) d(By, Ty) + \alpha_{12} d(By, Sx) d(Ax, Ty)\end{aligned}$$

for all x, y in X where α_i, $i = 1, 2, \ldots, 12$ are some non-negative constants satisfying (E_1) and (E_2).

REMARK. If we set $\alpha_1 = \alpha_2 = \beta_1$, $\alpha_3 = \alpha_4 = \beta_2$, $\alpha_5 = \alpha_6 = \beta_3$, $\alpha_7 = \alpha_8 = \beta_4$, $\alpha_9 = \alpha_{10} = \beta_5$ and $\alpha_{11} = \alpha_{12} = \beta_6$, the conditions (E_1) and (E_2) will be reduced to the single one

$$2\sum_{i=1}^{5}\beta_i + \beta_6 < 1.$$

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DECOMPOSITION OF QUASINORMAL OPERATORS

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(Received July 17, 1985)

I. Introduction

Let H be a Hilbert space, T be a bounded linear operator on H. T is *normal* if $TT^* = T^*T$, T is *hyponormal* if $TT^* \leq T^*T$, T is *quasinormal* if T commutes with T^*T, and T is *subnormal* if T has a normal extension i.e. there is a Hilbert space K, $H \subset K$ and a normal operator N on K such that $N|H = T$.

It is well known that:

T is normal iff $\|Th\| = \|T^*h\|$,
T is hyponormal iff $\|T^*h\| \leq \|Th\|$,

"Every quasinormal operator is subnormal", (see [1]. Ch. III, prop. 1.7, [2], Problem 154.).

"Every subnormal operator is hyponormal", (see [1], Ch. III, prop. 4.2, [2], Problem 160.).
It follows that: "Every quasinormal operator is hyponormal".

It may be of interest to give the following direct proof of this result.

It is known that if $T = UA$ is the polar decomposition of an operator T, then T is quasinormal if and only if $UA = AU$ ([1] Ch. III, prop. 1.6, [2], Problem 108.), where $T^*T = A^2$ and U is a patial isometry with initial space $(\ker T)^\perp = \operatorname{ran} T^*$, and final space (ran T). Now,

$$T^*T - TT^* = A^2 - UA^2U^* = A^2 - A^2UU^* = A^2(I - UU^*) =$$

$$= A^2E = A^2E^2 = EA^2E = ET^*TE = (TE)^*TE \geqq 0,$$

where E is an orthogonal projection commuting with A.

An operator T on H is called pure or completely non-normal if there is no non-zero reducing subspace H_0 such that $T|H_0$ is normal.

It is known that: "If T is a bounded linear operator on H, then there is a reducing subspace H_0 for T such that:

a) $T_0 = T|H_0$ is normal,

b) $T_1 = T|H_1$ is pure, where $H_1 = H_0^\perp$, (see [1], Ch. III, prop. 2.1)"

II. Results

Lemma 1. If T is hyponormal, and M is an invariant subspace of T, i.e. $M \in \mathrm{Lat}\,(T)$, such that $T|M$ is normal, then M reduces T.

Proof. If P is the projection of H onto M, and $S = T|M$, then $S^* = PT^*|M$. So for $f \in M$,

$$\|S^*f\| = \|PT^*f\| \leq \|T^*f\| \leq \|Tf\| = \|Sf\|.$$

Now $\|S^*f\| = \|Sf\|,\ \forall f \in M \Rightarrow \|PT^*f\| = \|T^*f\|\ \forall f \in M$.

It follows that $T^*f \in M$, and M reduces T. □

Let T be a hyponormal operator on H. Then $[T^*,T]:= T^*T - TT^*$ is a positive operator on H. $(T^*T-TT^*)^{1/2}$ is denoted by D, so that $D^2 = (T^*T-TT^*)$.

Lemma 2. For a hyponormal operator T on H,

$$\eta(D) = \{h{:}h \in H,\ Dh = 0\} = \{h{:}h \in H,\ \|T^*h\| = \|Th\|\}.$$

Proof. $\|Dh\|^2 = \langle D^2h,\ h\rangle = \langle T^*Th,\ h\rangle - \langle TT^*h,\ h\rangle = \|Th\|^2 - \|T^*h\|^2 \geq 0$. It follows that:

$$Dh = 0 \Leftrightarrow \|Th\| = \|T^*h\|.\ \square$$

We notice that $\eta(D^2) = \eta(D)$.
In fact $Dh = 0 \Rightarrow D^2h = 0$ so that

$$\eta(D) \subset \eta(D^2).$$

Now $\|Dh\|^2 = \langle Dh,\ Dh\rangle = \langle D^2h,\ h\rangle$ and $D^2h = 0 \Rightarrow Dh = 0$, which gives

$$\eta(D^2) \subset \eta(D).$$

Let T be a bounded linear operator on H, and let H_0 be the set:

$$H_0 = \{h{:}h \in H,\ \|T^*h\| = \|Th\|\}.$$

Then H_0 is a subspace of H.

Lemma 3. Let T be a quasinormal operator on H. Then H_0 is invariant under T.

Proof. We show that

$$\|D(Th)\|^2 = \langle D^2(Th),\ Th\rangle = \langle (T^*T-TT^*)\ Th,\ Th\rangle =$$
$$= \langle T^*(T^*T-TT^*)\ Th,\ h\rangle = \langle T^{*2}T^2h,\ h\rangle - \langle (T^*T)^2h,\ h\rangle.$$

Now $T \sim T^*T \Rightarrow TT^*T = T^*TT$, and so $(T^*T)^2 = T^{*2}T^2$. It follows that $D(Th) = 0$, and $Th \in H_0$. □

Lemma 4. Let T be a quasinormal operator on H. Then H_0 reduces T.

Proof. For $h \in H_0$, we have $\|T^*h\| = \|Th\|$. It follows that $T|H_0$ is normal.

Since H_0 is invariant under T, it follows that H_0 reduces T. □

PROPOSITION. Let T be a quasinormal operator on H. Then,

(i) H_0 and $H_1 = H \ominus H_0$ reduce T,

(ii) $T_0 = T|H_0$ is normal, $T_1 = T|H_1$ is pure, and $T = T_0 \oplus T_1$.

PROOF. By lemmas 3 and 4, H_0 reduces T, and $T_0 = T|H_0$ is normal. Let $H_1 = H \ominus H_0$.

Suppose that there is a non-zero subspace H_2 of H_1 reducing T, and $T|H_2$ is normal. It follows that $\|Th\| = \|T^*h\|$ for $h \in H_2$. This means that $H_2 \subset H_0$, which is a contradiction, and so $T|H_1$ is pure.

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SYNTOPOGENE UND TOPOGENE KONVERGENZRÄUME

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(*Eingegangen am 2. September 1985*)

0. Einführung

Syntopogene Konvergenzräume sind bereits vom Verfasser unter dem Namen „syntopoforme Räume" in die Literatur eingeführt worden (vergl. [7]). Sie dienen der gemeinsamen Untersuchung der uniformen Konvergenzräume in Sinne von COOK und FISCHER [2] mit den darin enthaltenen Unterkategorien der proximalen Limesräume in Sinne von MARNY [9], der uniformen Räume in Sinne von BOURBAKI [1] sowie der Proximitätsräume im Sinne von EFREMOVIČ [10] und der syntopogenen Räume im Sinne von CSÁSZÁR [3] mit den darin enthaltenen Unterkategorien der quasiuniformen Räume im Sinne von PERVIN und SIEBER [12], der Proximitätsräume im Sinne von PERVIN [10] und der topologischen Räume.

Implizit wurden die topogenen Konvergenzräume bereits in den folgenden Arbeiten behandelt (vergl. [5], [6] und [8]). Eine präzise Axiomatisierung findet nun im vorliegenden Artikel statt. Es wird gezeigt, daß jede S_1-Limitierung (vergl. [13]) einen kompatiblen topogenen Konvergenzoperator besitzt. Dabei enthalten die topogenen Konvergenzräume als Unterkategorien die Kategorie der topologischen Räume, die Kategorie der Proximitätsräume und die Kategorie der topogenen Räume im Sinne von CSÁSZÁR [3].

Kompaktheitsbegriffe lassen sich mit Hilfe der eingeführten „Császár-raster" auf topogene Konvergenzoperatoren übertragen, und ein Charakterisierungssatz stellt den Zusammenhang zu dem üblichen in der Topologie verwendeten Begriff her.

Eine wichtige Unterklasse der syntopogenen Konvergenzräume stellen die syntopologischen dar, die in „natürlicher Weise" „unterliegende" pseudotopogene Konvergenzoperatoren erzeugen. Deren Kompaktheit zieht sowohl die Präkompaktheit als auch die „Cauchy-Vollständigkeit" der Ausgangsstrukturen nach sich. Im Falle der „Erzeugung" von schwach symmetrischen syntopologischen Konvergenzstrukturen gilt auch die Umkehrung, womit eine Verallgemeinerung des „klassischen Satzes" für uniforme Räume erreicht wird.

1. Einige Symbole

Für eine Menge X bezeichne
$\mathcal{P}X$ die Potenzmenge von X; $\subset$ bezeichne die Inklusion;
für $A \subset X$ bezeichne X/A das Komplement von A bezüglich X;
$\forall$ bezeichne den Quantor „für alle";
$\exists$ bezeichne den Quantor „es existiert";
für $\mathcal{F}, \mathcal{N} \subset \mathcal{P}X$ setze:
$\mathcal{F} \ll \mathcal{N} :\Leftrightarrow \forall F \in \mathcal{F} \, \exists N \in \mathcal{N} \, F \supset N$;
$\mathcal{F} \vee \mathcal{N} := \{F \cup N \mid F \in \mathcal{F}, \; N \in \mathcal{N}\}$;
stack $\mathcal{N} := \{F \subset X \mid \exists N \in \mathcal{N} \, F \supset N\}$;
sec $\mathcal{N} := \{F \subset X \mid \forall N \in \mathcal{N} \, F \cap N \neq \emptyset\}$;
$\cap \mathcal{N} := \cap \{N \mid N \in \mathcal{N}\}$;
für eine Abbildung $f: X \to Y$ setze:
$f\mathcal{N} := \{f[F] \mid F \in \mathcal{N}\}$;
für Abbildungen $t_1, t_2 : \mathcal{P}X \to \mathcal{P}(\mathcal{P}X)$ setze:
$(t_1 \cap t_2)(A) := t_1(A) \cap t_2(A)$;
$(t_1 \cup t_2)(A) := t_1(A) \cup t_2(A)$;
$t_1 \leq t_2 :\Leftrightarrow \forall A \in \mathcal{P}X \, t_1(A) \subset t_2(A)$;
für Mengen Ω, Λ von Abbildungen $t: \mathcal{P}X \to \mathcal{P}(\mathcal{P}X)$ setze:
$\Omega \leq \Lambda :\Leftrightarrow \forall t_1 \in \Omega \, \exists \, t_2 \in \Lambda \; t_1 \leq t_2$;
$\Omega \sim \Lambda :\Leftrightarrow \Omega \leq \Lambda \leq \Omega$;
$\Omega \wedge \Lambda := \{t_1 \cap t_2 \mid t_1 \in \Omega, \; t_2 \in \Lambda\}$;

für eine Abbildung $t: \mathcal{P}X \to \mathcal{P}(\mathcal{P}X)$ und für $A \in \mathcal{P}X$ gibt es einen fundamentalen Unterschied zwischen folgenden Bezeichnungen:
$\cap t(A) \subset X$ und
$\cap \{t(\{x\}) \mid x \in A\} \subset \mathcal{P}X$.

2. Vorbereitungen

2.1. Definition. X sei eine Menge. $X \times X$ bezeichne das kartesische Produkt von X mit sich selbst. Eine Menge Q von Präuniformitäten $\mathcal{U}$ auf X im Sinne von [9] heißt *uniforme Konvergenzstruktur* auf X, und das Paar (X, Q) heißt *uniformer Konvergenzraum*, wenn die folgenden Axiome erfüllt sind:

(uK_1) $\mathcal{U} \cap \mathcal{V} \in Q$ genau dann, wenn $\mathcal{U}, \mathcal{V} \in Q$,
(uK_2) $\mathcal{U}, \mathcal{V} \in Q$ implizieren $\mathcal{U} \circ \mathcal{V} \in Q$,
(uK_3) $\dot{\Delta} \in Q$,
(uK_4) $\mathcal{U} \in Q$ impliziert $\mathcal{U}^{-1} \in Q$.

Für uniforme Konvergenzräume (X, Q_1), (Y, Q_2) heißt eine Abbildung $f: X \to Y$ *gleichmäßig stetig von* (X, Q_1) *nach* (Y, Q_2) oder kurz *gleichmäßig stetig*, wenn gilt: $\mathcal{U} \in Q_1$ impliziert $f \times f(\mathcal{U}) \in Q_2$. Ein uniformer Konvergenzraum (X, Q) und seine Struktur Q heißen *erzeugt*, wenn gilt:
(uK_5) Es gibt $\mathcal{V} \in Q$, so daß für alle $\mathcal{U} \in Q$ die Inklusion $\mathcal{V} \subset \mathcal{U}$ erfüllt ist.

2.2. Bemerkung. Cook und Fischer zeigen nun in ihrer Arbeit (vergl. [2]), daß die uniformen Räume in Sinne von Bourbaki [1] im Wesentlichen (bis auf Isomorphie!) erzeugte uniforme Konvergenzräume sind. Im Folgenden heißen erzeugte uniforme Konvergenzräume auch *uniforme Räume*.

2.3. DEFINITIONEN. X sei eine Menge. ppX bezeichne die Menge aller Präproximitäten auf X im Sinne von [9]. Eine Teilmenge $\mathcal{P}\subset ppX$ heißt *proximale Konvergenzstruktur* auf X, und das Paar $(X, \mathcal{P})$ heißt *proximaler Konvergenzraum*, wenn die folgenden Axiome erfüllt sind:

(pK_1) $p\cup q\in\mathcal{P}$ genau dann, wenn $p, q\in\mathcal{P}$,
(pK_4) $p\in\mathcal{P}$ und $q\in\mathcal{P}$ implizieren $p\circ q\in\mathcal{P}$,
(pK_3) $d_x\in\mathcal{P}$,
(pK_2) $p\in\mathcal{P}$ impliziert $p^{-1}\in\mathcal{P}$.

Für proximale Konvergenzräume $(X, \mathcal{P}_1)$, $(Y, \mathcal{P}_2)$ heißt eine Abbildung $f{:}X\to Y$ *proximal stetig von* $(X, \mathcal{P}_1)$ *nach* $(Y, \mathcal{P}_2)$ oder kurz *proximal stetig*, wenn gilt: $p\in\mathcal{P}_1$ impliziert $f_{pp}p\in\mathcal{P}_2$; wobei $f_{pp}p=\{(A, B)\in\mathcal{P}Y\times\mathcal{P}Y\,|\,(f^{-1}[A], f^{-1}[B])\in p\}$.

Ein proximaler Konvergenzraum $(X, \mathcal{P})$ und seine Struktur $\mathcal{P}$ heißen *erzeugt*, wenn gilt:
(pK_5) Es gibt $q\in\mathcal{P}$, so daß für alle $p\in\mathcal{P}$ die Inklusion $p\subset q$ erfüllt ist.

2.4. BEMERKUNG. Marny zeigt in seiner Arbeit (vergl. [9]), daß die Proximitätsräume in Sinne von EFREMOVIČ im Wesentlichen (bis auf Isomorphie!) erzeugte proximale Konvergenzräume sind. Im Folgenden heißen erzeugte proximale Konvergenzräume auch *proximale Räume (Proximitätsräume)*. Darüberhinaus wird bewiesen, daß die Kategorie der proximalen Konvergenzräume mit den proximal stetigen Abbildungen eine bireflektive Unterkategorie der topologischen Kategorie der uniformen Konvergenzräume ist.

2.5. DEFINITIONEN. Für eine Menge X heißt eine Relation $<\ \subset\mathcal{P}X\times\mathcal{P}X$ *topogene Ordnung* auf X, wenn die folgenden Axiome erfüllt sind:

(tO_1) $\varnothing<\varnothing$ und $X<X$,
(tO_2) $A'\subset A<B\subset B'$ implizieren $A'<B'$,
(tO_3) $A<B$ impliziert $A\subset B$,
(tO_4) $A<B$ und $A'<B'$ implizieren $A\cap A'<B\cap B'$ und $A\cup A'<B\cup B'$.

Eine Menge $\mathcal{S}$ von topogenen Ordnungen auf einer Menge X heißt *syntopogene Struktur* auf X, und das Paar $(X, \mathcal{S})$ heißt *syntopogener Raum*, wenn die folgenden Axiome erfüllt sind:
(st_1) $\mathcal{S}\neq\varnothing$,
(st_2) $<_1$, $<_2\in\mathcal{S}$ implizieren die Existenz einer topogenen Ordnung $<_3\in\mathcal{S}$ mit der Eigenschaft $<_1\cup<_2\subset<_3$,
(st_3) $<\ \in\mathcal{S}$ impliziert die Existenz einer topogenen Ordnung $<^+\in\mathcal{S}$ mit der Eigenschaft $<\ \subset\ <^{+2}$.

Für syntopogene Räume $(X, \mathcal{S}_1)$, $(Y, \mathcal{S}_2)$ heißt eine Abbildung $f{:}X\to Y$ *syntopogene Abbildung von* $(X, \mathcal{S}_1)$ *nach* $(Y, \mathcal{S}_2)$ oder kurz *syntopogene Abbildung*, wenn gilt:

$$\forall <_2\in\mathcal{S}_2\ \exists <_1\in\mathcal{S}_1 (A<_2B \text{ impliziert } f^{-1}[A]<_1f^{-1}[B]).$$

Ein syntopogener Raum $(X, \mathcal{S})$ und seine Struktur heißen *topogen*, wenn gilt:
(st_4) $\mathcal{S}$ ist einelementig.

2.6. Bemerkung. Császár zeigt nun in seiner Monographie (vergl. [3]), daß sich quasiuniforme Räume (vergl. [12]), Pervinproximitätsräume (vergl. [10]) und topologische Räume als spezielle syntopogene Räume auffassen lassen. Es wird weiter gezeigt, daß eine syntopogene Struktur $\mathcal{S}$ genau dann topogen ist, wenn ihre einzige topogene Ordnung $<$ das folgende Axiom erfüllt: (tO_5) $A < B$ impliziert die Existenz einer Menge $C \subset X$ mit den Eigenschaften $A < C < B$.

3. Syntopogene Konvergenzräume

3.1. Definitionen. Für eine Menge X heißt eine Abbildung $t: \mathcal{P}X \to \mathcal{P}(\mathcal{P}X)$ *topoformer Operator* auf X, wenn die folgenden Axiome erfüllt sind:

(tf_1) $A \in \mathcal{P}X$ impliziert stack $t(A) \subset t(A)$,
(tf_2) $A_1, A_2 \in \mathcal{P}X$ und $A_1 \subset A_2$ implizieren $t(A_2) \subset t(A_1)$,
(tf_3) $X \in t(X)$ und $\varnothing \in t(\varnothing)$,
(tf_4) $A \in \mathcal{P}X$ und $B_1, B_2 \in t(A)$ implizieren $B_1 \cap B_2 \in t(A)$,
(tf_5) $A_1, A_2 \in \mathcal{P}X$ implizieren $t(A_1) \cap t(A_2) \subset t(A_1 \cup A_2)$.
Ein topoformer Operator t auf einer Menge X heißt
(i) *holoform*, wenn gilt:
(tf_6) $A \in \mathcal{P}X$ impliziert $\bigcap t(A) \in t(A)$.
Ein holoformer Operator t auf einer Menge X heißt
(ii) *homoform*, wenn gilt:
(tf_7) $A \in \mathcal{P}X$ impliziert $\bigcap \{t(\{x\}) \mid x \in A\} = t(A)$.

3.2. Definitionen. Für eine Menge X sei t ein topoformer Operator auf X. t heißt
(i) *schwach symmetrisch*, wenn gilt:
(ss) $\forall x, y \in X (y \in \bigcap t(\{x\})$ impliziert $x \in \bigcap t(\{y\}))$;
(ii) *punktsymmetrisch*, wenn gilt:
(ps) $\forall B \subset X\ \forall x \in X (B \notin t(\{x\})$ impliziert $X / \{x\} \notin t(X/B))$;
(iii) *symmetrisch*, wenn gilt:
(s) $A \in PX$ und $B \in t(A)$ implizieren $X/A \in t(X/B)$.

3.3. Bemerkung. Beachte, daß jeder symmetrische topoforme Operator punktsymmetrisch und jeder punktsymmetrische schwach symmetrisch ist. Für homoforme Operatoren sind die drei Begriffe äquivalent. Ein schwach symmetrischer holoformer Operator ist automatisch punktsymmetrisch.

3.4. Resultate (vergl. [7]). Für eine Menge X sei t ein topoformer Operator auf X; setze für jedes $A \in \mathcal{P}X$:
(i) $t_{ss}(A) := \{B \subset X \mid B \in t(A)$ und $\forall x, y \in X (x \in A$ und $y \in X/B$ implizieren $X/\{x\} \in t(\{y\}))\}$;
(ii) $t_{ps}(A) := \{B \subset X \mid B \in t(A)$ und $\forall x \in X (x \in X/B$ impliziert $X/A \in t(\{x\}))\}$;
(iii) $t_d(A) := \{B \subset X \mid X/A \in t(X/B)\}$, dann gelten die folgenden Aussagen:

(1) t_{ss} ist der größte schwach symmetrische topoforme Operator auf X, der kleiner ist als t.
(2) t_{ps} ist der größte punktsymmetrische topoforme Operator auf X, der kleiner ist als t.
(3) t_d ist ein topoformer Operator auf X mit $(t_d)_d = t$.
(4) Für einen topoformen Operator t auf einer Menge X sind die folgenden Aussagen paarweise äquivalent:
(α) t ist schwach symmetrisch,
(α') $t \leq t_{ss}$;
(β) t ist punktsymmetrisch,
(β') $t \leq t_{ps}$;
(γ) t ist symmetrisch,
(γ') $t \leq t_d$.
(5) Für t schwach symmetrisch ist t_d schwach symmetrisch.
(6) Für topoforme Operatoren t_1, t_2 auf einer Menge X setze für jedes $A \in \mathcal{P}X$: $(t_1+t_2)(A) := \{B \subset X \mid \exists C \subset X (C \in t_2(A) \text{ und } B \in t_1(C))\}$, so gelten die folgenden Aussagen:
(i) t_1+t_2 ist ein topoformer Operator auf X mit der Eigenschaft $(t_1+t_2)_d = t_{2_d}+t_{1_d}$;
(ii) t_1, t_2 holoform implizieren t_1+t_2 ist holoform;
(iii) t_1, t_2 homoform implizieren t_1+t_2 ist homoform.
(7) Für ein Menge X setze für jedes $A \in \mathcal{P}X$: $I^X(A) := \{B \subset X \mid B \supset A\}$, dann ist I^X ein homoformer Operator auf X, der zusätzlich folgendes Axiom erfüllt:
(tf_8) $A \in \mathcal{P}X$ impliziert $A \subset \bigcap t(A)$.
(8) Für Mengen X, Y einer Abbildung $f: X \to Y$ und einem topoformen Operator t auf X setze für jedes $A \in \mathcal{P}Y$: $t_f(A) := \{B \subset Y \mid f^{-1}[B] \in t(f^{-1}[A])\}$, so ist t_f ein topoformer Operator auf Y mit $(t_d)_f = (t_f)_d$.

3.5. Definitionen. Eine Menge Ω von topoformen Operatoren auf einer Menge X heißt *Syntopoform* auf X, wenn gilt:
(Stf_1) $\Omega \neq \emptyset$,
(Stf_2) $t_1, t_2 \in \Omega$ implizieren die Existenz eines topoformen Operators $t_3 \in \Omega$ mit der Eigenschaft $t_1 \cup t_2 \leq t_3$.

Eine Syntopoform Ω auf einer Menge X heißt
(i) *Synholoform*, wenn gilt: $t \in \Omega$ impliziert t ist ein holoformer Operator auf X;
(ii) *Synhomoform*, wenn gilt: $t \in \Omega$ impliziert t ist ein homoformer Operator auf X;
(iii) *Topoform*, wenn gilt: Ω ist einelementig.

3.6. Resultate (vergl. [5], [7]).
(1) Für eine Menge X seien Ω, Λ Syntopoformen auf X, setze:
(i) $\Omega_{ss} := \{t_{ss} \mid t \in \Omega\}$;
(ii) $\Omega_{ps} := \{t_{ps} \mid t \in \Omega\}$;
(iii) $\Omega_d := \{t_d \mid t \in \Omega\}$;
(iv) $\Omega+\Lambda := \{t_1+t_2 \mid t_1 \in \Omega, t_2 \in \Lambda\}$, so sind Ω_{ss}, Ω_{ps}, Ω_d, $\Omega+\Lambda$ Syntopoformen auf X mit $(\Omega_d)_d = \Omega$ (vergl. Resultate 3.4.).

(2) Für Mengen X, Y, Z, Abbildungen $f: X \to Y$, $g: Y \to Z$ und einer Syntopoform Ω auf X setze: $\Omega_f := \{t_f | t \in \Omega\}$, so ist Ω_f eine Syntopoform auf Y mit $(\Omega_f)_g = \Omega_{f \circ g}$.

3.7. Definitionen. X sei ein Menge. Eine Menge $\mathcal{K}$ von Syntopoformen auf X heißt *syntopogene Konvergenzstruktur* auf X, und das Paar $(X, \mathcal{K})$ heißt *syntopogener Konvergenzraum*, wenn die folgenden Axiome erfüllt sind:

(sK_1) $\Omega_1, \Omega_2 \in \mathcal{K}$ implizieren die Existenz einer Syntopoform $\Omega_3 \in \mathcal{K}$ mit der Eigenschaft $\Omega_3 \leq \Omega_1 \wedge \Omega_2$,

(sK_2) $\Omega_1, \Omega_2 \in \mathcal{K}$ implizieren die Existenz einer Syntopoform $\Omega_3 \in \mathcal{K}$ mit der Eigenschaft $\Omega_3 \leq \Omega_1 + \Omega_2$,

(sK_3) Es existiert $\Omega \in \mathcal{K}$ mit $\Omega \leq \{I^X\}$.

Für syntopogene Konvergenzräume $(X, \mathcal{K}_1)$, $(Y, \mathcal{K}_2)$ heißt eine Abbildung $f: X \to Y$ *konvergenztreu von* $(X, \mathcal{K}_1)$ *nach* $(Y, \mathcal{K}_2)$ oder kurz *konvergenztreu*, wenn gilt:

$$\forall\, \Omega \in \mathcal{K}_1 \;\; \exists\, \Lambda \in \mathcal{K}_2 \;\; \Lambda \leq \Omega_f.$$

Ein syntopogener Konvergenzraum $(X, \mathcal{K})$ und seine Struktur $\mathcal{K}$ heißen *erzeugt*, wenn gilt:

(sK_4) Es gibt $\Lambda \in \mathcal{K}$, so daß für alle $\Omega \in \mathcal{K}$ die Aussage $\Lambda \leq \Omega$ erfüllt ist. Λ heißt dann auch *Erzeuger von* $\mathcal{K}$ (*erzeugendes Element von* $\mathcal{K}$).

3.8. Bemerkung. Wir weisen in diesem Zusammenhang auf einen Schreibfehler in [7] beziehungsweise in [6] hin, wo bei der Axiomatisierung der syntopogenen Konvergenzräume gefordert war, daß die Topoform $\{I^X\}$ zur Struktur $\mathcal{K}$ gehört. Dies gilt insbesondere bei den sogenannten *saturierten* syntopogenen Konvergenzräumen. (Eine syntopogene Konvergenzstruktur $\mathcal{K}$ heißt dabei *saturiert*, wenn gilt: $\forall\, \Omega \in \mathcal{K}\; \forall\, \Lambda$ Syntopoform auf $X(\Omega \leq \Lambda$ impliziert $\Lambda \in \mathcal{K}))$. Wir bemerken, daß zu jeder syntopogenen Konvergenzstruktur $\mathcal{K}$ eine saturierte syntopogene Konvergenzstruktur $\mathcal{K}^{sat}$ existiert, indem wir $\mathcal{K}^{sat}$ folgendermaßen definieren:

$$\mathcal{K}^{sat} := \{\Omega \text{ Syntopoform auf } X \,|\, \exists\, \Lambda \in \mathcal{K}\, \Lambda \leq \Omega\}.$$

$(X, \mathcal{K}^{sat})$ ist dann ein saturierter syntopogener Konvergenzraum derart, daß $1_X : (X, \mathcal{K}) \to (X, \mathcal{K}^{sat})$ in „beiden Richtungen" konvergenztreu ist.

In der vorliegenden Arbeit (vergl. [7]) wird gezeigt, daß die syntopogenen Räume im Sinne von Császár im Wesentlichen (bis auf Äquivalenz! Vergl. obige Bemerkung) erzeugte syntopogene Konvergenzräume sind. Im Folgenden heißen erzeugte syntopogene Konvergenzräume auch *syntopogene Räume*.

3.9. Bemerkung. Ein syntopogener Konvergenzraum $(X, \mathcal{K})$ und seine Struktur $\mathcal{K}$ heißen *synuniform*, wenn gilt:

(suK) $\Omega \in \mathcal{K}$ impliziert Ω ist eine Synhomoform auf X.

In der vorbezeichneten Arbeit (vergl. [7]) wird nun gezeigt, daß die uniformen Konvergenzräume (vergl. Definition 2.1.) im Sinne von Cook und Fischer im Wesentlichen (bis auf Äquivalenz!) symmetrische synuniforme Konvergenzräume sind (vergl. Definition 6.4. (iii)). Somit stellt das Konzept der syntopogenen Konvergenzräume eine gemeinsame Verallgemeinerung der Konzepte der uniformen sowie proximalen Konvergenzräume und der syntopogenen Räume in Sinne von CSÁSZÁR dar (vergl. auch Bemerkung 2.4.).

3.10. BEMERKUNG. Ein syntopogener Konvergenzraum $(X, \mathcal{K})$ und seine Struktur $\mathcal{K}$ heißen *syntopologisch*, wenn gilt:
(stK) $\Omega \in \mathcal{K}$ impliziert Ω ist eine Synholoform auf X.
Beachte dabei, daß jeder synuniforme Konvergenzraum ein syntopologischer Konvergenzraum ist. Syntopologien im Sinne von CSÁSZÁR (vergl. [3]) sind bis auf Äquivalenz erzeugte syntopologische Konvergenzräume. Im Folgenden heißen erzeugte syntopologische Konvergenzräume auch *syntopologische Räume*.

3.11. RESULTATE (vegl. [7]).
(1) Für einen syntopogenen Raum $(X, \mathcal{K})$ seien Ω, Λ zwei Erzeuger von $\mathcal{K}$, so gelten die folgenden Aussagen:
(i) $\Omega \sim \Lambda$;
(ii) Jeder topoforme Operator $t \in \Omega$ erfüllt das Axiom (tf_8), vergl. Resultat 3.4. (7);
(iii) $\Lambda \leq \Lambda^2$; wobei $\Lambda^2 := \{t+t \mid t \in \Lambda\}$.
(2) $(X, \mathcal{K}_1)$, $(Y, \mathcal{K}_2)$ seien syntopogene Räume; $\Lambda_{\mathcal{K}_1}$, $\Lambda_{\mathcal{K}_2}$ bezeichne zwei zu den jeweiligen Strukturen gehörende Erzeuger.

Für eine Abbildung $f: X \to Y$ sind äquivalent:
(i) f ist konvergenztreu von $(X, \mathcal{K}_1)$ nach $(Y, \mathcal{K}_2)$;
(ii) $\Lambda_{\mathcal{K}_2} \leq (\Lambda_{\mathcal{K}_1})_f$.

4. Topogene Konvergenzräume

4.1. DEFINITIONEN. Für eine Menge X bezeiche $\mathcal{F}X$ die Menge aller filtrierten Mengensysteme über X (vergl. [8]). Eine Abbildung $T: \mathcal{P}X \to \mathcal{P}(\mathcal{F}X)$ heißt *topogener Konvergenzoperator* auf X, und das Paar (X, T) heißt *topogener Konvergenzraum*, wenn die folgenden Axiome erfüllt sind:
(tK_1) $A \in \mathcal{P}X$, $\mathfrak{N}_1 \in T(A)$ und $\mathfrak{N}_1 \ll \mathfrak{N}_2 \in \mathcal{F}X$ implizieren $\mathfrak{N}_2 \in T(A)$,
(tK_2) $A_1, A_2 \in \mathcal{P}X$ und $A_1 \subset A_2$ implizieren $T(A_1) \subset T(A_2)$,
(tK_3) $A \in \mathcal{P}X$ impliziert $\{A\} \in T(A)$,
(tK_4) $\mathfrak{N} \in T(\emptyset)$ impliziert $\emptyset \in \mathfrak{N}$,
(tK_5) $A_1, A_2 \in \mathcal{P}X$ und $\mathfrak{N} \in T(A_1 \cup A_2)$ implizieren die Existenz von Mengensystemen $\mathfrak{N}_1 \in T(A_1)$, $\mathfrak{N}_2 \in T(A_2)$ mit $\mathfrak{N}_1 \vee \mathfrak{N}_2 \ll \mathfrak{N}$,
(tK_6) $A \in \mathcal{P}X$ und $\mathfrak{N}_1$ $\mathfrak{N}_2 \in T(A)$ implizieren $\mathfrak{N}_1 \vee \mathfrak{N}_2 \in T(A)$,
(tK_7) $A \in \mathcal{P}X$ und $\mathfrak{N} \in T(A)$ implizieren die Existenz eines Mengensystems $\mathfrak{N}^+ \in T(A)$ mit der Eigenschaft $(\forall F^+ \in \mathfrak{N}^+ \exists E^+ \subset X \exists \mathfrak{R} \in T(E^+))$ $(E^+ \in$ stack $\mathfrak{N}$ und $F^+ \in$ stack $\mathfrak{R})$.

Für einen topogenen Konvergenzoperator T auf X sei $\mathfrak{N} \in T(A)$, so heißt *$\mathfrak{N}$ T-konvergent gegen A* und *A heißt T-Limes von $\mathfrak{N}$*. Für topogene Konvergenzräume (X, T_1), (Y, T_2) heißt eine Abbildung $f: X \to Y$ *konvergenztreu von (X, T_1) nach (Y, T_2)* oder kurz *konvergenztreu*, wenn gilt: $A \in \mathcal{P}X$ und $\mathfrak{N} \in T_1(A)$ implizieren $f\mathfrak{N} \in T_2(f[A])$.

Beachte bei der obigen Definition insbesondere, daß das Bild eines filtrierten Mengensystems über einer Menge unter einer Abbildung wieder ein filtriertes Mengensystem über dem Wertevorrat der Abbildung ist.

Ein topogener Konvergenzraum (X, T) und sein Operator T heißen *erzeugt*, wenn gilt:

(tK_8) $A \in \mathcal{P}X$ impliziert $\mathcal{E} := \vee \{\mathcal{M} | \mathcal{M} \in T(A)\} \in T(A)$. Beachte für jedes Element $\mathcal{M} \in T(A)$ gilt dann $\mathcal{E} \ll \mathcal{M}$.

4.2. Bemerkung. In der Dissertation des Verfassers (vergl. [5]) wird gezeigt, daß die topogenen Räume im Sinne von Császár im Wesentlichen (bis auf Isomorphie!) erzeugte topogene Konvergenzräume sind. Im Folgenden heißen erzeugte topogene Konvergenzräume auch *topogene Räume*.

4.3. Beispiele.

(i) $(X, \mathcal{D})$ sei ein proximaler Konvergenzraum in Sinne von Marny (vergl. auch Definition 2.3.). Für $A \in \mathcal{P}X$ setze:

$$T_{\mathcal{D}}(A) := \{\mathcal{M} \in \mathcal{F}X \mid \exists p \in \mathcal{D}\ \forall F \notin \text{stack}\,\mathcal{M}(X/F, A) \in p\};$$

(ii) (X, h) sei ein topologischer Hüllenraum in Sinne von Kuratowski. Für $A \in \mathcal{P}X$ setze:

$$T_h(A) := \{\mathcal{M} \in \mathcal{F}X \mid h(A) \in \text{stack}\,\mathcal{M}\};$$

(iii) $(X, \{<\})$ sei ein topogener Raum in Sinne von Császár. Für $A \in \mathcal{P}X$ setze:

$$T_{\{<\}}(A) := \{\mathcal{M} \in \mathcal{F}X \mid \forall F \subset X (A < F \text{ impliziert } F \in \text{stack}\,\mathcal{M})\}.$$

4.4. Bemerkung. In der Arbeit von Marny (vergl. [9]) wird gezeigt, daß jeder S_1-Limesraum einen kompatiblen proximalen Konvergenzraum besitzt. Hierbei wird insbesondere auf die Arbeit [13] Bezug genommen.

Genauer: Für einen proximalen Konvergenzraum $(X, \mathcal{D})$ setze für jedes $x \in X$; $\tau_{\mathcal{D}}(x) := \{\mathcal{F}$ Filter auf $X \mid p_{(\mathcal{F}, x)} \in \mathcal{D}\}$; wobei $p_{(\mathcal{F}, x)} := \{(A, B) \in \mathcal{P}X \times \mathcal{P}X \mid A \in \text{sec}\,\mathcal{F},\ x \in B\}$, so ist $\tau_{\mathcal{D}}$ eine S_1 – Limitierung auf X. Umgekehrt sei α eine S_1-Limitierung auf X, so existiert eine proximale Konvergenzstruktur Q auf X mit $T_Q = \alpha$, Nun hat jeder proximale Konvergenzraum einen „unterliegenden" symmetrischen topogenen Konvergenzraum (vergl. Beispiel 4.3. (i) und [6]). Für einen topogenen Konvergenzraum (X, T) setze für jedes $x \in X$: $\tau_T(x) := \{\mathcal{F}$ Filter auf $X \mid \exists \mathcal{N} \in T(\{x\}) \mathcal{N} \ll \mathcal{F}\}$, so ist τ_T eine Limitierung auf X. Gilt $\mathcal{F} \in \tau_T(x)$, so sagen wir *$\mathcal{F}$ ist T-konvergent gegen x* oder kürzer *T-konvergent*. Es gilt nun folgendes Lemma.

4.5. Lemma. Für einen proximalen Konvergenzraum $(X, \mathcal{D})$ gilt die folgende Identität: $\tau_{T_{\mathcal{D}}} = \tau_{\mathcal{D}}$.

Beweis „$\leq$": Für $x \in X$ sei $\mathcal{F} \in \tau_{T_{\mathcal{D}}}(x)$, so existiert ein $\mathcal{M} \in T_{\mathcal{D}}(\{x\})$ mit stack $\mathcal{M} \subset \mathcal{F}$. Es folgt die Existenz einer Präproximität $q \in \mathcal{D}$; wir zeigen: $p_{(\mathcal{F}, x)} \subset q$. $(A, B) \in p_{(\mathcal{F}, x)}$ impliziert $A \in \text{sec}\,\mathcal{F}$ und $x \in B$, mithin gilt $X/A \notin$ stack $\mathcal{F}$. Da stack $\mathcal{M} \subset \mathcal{F}$, folgt $X/A \notin$ stack $\mathcal{M}$, also $(A, \{x\}) \in q$, weil $\mathcal{M} \in T_{\mathcal{D}}(\{x\})$ gilt. Es folgt $(A, B) \in q$, womit $p_{(\mathcal{F}, x)} \in \mathcal{D}$ realisiert ist. Damit gilt $\mathcal{F} \in \tau_{\mathcal{D}}(x)$.

„$\geq$": Umgekehrt sei $\mathcal{F} \in \tau_{\mathcal{P}}(x)$, so gilt $p_{(\mathcal{F}, x)} \in \mathcal{P}$. Setze $\mathcal{N} := \{F \subset X \mid (X/F, \{x\}) \notin p_{(\mathcal{F}, x)}\}$, so ist $\mathcal{N} \in T_{\mathcal{P}}(\{x\})$ (da $p_{(\mathcal{F}, x)}$ eine Präproximität ist, folgt $\mathcal{N}$ ist filtriert). Für $F \in \mathcal{N}$ gilt $(X/F, \{x\}) \notin p_{(\mathcal{F}, x)}$, d.h. $X/F \notin \sec \mathcal{F}$, damit folgt $F \in \operatorname{stack} \mathcal{F}$.

4.6. Bemerkung. Aus dem bisher Gesagten folgt nun, daß jeder S_1-Limesraum einen kompatiblen symmetrischen topogenen Konvergenzraum besitzt.

Genauer: Zu jeder S_1-Limitierung α auf einer Menge X existiert ein symmetrischer topogener Konvergenzoperator T auf X mit $\tau_T = \alpha$. Wir bemerken noch, daß bei einem topogenen Raum (X, T) die „unterliegende" Limitierung τ_T prätopologisch ist (vergl. [4], [11]). Schließlich hat jeder topogene Konvergenzoperator T auf einer Menge X einen „unterliegenden" Hüllenoperator cl_T, der für jedes $A \in \mathcal{P}X$ wie folgt erklärt ist:

$\mathrm{cl}_T(A) := \{x \in X \mid \exists \mathcal{N} \in T(A) x \in \bigcap \mathcal{N}\}$. Unter Beachtung von Beispiel 4.3. (ii) gilt nun, *daß die Kategorie der topologischen Räume isomorph ist zu einer Unterkategorie der Kategorie der topogenen Konvergenzräume.*

5. Zusammenhang zwischen syntopogenen und topogenen Konvergenzräumen

5.1. Bemerkung. Ein Zusammenhang zwischen beiden in der Überschrift genannten Raumtypen studieren wir zunächst in einer allgemeineren Fassung. Genauer wollen wir im Folgenden unter einer *pseudosyntopogenen Konvergenzstruktur* auf einer Menge X eine Menge $\mathcal{K}$ von Syntopoformen verstehen, welche die Axiome (sK_1) und (sK_3) erfüllt aber nicht notwendigerweise das Axiom (sK_2) in Definition 3.7. (vergl. auch [7]). Analog heißt ein Operator $T: X \to \mathcal{P}(\mathcal{F}X)$ *pseudotopogener Konvergenzoperator* auf X, sofern er die Axiome (tK_1) bis (tK_6) erfüllt aber nicht notwendigerweise das Axiom (tK_7) in Definition 4.1.. Wir liefern nun ein Beispiel eines pseudotopogenen Konvergenzoperators, der kein topogener Konvergenzoperator ist.

5.2. Beispiel. Setze: $X := \mathbf{N} \cup \{\infty\}$. Für $A \in \mathcal{P}X$ definiere:

$$T^{-}(A) := \begin{cases} \{\mathcal{N} \in \mathcal{F}X \mid A \in \operatorname{stack} \mathcal{N}\} & \text{falls } \infty \notin A \\ \left\{\mathcal{N} \in \mathcal{F}X \mid \begin{array}{l}\exists E \subset \mathbf{N} \text{ endlich} \\ E \cup A \in \operatorname{stack} \mathcal{N}\end{array}\right\} & \text{falls } \infty \in A. \end{cases}$$

5.3. Satz. *Für einen pseudosyntopogenen Konvergenzraum* $(X, \mathcal{K})$ *setze für* $A \in \mathcal{P}X: T_{\mathcal{K}}(A) := \{\mathcal{N} \in \mathcal{F}X \mid \exists \Omega \in \mathcal{K}\ \mathcal{M}_{\Omega}^{A} \ll \mathcal{N}\}$, *wobei* $\mathcal{M}_{\Omega}^{A} := \{\bigcap t(A) \mid t \in \Omega\}$, *so ist* $T_{\mathcal{K}}$ *ein pseudotopogener Konvergenzoperator auf* X.

Beweis. Zu (tK_1): trivial.

Zu (tK_2): *Für* $A_1 \subset A_2$ sei $\mathcal{N} \in T_{\mathcal{K}}(A_1)$, so gibt es eine Syntopoform $\Omega \in \mathcal{K}$ mit $\mathcal{M}_{\Omega}^{A_1} \ll \mathcal{N}$. Nun gilt mit ($tf_2$) (vergl. Definition 3.1.) $\mathcal{M}_{\Omega}^{A_2} \ll \mathcal{M}_{\Omega}^{A_1}$, und damit folgt $\mathcal{M}_{\Omega}^{A_2} \ll \mathcal{N}$; also ist $\mathcal{N} \in T_{\mathcal{K}}(A)$.

Zu (tK_3): Sei $A \in \mathcal{P}X$; wähle $\Omega \in \mathcal{X}$ mit $\Omega \leq \{I^X\}$ unter Anwendung von (sK_3); zeige: $\mathcal{M}_\Omega^A \ll \{A\}$. Sei $\cap t(A) \in \mathcal{M}_\Omega^A$ für $t \in \Omega$, so folgt $t(A) \subset I^X(A)$ und somit gilt $\cap t(A) \supset A$.

Zu (tK_4): Sei $\mathcal{N} \in T_\mathcal{X}(\emptyset)$, so wähle $\Omega \in \mathcal{X}$ mit $\mathcal{M}_\Omega^\emptyset \ll \mathcal{N}$. Da $\Omega \neq \emptyset$ wähle $t \in \Omega$, so folgt $\cap t(\emptyset) \in \mathcal{M}_\Omega^\emptyset$. Mit (tf_3) in Definition 3.1. gilt $\emptyset \in \mathcal{N}$.

Zu (tK_5): Sei $\mathcal{N} \in T_\mathcal{X}(A \cup B)$, so gibt es eine Syntopoform $\Omega \in \mathcal{X}$ mit $\mathcal{M}_\Omega^{A \cup B} \ll \mathcal{N}$. Da Ω eine Syntopoform auf X ist, folgt $\mathcal{M}_\Omega^A$, $\mathcal{M}_\Omega^B$ sind filtrierte Mengensysteme über X, und es gelten die Aussagen: $\mathcal{M}_\Omega^A$ ist $T_\mathcal{X}$-konvergent gegen A beziehungsweise $\mathcal{M}_\Omega^B$ ist $T_\mathcal{X}$-konvergent gegen B (vergl. Definition 4.1.). Es bleibt zu zeigen: $\mathcal{M}_\Omega^A \vee \mathcal{M}_\Omega^B \ll \mathcal{N}$. Sei $(\cap t(A)) \cup (\cap l(B)) \in \mathcal{M}_\Omega^A \vee \mathcal{M}_\Omega^B$ für $t, l \in \Omega$, so existiert ein $b \in \Omega$ mit $t \cup l \leq b$; mithin gelten die Aussagen $\cap b(A) \subset \cap t(A)$ und $\cap b(B) \subset \cap l(B)$. Daraus folgt $(\cap b(A)) \cup (\cap b(B)) \subset (\cap t(A)) \cup (\cap l(B))$. Mit dem Axiom (tf_5) in Definition 3.1. gilt $b(A) \cap \cap b(B) \subset b(A \cup B)$; mithin ergibt sich, daß $\cap b(A \cup B) \subset \cap(b(A) \cap b(B)) \subset \subset (\cap b(A)) \cup (\cap b(B))$. $\cap b(A \cup B) \in \mathcal{M}_\Omega^{A \cup B}$ und umfaßt daher nach Voraussetzung eine Menge $F \in \mathcal{N}$. Das bedeutet aber $(\cap t(A)) \cup (\cap l(B)) \in \text{stack}\, \mathcal{N}$.

Zu (tK_6): Seien $\mathcal{N}_1, \mathcal{N}_2 \in T_\mathcal{X}(A)$. Wähle Syntopoformen $\Omega_1, \Omega_2 \in T_\mathcal{X}(A)$ mit $\mathcal{M}_{\Omega_1}^A \ll \mathcal{N}_1$ und $\mathcal{M}_{\Omega_2}^A \ll \mathcal{N}_2$. Wegen (sK_1) gibt es eine Syntopoform $\Omega \in \mathcal{X}$ mit $\Omega \leq \Omega_1 \wedge \Omega_2$. Es folgt $\mathcal{M}_\Omega^A \ll \mathcal{M}_{\Omega_1}^A \vee \mathcal{M}_{\Omega_2}^A \ll \mathcal{N}_1 \vee \mathcal{N}_2$. Somit gilt $\mathcal{N}_1 \vee \mathcal{N}_2 \in T_\mathcal{X}(A)$. Im Anschluß bemerken wir, daß $T_\mathcal{X}$ das Axiom (tK_8) in Definition 4.1. erfüllt, sofern für $\mathcal{X}$ das Axiom (sK_4) in Definition 3.7. vorausgesetzt wird.

5.4. Bemerkung. Aufgrund des eben Bewiesenen besitzt jede pseudosyntopogene Konvergenzstruktur auf einer Menge X in naheliegender Weise einen „unterliegenden" pseudotopogenen Konvergenzoperator. Wie in Bemerkung 4.4. ausgeführt besitzt jeder topogene Konvergenzoperator auf einer Menge X eine „unterliegende" Limitierung. Wir weisen darauf hin, daß τ_T bereits unter der schwächeren Voraussetzung eines pseudotopogenen Konvergenzoperators T eine Limitierung ist. In der Arbeit des Verfassers (vergl. [7]) wurde schon die „$\mathcal{X}$-Konvergenz" in syntopogenen Konvergenzstrukturen eingeführt und untersucht. Wir geben hier die Definition wieder und fragen uns, ob die $\mathcal{X}$-Konvergenz in pseudosyntopogenen Konvergenzräumen mit der oben genannten Limitierung übereinstimmt.

5.5. Definition. $(X, \mathcal{X})$ sei ein pseudosyntopogener Konvergenzraum. Ein Filter $\mathcal{F}$ über X heißt $\mathcal{X}$-konvergent gegen ein $x \in X$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ gibt, so daß für alle $t \in \Omega t(\{x\}) \subset \text{stack}\, \mathcal{F}$ gilt. Wir sagen dann auch $\mathcal{F}$ ist $\mathcal{X}$-konvergent. Setze $\tau_\mathcal{X}(x) = \{\mathcal{F} \mid \mathcal{F}$ ist $\mathcal{X}$-konvergent gegen $x\}$ für jedes $x \in X$.

5.6. Satz. *Für einen pseudosyntopologischen Konvergenzraum* $(X, \mathcal{X})$ *gilt die folgende Identität:* $\tau_\mathcal{X} = \tau_{T_\mathcal{X}}$ (vergl. auch Bemerkung 3.10.).

Beweis: „$\leq$": Für $x \in X$ sei $\mathcal{F} \in \tau_{\mathcal{X}}(x)$; wähle eine Synholoform $\Omega \in \mathcal{X}$ (vergl. Bemerkung 3.10.), so gilt $\mathcal{M}_{\Omega}^{\{x\}} \in T_{\mathcal{X}}(\{x\})$. Für $\cap t(\{x\}) \in \mathcal{M}_{\Omega}^{\{x\}}$ mit geeignetem $t \in \Omega$ gilt mit Voraussetzung $\cap t(\{x\}) \in t(\{x\}) \subset \text{stack } \mathcal{F}$, und damit folgt $\mathcal{F} \in \tau_{T_{\mathcal{X}}}(x)$.

„$\geq$": Umgekehrt sei $\mathcal{F} \in \tau_{T_{\mathcal{X}}}(x)$, so existiert ein $\mathcal{N} \in T_{\mathcal{X}}(\{x\})$ mit $\mathcal{N} \ll \mathcal{F}$. Wähle eine Synholoform $\Omega \in \mathcal{X}$ mit $\mathcal{M}_{\Omega}^{\{x\}} \ll \mathcal{N}$, so gilt $\mathcal{M}_{\Omega}^{\{x\}} \ll \mathcal{F}$. Für $t \in \Omega$ und $B \in t(\{x\})$ gilt $B \supset \cap t(\{x\}) \supset F \in \mathcal{F}$, und damit folgt $\mathcal{F} \in \tau_{\mathcal{X}}(x)$.

5.7. Bemerkung. Die in Satz 5.6. gemachten Aussagen bleiben richtig, wenn wir anstatt „konvergente" Filter „konvergente" Raster im Sinne von Császár (vergl. [3]) betrachten. Dies wollen wir nun im Folgenden tun!

5.8. Satz. *Für pseudosyntopologische Konvergenzräume $(X, \mathcal{X}_1)$, $(Y, \mathcal{X}_2)$ sei eine Abbildung $f: X \to Y$ konvergenztreu von $(X, \mathcal{X}_1)$ nach $(Y, \mathcal{X}_2)$, so ist f konvergenztreu von $(X, T_{\mathcal{X}_1})$ nach $(Y, T_{\mathcal{X}_2})$.*

Beweis. Für $A \in \mathcal{P}X$ sei $\mathcal{N} \in T_{\mathcal{X}_1}(A)$, so existiert eine Synholoform $\Omega \in \mathcal{X}_1$ mit $\mathcal{M}_{\Omega}^{A} \ll \mathcal{N}$. Da f konvergenztreu ist, existiert eine Synholoform $\Lambda \in \mathcal{X}_2$ mit $\Lambda \leq \Omega_f$. Wir zeigen: $\mathcal{M}_{\Lambda}^{f[A]} \ll f\mathcal{N}$. Sei $\cap l(f[A]) \in \mathcal{M}_{\Lambda}^{f[A]}$ für ein $l \in \Lambda$, so folgt $l \leq t_f$ für ein $t \in \Omega$. Da $\cap l(f[A]) \in l(f[A])$, folgen die Aussagen $f^{-1}[\cap l(f[A])] \in t(f^{-1}[f[A]])$ und $f^{-1}[f[A]] \supset A$; mithin gilt $f^{-1}[\cap l(f[A])] \in \in t(A)$ unter Beachtung von (tf_2) in Definition 3.1.. $f^{-1}[\cap l(f[A])] \supset \cap t(A) \supset \supset F \in \mathcal{N}$, da $\mathcal{M}_{\Omega}^{A} \ll \mathcal{N}$. Es folgen die Aussagen $f[f^{-1}[\cap l(f[A])]] \supset f[F] \in f\mathcal{N}$ und $\cap l(f[A]) \supset f[^{-1}[\cap l(f[A])]]$; mithin gilt $\mathcal{M}_{\Lambda}^{f[A]} \ll f\mathcal{N}$.

5.9. Bemerkung. In der Arbeit des Verfassers (vergl. [8]) wird gezeigt, daß konvergenztreue Abbildungen zwischen pseudotopogenen Konvergenzräumen stetig bezüglich der „unterliegenden" Limitierungen sind.

6. Raster, die "beliebig kleine Elemente" enthalten

6.1. Definitionen. $(X, \mathcal{X})$ sei ein pseudosyntopogener Konvergenzraum. Ein Raster $\mathcal{R}$ über X heißt

(i) *Mikroraster* in $\mathcal{X}$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ so gibt, daß für alle $t \in \Omega$ ein Punkt $x \in X$ existiert mit $t(\{x\}) \ll \mathcal{R}$;

(ii) *Pervinraster* in $\mathcal{X}$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ so gibt, daß für alle $t \in \Omega$ ein Punkt $x \in X$ existiert mit $\cap t(\{x\}) \in \text{stack } \mathcal{R}$;

(iii) *Cauchyraster* in $\mathcal{X}$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ so gibt, daß für alle $t \in \Omega$ eine Menge $R \in \mathcal{R}$ existiert, so daß für Punkte $x, y \in R$ die Aussage $y \in \cap t(\{x\})$ erfüllt ist;

(iv) *Toporaster* in $\mathcal{X}$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ so gibt, daß für alle $t \in \Omega$ und für alle Mengen $A \in \sec \mathcal{R}$ die Aussage $\cap t(A) \in \text{stack } \mathcal{R}$ erfüllt ist;

(v) *Császárraster* in $\mathcal{X}$ genau dann, wenn es eine Syntopoform $\Omega \in \mathcal{X}$ so gibt, daß für alle $t \in \Omega$ und für alle Mengen $A \in \sec \mathcal{R}$ die Aussage $t(A) \ll \mathcal{R}$ erfüllt ist.

6.2. Bemerkung. Für einen pseudosyntopogenen Konvergenzraum $(X, \mathcal{K})$ ist offensichtlich ein $\mathcal{K}$-konvergenter Raster über X ein Mikroraster in $\mathcal{K}$ (vergl. auch Definition 5.5. bzw. Bemerkung 5.7.). Offenkundig ist auch jeder Cauchyraster in $\mathcal{K}$ sowohl ein Pervinraster als auch ein Toporaster. Jeder Pervinraster in $\mathcal{K}$ ist ein Mikroraster und jeder Toporaster in $\mathcal{K}$ ist ein Császárraster. Jeder Ultrafilter über X ist ein Toporaster in $\mathcal{K}$ unter Anwendung von (sK_3) (vergl. Definition 3.7.). Für pseudosyntopologische Konvergenzräume sind sowohl die Begriffe Mikroraster und Pervinraster als auch die Begriffe Császárraster und Toporaster äquivalent (vergl. auch Bemerkung 3.10.).

6.3. Bemerkung. Werden gewisse Symmetriebedingungen an eine syntopogene Konvergenzstruktur gestellt, so wird im Folgenden bewiesen, daß auch die Begriffe Pervinraster und Cauchyraster äquivalent sind.

6.4. Definitionen. $(X, \mathcal{K})$ sei ein pseudosyntopogener Konvergenzraum. $\mathcal{K}$ heißt

(i) *schwach symmetrisch*, wenn gilt: $\forall\, \Omega \in \mathcal{K}\, \exists\, A \in \mathcal{K}\, A \leq \Omega_{ss}$;
(ii) *punktsymmetrisch*, wenn gilt: $\forall\, \Omega \in \mathcal{K}\, \exists\, A \in \mathcal{K}\, A \leq \Omega_{ps}$;
(iii) *symmetrisch*, wenn gilt: $\forall\, \Omega \in \mathcal{K}\, \exists\, A \in \mathcal{K}\, A \leq \Omega_{d}$ (vergl. Resultat 3.6. (1)).

6.5. Lemma. Jeder symmetrische pseudosyntopogene Konvergenzraum $(X, \mathcal{K})$ ist punktsymmetrisch.

Beweis. Sei $\Omega \in \mathcal{K}$, so existiert nach Voraussetzung eine Syntopoform $A \in \mathcal{K}$ mit $A \leq \Omega_d$. Wähle nach Axiom (sK_1) eine Syntopoform $A' \in \mathcal{K}$ mit $A' \leq \Omega \wedge A$. Wir zeigen: $A' \leq \Omega_{ps}$. Sei dazu $t' \in A'$, so gilt $t' \leq t \cap l$ für geeignete $t \in \Omega$ und $l \in A$; mithin folgt $l \leq b_d$ für $b \in \Omega$ nach Voraussetzung. Für $t, b \in \Omega$ existiert ein topoformer Operator $h \in \Omega$ mit $t \cup b \leq h$. Wir beweisen nun $t' \leq h_{ps}$. Für $A \in \mathcal{P}X$ sei $B \in t'(A)$, so folgen die Aussagen $B \in t(A)$ und $B \in l(A)$, mithin gilt $B \in h(A)$. Sei nun $x \in X$ mit $x \in X \setminus B$; wir zeigen: $X/A \in h(\{x\})$ (vergl. Resultate 3.4.). Nun gilt: $X/A \in b(X \setminus B)$, weil die Aussage $B \in l(A)$ erfüllt ist, mithin folgt $X/A \in h(X/B)$. Da nach Axiom (tf_2) $h(X/B) \subset h(\{x\})$, gilt $X/A \in$ $\in h(\{x\})$, und damit folgt insgesamt $B \in h_{ps}(A)$.

6.6. Lemma. Jeder punktsymmetrische pseudosyntopogene Konvergenzraum ist schwach symmetrisch.
Beweis: Analog zum Vorherigen!

6.7. Korollar. Jeder symmetrische pseudosyntopogene Konvergenzraum ist schwach symmetrisch (vergl. vorangehende Lemmata).

6.8. Satz. *Für einen schwach symmetrischen syntopogenen Konvergenzraum $(X, \mathcal{K})$ ist jeder Pervinraster in $\mathcal{K}$ ein Cauchyraster.*

Beweis. $\mathcal{R}$ sei ein Pervinraster in $\mathcal{K}$. Wähle dazu eine Syntopoform $\Omega \in \mathcal{K}$. Nach Voraussetzung ist $\mathcal{K}$ schwach symmetrisch, und somit existiert eine Syntopoform $A \in \mathcal{K}$ mit $A \leq \Omega_{ss}$. Mit Axiom (sK_2) gibt es eine weitere Syntopoform $\Omega' \in \mathcal{K}$ mit $\Omega' \leq \Omega + A$. Sei nun $t' \in \Omega'$, so folgt $t' \leq t_1 + l$ für $t_1 \in \Omega$ und $l \in A$; damit gilt $l \leq t_{2_{ss}}$ für ein $t_2 \in \Omega$. Da Ω eine Syntopoform ist, wähle $t_3 \in \Omega$, so daß die Aussage $t_1 \cup t_2 \leq t_3$ gilt. Da $\mathcal{R}$ Pervinraster in $\mathcal{K}$ ist, gibt es zu t_3 einen Punkt $x \in X$ mit $\bigcap t_3(\{x\}) \supset R \in \mathcal{R}$. Für Punkte $z, y \in R$ und $B \in t'(\{z\})$ zeige $y \in B$ (vergl. Definition 6.1. (iii)). Zunächst existiert eine Menge $C \in l(\{z\})$ mit $B \in t_1(C)$ (vergl. Resultat 3.4. (6)). Es gelten die Aussagen: $C \in t_{3_{ss}}(\{z\})$ und $B \in t_3(C)$. Da $z \in \bigcap t_3(\{x\})$, folgt $x \in \bigcap t_{3_{ss}}(\{z\})$; andernfalls existiert eine Menge $D \in t_{3_{ss}}(\{z\})$ mit $x \in X/D$; daraus folgt $X/\{z\} \in t_3(\{x\})$ und $z \notin X/\{z\}$; damit gilt $z \notin \bigcap t_3(\{x\})$, was zum Widerspruch führt. Unsere obige Argumentation fortsetzend, folgt sofort $x \in C$; mithin ist $B \in t_3(\{x\})$. Da $y \in \bigcap \bigcap t_3(\{x\})$ folgt $y \in B$, und somit ist alles gezeigt!

6.9. Bemerkung. Raster, die „beliebig kleine Elemente" enthalten, dienen in der Arbeit des Verfassers (vergl. [7]) nun dazu, Kompaktheits- und Vollständigkeitsbegriffe einzuführen. Wir listen unter Hinzufügung eines neuen Begriffes diese noch einmal auf.

Für einen pseudosyntopogenen Konvergenzraum $(X, \mathcal{K})$ heißt K

(i) *kompakt*, wenn jeder Császárraster $\mathcal{K}$-konvergent ist;
(ii) *t-kompakt*, wenn jeder Toporaster $\mathcal{K}$-konvergent ist;
(iii) *präkompakt*, wenn jeder Császárraster ein Mikroraster ist;
(iv) *vollständig*, wenn jeder Mikroraster einen Adhärenzpunkt besitzt;
(v) *p-vollständig*, wenn jeder Pervinraster einen Adhärenzpunkt besitzt;
(vi) *c-vollständig*, wenn jeder Cauchyraster $\mathcal{K}$-konvergent ist.

Bei diesen Definitionen heißt ein Punkt $x \in X$ *Adhärenzpunkt* eines Rasters $\mathcal{R}$ über X, wenn gilt: Es existiert eine Syntopoform $\Omega \in \mathcal{K}$, so daß für jedes $t \in \Omega$ die Aussage $t(\{x\}) \subset \sec \mathcal{R}$ erfüllt ist. Im Folgenden seien die genannten Räume syntopogene Konvergenzräume.

Aufgrund von Bemerkung 6.2. ist jeder kompakte Raum t-kompakt. Beide Begriffe stimmen überein, sofern $\mathcal{K}$ eine syntopologische Konvergenzstruktur auf X ist. Jeder kompakte Raum ist präkompakt, wenn man beachtet, daß jeder $\mathcal{K}$-konvergente Raster ein Mikroraster ist. Jeder kompakte Raum ist vollständig (vergl. [7]). Jeder vollständige Raum ist p-vollständig und jeder p-vollständige Raum ist c-vollständig. Somit ist auch jeder kompakte Raum c-vollständig. Für syntopologische Konvergenzräume stimmen die Begriffe vollständig und p-vollständig überein. Für schwach symmetrische syntopogene Konvergenzräume stimmen die Begriffe p-vollständig und c-vollständig überein (vergl. Satz 6.8.). Schließlich sind für schwach symmetrische syntopologische Konvergenzräume die Begriffe vollständig, p-vollständig und c-vollständig äquivalent (vergl. Satz 6.8.). Beachte, daß in der genannten Arbeit (vergl. [7]) gezeigt wird, daß jeder präkompakte und vollständige Raum bereits kompakt ist.

7. Fundamentalkonzepte in topogenen Konvergenzräumen

7.1. Definitionen. (X, T) sei ein pseudotopogener Konvergenzraum. Ein Raster $\mathcal{R}$ über X heißt

(i) *Császárraster* in T, wenn gilt: $\forall A \in \sec \mathcal{R} \, \exists \mathcal{M} \in T(A) \, \mathcal{M} \ll \mathcal{R}$;
(ii) *Toporaster* in T, wenn gilt: $\forall A \in \sec \mathcal{R} \, \exists \mathcal{M} \in T(A) \cap \mathcal{M} \in \text{stack}\, \mathcal{R}$.

7.2. Bemerkung. Beachte, daß jeder Ultrafilter über X ein Toporaster in T ist bei Anwendung des Axioms (tK_3) in Definition 4.1.. Für einen pseudosyntopogenen Konvergenzraum $(X, \mathcal{K})$ ist jeder Toporaster in $\mathcal{K}$ ein Császárraster in $T_{\mathcal{K}}$ und damit erst recht jeder Cauchyraster in $\mathcal{K}$ (vergl. Bemerkung 6.2.). Offenkundig ist jeder Toporaster in T ein Császárraster. Die Begriffe Toporaster und Császárraster stimmen in den minorisierten pseudotopogenen Konvergenzräumen überein (zur Definition eines minorisierten Operators vergleiche [6] und beachte Beispiel 4.3. (ii)). In Bemerkung 4.4. haben wir für einen topogenen Konvergenzoperator T die „unterliegende“ Limitierung τ_T eingeführt. In Übereinstimmung mit den Bemerkungen 5.4., 5.7. „übertragen“ wir diese Definitionen für Raster. Wir sagen daher, daß ein Punkt $x \in X$ *T-Adhärenzpunkt* eines Rasters $\mathcal{R}$ über X heißt, wenn es einen Raster $\mathcal{F}$ über X gibt mit $\mathcal{R} \ll \mathcal{F}$ und $\mathcal{F} \in \tau_T(x)$.

7.3. Lemma. Für einen pseudotopogenen Konvergenzraum (X, T) sei $\mathcal{R}$ ein Raster über X. Dann sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:

(i) x ist T-Adhärenzpunkt von $\mathcal{R}$;

(ii) $\exists\ \mathcal{N} \in T(\{x\})$ mit stack $\mathcal{N} \subset$ sec $\mathcal{R}$.

(i) impliziert (ii): Wähle einen Raster $\mathcal{F}$ über X mit $\mathcal{R} \ll \mathcal{F}$ und $\mathcal{F} \in \in \tau_T(x)$. Es existiert $\mathcal{N} \in T(\{x\})$ mit $\mathcal{N} \ll \mathcal{F}$. Nun folgt stack $\mathcal{N} \subset$ stack $\mathcal{F} \subset$ sec $\mathcal{F} \subset$ sec $\mathcal{R}$, so daß die Behauptung gilt.

(ii) impliziert (i): Nach Voraussetzung wähle $\mathcal{N} \in T(\{x\})$ mit stack $\mathcal{N} \subset \subset$ sec $\mathcal{R}$. Setze $\mathcal{F}$: $\{F \cap R \mid F \in \mathcal{N}, R \in \mathcal{R}\}$, so ist $\mathcal{F}$ ein Raster über X, der die gewünschten Bedingungen erfüllt.

7.4. Satz. *Für einen topogenen Raum (X, T) sei $\mathcal{R}$ ein Császárraster in T. Dann sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:*

(i) *x ist T-Adhärenzpunkt von $\mathcal{R}$;*

(ii) $\mathcal{R} \in \tau_T(x)$.

Beweis. (ii) impliziert (i): Trivial.

(i) impliziert (ii): Nach Voraussetzung wähle unter Beachtung von Lemma 7.3. $\mathcal{N} \in T(\{x\})$ mit stack $\mathcal{N} \subset$ sec $\mathcal{R}$. Mit Axiom (tK_7) in Definition 4.1. wähle $\mathcal{N}^+ \in T(\{x\})$ mit den dortgenannten Eigenschaften. Ziel: $\mathcal{N}^+ \ll \mathcal{R}$. Sei $F^+ \in \mathcal{N}^+$, so existiert eine Menge $E^+ \subset X$ mit den Eigenschaften $E^+ \in$ stack $\mathcal{N}$ und $F^+ \in \bigcap \{\text{stack } \mathcal{M} \mid \mathcal{M} \in T(E^+)\}$. Es folgt $E^+ \in$ sec $\mathcal{R}$. Da $\mathcal{R}$ Császárraster in T ist, wähle $\mathcal{M}' \in T(E^+)$ mit $\mathcal{M}' \ll \mathcal{R}$. Wir haben $F^+ \in$ stack $\mathcal{M}'$ und stack $\mathcal{M}' \subset$ stack $\mathcal{R}$, was die Behauptung beweist (vergl. auch [5]).

7.5. Definitionen. Für einen pseudotopogenen Konvergenzraum (X, T) und einen Punkt $x \in X$ heißt eine Menge $U \subset X$ *T-Umgebung von x*, wenn gilt: $U \in \bigcap \{\text{stack } \mathcal{N} \mid \mathcal{N} \in T(\{x\})\}$. Setze: $\mathcal{U}_T(x) := \{U \subset X \mid U$ ist T-Umgebung von $x\}$, so heißt $\mathcal{U}_T(x)$ *T-Umgebungssystem von x*.

7.6. Lemma. **Für einen pseudotopogenen Konvergenzraum (X, T), welcher zusätzlich das Axiom (tK_8) in Definition 4.1. erfüllt** und einen Raster $\mathcal{R}$ über X sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:

(i) $\mathcal{R} \in \tau_T(x)$;
(ii) $\mathcal{U}_T(x) \subset \text{stack } \mathcal{R}$.

Beweis. (ii) impliziert (i): Nach (tK_8) gilt insbesondere $\mathcal{E} := \vee \{\mathcal{N} | \mathcal{N} \in T(\{x\})\} \in T(\{x\})$. Nun gilt $\mathcal{E} \subset \text{stack } \mathcal{E} = \bigcap \{\text{stack } \mathcal{N} | \mathcal{N} \in T(\{x\})\}$; mithin folgt $\mathcal{E} \subset U_T(x)$, und nach Voraussetzung gilt $U_T(x) \subset$ stack R; damit ist $\mathcal{E} \subset \text{stack } \mathcal{R}$, also $\mathcal{R} \in \tau_T(x)$.

(i) impliziert (ii): Für $U \in \mathcal{U}_T(x)$ gilt $U \in \bigcap \{\text{stack } \mathcal{N} | \mathcal{N} \in T(\{x\})\} = \text{stack } \vee \{\mathcal{N} | \mathcal{N} \in T(\{x\})\} = \text{stack } \mathcal{E}$. Nach Voraussetzung existiert $\mathcal{N}' \in T(\{x\})$ mit $\mathcal{N}' \ll \mathcal{R}$. Da $\mathcal{E} \ll \mathcal{N}'$ folgt $\mathcal{E} \ll \mathcal{R}$, und damit gilt $U \in \text{stack } \mathcal{R}$.

7.7. Lemma. **Für einen pseudotopogenen Konvergenzraum (X, T), welcher zusätzlich das Axiom (tK_8) in Definition 4.1. erfüllt und einen Raster $\mathcal{R}$ über X sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:**

(i) x ist T-Adhärenzpunkt von $\mathcal{R}$;

(ii) $\mathcal{U}_T(x) \subset \sec \mathcal{R}$.

Beweis. Analog!

7.8. Definition. (X, T) sei ein pseudotopogener Konvergenzraum. T heißt

(i) *kompakt*, wenn gilt: Jeder Császárraster $\mathcal{R}$ in T ist T-konvergent;

(ii) *t-kompakt*, wenn gilt: Jeder Toporaster $\mathcal{R}$ in T ist T-konvergent (vergl. in diesem Zusammenhang auch die Bemerkungen 4.4., 5.4. und 5.7.).

7.9. Bemerkung. Offensichtlich ist jeder kompakte Raum t-kompakt. **Mit Bemerkung 7.2. stimmen beide** Kompaktheitsbegriffe in minorisierten **pseudotopogen Konvergenzräumen** überein (vergl. auch Beispiel 4.3. (ii)). Für eine endliche Menge ist jeder pseudotopogene Konvergenzoperator kompakt.

7.10. Satz. *Für einen topogenen Raum (X, T) sind die folgenden Aussagen äquivalent:*

(i) *T ist kompakt;*

(ii) *Jeder Raster $\mathcal{R}$ über X besitzt einen T-Adhärenzpunkt.*

Beweis. (i) impliziert (ii): Sei $\mathcal{R}$ ein Raster über X. Wähle einen Ultrafilter $\mathcal{F}$ über X mit $\mathcal{R} \subset \mathcal{F}$; $\mathcal{F}$ ist ein Császárraster nach Bemerkung 7.2. und damit T-konvergent gegen ein $x \in X$. Nach Definition ist damit x T-Adhärenzpunkt von $\mathcal{R}$.

(ii) impliziert (i): Für einen Császárraster $\mathcal{R}$ in T sei $x \in X$ eine T-Adhärenzpunkt von $\mathcal{R}$. **Mit Satz 7.4. gilt dann die Behauptung.**

7.11. SATZ. *Für einen topogenen Raum* (X, T) *sind die folgenden Aussagen äquivalent:*

(i) *T ist kompakt;*
(ii) *Jeder Raster über X besitzt einen T-Adhärenzpunkt;*
(iii) *Für $x \in X$ sei U_x eine T-Umgebung von x, so gibt es eine endliche Menge E mit $X = \bigcup\{U_{x_i} | i \in E\}$.*

BEWEIS. Nach Satz 7.10. sind die ersten beiden Aussagen bereits äquivalent. Wir zeigen, daß die Aussagen (ii) und (iii) äquivalent sind.

(ii) impliziert (iii): Sei U_x eine T-Umgebung von x für jedes $x \in X$. Angenommen für jede endliche Teilmenge $E \subset X$ gilt $X \not\subset \bigcup\{U_x | x \in E\}$; setze: $\mathcal{R} := \{X / \bigcup\{U_x | x \in E\} | E \subset X,\ E \text{ endlich}\}$, so ist $\mathcal{R}$ ein Raster über X. Sei $z \in X$ T-Adhärenzpunkt von $\mathcal{R}$, so folgt mit Lemma 7.7. $U_z \in \sec \mathcal{R}$; d.h. $U_z \cap (X/U_z) \neq \emptyset$, was einen Widerspruch zur Folge hat.

(iii) impliziert (ii): Angenommen es gäbe einen Raster $\mathcal{R}$ über X, der keinen T-Adhärenzpunkt besitzt. Für $x \in X$ sei $F_x \in \mathcal{U}_T(x)$ mit $X/F_x \supset R_x \in \mathcal{R}$ (vergl. Lemma 7.7.), so ist X/R_x eine T-Umgebung von x. Nach Voraussetzung existiert eine endliche Teilmenge E mit $X = \bigcup\{X/R_{x_i} | i \in E\} = X / \bigcap \{R_{x_i} | i \in E\}$. $\bigcap \{R_{x_i} | i \in E\} \supset R \in \mathcal{R}$, was einen Widerspruch zur Folge hat.

8. Anwendungen

8.1. SATZ. *$(X, \mathcal{K})$ sei ein syntopologischer Konvergenzraum. Für $T_{\mathcal{K}}$ kompakt ist $\mathcal{K}$ präkompakt und c-vollständig.*

BEWEIS. Sei $\mathcal{R}$ Cauchyraster in $\mathcal{K}$, so ist mit Bemerkung 7.2. $\mathcal{R}$ ein Császárraster in $T_{\mathcal{K}}$, der nach Voraussetzung $T_{\mathcal{K}}$-konvergent gegen ein $x \in X$ ist. Mit Satz 5.6. ist $\mathcal{R}$ $\mathcal{K}$-konvergent gegen x (vergl. auch Definition 5.5.). Damit ist $\mathcal{K}$ c-vollständig. Sei nun weiter $\mathcal{R}$ ein Raster über X, so gibt es einen Ultrafilter $\mathcal{F}$ mit $\mathcal{R} \subset \mathcal{F}$. Als Ultrafilter ist $\mathcal{F}$ ein Császárraster in $T_{\mathcal{K}}$ (vergl. Bemerkung 7.2.) und ist nach Voraussetzung $T_{\mathcal{K}}$-konvergent gegen ein $x \in X$. Mit Satz 5.6. ist $\mathcal{F}$ $\mathcal{K}$-konvergent gegen x. Somit ist $\mathcal{F}$ ein Mikroraster in $\mathcal{K}$ (vergl. Bemerkung 6.2.). Dies bedeutet jedoch, daß $\mathcal{K}$ präkompakt ist (siehe auch [7]).

8.2. SATZ. *Für einen syntopologischen Raum $(X, \mathcal{K})$ ist $(X, T_{\mathcal{K}})$ ein topogener Raum.*

BEWEIS. Wegen Satz 5.3. genügt es zu zeigen, daß $T_{\mathcal{K}}$ das Axiom (tK_7) in Definition 4.1. erfüllt (trivialerweise ist $T_{\mathcal{K}}$ erzeugt, vergl. Bemerkung im Beweis zu 5.3. Satz). Sei $\mathcal{M} \in T_{\mathcal{K}}(A)$, so gibt es eine Synheloform $\Omega \in \mathcal{K}$ mit $\mathcal{M}_{\Omega}^{A} \ll \mathcal{M}$; Λ bezeichne einen Erzeuger von $\mathcal{K}$. Es folgen $\Lambda \leq \Lambda^2$ (vergl. Resultat 3.11. (iii)) und $\Lambda \leq \Omega$. Da $\mathcal{M}_{\Lambda}^{A} \in T_{\mathcal{K}}(A)$, sei nun $\bigcap l(A) \in \mathcal{M}_{\Lambda}^{A}$ für ein $l \in \Lambda$, so gilt $l \leq l'^2$ für ein $l' \in \Lambda$ und weiter $l' \leq t$ für ein $t \in \Omega$. Wir haben $\bigcap l(A) \in l(A)$, was die Existenz einer Menge $E \subset X$ zur Folge hat mit den Eigenschaften $E \in l'(A)$ und $\bigcap l(A) \in l'(E)$. Es folgt $E \in t(A)$, und da $\bigcap l(A) \supset F \in \mathcal{M}$, gilt $E \in \operatorname{stack} \mathcal{M}$.

Sei $\mathcal{M} \in T_{\mathcal{K}}(E)$, so existiert eine Synholoform $\Omega' \in \mathcal{K}$ mit $\mathcal{M}^E_{\Omega'} \ll \mathcal{M}$; wir zeigen $\cap l(A) \in$ stack $\mathcal{M}$. Da $A \leq \Omega'$, wähle $t' \in \Omega'$ mit $l' \leq t'$, so folgt $\cap l(A) \in t'(E)$ unter Beachtung der Aussage $\cap l(A) \in l'(E)$. Somit $\cap t'(E) \supset$ $\supset F \in \mathcal{M}$, so daß $\cap l(A) \in$ stack $\mathcal{M}$ gilt.

8.3. SATZ. *Für einen schwach symmetrischen syntopologischen Raum* $(X, \mathcal{K})$ *sind die folgenden Aussagen äquivalent:*

(i) $\mathcal{K}$ *ist präkompakt und c-vollständig;*
(ii) $T_{\mathcal{K}}$ *ist kompakt.*

BEWEIS. (ii) impliziert (i): vergl. Satz 8.1..

(i) impliziert (ii): Sei $\mathcal{R}$ ein Császárraster in $T_{\mathcal{K}}$, so gibt es einen Mikroraster $\mathcal{M}$ in $\mathcal{K}$ mit $\mathcal{R} \ll \mathcal{M}$, da $\mathcal{K}$ nach Voraussetzung präkompakt ist. Nach Bemerkung 6.2. ist $\mathcal{M}$ ein Pervinraster in $\mathcal{K}$, der wegen Satz 6.8. sogar ein Cauchyraster ist. Da $\mathcal{K}$ c-vollständig ist, folgt $\mathcal{M}$ ist $\mathcal{K}$-konvergent und mit Satz 5.6. auch $T_{\mathcal{K}}$-konvergent. Somit besitzt $\mathcal{R}$ einen T-Adhärenzpunkt gegen diesen ist $\mathcal{R}$ $T_{\mathcal{K}}$-konvergent unter Beachtung der Sätze 7.4. beziehungsweise 8.2.. Damit ist aber $T_{\mathcal{K}}$ kompakt.

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[1] **Correction.** On page 115 of this paper the axiom (T08) should be explained as
(T08) $A \in \mathbf{P}X$ and $\varrho \in T(A)$ imply there exists $\varrho^* \in T(A)$ such that for each $F^* \in \varrho^*$ we can find $E \subset X$ and $\varepsilon \in T(E)$ with the properties $E \in$ stack ϱ and $F^* \in$ stack ε.

ON SUCCESSIVE COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS OF ORDER β

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(Received October 2, 1985)

1. Introduction.

Let $K(\beta/k)$ denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{nk+1} z^{nk+1} \tag{1}$$

which are regular and satisfy the condition $|\arg Czf'(z)/g(z)| \leq \pi\beta/2$ in $|z|<1$ for some normalized starlike function g and some constant C. Then $f(z)$ is called close-to-convex of order β in $|z|<1$. It is known that if $\beta>1$ then $K(\beta/k)$ contains functions of arbitrary high valence, but for $0\leq\beta\leq 1$, $K(\beta/k)$ contains only univalent functions and that $K(1/k)$ is the ordinary class of k-fold close-to-convex functions ([2], [7]). It is also known that if $\beta = -1+ +m/2$ $(2\leq m<\infty)$ then, $K(-1+m/2)$ contains the class of functions of bounded boundary rotation at most πm which is usually denoted by V_m, [2].

In this paper we shall generalise recent results due to the author [5], HAMILTON [8] and LEUNG [12].

We shall mean by $A(k, \ldots)$ constants depending on $k, \ldots$ and not necessarily the same on each occurrence.

2. Coefficient difference

THEOREM 1. *Let $f\in K(\beta)$ and* (1) *holds. Then for $n\geq 1$ and $k\geq 1$ we have*

$$\big||a_{nk+1}| - |a_{(n-1)k+1}|\big| \leq A(\beta, k)\begin{cases} n^{\beta-2+1/k} & (\beta\geq 1), \\ n^{-1+1/k}, & (0\leq\beta<1). \end{cases}$$

The function $f'(z) = (1+z^{2k})^{\beta}/(1-z^{2k})^{\beta+1/k}$ shows that these estimates are the best possible.

We note that some related results have been obtained by several authors (see for example [3], [4], [14] and [15].

PROOF of THEOREM 1. Since there is no loss in generality by assuming $C = 1$ in our case then we may, in view of [2] and [10], write

$$zf'(z) = g(z)p(z)^{\beta},\ zg'(z) = g(z)G(z) \tag{2}$$

where $g(z)$ is k-fold symmetric and starlike and $|f'(0)| = |g'(0)| = 1$ so hat $\operatorname{Re} p(z) \geq 0$, $\operatorname{Re} G(z) \geq 0$ and $|p(0)| = |G(0)| = 1$. Now following [5] we set, since $g(z)$ is starlike, $g(z) = \psi(z^k)^{1/k}$ and deduce from [6, p. 193] for a fixed z_1 and all z such that $|z| = |z_1| = r$ that

$$|z^k - z_1^k|^{1/k}|g(z)| = \left(|z^k - z_1^k|\,|\psi(z^{1/k})|\right)^{1/k} \leq \left(2r^{2k}/(1-r^{2k})\right)^{1/k}. \tag{3}$$

We now prove the case $\beta \geq 1$:

We see from (1) and (2) that

$$(z^k - z_1^k)\frac{d}{dz}(zf'(z)) = (z^k - z_1^k)\frac{d}{dz}(g(z)p^{\beta}(z)) =$$

$$= -a_1 z_1^k + \sum_{n=0}^{\infty}[((n-1)k+1)^2 a_{(n-1)k+1} - (nk+1)^2 z_1^k a_{nk+1}]z^{nk}.$$

Applying the coefficient formula to this we deduce, by (2), (3), setting $r = [((n-1)k+1)/(nk+1)]^{2/k}$, that

$$||a_{nk+1}| - |a_{(n-1)k+1}|| \leq$$

$$\leq A(k)n^{-2}\max_{|z|=r}\left(|z^k - z_1^k|\,|g(z)|\right)\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{2}{1-r^k}|p(z)|^{\beta} + \frac{2^{\beta-1}}{1-r^k}|p'(z)|\right)d\Theta \leq$$

$$\leq A(k,\beta)/n^2(1-r^k)^{\beta+1/k} \leq A(k,\beta)n^{\beta-2+1/k},$$

as required, where we have used the distortion theorem and MacGregor's inequalities [13, p. 373].

To prove the second part we see in the same way from (1) and (3) that the coefficients of the expansion $(z^k - z_1^k)f'(z)$ are subject to the inequality

$$||a_{nk+1}| - |a_{(n-1)k+1}|| \leq \frac{A(k,\beta)}{n(1-r^k)^{1/k}}\int_0^{2\pi}|p(re^{i\Theta})|^{\beta}\,d\Theta \leq A(k,\beta)n^{-1+1/k},$$

since the integral involving p is bounded above for $0 \leq \beta < 1$. This completes the proof of Theorem 1.

3. Robertson's Conjecture

THEOREM 2. *Let $f \in K(\beta)$ and* (1) *hold for $k = 1$. Let also*

$$(1+z)^{\beta}/(1-z)^{1+\beta} = \sum_{n=0}^{\infty} A_n(\beta) z^n.$$

Then for $n \geq 1$ and $\beta \geq 1$ we have

$$\big|n|a_n| - (n-1)|a_{n-1}|\big| \leq A_n(\beta). \tag{4}$$

If $\beta = 1$ then $A_n(1) = 2n-1$ and this is due to LEUNG [11] and HAMILTON [8].

PROOF of THEOREM 2. We see from [2], [11] and (2) for $k = 1$ and some ξ such that $|\xi| = 1$, that

$$(1-\xi z)f'(z) = \big((1-\xi z)g(z)/z\big)p(z)^{\beta} \ll (1+z)^{\beta}/(1-z)^{1+\beta} \tag{5}$$

where $\ll$ means that if $\Sigma b_n z^n \ll \Sigma C_n z^n$ then $|b_n| \leq C_n$ as in ([2], [7], [10]). Theorem 2 is now an immediate consequence of (5).

4. Remarks

1. The case whether (4) holds for $0 \leq \beta < 1$ remains open. However one can use [16] (see also [2, p. 8]) to deduce, as above, that

$$\big|n|a_n| - (n-1)|a_{n-1}|\big| \leq 2n\beta - 1 \text{ for } n \geq 1.$$

2. Replacing β by $(m/2)-1$ in the above results we obtain the analogous estimates for the class V_m defined above.

3. It is clear that both theorems give partial answers to questions raised in [1], [4] and [9].

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ON THE CONJUGATE OF THE WALSH SERIES

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1. Introduction

An interesting theorem bearing on the conjugate trigonometric Fourier series of a function f is the following theorem, due to F. LUKÁCS [1] (see also [2], [3]).

THEOREM. *If $f(x_0 \pm 0)$ exist, and if $f(x_0+0)-f(x_0-0) = d$, then for the partial sums of the conjugate trigonometric Fourier series the following is satisfy:*

$$\lim_{n\to\infty} \frac{(\tilde{S}_n f)(x_0)}{\log n} = \frac{-d}{\pi}.$$

The theorem yields an interesting means of determining the jump of the function f.

In this paper we give an analogue for this theorem in the Walsh case.

I express my acknowledgements to Professor F. SCHIPP for his useful advices and for his attention paid to my work.

2. Definitions and properties

We recall some definitions and properties of Walsh functions.

Let r_n be the n-th Rademacher function. For any nonnegative integer n, with $n = \sum_{j=1}^{\infty} n_j 2^j$, $n_j = 0$ or 1, the n-th Walsh function is defined by [4]

$$w_n = \prod_{j=0}^{\infty} r_j^{n_j}.$$

If $x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}$ and $t = \sum_{i=0}^{\infty} \frac{t_i}{2^{i+1}}$, $x_i, t_i = 0$ or 1, let $x \dotplus t = \sum_{i=0}^{\infty} \frac{|x_i - t_i|}{2^{i+1}}$.

Then $w_n(x \dotplus t) = w_n(x) w_n(t)$, whenever $x \dotplus t$ not belongs to the set of dyadic rationals in the interval $[0, 1)$.

The Walsh-Fourier series of an $f\in L^1[0,1)$ is the Walsh series

$$Sf=\sum_{k=0}^{\infty}\hat{f}(k)w_k,$$

where $\hat{f}(k)=\int_0^1 f(t)w_k(t)dt$ are the Walsh-Fourier coefficients.

The n-th partial sums of this series will be denoted by

$$S_nf=\sum_{k=0}^{n-1}\hat{f}(k)w_k,\quad n\in\mathbf{P}=\{1,2,\ldots\}.$$

Notice for $n\in\mathbf{P}$ and $x\in[0.1)$, that

$$(S_nf)(x)=\sum_{k=0}^{n-1}\left(\int_0^1 f(t)w_k(t)dt\right)w_k(x)=\int_0^1 f(t)D_n(x\dot{+}t)dt,$$

where $D_n=\sum_{k=0}^{n-1}w_k$ is the n-th Dirichlet kernel.

Let $A_i=\{n\in\mathbf{N}|2^i\leq n<2^{i+1}\}$, $i\in\mathbf{N}=\mathbf{P}\cup\{0\}$, $\alpha\in[0,1)$ with binary coefficients $(\alpha_i,\ i\in\mathbf{N})$. The conjugate series $\tilde{S}^\alpha f$ for fixed α is define as

$$\tilde{S}^\alpha f\sim-\sum_{n\in P_{(\alpha)}}\hat{f}(n)w_n+\sum_{n\in N_{(\alpha)}}\hat{f}(n)w_n,$$

where $P_{(\alpha)}=\bigcup_{\alpha_i=1}A_i$ and $N_{(\alpha)}=\bigcup_{\alpha_i=0}A_i$.

An elementary calculation shows, that the conjugate series $\tilde{S}^\alpha f$ can be written in the following form

$$\tilde{S}^\alpha f\sim\sum_{k=0}^{\infty}(-1)^{\alpha_k}\Delta_kf,$$

where $\Delta_kf=E_{k+1}f-E_kf$ $(k\in\mathbf{N})$ and E_kf denotes the 2^k-th partial sum of the Walsh-Fourier series of f. The 2^n-th partial sums of this series will be denoted by

$$\tilde{S}^\alpha_{2^n}f=\sum_{k=0}^{n-1}(-1)^{\alpha_k}\Delta_kf.$$

Then

$$(1)\qquad (\tilde{S}^\alpha_{2^n}f)(x)=\int_0^1 f(t)\tilde{D}^\alpha_{2^n}(t\dot{+}x)dt.$$

where

$$\widetilde{D}^{\alpha}_{2^n} = \sum_{k=0}^{n-1} (-1)^{\alpha_k} r_k D_{2^k} \tag{2}$$

is the 2^n-th conjugate Direchlet kernel.

In the following lemma we give a pointwise estimates for the $\widetilde{D}^{\alpha}_{2^n}$.

LEMMA 1. If $x \in (0, 1)$ and $n \in \mathbf{N}$, then

$$|\widetilde{D}^{\alpha}_{2^n}(x)| \leqq \frac{2}{x}. \tag{3}$$

PROOF. From (2) $|\widetilde{D}^{\alpha}_{2^n}(x)| \leqq \sum_{k=0}^{n-1} D_{2^k}(x)$. We choose $j \in \mathbf{N}$ such that $2^{-j} \leqq x < 2^{-j+1}$.

Since

$$D_{2^k}(x) = \begin{cases} 2^k, & \text{if } 0 \leqq x < 2^{-k}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then $|\widetilde{D}^{\alpha}_{2^n}(x)| \leqq \sum_{k=0}^{j-1} 2^k < 2^j$. Since $2^j < \frac{2}{x}$, the proof is complete.

From (4) it follows that, in the case $x \neq 0$, the series $\sum_{k=0}^{\infty} (-1)^{\alpha_k} r_k(x) D_{2^k}(x)$ contains only a finite many terms not equal to zero.

COROLLARY. If $x \neq 0$, then there exists $\widetilde{D}^{\alpha}(x) = \lim_{n \to \infty} \widetilde{D}^{\alpha}_{2^n}(x)$ and $|\widetilde{D}^{\alpha}(x)| \leqq \leqq \frac{2}{x}$.

In the next lemma we give an upper and lower estimate for the conjugate Lebesque constants by using the variations

$$V_n(\alpha) = \sum_{k=0}^{n-1} |\alpha_k - \alpha_{k+1}| + \alpha_0 \quad (n \in \mathbf{P})$$

of the sequence $\alpha = (\alpha_k,\ k \in \mathbf{N})$.

LEMMA 2. There is an absolute constant $c > 0$ such that

$$2cV_n(\alpha) - 2 \leqq \int_0^1 |\widetilde{D}^{\alpha}_{2^n}(t)|\,dt \leqq 2V_n(\alpha) + 2 \quad (n \in \mathbf{P}). \tag{5}$$

PROOF. In the proof we shall use the following identity fot the m-th Walsh-Fourier kernel:

$$w_m D_m = \sum_{k=0}^{n-1} m_k r_k D_{2^k} = \sum_{k=0}^{n-1} \frac{1-(-1)^{m_k}}{2} r_k D_{2^k},$$

where $m = \sum_{k=0}^{n-1} m_k 2^k$, $m_k = 0$ or 1 (see [4]). Then

$$w_m D_m = \frac{1}{2} \sum_{k=0}^{n-1} r_k D_{2^k} - \frac{1}{2} \sum_{k=0}^{n-1} (-1)^{m_k} r_k D_{2^k}. \tag{6}$$

Since

$$D_{2^n} = \sum_{j=0}^{2^n-1} w_j = 1 + \sum_{k=0}^{n-1} \sum_{j=2^k}^{2^{k+1}-1} w_j = 1 + \sum_{k=0}^{n-1} r_k D_{2^k},$$

from (6), we have

$$w_m D_m = \frac{1}{2}(D_{2^n} - 1) - \frac{1}{2} \sum_{k=1}^{n-1} (-1)^{m_k} r_k D_{2^k}.$$

Thus on the basis of (4) we get

$$2\int_0^1 |D_m| - 2 \leqq \int_0^1 \left| \sum_{k=0}^{m-1} (-1)^{m_k} r_k D_{2^k} \right| \leqq 2 \int_0^1 |D_m| + 2.$$

The Lebesgue constants $L_m = \int_0^1 |D_m|$ can be estimate by the variation

$$V(m) = \sum_{k=0}^{m-1} |m_{k+1} - m_k| + m_0$$

of the number m in the following way:

$$c\ V(m) \leqq L_m \leqq V(m),$$

where c is an absolute constant (see [4]). Using this formula for $m = \sum_{k=0}^{n-1} \alpha_k 2^k$ we get (5).

3. Proof of the theorem

To formulate the analogue of Lukács's theorem we consider the characteristic function of the interval $[0, \alpha)$,

$$F_\alpha = \chi_{[0,\alpha)}, \quad \alpha \in [0, 1).$$

The Walsh series of F_α is

$$SF_\alpha = \sum_{k=0}^{\infty} J_k(\alpha) w_k.$$

where $J_k(\alpha)$ represents the indefinite integral of w_k for $k \in \mathbf{N}$ (see [4]).

For the Walsh-Fourier coefficients $J_k(\alpha)$, we have

(7) $$J_{2^k+j}(\alpha) = w_j(\alpha) J_{2^k}(\alpha) \quad (0 \leqq j < 2^k,\ k \in \mathbf{N})$$

and

(8) $$2^k J_{2^k}(\alpha) = J_1(2^k \alpha) \quad (k \in \mathbf{N},\ \alpha \geqq 0).$$

The m-th partial sum is

$$(S_m F_\alpha)(x) = \sum_{k=0}^{m-1} J_k(\alpha) w_k(x) = \int_0^\alpha D_m(t \dotplus x)\,dt.$$

On the basis of (7) the 2^n-th partial sum of the conjugate series of F_α can be written in the form

$$(\tilde{S}_{2^n}^a F_\alpha)(x) = \sum_{k=0}^{n-1} (-1)^{a_k} \Delta_k f = \sum_{k=0}^{n-1} (-1)^{a_k} \sum_{j=0}^{2^k-1} J_{2^k+j}(\alpha) w_{2^k+j}(x) =$$

$$= \sum_{k=0}^{n-1} (-1)^{a_k} r_k(x) J_{2^k}(\alpha) D_{2^k}(\alpha \dotplus x).$$

Hence by (4) and (8)

$$(\tilde{S}_{2^n}^a F_\alpha)(\alpha) = \sum_{k=0}^{n-1} J_1(2^k \alpha).$$

Let

(9) $$\lambda_n(\alpha) = \sum_{k=0}^{n-1} J_1(2^k \alpha).$$

Obviously, since $0 \leqq J_1(x) \leqq \frac{1}{2}$ for each $x \in [0, 1)$, $\lambda_n(\alpha) = 0(n)$.

In the next lemma we give an upper and lower estimate for $\lambda_n(\alpha)$ using the variations $V_n(\alpha)$.

LEMMA 3. For fixed $\alpha \in [0, 1]$, we have

(10) $$\frac{1}{4}\left(V_n(\alpha) - 1\right) \leqq \lambda_n(\alpha) \leqq V_n(\alpha) + 1.$$

Proof. Using the binary expansion of α,

$$\alpha = \sum_{j=0}^{\infty} \frac{\alpha_j}{2^{j+1}} \quad (\alpha_j = 0 \text{ or } 1,\ j \in \mathbf{N}),$$

and Abel's transformation, we have

$$\alpha = \alpha_0 + \sum_{j=1}^{\infty} \frac{\alpha_j - \alpha_{j-1}}{2^j}. \tag{11}$$

For the function J_1

$$J_1(\alpha) = \begin{cases} \alpha, & \text{if } \alpha_0 = 0, \\ 1-\alpha, & \text{if } \alpha_0 = 1, \end{cases}$$

hold. Applying (11) for α and

$$1-\alpha = \sum_{j=0}^{\infty} \frac{1-\alpha_j}{2^{j+1}}$$

we get

$$J_1(\alpha) = (-1)^{\alpha_0} \sum_{j=1}^{\infty} \frac{\alpha_j - \alpha_{j-1}}{2^j},$$

and consequently

$$J_1(2^k\alpha) = (-1)^{\alpha_k} \sum_{j=1}^{\infty} \frac{\alpha_{j+k} - \alpha_{j+k-1}}{2^j}.$$

From (9), we have

$$\lambda_n(\alpha) \leqq \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} \frac{|\alpha_{j+k} - \alpha_{j+k-1}|}{2^j} = \sum_{k=0}^{n-1} \left(\left(\sum_{j=1}^{n-k} + \sum_{j=n-k+1}^{\infty} \right) \frac{|\alpha_{j+k} - \alpha_{j+k-1}|}{2^j} \right) \leqq$$

$$\leqq \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-k} \frac{|\alpha_{j+k} - \alpha_{j+k-1}|}{2^j} + \frac{1}{2^{n-k}} \right) \leqq \sum_{j=1}^{n} \left(\frac{1}{2^j} \sum_{k=0}^{n-j} |\alpha_{j+k} - \alpha_{j+k-1}| \right) +$$

$$+ 1 \leqq V_n(\alpha) \sum_{j=1}^{n} \frac{1}{2^j} + 1 \leqq V_n(\alpha) + 1.$$

To find a lower estimate for $\lambda_n(\alpha)$, we define the function G as follows:

$$G(\alpha) = \begin{cases} 0, & \text{if } 0 \leqq \alpha < \dfrac{1}{4} \text{ or } \dfrac{3}{4} \leqq \alpha < 1, \\ \dfrac{1}{4}, & \text{if } \dfrac{1}{4} \leqq \alpha < \dfrac{3}{4}, \end{cases}$$

and let $G(\alpha) = G(\alpha+1)$ $(\alpha > 0)$.

Then G can be expressed by the digits of α in the following way:

$$G(\alpha) = \frac{\alpha_0 \dotplus \alpha_1}{4} = \frac{|\alpha_1 - \alpha_0|}{4}.$$

Obviously $G(\alpha) \leqq J_1(\alpha)(\alpha \in [0,1))$ and consequently

$$\frac{|\alpha_{k+1} - \alpha_k|}{4} = G(2^k\alpha) \leqq J_1(2^k\alpha).$$

Therefore,

$$\lambda_n(\alpha) = \sum_{k=0}^{n-1} J_1(2^k\alpha) \geqq \sum_{k=0}^{n-1} G(2^k\alpha) \geqq \frac{1}{4}(V_n(\alpha) - 1).$$

Thus the proof of the Lemma is completed.

The sequence $(\lambda_n(\alpha),\ n \in \mathbf{P})$ and the Walsh conjugate series $\tilde{S}^\alpha f$ can be used to determine the jump of the function f at the point α. Namely the following analogue of the Lukács's theorem is true.

THEOREM. *Let $\alpha \in [0,1)$ and $\lim_{n\to\infty} V_n(\alpha) = +\infty$. If f has a jump d at α then*

$$\lim_{n\to\infty} \frac{(\tilde{S}^\alpha_{2^n} f)(\alpha)}{\lambda_n(\alpha)} = -d.$$

PROOF. First we suppose that $d = 0$, i.e., f is continuous in the usual sense at α. Then we have to show that

$$\lim_{n\to\infty} \frac{(\tilde{S}^\alpha_{2^n} f)(\alpha)}{\lambda_n(\alpha)} = 0.$$

We may suppose that $f(\alpha) = 0$. Then for every $\varepsilon > 0$ there eixsts a dyadic interval $I = [\beta, \gamma)$ with lenght 2^{-s} containing α such that $|f(t)| < \varepsilon$ whenever $t \in I$.

We have by (1),

$$\frac{(\tilde{S}^\alpha_{2^n} f)(\alpha)}{\lambda_n(\alpha)} = \int_0^1 f(t) \frac{\tilde{D}^\alpha_{2^n}(t \dotplus \alpha)}{\lambda_n(\alpha)}\, dt = \int_0^\beta + \int_I + \int_\gamma^1 = I_1 + I_2 + I_3.$$

To estimate I_2 we use

$$|I_2| = \left| \int_I f(t) \frac{\tilde{D}^\alpha_{2^n}(t \dotplus \alpha)}{\lambda_n(\alpha)}\, dt \right| \leqq \varepsilon \int_I \frac{|\tilde{D}^\alpha_{2^n}(t \dotplus \alpha)|}{\lambda_n(\alpha)}\, dt.$$

Using (5) and (10), we obtain the following inequality

$$|I_2| \leqq \frac{8\varepsilon(V_n(\alpha) + 1)}{(V_n(\alpha) - 1)} \leqq \varepsilon K,$$

where K is an absolute constant.

To estimate I_1, and I_3 we consider

$$|I_1| = \left|\int_0^\beta f(t)\frac{\widetilde{D}^\alpha_{2^n}(t \dotplus \alpha)}{\lambda_n(\alpha)}\,dt\right| \leqq \int_0^\beta |f(t)|\,\frac{|\widetilde{D}^\alpha_{2^n}(t \dotplus \alpha)|}{\lambda_n(\alpha)}\,dt.$$

From (3),

$$|\widetilde{D}^\alpha_{2^n}(t \dotplus \alpha)| \leqq \frac{2}{t \dotplus \alpha} \leqq \frac{2}{2^{-s}} \quad (0 \leqq t < \beta). \tag{12}$$

Using (10) and (12) we obtain

$$|I_1| \leqq \frac{8\cdot 2^s}{(V_n(\alpha)-1)}\int_0^1 |f(t)|\,dt.$$

Since $\lim_{n\to\infty} V_n(\alpha) = +\infty$, then $|I_1|$ tends to zero as $n\to\infty$.

In similar way $|I_3|$ tends to zero as $n\to\infty$. Therefore

$$\lim_{n\to\infty}\frac{(\widetilde{S}^\alpha_{2^n}f)(\alpha)}{\lambda_n(\alpha)} = 0.$$

The case $d\neq 0$ can be reduced to the preceding one by considering the function $g = f + d\chi_{[0,\alpha)}$.

Then g is continuous at α and consequently

$$\frac{(\widetilde{S}^\alpha_{2^n}g)(\alpha)}{\lambda_n(\alpha)} = \frac{(\widetilde{S}^\alpha_{2^n}f)(\alpha)}{\lambda_n(\alpha)} + d\,\frac{(\widetilde{S}^\alpha_{2^n}F_\alpha)(\alpha)}{\lambda_n(\alpha)}$$

tends to zero as $n\to\infty$.

Since from (9)

$$(\widetilde{S}^\alpha_{2^n}F_\alpha)(\alpha) = \lambda_n(\alpha).$$

This completes the proof.

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ON NON-LINEAR ELLIPTIC EQUATIONS IN AN UNBOUNDED DOMAIN WITH QUADRATIC GROWTH CONDITIONS

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(Received September 9, 1986)

0. Introduction

Authors [1] proved existence of solutions for a quasi-linear equation in a bounded domain Ω of $\mathbf{R}^n$ with quadratic growth conditions, by giving L^∞-estimates for the approximate solution.

The aim of this paper is to prove existence of weak solutions for a non-linear equation in an unbounded domain Ω of $\mathbf{R}^n$ by using methods which are analogous to [1]. Further we extend the result to prove some results on stability of solutions.

ACKNOWLEDGEMENT. The author is thankful to Prof. DR. LÁSZLÓ SIMON for useful discussions and suggestions during the prepartion of this paper.

1. Existence theorem

Let Ω be an unbounded domain of $\mathbf{R}^n$ and $\partial\Omega$ be its boundary which is bounded and smooth. The following type of problem is considered.

$$(1.1) \qquad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [a_i(x, u, \operatorname{grad} u)] + a_0 u + f(x, u, \operatorname{grad} u) = 0 \text{ in } \Omega,$$

$$(1.2) \qquad u = 0 \text{ on } \partial\Omega.$$

Here a_i's are considered as functions of x belonging to Ω, as well as of $\eta\in\mathbf{R}$, and $\xi\in\mathbf{R}^n$. The coefficients a_i's and a_0 satisfy the under mentioned conditions, called Carathéodory conditions:

(1.3) $(\eta, \xi)\to a_i(x, \eta, \xi)$ are continuous over $\mathbf{R}\times\mathbf{R}^n$ for a.e. $x\in\Omega$.
(1.4) $x\to a_i(x, \eta, \xi)$ are measurable over Ω for all $\eta\in\mathbf{R}$, $\xi\in\mathbf{R}^n$.
(1.5) a_0 is measurable in Ω.

Further we suppose that $\exists$ a constant $c_1>0$, a function $c_2\in L^2(\Omega)$ such

$$(\mathrm{A}_1) \qquad |a_i(x, \eta, \xi)| \le c_1(|\eta|+|\xi|)+c_2(x);$$

(A$_2$) $$\sum_{i=1}^{n} [a_i(x, \eta, \xi) - a_i, \eta, \xi^{(1)})](\xi_i - \xi_i^{(1)}) > 0 \text{ if } \xi \neq \xi^{(1)};$$

there exist constants $\nu > 0$, $\nu_0 > 0$ and $\nu_1 < 0$ such that

(A$_3$) $$\sum_{i=1}^{n} a_i(x, \eta, \xi)\xi_i \geq \nu|\xi|^2, \; \nu_0 \leq a_0(x) \leq \nu_1;$$

(A$_4$) $$a_i(x, \eta, \xi) = \tilde{a}_i(x, \eta,)\, \xi + q_i(x, \eta, \xi),$$

where $\tilde{a}_i(x, \eta)$ and $q_i(x, \eta, \xi)$ satisfy Carathéodory conditions, $\tilde{a}_i(x, \eta)$ is bounded if $|\eta|$ is bounded an $|q_i(x, \eta, \xi)| \leq \tilde{q}_i(x)$, where $\tilde{q}_i \in L^2(\Omega)$.

The other non-linear part f is such that $f(x, \eta, \xi)$ is a Carathéodory function and following conditions are fulfilled:

(A$_5$) $$f(x, \eta, \xi)\eta \geq 0,$$

(A$_6$) $$|f(x, \eta, \xi)| \leq c_0(x) + b(|\eta|)|\xi|^2,$$

where $c_0 \in L^\infty(\Omega) \cap L^1(\Omega)$ and $b\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a monotone function. Here we supposed that f grows quadratically in its variable ξ (= grad u).

(A$_7$) We suppose that the original equation can be expressed in the following form:

$$Pu + Q(u) = -f \text{ in } \Omega \subset \mathbf{R}^n,$$

where P is an elliptic linear operator with constant coefficients;

$$Q(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} b_i(x, u, \text{grad } u) + b_0(x)$$

is such that the functions $b_i(x, \eta, \xi)$, $b_0(x)$ are zero if x is out of the compact subset K of Ω.

Further, assume that for $h \in L^\infty(\Omega)$ all the weak solutions of $Pu + Q(u) = h$ belong to $L^\infty_{\text{loc}}(\Omega)$. (This assumption is satisfied e.g. if Q is linear with sufficiently smooth coefficients, See: *Comm. Pure & Appl. Math*, **12** (1959), 623–727.

Before formulating the main result of this paragraph we prove the following lemma.

Lemma 1.1. Let $h \in L^\infty(\Omega)$ with the property that $h(x) = 0$ out of a compact set of $\mathbf{R}^n$. If $u \in H_0^1(\Omega)$ is a solution of $Pu + Q(u) = h$, then

$$\lim_{|x| \to \infty} u(x) = 0.$$

Proof. We may write $Pu + Q(u) = h$ in the following form: $Pu = h - Q(u)$, and we put $v = h - Q(u)$. Further we define a function $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \text{ large and} \\ 0 & \text{in a neighbourhood of } \mathbf{R}^n/\Omega. \end{cases}$$

Let $\tilde{u}:=\psi u$, then $P\tilde{u}=\tilde{\nu}$ in $\mathbf{R}^n$, where $\tilde{\nu}$ is a distribution with compact support. Since

$$P(\psi u)=\psi(Pu)+\sum_{\substack{2\geq|\alpha|\geq 1\\ 0\leq|\beta|\leq 1}} c_\alpha \partial^\alpha\psi\cdot\partial^\beta u,$$

$Pu=\nu$ and ν is a distrubution with compact support. Here we also note that the second term will have compact support as well, since $\partial^\alpha\psi=0$, when $\psi(x)=1$, $|\alpha|\geq 1$ for $|x|$ being large. Further $P\tilde{u}=\tilde{\nu}\Rightarrow\tilde{u}=\tilde{\nu}*E$, where $\tilde{\nu}=\partial^\alpha g$ such that $g\in L^1(\mathbf{R}^n)$, supp g is compact. In the convolution above the second term E denotes the fundamental solution of P. Thus we may write

$$\tilde{u}=\partial^\alpha g*E=g*\partial^\alpha E,$$

and

$$\tilde{u}(x)=\int_K \partial^\alpha E(x-y)\cdot g(y)dy.$$

Therefore for $x\notin K$ we have

$$|\tilde{u}(x)|\leq\sup_{y\in K}|\partial^\alpha E(x-y)|\cdot\int_K|g(y)|dy.$$

This implies that $\tilde{u}(x)\to 0$ as $|x|\to\infty$ and $u(x)\to 0$ as $|x|\to\infty$, since

$$\sup_{y\in K}|\partial^\alpha E(x-y)|\to 0 \text{ as } x\to\infty$$

because by (A_3) $P(\xi)\neq 0$ for $\xi\in\mathbf{R}^n$.

We state following theorem:

THEOREM 1.2. *If the hypotheses* $(A_1)-(A_7)$ *are satisfied then there exists* $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ *such that for an arbitrary test function* $\varphi\in C_0^\infty(\Omega)$,

$$\sum_{i=1}^n\langle a_i(x,u,\text{grad }u),\ \frac{\partial\varphi}{\partial x_i}\rangle+\langle a_0,\varphi\rangle=\langle -f,\varphi\rangle. \tag{1.6}$$

To prove the above theorem, for an arbitrary $\varepsilon>0$ we define a function f_ε by

$$f_\varepsilon=\frac{f(x,\eta,\xi)}{1+\varepsilon|f(x,\eta,\xi)|}\,\eta(x),$$

where $\tilde{\eta}\in C_0^\infty(\mathbf{R}^n)$ with the property for $|y|<1$, $\tilde{\eta}(y)=1$. Then we have

$$|f_\varepsilon(x,\eta,\xi)|\leq|f(x,\eta,\xi)|\leq\frac{1}{\varepsilon}\cdot\tilde{\eta}(\varepsilon x).$$

Now we consider a sequence of functions u_ε which belong to $H_0^1(\Omega)$ and satisfy

$$-\sum_{i=1}^n\frac{\partial}{\partial x_i}[a_i(x,u_\varepsilon,\text{grad }u_\varepsilon)]+a_0u_\varepsilon+f_\varepsilon(x,u_\varepsilon,\text{grad }u_\varepsilon)=0. \tag{1.7}$$

This problem has solution in $H_0^1(\Omega)$ according to [3]. We know that $u_\varepsilon \in L^\infty(\Omega)$ by lemma 1.1 and the known local estimates (see [7]).

Consider $z_\varepsilon = u_\varepsilon - \dfrac{\sup c_0(x)}{\nu_0}$, equation (1.7) changes to

$$(1.8) \qquad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_i(x, u_\varepsilon, \operatorname{grad} z_\varepsilon)] + a_0 z_\varepsilon = $$

$$= \left\{-f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon) - a_0 \frac{\sup C_0(x)}{\nu_0}\right\}.$$

We prove following lemmas.

LEMMA 1.3. Under assumptions $(A_1)-(A_7)$, the solutions of the equation (1.7) are bounded in L^∞-norm.

PROOF. Since $\|u_\varepsilon(x)\|_{L^\infty(\Omega)} < \infty$, therefore we may write $b\|u_\varepsilon(x)\|_{L^\infty(\Omega)} = c_\varepsilon$.

By (A_3), (A_6), the right hand side (r.h.s.) of (1.8) can be estimated as follows:

$$-f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon) - a_0(x)\frac{\sup c_0(x)}{\nu_0} \leq \sup c_0(x) +$$

$$+ b(|u_\varepsilon|)|\operatorname{grad} u_\varepsilon(x)|^2 - a_0(x)\frac{\sup c_0(x)}{\nu_0}$$

or

$$(1.9) \qquad -f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon) - a_0(x)\cdot\frac{M}{\nu_0} \leq c_\varepsilon \cdot |\operatorname{grad} u_\varepsilon(x)|^2,$$

where $M = \sup c_0(x)$.

We define $\varphi_\varepsilon = e_\varepsilon \cdot z_\varepsilon^+$, where z_ε^+ is the positive part of z_ε, $e_\varepsilon = \exp(\lambda_\varepsilon |z_\varepsilon^+|^2)$, and $\lambda_\varepsilon = \dfrac{c_\varepsilon^2}{2\nu^2}$. Then $\varphi_\varepsilon \in H_0^1(\Omega)$ since its derivatives belong to $L^2(\Omega)$ and at the boundary $z_\varepsilon^+ = 0$ since $z_\varepsilon = -\dfrac{\sup c_0(x)}{\nu_0} = -\dfrac{M}{\nu_0} < 0$ on $\partial\Omega$. By φ_ε we multiply equation (1.8) and integrate over Ω, then we get on left hand side (l.h.s.):

$$(1.10) \qquad \sum_{i=1}^{n} \int_\Omega a_i(x, u_\varepsilon, \operatorname{grad} z_\varepsilon)\left\{e_\varepsilon \frac{\partial z_\varepsilon^+}{\partial x_i} + 2\lambda_\varepsilon e_\varepsilon |z_\varepsilon^+|^2 \frac{\partial z_\varepsilon^+}{\partial x_i}\right\} + \int_\Omega a_0 z_\varepsilon e_\varepsilon z_\varepsilon^+.$$

Thus by (A_3) and (1.8), (1.9) we have

$$\nu\int_\Omega e_\varepsilon|\operatorname{grad} z_\varepsilon^+|^2 + 2\lambda_\varepsilon \nu \int_\Omega e_\varepsilon |z_\varepsilon^+|^2 |\operatorname{grad} z_\varepsilon^+|^2 + \nu_0 \int_\Omega e_\varepsilon |z_\varepsilon^+|^2 \leq \int_\Omega c_\varepsilon |\operatorname{grad} z_\varepsilon^+|^2 e_\varepsilon z_\varepsilon^+.$$

where for right hand side by Young's inequality we have

$$\int_\Omega c_\varepsilon|\operatorname{grad} z_\varepsilon^+|^2 e_\varepsilon z_\varepsilon^+ \le \frac{\nu}{2}\int_\Omega e_\varepsilon|\operatorname{grad} z_\varepsilon^+|^2 + \frac{c_\varepsilon^2}{2\nu^2}\int_\Omega e_\varepsilon|z_\varepsilon^+|^2|\operatorname{grad} z_\varepsilon^+|^2.$$

Consequently,

$$(1.11)\qquad \frac{\nu}{2}\int_\Omega e_\varepsilon|\operatorname{grad} z_\varepsilon^+|^2 + \lambda_\varepsilon\nu\int_\Omega e_\varepsilon|z_\varepsilon^+|^2|\operatorname{grad} z_\varepsilon^+|^2 + \nu_0\int_\Omega e_\varepsilon|z_\varepsilon^+|^2 \le 0,$$

where we used $\lambda_\varepsilon = \dfrac{c_\varepsilon^2}{2\nu^2}$.

Since $e_\varepsilon = \exp(\lambda_\varepsilon|z_\varepsilon^+|^2) \ge 1$, thus from (1.11)it is clear that $z_\varepsilon^+ = 0$, i.e. $z_\varepsilon \le 0$, $u_\varepsilon \le M/\nu_0$. Similarly can be proved that $u_\varepsilon \ge -M/\nu_0$.

LEMMA 1.4. The solutions u_ε are bounded with respect to the norm of $H_0^1(\Omega)$.

PROOF. Since $|u_\varepsilon| \le M/\nu_0$, this implies that $b(|u_\varepsilon(x)|) \le C_4$ where C_4 is a constant. We consider $\varphi_\varepsilon = E_\varepsilon u_\varepsilon$, where $E_\varepsilon = \exp(\lambda u_\varepsilon^2) \ge 0$ and $\lambda = c_4^2/2\nu^2$. Clearly $\varphi_\varepsilon \in H_0^1(\Omega)$. Multiply equation (1.7) by φ_ε, integrating by parts we obtain

$$\sum_{i=1}^n \int_\Omega a_i(x, u_\varepsilon \operatorname{grad} u_\varepsilon)E_\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} + \sum_{i=1}^n \int_\Omega a_i(x, u_\varepsilon, \operatorname{grad} u_\varepsilon)2\lambda E_\varepsilon u_\varepsilon^2 \frac{\partial u_\varepsilon}{\partial x_i} +$$

$$(1.12)\qquad + \int_\Omega a_0 u_\varepsilon E_\varepsilon u_\varepsilon = -\int_\Omega f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon)E_\varepsilon u_\varepsilon.$$

Using (A_3), l.h.s. can be estimated by

$$(1.13)\qquad \nu\int_\Omega E_\varepsilon|\operatorname{grad} u_\varepsilon|^2 + 2\lambda\nu\int_\Omega E_\varepsilon|u_\varepsilon|^2|\operatorname{grad} u_\varepsilon|^2 + \nu_0\int_\Omega E_\varepsilon|u_\varepsilon|^2;$$

r.h.s. can be estimated as follows:

$$-\int_\Omega |f_\varepsilon(x, u_\varepsilon \operatorname{grad} u_\varepsilon)|E_\varepsilon|u_\varepsilon| \le \int_\Omega |f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon)|E_\varepsilon|u_\varepsilon| \le$$

$$\le \int_\Omega [c_0(x) + b(|u_\varepsilon(x)|)|\operatorname{grad} u_\varepsilon(x)|^2]E_\varepsilon|u_\varepsilon|,$$

or

$$-\int_\Omega f_\varepsilon(x, u_\varepsilon, \operatorname{grad} u_\varepsilon)E_\varepsilon u_\varepsilon \le \frac{M}{\nu_0}\sup E_\varepsilon \int_\Omega c_0(x)dx + \frac{\nu}{2}E_\varepsilon|\operatorname{grad} u_\varepsilon|^2 +$$

$$(1.14)\qquad + \frac{1}{2\nu}\int_\Omega c_4^2 E_\varepsilon|u_\varepsilon|^2|\operatorname{grad} u_\varepsilon|^2.$$

Using (1.13), (1.14), (1.12) changes to

$$\frac{\nu}{2}\int_\Omega E_\varepsilon|\text{grad } u_\varepsilon|^2+\lambda\nu\int_\Omega E_\varepsilon|\text{grad } u_\varepsilon|^2|u_\varepsilon|^2+\nu_0\int_\Omega E_\varepsilon|u_\varepsilon|^2\le$$

$$\le\frac{M}{\nu_0}\sup E_\varepsilon\int_\Omega c_0(x)dx. \tag{1.15}$$

Hence

$$\|u_\varepsilon\|_{H^1_0(\Omega)}\le\text{constant}.$$

We have proved that u_ε is bounded in $H^1_0(\Omega)$ which implies that there exists a subsequence of u_ε (again denoted by u_ε) such that:

$$u_\varepsilon\to u \quad\text{in}\quad H^1_0(\Omega)$$

weakly and

$$u_\varepsilon\to u \quad\text{a.e. in } \Omega,\ u_\varepsilon\in L^\infty(\Omega).$$

LEMMA 1.5. For the above defined sequence;

$$u_\varepsilon\to u \text{ in } H^1_0(\Omega) \text{ strongly.}$$

PROOF. Let us consider functions $\bar u_\varepsilon=u_\varepsilon-u$ belonging to $H^1_0(\Omega)\cap L^\infty(\Omega)$. Putting these functions in (1.7), we obtain

$$\sum_{i=1}^n\int_\Omega a_i(x,u_\varepsilon\text{ grad } u+\text{grad }\bar u_\varepsilon)\frac{\partial\bar\varphi_\varepsilon}{\partial x_i}+\int_\Omega a_0\bar u_\varepsilon\bar\varphi_\varepsilon=$$

$$=-\int_\Omega f_\varepsilon(x,u_\varepsilon,\text{grad } u_\varepsilon)\bar\varphi_\varepsilon-\int_\Omega a_0u\bar\varphi_\varepsilon, \tag{1.16}$$

where $\bar\varphi_\varepsilon\in H^1_0(\Omega)$, $\bar\varphi_\varepsilon=\bar E_\varepsilon\bar u_\varepsilon$ und $\bar E_\varepsilon=\exp(\bar\lambda\bar u_\varepsilon^2)$, $\bar\lambda=\frac{2c_4^2}{\nu^2}$. Using ($A_4$) we obtain

$$\sum_{i=1}^n\int_\Omega a_i(x,u_\varepsilon\text{ grad }\bar u_\varepsilon)\bar E_\varepsilon\frac{\partial\bar u_\varepsilon}{\partial x_i}+\sum_{i=1}^n 2\bar\lambda\int_\Omega a_i(x,u_\varepsilon,\text{grad }\bar u_\varepsilon)\bar E_\varepsilon\bar u_\varepsilon^2\frac{\partial\bar u_\varepsilon}{\partial x_i}+$$

$$+\sum_{i=1}^n\int_\Omega a_0\bar u_\varepsilon\bar E_\varepsilon\bar u_\varepsilon=-\int_\Omega f_\varepsilon(x,u_\varepsilon,\text{grad } u_\varepsilon)\bar E_\varepsilon\bar u_\varepsilon-\int_\Omega a_0u\bar E_\varepsilon\bar u_\varepsilon- \tag{1.17}$$

$$-\sum_{i=1}^n\int_\Omega\tilde a_i(x,u_\varepsilon)\text{ grad } u\bar E_\varepsilon\frac{\partial\bar u_\varepsilon}{\partial x_i}-\sum_{i=1}^n 2\bar\lambda\int_\Omega\tilde a_i(x,u_\varepsilon)\text{ grad } u\bar E_\varepsilon\bar u_\varepsilon^2\frac{\partial\bar u_\varepsilon}{\partial x_i}-$$

$$-\sum_{i=1}^n\Big\langle\frac{\partial}{\partial x_i}\cdot[q_i(x,u_\varepsilon,\text{grad } u+\text{grad }\bar u_\varepsilon)-$$

$$-q_i(x,u_\varepsilon\text{ grad }\bar u_\varepsilon)],\ \bar E_\varepsilon\bar u_\varepsilon\Big\rangle_{(H^{-1}(\Omega),\,H^1_0(\Omega))}.$$

Using (A_3), l.h.s. can be estimated by

$$(1.18)\qquad \nu\int_\Omega \bar E_\varepsilon|\text{grad}\,\bar u_\varepsilon|^2+2\bar\lambda\nu\int_\Omega \bar E_\varepsilon\bar u_\varepsilon^2|\text{grad}\,\bar u_\varepsilon|^2+\nu_0\int_\Omega \bar E_\varepsilon\bar u_\varepsilon^2;$$

further, by (A_6)

$$(1.19)\qquad \begin{aligned}-\int_\Omega |f_\varepsilon(x,u_\varepsilon,\text{grad}\,u_\varepsilon)|\bar E_\varepsilon|u_\varepsilon| &\le \int_\Omega [(c_0(x)+2c_4]|\text{grad}\,u|^2]\bar E_\varepsilon\bar u_\varepsilon+\\ &+\frac{\nu}{2}\int_\Omega \bar E_\varepsilon|\text{grad}\,\bar u_\varepsilon|^2+\frac{1}{2\nu}\int_\Omega \bar E_\varepsilon(2c_4)^2\bar u_\varepsilon^2|\text{grad}\,\bar u_\varepsilon|^2,\end{aligned}$$

where we have used following inequalities:

$$\text{(i)}\qquad |\text{grand}\,u_\varepsilon|^2\le 2|\text{grad}\,u|^2+2|\text{grad}\,\bar u_\varepsilon|^2$$

and

$$\text{(ii)}\qquad ab\le(\varepsilon a)^2/2+(b/\varepsilon)^2/2.$$

Using (1.18), (1.19), (1.17) changes to following form

$$(1.20)\qquad \begin{aligned}\frac{\nu}{2}\int_\Omega \bar E_\varepsilon|\text{grad}\,u_\varepsilon|^2+\bar\lambda\nu\int_\Omega \bar E_\varepsilon\bar u_\varepsilon^2|\text{grad}\,\bar u|^2+\nu_0\int_\Omega \bar E_\varepsilon\bar u_\varepsilon^2 &\le \int_\Omega c_0(x)\bar E_\varepsilon|\bar u_\varepsilon|+\\ +\int_\Omega (2c_4)|\text{grad}\,u|^2\bar E_\varepsilon|\bar u_\varepsilon|-\sum_{i=1}^n\int_\Omega \tilde a_i(x,u_\varepsilon)\,\text{grad}\,u\bar E_\varepsilon\frac{\partial\bar u_\varepsilon}{\partial x_i}&-\\ -\sum_{i=1}^n 2\bar\lambda\int_\Omega \tilde a_i(x,u_\varepsilon)\,\text{grad}\,u\bar E_\varepsilon\bar u_\varepsilon^2\frac{\partial u_\varepsilon}{\partial x_i}&+\\ +\sum_{i=1}^n\int_\Omega [q_i(x,u_\varepsilon,\text{grad}\,u+\text{grad}\,\bar u_\varepsilon)-q_i(x,u_\varepsilon\,\text{grad}\,\bar u_\varepsilon)]\bar E_\varepsilon\frac{\partial\bar u_\varepsilon}{\partial x_i}&+\\ +\sum_{i=1}^n 2\bar\lambda\int_\Omega [q_i(x,u_\varepsilon,\text{grad}\,u+\text{grad}\,\bar u_\varepsilon)-q_i(x,u_\varepsilon,\text{grad}\,\bar u_\varepsilon)]\bar E_\varepsilon\bar u_\varepsilon^2\frac{\partial\bar u_\varepsilon}{\partial x_i}&-\\ -\int_\Omega a_0u\bar E_\varepsilon\bar u_\varepsilon.&\end{aligned}$$

We show that all terms on r.h.s. tend to zero. This will certainly imply that l.h.s. tends to zero in L^2-norm. This will prove that $u_\varepsilon\to u$ in $H_0^1(\Omega)$ strongly. Since $u_\varepsilon\to u$ a.e. in Ω, u_ε is bounded in $L^\infty(\Omega)$ thus by Lebesque's dominated convergence theorem for the 1st and 2nd terms

$$\int_\Omega c_0(x)\bar E_\varepsilon|\bar u_\varepsilon|+\int_\Omega(2c_4)|\text{grad}\,u|^2\bar E_\varepsilon|\bar u_\varepsilon|\to 0,$$

because $c_0 \in L^1(\Omega)$ and $|\text{grad } u|^2 \in L^1(\Omega)$, $\tilde{E}_\varepsilon$ is uniformly bounded. Further $u_\varepsilon \to u$ in $H_0^1(\Omega)$ weakly implies that $\dfrac{\partial \bar{u}_\varepsilon}{\partial x_i} \to 0$ in $L^2(\Omega)$ weakly. By (A_4) $\bar{a}_i(x, u_\varepsilon)$ is bounded in $L^\infty(\Omega)$. Thus the third and fourth terms in the right of (1.20) converge to zero. Similarly, by (A_4) also the last terms in the right of (1.20) will tend to zero.

PROOF of THEOREM. By the preceding lemma u_ε converges to u in $H_0^1(\Omega)$ strongly. Thus grad $u_\varepsilon \to$ grad u a.e. in Ω for a subsequence. We know $u_\varepsilon \to u$ a.e. in Ω. Consequently,

$$f_\varepsilon(x, u_\varepsilon, \text{grad } u_\varepsilon) \to f(x, u, \text{grad } u) \text{ a.e. in } \Omega.$$

Since

$$|f_\varepsilon(x, u_\varepsilon, \text{grad } u_\varepsilon)| \leq c_0(x) + b(|u_\varepsilon(x)|).\ |\text{grad } u_\varepsilon(x)|^2,$$

where $c_0 \in L^1(\Omega)$, $b(|u_\varepsilon(x)|) \leq c_4$ and $|\text{grad } u_\varepsilon(x)|^2$ is convergent in $L^1(\Omega)$; thus by Vitali's convergence theorem

$$f_\varepsilon(x, u_\varepsilon, \text{grad } u_\varepsilon) \to f(x, u, \text{grad } u) \tag{1.21}$$

strongly in L^1-norm. Further by (1.3)

$$a_i(x, u_\varepsilon, \text{grad } u_\varepsilon) \to a_i(x, u, \text{grad } u) \text{ a.e. in } \Omega.$$

Thus from (A_1) and Vitali's convergence theorem it follows that

$$a_i(x, u_\varepsilon, \text{grad } u_\varepsilon) \to a_i(x, u, \text{grad } u) \text{ in } L^2(\Omega).$$

Hence from equations (1.7), (1.21) one obtains as $\varepsilon \to 0$ that u satisfies the equation (1.6) for any $\varphi \in c_0^\infty(\Omega)$.

2. Stability of solutions

The following type of non-linear differential equation is considered:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [a_i^k(x, u^k, \text{grad } u^k] + a_0^k u^k + f(x, u^k, \text{grad } u^k) = 0 \text{ in } \Omega, \tag{2.1}$$

$$u^k = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Here a_i^k's are considered as functions of $x \in \Omega$, $\eta \in \mathbf{R}$ and of $\xi \in \mathbf{R}^n$. They satisfy Carathéodory conditions, i.e.

$$(\eta, \xi) \to a_i^k(x, \eta, \xi) \text{ are continuons over } \mathbf{R} \times \mathbf{R}^n. \tag{2.3}$$

$$x \to a_i^k(x, \eta, \xi) \text{ are measurable.} \tag{2.4}$$

We assume that a_0^k is measurable in Ω.

Further we suppose that

$(A_1)'$ $\exists$ a constant $c_1>0$, a function $c_2\in L^2(\Omega)$ such that

$$|a_i^k(x, \eta, \xi)|\leq c_1[|n|+|\xi|]+c_2(x);$$

$(A_2)'$ $$\sum_{i=1}^{n}[a_i^k(x, \eta, \xi)-a_i^k(x, \eta, \xi^{(i)})](\xi_i-\xi_i^{(1)})>0 \text{ if } \xi\neq\xi^{(1)};$$

$(A_3)'$ $\exists$ constants $\nu>0$, $\nu_0>0$ and $\nu_1>0$ such that

$$\sum_{i=1}^{n} a_i^k(x, \eta, \xi)\xi_i\geq\nu|\xi|^2;\ \ \nu_0\leq a_0^k(x)\leq\nu_1.$$

$(A_4)'$ $$a_i^k(x, \eta, \xi) = \tilde{a}_i^k(x, \eta)\xi+q_i^k(x, \eta, \xi),$$

where a_i^k, q_i^k satisfy Carathéodory conditions, $\tilde{a}_i^k$ is uniformly bounded if $|\eta|$ is bounded and also we have

$$|q_i^k(x, \eta, \xi)|\leq\tilde{q}_i^k \text{ where } \tilde{q}_i^k\in L^2(\Omega).$$

The second non-linear part f^k satisfies:

$(A_5)'$ $$f^k(x, \eta, \xi, \eta\geq 0,$$

$(A_6)'$ $$|f^k(x, \eta, \xi)|\leq c_0(x)+b(|\eta|)|\xi|^2,$$

where $c_0\in L^1(\Omega)\cap L^\infty(\Omega)$ and b is a positive monotone function.

$(A_7)'$ Let us suppose that the original equations may be written in the following form:

$$P^k u^k+Q^k(u^k) = -f^k \text{ in } \Omega\subset\mathbf{R}^n$$

where P^k is an elliptic operator with constant coefficients;

$$Q^k(u^k)=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}b_i^k(x, u^k), \text{grad } u^k)+b_0^k(x)u^k$$

is such that the functions $b_i^k(x, \eta, \xi)$, $b_0^k(x)$ are zero if x is out of a compact subset K of Ω.

Further for $h\in L^\infty(\Omega)$ all the weak solutions of the equation

$$P^k u+Q^k(u) = h$$

belong to $L^\infty_{\text{loc}}(\Omega)$.

$(A_8)'$ $$\text{If } \xi^k\to\xi^0, \eta^k\to\eta^0$$

then

$$a_i^k(x, \eta^k, \xi^k)\to a_i^0(x, \eta^0, \xi^0),$$
$$a_0^k(x)\to a_0^0(x)$$

and

$$f^k(x, \eta^k, \xi^k), \ \to f^0(x, \eta^0, \xi^0).$$

THEOREM 2.1. *We assume that the conditions* $(A_1)'-(A_8)'$ *are satisfied and* $u^k \in H_0^1(\Omega)\cap L^\infty(\Omega)$ *is a solution of equation* (2.1). *Then there exists a subsequence* $(\tilde{u}^k)$ *of* (u^k) *such that* $(\tilde{u}^k)$ *converges in* $H_0^1(\Omega)$ *to a solution* $u^0 \in H_0^1(\Omega)\cap \cap L^\infty(\Omega)$ *of*

$$(2.5) \qquad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_i^0(x, u^0, \text{grad } u^0)] + a_i^0(x)u^0 + f^0(x, u^0 \text{ grad } u^0) \text{ in } \Omega.$$

PROOF. We have a sequence of solutions $u^k \in H_0^1(\Omega)$ of the equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_i^k(x, u^k, \text{grad } u^k)] + a_0^k(x)u^k + f^k(x, u^k, \text{grad } u^k) = 0 \text{ in } D'(\Omega). \tag{2.6}$$

First of all we prove boundedness of u^k, i.e. $\|u^k\|_{L^\infty(\Omega)} \leq M/\nu_0$, where $M = \sup c_0(x)$.

Let us consider a function $z^k = u^k - \sup c_0(x)/\nu_0$ and put this value in (2.6) then we obtain

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_i^k(x, u^k, \text{ grad } z^k)] + a_0^k z^k = -f^k(x, u^k, \text{grad } u^k) - a_0^k \sup c_0(x)/\nu_0. \tag{2.7}$$

According to assumption of theorem 2.1,

$$\|u^k\|_{L^\infty(\Omega)} < \infty$$

and so we can define constants c^k by

$$\|b(u^k(x)\|_{L^\infty(\Omega)} = c^k.$$

R.h.s. of (2.7) can be estimated as follows:

$$-f^k(x, u^k, \text{grad } u^k) - a_0^k(x)\frac{\sup c_0(x)}{\nu_0} \leq \sup c_0(x) + b(|u^k(x)|)|\text{grad } u^k(x)| -$$

$$-a_0^k(x)\frac{\sup c_0(x)}{\nu_0} \leq c^k|\text{grad } u^k(x)|^2 = c^k|\text{grad } z^k(x)|^2 \text{ a.e. in } \Omega,\ a_0^k \geq \nu_0.$$

Let $\varphi^k = e^k(z^k)^+$ where $(z^k)^+$ is the positive part of $z^k = u^k - \dfrac{\sup c_0(x)}{\nu_0}$, $e^k = \exp(\lambda^k|z^k|^{+2}$ and $\lambda^k = (c^k)^2/_2\nu^2$ is a constant. Then $\varphi^k \in H_0^1(\Omega)$ since the derivatives will belong to $L^2(\Omega)$ and at the boundary $(z^k)^+ = 0$ as

$$z^k = -\frac{\sup c_0(x)}{\nu_0} = -\frac{M}{\nu_0}.$$

on $\partial\Omega$. Furthermore, on l.h.s. after multiplication with φ^k and integration over Ω,

$$\sum_{i=1}^{n}\int_{\Omega} a_i^k(x, u^k, \text{grad } z^k)e^k \frac{\partial(z^k)^+}{\partial x_i} +$$

$$+\sum_{i=1}^{n}\int_{\Omega} a_i^k(x,u^k, \text{grad } z^k)2\lambda^k e^k|(z^k)^+|^2 \frac{\partial(z^k)^+}{\partial x_i} + \int_{\Omega} a_0^k z^k e^k(z^k)^+ =$$

$$= \int_{\Omega} c^k|\text{grad } u^k(x)|^2 e^k(z^k)^+.$$

Using $(A_3)'$ we have

$$\nu\int_{\Omega} e^k|\text{grad }(z^k)^+|^2 + 2\lambda^k\nu\int_{\Omega} e^k|(z^k)^+|^2|\text{grad}(z^k)^+|^2 + \nu_0\int_{\Omega} e^k|(z^k)^+|^2 =$$

(2.8)

$$= \frac{\nu}{2}\int_{\Omega} e^k|\text{grad }(z^k)^+|^2 + \frac{(c^k)^2}{2\nu}\int_{\Omega} e^k|(z^k)^+|^2|\text{grad }(z^k)^+|^2$$

where we have used Young's inequality. Therefore, we obtain:

$$(2.9)\quad \frac{\nu}{2}\int_{\Omega} e^k|\text{grad }(z^k)^+|^2 + \lambda^k\nu\int_{\Omega} e^k|(z^k)^+|^2|\text{grad }(z^k)^+|^2 + \nu_0\int_{\Omega} e^k|(z^k)^+|^2 \le 0,$$

where $\lambda^k = (c^k)^2/_2\nu^2$. Since $e^k \ge 1$, thus $(z^k)^+ = 0$. It follows that $z^k \le 0$, $u^k - \frac{M}{\nu_0} \le 0$. Similarly it can be shown that $u^k + \frac{M}{\nu_0} \ge 0$.

Next we show that u^k is bounded in $H_0^1(\Omega)$. We have proved that $\|u^k(x)\|_{L^\infty(\Omega)} \le$ constant. Therefore $b(|u^k(x)|) \le c_3$, where $c_3 = b(M/\nu_0)$ is a constant. Consider $\varphi^k = E^k u^k$ where $E^k = \exp(\lambda(u^k)^2)$ $\lambda = \frac{c_3^2}{2\nu^2}$. Obviously $\varphi^k \in H_0^1(\Omega)$. Multipling equation (2.6) by φ^k and integrating we obtain:

$$\sum_{i=1}^{n}\int_{\Omega} a_i^k(x, u^k, \text{grad } u^k)E^k \frac{\partial u^k}{\partial x_i} + \sum_{i=1}^{n}\int_{\Omega} a_i^k(x, u^k \text{grad } u^k)2\lambda E^k(u^k)^2 \frac{\partial u^k}{\partial x_i} +$$

(2.10)

$$+\int_{\Omega} a_0^k E^k u^k = -\int_{\Omega} f^k(x, u^k, \text{grad } u^k)E^k u^k.$$

Using $(A_3)'$ we find that l.h.s. is greater or equal than

$$(2.11)\quad \nu\int_{\Omega} E^k|\text{grad } u^k|^2 + 2\lambda\nu\int_{\Omega} E^k(u^k)^2|\text{grad } u^k|^2 + \nu_0\int_{\Omega} E^k(u^k)^2;$$

and for r.h.s.

$$(2.12)\qquad -\int_\Omega f^k(x,u^k,\text{grad }u^k)E^k u^k \le \int_\Omega |f^k(x,u^k,\text{grad }u^k)|E^k|u^k| \le$$

$$\le \int_\Omega c_0(x)E^k|u^k| + c_3\int_\Omega |\text{grad }u^k|^2 E^k|u^k|.$$

By Young's inequality it follows (see the proof of Lemma 1.4):

$$(2.13)\qquad \frac{\nu}{2}\int_\Omega E^k|\text{grad }u^k|^2 + \lambda\nu\int_\Omega E^k(u^k)^2|\text{grad }u^k|^2 + \nu_0\int_\Omega E^k(u^k)^2 \le$$

$$\le \frac{M}{\nu_0}\sup E^k\int_\Omega c_0(x)dx.$$

or

$$(2.14)\qquad \|u^k\|_{H^1_0(\Omega)} \le \text{const, for any } k.$$

The boundedness of u^k in $H^1_0(\Omega)$ implies that there exists a subsequence of u^k (denote again by u^k) such that $u^k \to u^0$ weakly in $H^1_0(\Omega)$ and $u^k \to u^0$ a.e. in Ω.

Next we prove that $u^k \to u^0$ strongly in $H^1_0(\Omega)$. Putting $\bar{u}^k = u^k - u^0$ in equation (2.6) and multiplying with $\bar{\varphi}^k$ we have after integration:

$$(2.15)\qquad \sum_{i=1}^n\int_\Omega a_i(x,u^k,\text{grad }u^0+\text{grad }u^k)\frac{\partial\bar{\varphi}^k}{\partial x_i} + \int_\Omega a_0^k u^k\varphi^k =$$

$$= -\int_\Omega f^k(x,u^k,\text{grad }u^k)\bar{\varphi}^k - \int_\Omega a_0^k u^0\bar{\varphi}^k,$$

where $\bar{\varphi}^k = \bar{E}^k\bar{u}^k$, $\bar{E}^k = \exp(\bar{\lambda}(\bar{u}^k)^2)$ and $\bar{\lambda} = 2c_3^2/\nu^2$. Here $b(|u^k|) \le c_3$. Using $(A_4)'$ we obtain:

$$\sum_{i=1}^n\int_\Omega a_i^k(x,u^k,\text{grad }\bar{u}^k)\bar{E}_k\frac{\partial\bar{u}^k}{\partial x_i} + \sum_{i=1}^n 2\bar{\lambda}\int_\Omega a_i^k(x,u^k,\text{grad }\bar{u}^k)\bar{E}^k(\bar{u}^k)^2\frac{\partial\bar{u}^k}{\partial x_i} +$$

$$(2.16)\qquad +\int_\Omega a_0^k\bar{u}^k\bar{\varphi}^k = -\int_\Omega f^k(x,u^k,\text{grad }u^k)\bar{E}^k\bar{u}^k - \int_\Omega a_0^k u^0\bar{E}^k\bar{u}^k -$$

$$-\sum_{i=1}^n\int_\Omega \tilde{a}_i^k(x,u^k)\,\text{grad }u^0\bar{E}^k\frac{\partial\bar{u}^k}{\partial x_i} - \sum_{i=1}^n 2\bar{\lambda}\int_\Omega \tilde{a}_i^k(x,u^k)\,\text{grad }u^0\bar{E}^k(\bar{u}^k)^2\frac{\partial\bar{u}^k}{\partial x_i} -$$

$$-\sum_{i=1}^n\left\langle\frac{\partial}{\partial x_i}[q_i^k(x,u^k,\text{grad }u^0+\text{grad }\bar{u}^k) - q_i^k(x,u^k,\text{grad }\bar{u}^k)], \bar{E}^k\bar{u}^k\right\rangle_{(H^{-1}(\Omega),H^1_0(\Omega))}.$$

Using $(A_3)'$, l.h.s. can be estimated by

$$(2.17)\qquad \nu\int_\Omega \bar{E}^k|\text{grad }\bar{u}^k|^2 + 2\bar{\lambda}\nu\int_\Omega \bar{E}^k(\bar{u}^k)^2|\text{grad }\bar{u}^k|^2 + \nu_0\int_\Omega \bar{E}^k(u^k)^2.$$

Further by $(A_6)'$ and Young's inequality

$$(2.18) \quad -\int_\Omega f^k(x, u^k, \text{grad } u^k)\bar{E}^k\bar{u}^k \leq \int_\Omega [c_0(x)+2c_3|\text{grad } u^0|^2]\bar{E}^k\bar{u}^k +$$

$$+\frac{\nu}{2}\int_\Omega \bar{E}^k|\text{grad } \bar{u}^k|^2 + \frac{1}{2\nu}\int_\Omega \bar{E}^k(2c_3)^2(\bar{u}^k)^2|\text{grad } \bar{u}^k|^2.$$

By (2.17), (2.18), (2.16) we have

$$\frac{\nu}{2}\int_\Omega \bar{E}^k|\text{grad } \bar{u}^k|^2 + \frac{2}{\nu}c_3^2\int_\Omega \bar{E}^k(\bar{u}^k)^2|\text{grad } \bar{u}^k|^2 + \nu_0\int_\Omega \bar{E}^k(u^k)^2 \leq$$

$$\leq \int_\Omega c_0(x)\bar{E}^k|\bar{u}^k| + 2c_3\int_\Omega \bar{E}^k|\bar{u}^k|\,|\text{grad } u^0|^2 - \sum_{i=1}^n \int_\Omega \tilde{a}_i^k(x, u^k)\,\text{grad } u^0 \bar{E}\frac{\partial \bar{u}^k}{\partial x_i} -$$

(2.19)

$$-\sum_{i=1}^n 2\bar{\lambda}\int_\Omega \tilde{a}_i^k(x, u^k)\,\text{grad } u^0\bar{E}^k(\bar{u}^k)^2\frac{\partial \bar{u}^k}{\partial x_i} +$$

$$+\sum 2\bar{\lambda}\int_\Omega [q_i^k(x, u^k, \text{grad } u^0 + \text{grad } \bar{u}^k) - q_i^k(x, u^k, \text{grad } \bar{u}^k)]\bar{E}^k\frac{\partial \bar{u}^k}{\partial x_i} +$$

$$+\sum_{i=1}^n 2\bar{\lambda}\int_\Omega [q_i(x, u^k, \text{grad } u^0 + \text{grad } \bar{u}^k) -$$

$$-q_i^k(x, u^k, \text{grad } \bar{u}^k)]\bar{E}^k(\bar{u}^k)^2\frac{\partial u^k}{\partial x_i} - \int_\Omega a_0^k u^0\bar{E}^k u^k.$$

We show that all, terms on r.h.s. tend to zero as $k\to\infty$. This will obviously imply that l.h.s. tends to zero. From this it will follow that $u^k\to u^0$ in $H_0^1(\Omega)$ strongly. So for example we show that the 3[rd] term on r.h.s. tends to zero as $k\to\infty$. $u^k\to u^0$ weakly in $H^1(\Omega)$. This implies that $\frac{\partial \bar{u}^k}{\partial x_i}\to 0$ weakly in $L^2(\Omega)$. By $(A_4)'$ we know that $\tilde{a}_i^k(x, u^k)$ is bounded in $L^\infty(\Omega)$. Thus the 3[rd] and fourth terms in the r.h.s. of equation (2.19) converge to zero. In a similar manner we can show that last terms also converge to zero as $k\to\infty$.

We have shown that $u^k\to u^0$ strongly in $H_0^1(\Omega)$ and therefore grad $u^k\to$ grad u^0 a.e. in Ω for a subsequence. Consequently, by $(A_8)'$ $f^k(x, u^k, \text{grad } u^k)\to$ $\to f^k(x, u^0, \text{grad } u^0)$ a.e. in Ω. Further, according to $(A_6)'$,

$$|f^k(x, u^k, \text{grad } u^k)| \leq c_0(x) + b(|u^k(x)|)|\text{ grad } u^k(x)|^2,$$

where $c_0(x)\in L^1(\Omega)$, $b|(u^k(x)|)\leq c_3$ and $|\text{grad } u^k(x)|^2$ is convergent in $L^1(\Omega)$.

Thus by Vitali's convergence theorem

$$(2.20) \qquad f^k(x, u^k, \text{grad } u^k)\to f(x, u^0, \text{grad } u^0) \text{ strongly in } L^1(\Omega).$$

Further by the assumption $(A_8)'$:

(2.21) $$a_i^k(x, u^k, \text{grad } u^k) \to a_i^0(x, u^0, \text{grad } u^0) \text{ for } a.e. \text{ in } \Omega,$$

$$a_0^k(x) \to a_0^0(x) \text{ for a.e. in } \Omega.$$

Thus from $(A_1)'$ and Vitali's convergence theorem it follows that

(2.22) $$a_i^k(x, u^k, \text{grad } u^k, \to a_i^0(x, u^0, \text{grad } u^0) \text{ in } L^2(\Omega).$$

From (2.6) and (2.20) − (2.22) one obtains as $k \to \infty$ that satisfies the equation (2.5). So Theorem 2.1 is proved.

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AN INEQUALITY OF PEANO'S TYPE AND ITS CONSEQUENCES

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(Received August 31, 1983)

In the present paper a lemma and a theorem will be proven. The lemma presents a generalization of the wellknown Peano inequality (see [1], p. 8., Theorem 2.1).

Our theorem is an existence theorem concerning the variational equation of an ordinary differential equation system generated by its initial value. The statement in it is a known one for a continously differentiable righthand side (see eg. [2], pp. 188–193). We show that the statement will remain valid under a much weaker assumption, too; for the case when the right-hand side expression is only a function from L_1 in its first variable. This theorem is of crucial importance for the existence investigations in the duality theory of control problems. Namely, the right-hand side function of the differential equations used here satisfies generally only the assumptions of the theorem to be proved here. E.g. the state equation and the constraint on the control:

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ and } u(t) \in U(t, x(t))$$

can be reformulated under appropriate assumptions on f and U as $\dot{x}(t) = u(t)$, $u(t) = g(t, x(t))$ or

$$u(t) = \sum_{i=1}^{N} \lambda_i(t, x(t)) \cdot g_i(t, x(t)).$$

It can be seen that if $u \in L_p$ then g resp. g_i are only functions of L_p concerning their first variable.

Consider the following differential equation:

$$\dot{x}(\tau) = g(\tau, x(\tau)) \text{ for a.e. } \tau \in I_1 \text{ with } I_1 = [0, a], \tag{1}$$

$$(a \leqq +\infty),\ g{:}\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n,\ D(g) = I_1 \times I_2.$$

DEFINITION. We call a function $\varphi{:}I_1 \to \mathbf{R}^n$ a weakly ε-approximating solution of (1), if

$$\left\| \int_0^t (\dot{\varphi}(\tau) - g(\tau, \varphi(\tau)))d\tau \right\|_{\mathbf{R}^n} < \varepsilon \text{ for all } t \in I_1. \tag{2}$$

LEMMA. Let $g: I_1 \times I_2 \to \mathbf{R}^n$, $I_1 \subset \mathbf{R}$, $I_2 \subset \mathbf{R}^n$ a given function with the following properties:

(i) $g(., \xi): I_1 \to \mathbf{R}^n$ is from L_1

(ii) $\exists K \in L_1(I_1)$ so that for a.e. $t \in I_1$ holds

$$\|g(t, x_1) - g(t, x_2)\| \leqq K(t)\|x_1 - x_2\|, \quad x_1, x_2 \in I_2.$$

Assume moreover that functions $x_1, x_2: I_1 \to \mathbf{R}^n$ are weakly $\frac{\varepsilon_1}{2}$, resp. $\frac{\varepsilon_2}{2}$ approximating solutions of (1). In this case following inequality is valid for a.e. $t, \tau \in I_1$ $t \leqq \tau$,

$$\|x_1(\tau) - x_2(\tau)\| \leqq \|x_1(t) - x_2(t)\| \exp\left(\int_t^\tau K(s)ds\right) + (\varepsilon_1 + \varepsilon_2) \exp\left(\int_t^\tau K(s)\,ds + 1\right). \tag{3}$$

PROOF. Introduce the notations:

$$r_i(\tau) := \dot{x}_i(\tau) - g(\tau, x_i(\tau)), \; i = 1, 2, \; \tau \in I_1. \tag{4}$$

As x_i is a weakly $\frac{\varepsilon_i}{2}$ approximating solution of (1) so we have

$$\left\| \int_0^t r_i(s)ds \right\|_{\mathbf{R}^n} < \frac{\varepsilon_i}{2}, \quad i = 1,2.$$

$$\int_t^\tau \dot{x}_i(s)ds = \int_t^\tau g(s, x_i(s))ds + \int_t^\tau r_i(s)ds. \tag{5}$$

But we also know that

$$\int_t^\tau \dot{x}_i(s)ds = x_i(\tau) - x_i(t).$$

So (5) can be written in this form:

$$x_i(\tau) = x_i(t) + \int_t^\tau g(s, x_i(s))ds + \int_t^\tau r_i(s)ds.$$

Hence we obtain

$$\|x_1(\tau) - x_2(\tau)\| \leqq \|x_1(t) - x_2(t)\| + \int_t^\tau \|g(s, x_1(s)) - g(s, x_2(s))\|ds +$$

$$+ \left\| \int_t^\tau r_1(s)ds \right\| + \left\| \int_t^\tau r_2(s)ds \right\| \leqq \|x_1(t) - x_2(t)\| + \int_t^\tau K(s)\|x_1(s) - x_2(s)\|\,ds +$$

$$+ \left\| \int_0^t r_1(s)ds \right\| + \left\| \int_0^\tau r_1(s)ds \right\| + \left\| \int_0^t r_2(s)ds \right\| + \left\| \int_0^\tau r_2(s)ds \right\| \leqq$$

$$\leqq \|x_1(t) - x_2(t)\| + \int_t^\tau K(s)\|x_1(s) - x_2(s)\|ds + (\varepsilon_1 + \varepsilon_2).$$

Let $\alpha(\tau)$ and ε defined by:

$$\alpha(\tau) = \|x_1(\tau)-x_2(\tau)\|, \quad \varepsilon = \varepsilon_1+\varepsilon_2.$$

Then we have

$$\alpha(\tau) \leqq \alpha(t) + \int_t^\tau K(s)\alpha(s)ds + \varepsilon .$$

We can now apply the lemma of GRONWALL ([3], p. 35, Lemma 1), obtaining in this case:

$$\alpha(\tau) \leqq (\alpha(t)+\varepsilon) \exp\left(\int_t^\tau K(s)ds\right).$$

Substituting here the actual values of $\alpha(\tau)$, $\alpha(t)$ and ε we will obtain exactly the required statement.

We remark here that our theorem is more general that the wellknown inequality of Peano (see in [1] p. 8., Theorem 2.1) in so far that

1. function g is not prescribed to be continuous in its first variable, it is enough if it is from L_1,
2. K is not a constant in the Lipschitz condition, instead it is a function from L_1,
3. in the interval $I_1 = [0, a]$ $a = +\infty$ is allowed,
4. for the functions x_i, $i = 1, 2$, instead of the usual assumption:

$$\|\dot{x}_i(\tau)-g(\tau, x(\tau))\| < \varepsilon_i$$

an essentially weaker condition is prescribed, namely that they should be weakly ε_i — approximating solutions.

It can easily be seen that if $I_1 = [0, a_1]$ is a finite interval and φ is an ε-approximating solution of (1) in the usual sense then it is at the same time a weakly $a\cdot\varepsilon$-approximating solution too, as

$$\left\|\int_0^t (\dot{\varphi}(\tau)-g(\tau, \varphi(\tau)))d\tau\right\| \leqq \int_0^t \|\dot{\varphi}(\tau)-g(\tau, \varphi(\tau))\|d\tau < t\cdot\varepsilon < a\cdot\varepsilon.$$

A consequence of this theorem is the following existence theorem concerning the variational differential equation of differential equation (1).

We remark that the existence theorem to be presented here plays an important role in the investigations of existence properties of solutions to dual problems of control theory.

THEOREM. (Existence theorem concerning the variational differential equation.) *Let $I_1\subset \mathbf{R}^1$ be an arbitrary open, bounded interval, $I_2\subset \mathbf{R}^n$ be a bounded set. Let $g: I_1\times I_2 \to \mathbf{R}^n$ be a function with the following properties:*

(i) $g(., \xi)\in L_1(I_1)$ for all $\xi\in I_2$,

(ii) *it satisfies the global Lipschitz condition in its second variable:*
$\exists K_1\in L_1(I_1)$ so that for $\forall x_1, x_2\in I_2$ holds
$\|g(t, x_1)-g(t, x_2)\| \leqq K_1(t)\|x_1-x_2\|$,

(iii) $\partial_2 g(., \xi) \in L_1(I_1)$ *for all* $\xi \in I_2$,

(iv) $\exists K_2 \in L_1(I_1)$ *so that*

$\|\partial_2 g(t, x_1) - \partial_2 g(t, x_2)\| \leqq K_2(t) \|x_1 - x_2\| \ \forall x_1, x_2 \in I_2$,

(v)
$$\exists M_1 \in L_p(I_1),\ \exists M_2 \in L_q(I_1),\ \frac{1}{p} + \frac{1}{q} = 1:$$

$$\|g(t, x)\| \leqq M_1(t),\ \|\partial_2 g(t, x)\| \leqq M_2(t)\ \ \forall x \in I_2$$

and $\forall t \in I_1$ *for which it has a meaning.*

Under these assumption the characteristic function $\varphi(\tau, t, \xi)$ *of the initial value problem:*

(6)
$$\dot{x}(\tau) = g(\tau, x(\tau)),\ x(t) = \xi$$

possesses the following properties:

1. $\exists\ \partial_3 \varphi(., t, \xi)$ *for all* ξ *and it is a solution of the initial value problem* (2):

(7)
$$\dot{x}(\tau) = \partial_2 g(\tau, \varphi(\tau, t, \xi)) x(\tau),\ x(t) = E,$$

where E *denotes the* n *dimensional unit-matrix.*

2. $\exists\ \partial_2 \varphi(., t, \xi)$ *for a.e.* $t \in I_1$ *and it is a solution of the following initial value problem:*

(8)
$$\dot{x}(\tau) = \partial_2 g(\tau, \varphi(\tau, t, \xi)) x(\tau),\ x(t) = -g(t, \xi).$$

Proof. The first part of the theorem will be proved by means of the implicit function theorem ([4], 41.o.)

Introduce the following transformation:

$$\Phi : \mathbf{R} \times \mathbf{R}^n \times W_p^{1,n}(I_1) \to L_p^n(I_1) \times \mathbf{R}^n$$

$$\Phi(t, \xi, x) := (\dot{x} - g \circ (\Theta, x), x(t) - \xi).$$

a) Transformation Φ is continuously differentiable with respect to ξ and it is continuously differentiable in x, too.

Let (t, ξ) fixed, $h \in C^n(I_1)$ arbitrary. Consider the following expression:

$$\frac{\|\Phi(t, \xi, x+h) - \Phi(t, \xi, x) - (\dot{h} + \partial_2 g \circ (\Theta, x)h, 0)\|_{L_p^n \times \mathbf{R}^n}}{\|h\|_{W_p^{1,n}}}$$

It should be shown that this tends to 0 as $\|h\|_{W_p^{1,n}} \to 0$. It is enough to look at the following expression

$$\frac{\|g \circ (\Theta, x+h) - g \circ (\Theta, x) - \partial_2 g \circ (\Theta, x)h\|_{L_p^n}}{\|h\|_{W_p^{1,n}}}.$$

Let's take the transformation

$$\Psi : \alpha \to g(t, x(t) + \alpha) - \partial_2 g(t, x(t)),$$

$$\Psi : \mathbf{R}^n \to \mathbf{R}^n.$$

This transformation is continuous, differentiable and its derivative is the following:

$$d\Psi\colon \alpha \mapsto \partial_2 g(t, x(t)+\alpha) - \partial_2 g(t, x(t))$$

$$\|g\circ(\Theta, x+h) - g\circ(\Theta, x) - \partial_2 g\circ(\Theta, x)h\|^p_{L^n_p} =$$

$$= \int_{I_1} \|g(t, x(t)+h(t)) - g(t, x(t)) - \partial_2 g(t, x(t))h(t)\|^p dt =$$

$$= \int_{I_1} \|\Psi(h(t)) - \Psi(0)\|^p dt \leqq \int_{I_1} \Big(\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) -$$

$$- \partial_2 g(t, x(t))\|\Big)^p \|h(t)\|^p dt \leqq \|h\|^p_{W^{1,n}_p} \int_{I_1} \Big(\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) - \partial_2 g(t, x(t))\|\Big)^p dt.$$

So

$$\lim_{\|h\|_{W^{1,n}_p}\to 0} \frac{1}{\|h\|_{W^{1,n}_p}} \cdot \|g\circ(\Theta, x+h) - g\circ(\Theta, x) - \partial_2 g\circ(\Theta, x)h\|_{L^n_p} \leqq$$

$$\leqq \lim_{\|h\|_{W^{1,n}_p}\to 0} \left\{ \int_{I_1} \Big(\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) - \partial_2 g(t, x(t))\|\Big)^p dt \right\}^{1/p}. \tag{9}$$

We shall show that this limit-value is zero. This will be done by making use of the Lebesque theorem about the calculation of the integral of a function-sequence as the sum of the integrals of the individual additive terms.

To start with we show that for a.e. $t\in I_1$

$$\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) - \partial_2 g(t, x(t))\| \to 0,$$

if $\|h\|_{W^{1,n}_p} \to 0$. If $\|h\|_{W^{1,n}_p} \to 0$ then $\|h(t)\| \to 0$ for all $t\in I_1$.

Moreover the function $\partial_2 g$ is continuous in this second variable, so

$$\partial_2 g(t, x(t)+\vartheta h(t)) \to \partial_2 g((t, x(t)) \text{ for a.e. } t\in I_1.$$

This involves already that

$$\lim_{\|h\|_{W^{1,n}_p}\to 0} \Big(\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) - \partial_2 g(t, x(t))\|\Big) = 0.$$

The second assumption of Lebesque's theorem is also fulfilled as there exists an upper bounding function taken from L_1.

Namely because of assumption (v):

$$\max_{\vartheta\in[0,1]} \|\partial_2 g(t, x(t)+\vartheta h(t)) - \partial_2 g(t, x(t))\| \leqq 2M_2(t).$$

If $M_2\in L_p(I_1)$ then $M_2\in L_1(I_2)$ as I_1 is a bounded interval.

So the sequence of the integration and the limiting procedure can be exchanged on therefore the right-hand side of expression (9) will be zero.

With this we have shown that $\partial_x \Phi$ exists and has the following form:

$$\partial_x \Phi(t, \xi, .) : h \mapsto \dot{h} + \partial_2 g \circ (\Theta, x)h,$$

$$\partial_x \Phi(t, \xi, .) : w_p^{1,n} \to L_p^n(I_1).$$

It is linear and therefore continuous, too.

b) The implicit-function theorem can be applied to Φ in the following manner. Let $\tilde{t}$ be arbitrarily fixed. Let's assume that the function $x_0 \in W_p^{1,n}(I_1)$ is a solution of the initial value problem $\dot{x} = g \circ (\Theta, x)$, $x(\tilde{t}) = \xi_0$

Then it evidently holds that

$$\Phi(\tilde{t}, \xi_0, x_0) = 0 \in L_p^n(I_1) \times \mathbf{R}^n.$$

Transformation Φ is in the neighbourhood of the point (ξ_0, x_0), in $U \subset \mathbf{R}^n \times W_p^{1,n}(I_1)$ continuously differentiable as a consequence of our results so for. Moreover the transformation $\partial_x \Phi(t, \xi_0, x_0) : W_p^{1,n}(I_1) \to L_p^n(I_1)$ is a linear homeomorphism. So the assumptions of the implicit function theorem are fulfilled and therefore $\exists$ a neighbourhood U_0 of ξ_0 and a neighbourhood V_0 of x_0, moreover a transformation $x: U_0 \to V_0$ that $x(\xi_0) = x_0$ and x is continuously differentiable in U_0, and $\forall \xi \in U_0$

$$\partial_\xi x(\tilde{t}, \xi, \tau) = -[\partial_x \Phi(\tilde{t}, \xi, x(\tilde{t}, \xi, \tau))]^{-1} \circ \partial_\xi \Phi(\tilde{t}, \xi, x(\tilde{t}, \xi, \tau)).$$

As $x(\tilde{t}, \xi, \tau) = \varphi(\tau, \tilde{t}, \xi)$ so we have shown that the characteristic function φ is continuously differentiable in ξ for a fixed $\tilde{t}$.

It has to be shown yet that φ satisfies the conditions of the initial value problem (7). Because of the definition of the characteristic function we know that

$$\partial_1 \varphi(\tau, t, \xi) = g(\tau, \varphi(\tau, t, \xi)), \; \varphi(t, t, \xi) = \xi.$$

By integrating here we obtain

$$\varphi(\tau, t, \xi) = \xi + \int_t^\tau g(s, \varphi(s, t, \xi)) ds.$$

Differentiating here by ξ:

$$\partial_3 \varphi(\tau, t, \xi) = E + \int_t^\tau \partial_2 g(s, \varphi(s, t, \xi)) \partial_3 \varphi(s, t, \xi) ds.$$

And this is exactly the integral-form of (7). It is also evident now that the function $\partial_3 \varphi(\tau, t, \xi)$ is absolutely continuous in its first two variables $-\tau, t-$ and continuously differentiable in the third one.

Now we turn to the proof of the second statement of the theorem. The difficulties are mainly encountered in showing that $\exists \partial_2 \varphi(., t, \xi)$ for a.e. $t \in I_1$.

Introduce the following auxiliary function:

$$\psi(\tau, t, \vartheta, \xi) := \varphi(\tau, t+\vartheta, \xi) - \varphi(\tau, t, \xi).$$

$$\partial_1\psi(\tau, t, \vartheta, \xi) = \partial_1\varphi(\tau, t+\vartheta, \xi) - \partial_1\varphi(\tau, t, \xi) =$$

$$= g\big(\tau, \varphi(\tau, t+\vartheta, \xi)\big) - g\big(\tau, \varphi(\tau, t, \xi)\big). \tag{10}$$

As g is differentiable in its second variable, the right-hand side can be written here:

$$g(\tau, \varphi(\tau, t+\vartheta, \xi)) - g(\tau, \varphi(\tau, t, \xi)) =$$

$$= \partial_2 g\big(\tau, \varphi(\tau, t, \xi)\big)\psi(\tau, t, \vartheta, \xi) + \sigma(\tau, t, \vartheta, \xi). \tag{11}$$

From here we get

$$\partial_1\left(\frac{\psi(\tau, t, \vartheta, \xi}{\vartheta}\right) = \partial_2 g(\tau, \varphi(\tau, t, \xi))\,\frac{\psi(\tau, t, \vartheta, \xi)}{\vartheta} + \frac{\sigma(\tau, t, \vartheta, \xi)}{\vartheta}. \tag{12}$$

This means that the function $\frac{1}{\vartheta}\psi(\tau, t, \vartheta, \xi)$ is an approximating solution to the variational differential equation

$$\dot{x}(\tau) = \partial_2 g\big(\tau, \varphi(\tau, t, \xi)\big)x(\tau).$$

We shall show that if $\vartheta \to 0$ then

$$\int_{I_t} \frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta}\, d\tau \to 0.$$

Let $(t, \xi) \in \mathcal{D}(g)$ be fixed. We express $\frac{1}{\vartheta}\sigma(\tau, t, \vartheta, \xi)$ from equation (11)

$$\frac{\|\sigma(\tau, t, \vartheta, \xi\|}{\vartheta} = \frac{1}{\vartheta}\,\|g\big(\tau, \varphi(\tau, t+\vartheta, \xi) - g\big(\tau, \varphi(\tau, t, \xi)\big) - \tag{13}$$

$$- \partial_2 g\big(\tau, \varphi(\tau, t, \xi)\big)\cdot\big(\varphi(\tau, t+\vartheta, \xi) - \varphi(\tau, t, \xi)\big)\|.$$

Introduce transformation $\mathcal{F}$ in the following way:

$$\mathcal{F}: h \to g\big(\tau, \varphi(\tau, t, \xi) + h\big) - \partial_2 g\big(\tau, \varphi(\tau, t, \xi)\big)\cdot h.$$

This transformation, $\mathbf{R}^n \to \mathbf{R}^n$, is continuously differentiable and $\mathcal{F}'(h) = \partial_2 g\big(\tau, \varphi(\tau, t, \xi) + h\big) - \partial_2 g\big(\tau, \varphi(\tau, t, \xi)\big)$.

Let now be $h := \varphi(\tau, t+\vartheta, \xi) - \varphi(\tau, t, \xi)$. Then equation (13) can be written in the following way, too:

$$\frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta} = \frac{\|\mathcal{F}(h) - \mathcal{F}(0)\|}{\vartheta}.$$

Applying the Lagrange inequality for $\mathcal{F}$ in the interval $[0, h]$:

$$\frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta} \leqq \max_{\alpha\in[0,1]} \|\mathcal{F}'(\alpha\cdot h)\| \frac{\|h\|}{\vartheta} = \max_{\alpha\in[0,1]} \|\partial_2 g(\tau, \varphi(\tau, t, \xi)+$$

$$(14) \qquad +\alpha(\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)))-\partial_2 g(\tau, \varphi(\tau, t, \xi))\|\cdot$$

$$\cdot\frac{1}{\vartheta}\|\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)\|.$$

We want to show that the right-hand side of (14) tends to 0, if $\vartheta\to 0$. For thi it is enough to see that the second expression on the right-hand side is fo a.e. $\tau\in I_1$ bounded, and the first expression tends to zero.

The functions $\varphi(., t+\vartheta, \xi)$, resp. $\varphi(., t, \xi)$ are exact solutions to the differential equation $\dot{x} = g\circ(\Theta, x)$ with initial-value assumptions $x(t+\vartheta) = \xi$ and $x(t) = \xi$. Therefore Peano's inequality with $\varepsilon = 0$ can be applied.

$$\frac{1}{\vartheta}\|\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)\| \leq \frac{1}{\vartheta}\|\varphi(t+\vartheta, t+\vartheta, \xi)-\varphi(t, t, \xi)\|\cdot$$

$$\cdot\exp\left(\int_{t+\vartheta}^{\tau} K_1(s)dt\right) \leqq \frac{1}{\vartheta}\int_t^{t+\vartheta} \|g(s, \varphi(s, t, \xi))\|ds\cdot\exp\|K_1\|_{L_1}.$$

$$\lim_{\vartheta\to 0}\frac{1}{\vartheta}\int_t^{t+\vartheta} \|g(s, \varphi(s, t, \xi))\|ds = \|g(t, \xi)\| < +\infty \text{ for e.a. } t\in I_1.$$

So

$$(15) \qquad \lim_{\vartheta\to 0}\frac{1}{\vartheta}\|\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)\| \leqq \|g(t, \xi)\|\cdot\exp\|K_1\|_{L_1} < +\infty$$

for a.e. $t\in I_1$ — for such t values where $g(t, \xi)$ makes sense —. From (15) we obtain

$$\|\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)\| \leqq \vartheta\cdot\text{constant, for a.e. } t\in I_1,\ \tau\in I_1.$$

So

$$\lim_{\vartheta\to 0}\|\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)\| = 0 \text{ for a.e. } \tau\in I_1, t\in I_1.$$

This gives that

$$\lim_{\vartheta\to 0}(\max\|\partial_2 g(\tau, \varphi(\tau, t, \xi)+\alpha(\varphi(\tau, t+\vartheta, \xi)-\varphi(\tau, t, \xi)))-\partial_2 g(\tau, \varphi(\tau, t, \xi))\|) = 0.$$

We have shown so that

$$\lim_{\vartheta\to 0}\frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta} = 0 \text{ for a.e. } \tau\in I_1, t\in I_1.$$

In addition it will be shown that φ has an upper bounding function which can be integrated.

By making use of expression (14) and assumption (v) the following estimation is obtained:

$$\frac{\sigma(\tau, t, \vartheta, \xi)}{\vartheta} \leqq 2M_2(t)(\|g(t, \xi)\| \cdot \exp\|K_1\|_{L_1} + \varepsilon) \leqq$$

$$\leqq 2M_2(t)(M_1(t) \cdot \exp\|K_1\|_{L_1} + \varepsilon) \in L_1(I_1),$$

as $M_2 \in L_q$, $M_1 \in L_p$, and $\frac{1}{p} + \frac{1}{q} = 1$.

So Lebesgue's theorem can be applied which gives

$$\lim_{\vartheta \to 0} \int_{I_1} \frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta} d\tau = \int_{I_1} \lim_{\vartheta \to 0} \frac{\|\sigma(\tau, t, \vartheta, \xi)\|}{\vartheta} d\tau = 0.$$

But this means exactly that for $\forall \varepsilon < 0 \exists \sigma_\varepsilon > 0$, so that if $\vartheta < \delta_\varepsilon$ then

$$\int_{I_1} \frac{1}{\vartheta} \|\sigma(\tau, t, \vartheta, \xi)\| d\tau < \varepsilon. \tag{16}$$

Let now $\varepsilon > 0$ arbitrarily fixed and $\vartheta < \delta_\varepsilon$. Then (16) means that the function $\frac{1}{\vartheta}\psi(., t, \vartheta, \xi)$ is also a weakly ε-approximating solution in a generalized sense (see Def.) of the variational differential equation:

$$\dot{x}(\tau) = \partial_2 g(\tau, \varphi(\tau, t, \xi))x(\tau).$$

Let $y(\tau)$ the exact solution of the initial-value problem (8). Applying the generalized Peano inequality for y and $\frac{1}{\vartheta}\psi(., t, \vartheta, \xi)$.

$$\left\| \frac{1}{\vartheta}\psi(\tau, t, \vartheta, \xi) - y(\tau) \right\| \leqq \left\| \frac{1}{\vartheta}\psi(t, t, \vartheta, \xi) - y(t) \right\| \cdot \exp \int_t^\tau K_2(s)ds + \tag{13}$$

$$+ \varepsilon\left(\exp \int_t^\tau K_2(s)ds + 1\right) = \left\| \frac{\varphi(t, t+\vartheta, \xi) - \varphi(t, t, \xi)}{\vartheta} + g(t, \xi) \right\| \cdot$$

$$\cdot \exp \int_t^\tau K_2(s)ds + 2\varepsilon \cdot \left(\exp \int_t^\tau K_2(s)ds + 1\right).$$

$$\lim_{\vartheta \to 0} \left\| \frac{\varphi(t, t+\vartheta, \xi) - \varphi(t, t, \xi)}{\vartheta} + g(t, \xi) \right\| =$$

$$= \lim_{\vartheta \to 0} \left\| \frac{1}{\vartheta} \int_{t+\vartheta}^{t} g(\tau, \varphi(s, t, \xi))\, ds + g(t, \xi) \right\| = 0.$$

So making use of (13) we obtain for $\varepsilon \to 0$:

$$\lim_{\vartheta \to 0} \left\| \frac{1}{\vartheta} \psi(\tau, t, \vartheta, \xi) - y(\tau) \right\| = 0 \text{ for a.e. } \tau \in I_1,\ t \in I_1.$$

That is

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \psi(\tau, t, \vartheta, \xi) = y(\tau).$$

This gives that

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \psi(\tau, t, \vartheta, \xi) = \partial_2 \varphi(\tau, t, \xi)$$

is the exact solution of (8). This completes our proof.

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THE FOCAL LOCUS OF A SUBMANIFOLD IN A RIEMANNIAN MANIFOLD

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(Received September 12, 1983)

Let M be a complete Riemannian manifold, $L \subset M$ a submanifold then $N(L)$, the normal bundle of L, is a submanifold of the tangent bundle TM and therefore the exponential map $\exp\colon TM \to M$ of the Riemannian manifold can be restricted to $N(L)$. A singular point of this restricted exponential map is called focal point of the submanifold L and the set of these focal points is called the focal locus of L. In the special case when the submanifold reduces to a single point, i.e. $L = \{x\}$, then $N(L) = T_xM$ and the restricted exponential map is $\exp_x : T_xM \to M$. In this case the focal points are the points conjugate to x and the focal locus is the conjugate locus of x in T_xM. As the conjugate locus of a point has a perspicuous structure according to results of F. W. WARNER [4], the question concerning the structure of the focal locus naturally arises. The first steps to answer this question are made below. In fact, it is shown that the fundamental results of WARNER concerning conjugate loci generalize almost completely to focal loci of submanifolds in Riemannian manifolds.

1. Some basic facts and concepts concerning submanifolds of Riemannian manifolds and the associated Jacobi-fields

Let M be an m-dimensional complete Riemannian manifold with a given metric g. Then a metric $\bar{g}$ of its tangent bundle TM can be given canonically as follows ([1] pp. 47)

$$\bar{g}(Z, Z) = g(Tp_M(Z), Tp_M(Z)) + g(K(Z), K(Z))$$

for any $Z \in TTM$, where $Tp_M\colon TTM \to TM$ is the tangent linear map belonging to the canonical projection $p_M\colon TM \to M$ and $K\colon TTM \to TM$ is the unique (Levi-Civita) connexion map corresponding to g.

It is well-known that for every $v \in TM$ the fibre T_vTM of the canonical projection

$$P_{TM}\colon TTM \to TM$$

has a unique (Levi-Civita) orthogonal decomposition

$$T_v TM = H_v TM \oplus V_v TM$$

where the subspaces

$$H_v TM = \operatorname{Ker}(K \restriction T_v TM),$$
$$V_v TM = \operatorname{Ker}(Tp_M \restriction T_v TM)$$

are called the horizontal and vertical subspaces of $T_v TM$, respectively. Moreover the following conditions are satisfied:

1. Both of the canonical isomorphisms

$$h_v^{-1} = Tp_M \restriction H_v TM : H_v TM \to T_x M$$

and

$$\iota_v = K \restriction V_v TM : V_v TM \to T_x M$$

are isometries for any $v \in TM$, where $x = p_M(v)$.

2. The canonical projection $p_M : TM \to M$ is a Riemannian submersion.

Let now $v: V \to TM$ be a vectorfield on a neighborhood V of $x \in M$ then for the covariant derivative of v in the direction $v_1 \in T_x M$ we have

$$v_2 = \nabla_{v_1} v(x) = K \cdot Tv(v_1)$$

where Tv is the tangent linear map defined by v.

Since $Tv(v_1) \in T_v TM$

$$Tv(v_1) = h_v(v_1) + \iota_v^{-1}(v_2)$$

holds by the above decomposition. More generally, for any $Z \in TTM$

$$Z = h_v(v_1) + \iota_v^{-1}(v_2) \text{ holds where}$$
$$v_1 = Tp_M(Z),\ v_2 = K(Z) \text{ and } v = p_{TM}(Z).$$

It has to be mentioned here that any $Z \in TTM$ $(p_{TM}(Z) \neq 0)$ defines uniquely a Jacobi-field

$$J: [0, \infty) \to TM \quad \text{along the geodesic}$$
$$\gamma: [0, \infty) \to M \quad \text{starting with}$$
$$\dot\gamma(0) = v = p_{TM}(Z)$$

such that the following initial conditions are considered:

$$J(0) = v_1 = Tp_M(Z),$$
$$J'(0) = v_2 = K(Z).$$

Let now $L \subset M$ be a k-dimensional compact submanifold of M where $0 \leq k < m$. Then the normal bundle $N(L)$ of L defined by

$$N(L) = \cup \{N_x L | x \in L\}$$

where $N_x L \subset T_x M$ is the normal space to L at $x \in L$, is a subbundle of TM.

In order to describe the canonical metric $\tilde{g}$ of the normal bundle $N(L)$ — it was used in [2] by E. HEINTZE and H. KARCHER — first we make some further preparations.

Let $\nu: V \to N(L)$ be a section of the normal bundle $N(L)$ on a neighborhood V of $x \in L$, then for the covariant derivative of ν in the direction $\nu_1 \in T_x L$ we have that

$$\nu_2 = \nabla_{\nu_1} \nu(x) = K \cdot T\nu(\nu_1)$$

holds as before. But considering the orthogonal decomposition

$$T_x M = T_x L \oplus N_x L$$

of the tangent space $T_x M$

$$\nu_2 = \sigma_\nu(\nu_1) + \tilde{\nu}_2$$

is valid, where

$$\sigma_\nu: T_x L \to T_x L$$

s the Weingarten map corresponding to the normal $\nu(x) \in N_x L$ and $\tilde{\nu}_2 = \tilde{\nabla}_{\nu_1} \nu(x) \in N_x L$.

Now the inclusions $N(L) \subset TM$ and $TN(L) \subset TTM$ permit to consider an induced connexion map

$$\tilde{K}: TN(L) \to N(L)$$

given by the definition

$$\tilde{K}(Z) = K(Z) - \sigma_\nu(\nu_1), \text{ for any } Z \in TN(L)$$

where

$$\nu = p_{TM} \restriction TN(L)\,(Z) \in N_x L \text{ and } \nu_1 = Tp_M \restriction TN(L)\,(Z) \in T_x L$$

Thus the induced covariant derivative of ν in the direction ν_1 is given simply by

$$\tilde{\nu}_2 = \tilde{\nabla}_{\nu_1} \nu(x) = \tilde{K} \cdot T\nu(\nu_1) \in N_x L.$$

Now the canonical metric $\tilde{g}$ of the normal bundle $N(L)$ can be given as follows:

$$\tilde{g}(Z, Z) = g\big(Tp_M(Z), Tp_M(Z)\big) + g\big(\tilde{K}(Z), \tilde{K}(Z)\big)$$

for any $Z \in TN(L)$.

It is easy to see that for every $\nu\in N(L)$ the fibre $T_\nu NL$ of the canonical projection

$$P_{TM}\restriction TN(L):TN(L)\to N(L)$$

has a unique orthogonal decomposition

$$T_\nu N(L)=\tilde{H}_\nu N(L)\oplus\tilde{V}_\nu N(L),$$

where the subspaces

$$\tilde{H}_\nu N(L)=\mathrm{Ker}\,(\tilde{K}\restriction T_\nu N(L)),$$
$$\tilde{V}_\nu N(L)=\mathrm{Ker}\,(Tp_M\restriction T_\nu N(L))$$

are called the horizontal and vertical subspaces of $T_\nu NL$, respectively. Moreover the following conditions are satisfied:

1. Both of the isomorphisms

$$\tilde{h}_\nu^{-1}=Tp_M\restriction\tilde{H}_\nu N(L):\tilde{H}_\nu N(L)\to T_xL$$

and

$$\tilde{\iota}_\nu=\tilde{K}\restriction\tilde{V}_\nu N(L):\tilde{V}_\nu N(L)\to N_xL$$

are isometries for any $\nu\in N(L)$, where $x=p_M(\nu)\in L$,

2. The canonical projection

$$p_M\restriction N(L):N(L)\to L$$

is a Riemannian submersion.

It should be noticed that for any $Z\in TN(L)\subset TTM$ we have already two orthogonal decompositions. In the first case the components of Z

$$Z_{\mathrm{hor}}=h_\nu(\nu_1)\text{ and }Z_{\mathrm{vert}}=\iota_\nu^{-1}(\nu_2)$$

are orthogonal in the metric $\bar{g}$ of TM; in the second case Z has the following components:

$$\tilde{Z}_{\mathrm{hor}}=Z_{\mathrm{hor}}+\iota_\nu^{-1}(\sigma_\nu(\nu_1))$$

and

$$\tilde{Z}_{\mathrm{vert}}=Z_{\mathrm{vert}}-\iota_\nu^{-1}(\sigma_\nu(\nu_1)),$$

which are evidently orthogonal in the metric $\tilde{g}$ of the normal bundle $N(L)$.

Let $\varphi:TN(L)\to TM$ be the map defined for any $Z\in TN(L)$ as follows:

$$\varphi(Z)=\nu_1+\nu_2-\sigma_\nu(\nu_1)\in T_xM,\ \text{where } x=p_M(\nu).$$

Then it is easy to see that for any $\nu\in N(L)$ the map

$$\varphi\restriction T_\nu N(L):T_\nu N(L)\to T_xM$$

is an isometry if the metrics $\tilde{g}$ and g are considered.

Now a characterization of the so-called L-Jacobi-fields will be given here.

It is clear that if $\nu \in N(L)$ is a non-zero and $\varrho:[0,\infty)\to N(L)$ is a ray of $N(L)$ defined by $\varrho(\tau)=\tau\cdot\nu$, for $\tau\in[0,\infty)$ then $\dot\varrho(1)\in T_\nu N(L)$ holds.

Let $Z\in TN(L)$ such that $\nu = p_{TM}(Z)\neq 0$ and $\tilde g(\dot\varrho(1), Z) = 0$ hold.

Then Z defines uniquely an L-Jacobi-field

$J:[0,\infty)\to TM$ along the geodesic $\gamma:[0,\infty)\to M$ starting with $\dot\gamma(0) = = \nu = p_{TM}(Z)\in N_x L$ such that the following initial conditions are considered:

$$J(0) = \nu_1 = Tp_M(Z)\in T_x L,$$

$$J'(0) = \nu_2 = K(Z).$$

Notice that the conditions

$$g(\nu, \nu_1) = 0 \text{ and } g(\nu, \nu_2) = \tilde g(\dot\varrho(1), Z) = 0$$

imply that $g(\dot\gamma, J) = 0$ is valid for every parameter along the geodesic γ. On the other hand

$$\tilde K(Z) = \nu_2 - \sigma_\nu(\nu_1)\in N_x L$$

holds, as well.

The following proposition reduces to the Gauss Lemma ([3] pp 136 – 137) in that special case when the submanifold L is a single point.

PROPOSITION 1. Let L be a submanifold of a Riemannian manifold M and

$$\varepsilon: N(L)\to M$$

the restricted exponential map. If $\nu\in N(L)$ is a non-zero vector and ϱ is the ray of $N(L)$ defined by ν then

$$\tilde g(\dot\varrho(1), Z) = g(T_\nu\varepsilon(\dot\varrho(1)),\ T_\nu\varepsilon(Z))$$

holds for any $Z\in T_\nu N(L)$, where $T_\nu\varepsilon$ denotes the tangent linear map of ε at ν.

PROOF. The tangent vector $Z\in T_\nu N(L)$ can be represented by a smooth variation Φ of the geodesic $\gamma = \varepsilon\circ\varrho$

$$\Phi(\tau, \sigma) = \varepsilon(\tau\cdot\nu(\sigma)), \text{ for } \tau, \sigma\in\mathbf{R}$$

where $Z = \dfrac{d}{d\sigma}\nu(\sigma)|_{\sigma=0}$. In other words, $Z\in T_\nu N(L)$ defines a Jacobi-field J along the geodesic γ such that

$$\dot\gamma(0) = \nu = p_{TM}(Z),$$

$$J(0) = \nu_1 = Tp_M(Z),$$

$$J'(0) = \nu_2 = K(Z)$$

are the initial conditions and

$$J(1) = T_v\varepsilon(Z),$$
$$\dot{\gamma}(1) = T_v\varepsilon(\dot{\varrho}(1))$$

are also valid. The Jacobi equation implies that

$$g(\dot{\gamma}, J'') = 0 \text{ and so } g(\dot{\gamma}, J) = c_1\tau + c_2 \text{ holds.}$$

Now, applying the definition of the metric $\tilde{g}$

$$\tilde{g}(\dot{\varrho}(1), Z) = g(0, v_1) + g(v, v_2 - \sigma_v(v_1)) = g(\dot{\gamma}(0), J'(0))$$

is valid after a short computation.

Thus, substituting $\tau = 1$ and

$$c_1 = g(\dot{\gamma}(0), J'(0)) = \tilde{g}(\dot{\varrho}(1), Z),$$
$$c_2 = g(\dot{\gamma}(0), J(0)) = 0$$

we get the desired result

$$g(\dot{\gamma}(1), J(1)) = \tilde{g}(\dot{\varrho}(1), Z).$$

Corollary. Let L be a submanifold of a Riemannian manifold M and let

$$\varepsilon: N(L) \to M$$

be the restricted exponential map. If $v \in N(L)$ is a nonzero vector then the kernel of the map $T_v\varepsilon$ is orthogonal to $\dot{\varrho}(1)$ where ϱ is the ray defined by v.

The following proposition yields a simple fact which will be essential for subsequent arguments.

Proposition 2. Let L be a submanifold of a Riemannian manifold M, $\gamma: \mathbf{R} \to M$ a geodesic with $\gamma(0) = x \in L$, $\dot{\gamma}(0) \in N_xL$ and $\mathcal{J}(\gamma; L)$ the space of L-Jacobi fields defined along γ. Consider for any $z = \gamma(\xi)$, $\xi \in \mathbf{R}$ the subspaces

$$A_z = \{X(\xi) \mid X \in \mathcal{J}(\gamma; L)\},$$
$$B_z = \{X'(\xi) \mid X(\xi) = 0,\ X \in \mathcal{J}(\gamma; L)\}$$

of T_zM. Then A_z and B_z are orthogonal and they span the orthogonal complement of $\dot{\gamma}(\xi)$ in T_zM.

Proof. If $u \in A_z$, $v \in B_z$ then $u = X(\xi)$ $v = Y'(\xi)$ with $X, Y \in \mathcal{J}(\gamma; L)$ such that $Y(\xi) = 0$. By basic properties of Jacobi fields

$$g(X(\tau), Y'(\tau)) - g(X'(\tau), Y(\tau)) = \text{const for } \tau \in \mathbf{R}.$$

On the other hand the symmetry of the Weingarten map and definition of L-Jacobi fields yield

$$g(X(0), Y'(0)) - g(X'(0), Y(0)) = 0$$

which means that the above constant is zero.

Thus $g(u, v) = 0$ and therefore $A_z \perp B_z$ is valid. Assume now that $\xi \cdot \dot{\gamma}(0) \in N_x L$ is a focal point of order r i.e.

$$r = \dim \operatorname{Ker} T_{\xi \cdot \dot{\gamma}(0)} \varepsilon$$

holds. Then $\dim B_z = r$ and $\dim A_z = m-1-r$ are also valid, where $m = \dim M$.

Proposition 3. Let L be a submanifold of a Riemannian manifold M, $v \in N(L)$ a non-zero vector, ϱ the ray defined by v and $A:\mathbf{R} \to TN(L)$ a non-vanishing vector field along ϱ. Consider now the vector field $B:\mathbf{R} \to TM$ given by $B(\tau) = T_{\varrho(\tau)}\varepsilon(A(\tau))$ along the geodesic $\gamma = \varepsilon \circ \varrho$, where $\tau \in \mathbf{R}$.

Then there is a non-vanishing C^∞ vector field

$$E:\mathbf{R} \to TM$$

along γ and a C^∞ function

$$\varphi:\mathbf{R} \to \mathbf{R}$$

such that $B(\tau) = \varphi(\tau) \cdot E(\tau)$, $\tau \in \mathbf{R}$ and $\varphi'(\tau) \neq 0$ holds whenever $\varphi(\tau) = 0$ for some $\tau \in \mathbf{R}$.

Proof. Let $E_1, \dots, E_m:\mathbf{R} \to TM$ be orthonormal and parallel vector fields along γ where $m = \dim M$. Then there are C^∞ functions $\varphi_1, \dots, \varphi_m:\mathbf{R} \to \mathbf{R}$ such that

$$B(\tau) = \sum_{i=1}^{m} \varphi_i(\tau) E_i(\tau), \quad \tau \in \mathbf{R}$$

holds. Suppose that

$$B(\tau_0) = T_{\varrho(\tau_0)}\varepsilon\, A(\tau_0) = 0$$

for some $\tau_0 \in \mathbf{R}$, then $A(\tau_0) \neq 0$ implies that $B'(\tau_0) \neq 0$.

In fact, let $A_1, \dots, A_m:\mathbf{R} \to TN(L)$ be independent vector fields along ϱ such that $B_i(\tau) = T_{\varrho(\tau)}\varepsilon A_i(\tau)$, $i = 1, 2, \dots, m$, $\tau \in \mathbf{R}$ are Jacobi fields. Moreover, suppose that for $A_1(\tau_0) = A(\tau_0)$ holds A_1 at $\tau_0 \in \mathbf{R}$.

Consider now the linear combinations

$$A(\tau) = \sum_{i=1}^{m} \xi_i(\tau) A_i(\tau) \quad \text{and} \quad B(\tau) = \sum_{i=1}^{m} \xi_i(\tau) B_i(\tau)$$

$$\text{where} \quad \xi_i(\tau_0) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } i = 2, \dots, m. \end{cases}$$

For the derivative of $B(\tau)$ at $\tau_0 \in \mathbf{R}$ we have

$$B'(\tau_0) = \sum_{i=1}^{m} \xi_i'(\tau_0) \cdot B_i(\tau_0) + \sum_{i=1}^{m} \xi_i(\tau_0) \cdot B_i'(\tau_0),$$

where the second sum is equal to $B_1'(\tau_0)$. Since the Jacobi field $B_1:\mathbf{R} \to TM$ is not identically zero $B_1(\tau_0) = 0$ implies that $B_1'(\tau_0) \neq 0$.

On the other hand the first sum of vectors is orthogonal to $B_1'(\tau_0)$ by proposition 2. Thus $B'(\tau_0)\neq 0$ holds, as well.

But from

$$B(\tau)=\sum_{i=1}^{m}\varphi_i(\tau)E_i(\tau)\quad B'(\tau_0)=\sum_{i=1}^{m}\varphi_i'(\tau_0)E_i(\tau_0)\neq 0,$$

hence not all of the $\varphi_1'(\tau_0)$ are zero. Consequently the zeros of B are isolated. This means that there exists a neighborhood U of τ_0 where $\varphi_i(\tau)=(\tau-\tau_0)g_i(\tau)$ is valid with C^∞ functions $g_i(i=1,2,\dots,m)$ and if $g_i(\tau_0)=\varphi_i'(\tau_0)\neq 0$ holds then $g_i(\tau)$ nowhere vanish on U.

Consider now on U the following C^∞ scalar function

$$\varphi(\tau)=(\tau-\tau_0)\cdot\sqrt{\sum_{i=1}^{m}g_i^2(\tau)}=\begin{cases}-\sqrt{\sum\limits_{i=1}^{m}\varphi_i^2(\tau)} & \text{for } \tau\leq\tau_0,\ \tau\in U,\\ \sqrt{\sum\limits_{i=1}^{m}\varphi_i^2(\tau)} & \text{for } \tau\geq\tau_0,\ \tau\in U.\end{cases}$$

Then we have to show that $E(\tau)=\dfrac{B(\tau)}{\varphi(\tau)}$ is a C^∞ unit vector field on U.

It is trivial for $\tau\neq\tau_0$ since

$$E(\tau)=\begin{cases}-\dfrac{B(\tau)}{\|B(\tau)\|} & \text{for } \tau<\tau_0,\\ \dfrac{B(\tau)}{\|B(\tau)\|} & \text{for } \tau>\tau_0.\end{cases}$$

But for $\tau=\tau_0$ we have

$$\lim_{\tau\to\tau_0}E(\tau)=\lim_{\tau\to\tau_0}\frac{\sum\limits_{i=1}^{m}(\tau-\tau_0)g_i(\tau)\cdot E_i(\tau)}{(\tau-\tau_0)\sqrt{\sum\limits_{i=1}^{m}g_i^2(\tau)}}=\frac{B'(\tau_0)}{\|B'(\tau_0)\|}.$$

Corollary. If $B(\tau)=0$ then $E(\tau)\|B'(\tau)$.

Proposition 4. Let L be a submanifold of the Riemannian manifold M and $v\in F(L)\subset N(L)$ a focal point of order r and $\varrho:\mathbf{R}\to N(L)$ the ray defined by v. Then there is a coordinate system $(x^1,\dots,x^m)$ on a regular neighborhood V of v and a coordinate system $(y^1,\dots,y^m)$ of M defined on the neighborhood $\varepsilon(V)$ of $\varepsilon(v)$ such that the following condition is satisfied:

$$T_{\varrho(\tau)}\varepsilon\frac{\partial}{\partial x_j}=\varphi_j(\tau)\cdot\frac{\partial}{\partial y_j}\ \text{ for }\ \tau\in I,\ j=1,2,\dots,m,$$

where φ_j, $j = 1, \ldots, m$ are C^∞ functions such that

1. $\varphi_j(\tau) \neq 0$, $\tau \in I$ for $j = 1, \ldots, m-r$;
2. $\varphi_j(1) = 0$ and $\varphi_j(\tau) \neq 0$ for $\tau \in I \setminus \{1\}$, where $j = m-r+1, \ldots, m$;
3. $\varphi_j'(1) > 0$ where $j = m-r+1, \ldots, m$.

PROOF. Consider a base $(a_1, \ldots, a_m)$ of $T_\nu N(L)$ such that

$$(a_{m-r+1}, \ldots, a_m)$$

is a base of the kernel of $T_\nu \varepsilon$. Let then $A_1, \ldots, A_m: \mathbf{R} \to TN(L)$ be independent vector fields with $A_i(1) = a_i$ for $i = 1, \ldots, m$ along the ray ϱ such that the vector fields $B_1, \ldots, B_m: I' \to TM$ given by

$$B_i(\tau) = T_{\varrho(\tau)}\varepsilon(A_i(\tau)),\ \tau \in I' \text{ for } i = 1, \ldots, m$$

are Jacobi fields along $\gamma \restriction I'$ where I' is a sufficiently small neighborhood of 1 in $\mathbf{R}$ such that γ is injective on I'. Then there are non-vanishing vector fields $E_1, \ldots, E_m: I' \to TM$ and functions $\varphi_1, \ldots, \varphi_m: I' \to \mathbf{R}$ such that $B_i(\tau) = \varphi_i(\tau) \cdot E_i(\tau)$, $\tau \in I'$ for $i = 1, \ldots, m$ and $\varphi_i'(\tau) \neq 0$ whenewer $\varphi_i(\tau) = 0$.

Now $\varphi_1(1) \neq 0, \ldots, \varphi_{m-r}(1) \neq 0$, $\varphi_{m-r+1}(1) = \ldots = \varphi_m(1) = 0$. Then $\varphi_{m-r+1}'(1), \ldots, \varphi_m'(1) > 0$ can be also achieved by changing sign E_j, φ_j. Then $(E_1(1), \ldots, E_m(1))$ is a base of $T_{\varepsilon(\nu)}M$ by Proposition 3. But then there is a coordinate system $(x_1, \ldots, x_m)$ on a neighborhood of ν such that

$$\frac{\partial}{\partial x^j}(\varrho(\tau)) = A_j(\tau)$$

and a coordinate system $(y_1, \ldots, y_m)$ on a neighborhood of $\varepsilon(\nu)$ with

$$\frac{\partial}{\partial y^j}(\varepsilon \circ \varrho(\tau)) = E_j(\tau).$$

Then the assertions follow.

2. Some fundamenthal properties of the focal locus of a Riemannian manifold

PROPOSITION 5. Let L be a submanifold of a Riemannian manifold M and $\nu \in N_z L$ a non-zero vector which is not a focal point of L. Then there is a neighborhood V of ν in $N(L)$ such that for any $\bar{\nu} \in V$ the geodesic

$$\bar{\gamma}(\tau) = \varepsilon(\tau\bar{\nu}),\ \ \tau \in [0,1]$$

has the same index with respect to L; in other words the index

$$\operatorname{Ind}(I_{\mathcal{V}^0(\bar{\gamma};\, L)})$$

is constant for $\bar{\nu} \in V$.

PROOF. Consider first the map $T\varepsilon: TN(L) \to TM$. Since $\nu \in N(L)$ is not a focal point of L, the Jacobi determinant of $T_\nu \varepsilon$ is non-zero. So its continuity yields a neighborhood W of ν in $N(L)$ such that for any $\bar{\nu} \in W$ the nap $T_{\bar{\nu}}\varepsilon$ has maximal rank, as well i.e. $\bar{\nu}$ is not a focal point of L.

Consider now the compact set of points of all geodesics given by

$$\Gamma = \{p \mid p = \bar{\gamma}(\tau) = \varepsilon(\tau\bar{v}),\ \bar{v}\in W,\ \tau\in[0,1]\}.$$

At every point $p = \bar{\gamma}(\tau)$ of Γ we can take the compact set of all bivectors defined by the twodimensional subspaces of $T_{\bar{\gamma}(\tau)}M$ containing $\bar{\gamma}'(\tau)$. Thus, from compactness the curvature $\varkappa$ remains bounded, i.e. $\varkappa < K > 0$ holds, and so on account of the Morse-Schoenberg theorem ([3] pp. 176) we have that two conjugated points on any $\bar{\gamma}$ are in a distance no less than $\frac{\pi}{\sqrt{K}}$.

On the other hand in case of a compact submanifold $L\subset M$ there is a tubular neighborhood of the zero-section of $N(L)$ where the restriction of ε is a diffeomorphism.

These facts imply that there is a universal subdivision

$$0 = \tau_0 < \tau_1 \ldots < \tau_s = 1$$

of the parameter interval $[0,1]$ in the sense that it is admissible for every geodesic

$$\bar{\gamma}(\tau) = \varepsilon(\tau\bar{v}),\ \bar{v}\in W,\ \tau\in[0,1].$$

We call here a subdivision admissible for the corresponding geodesic $\bar{\gamma}$ if there is no focal point on the first segment belonging to $[\tau_0, \tau_1]$ and there are no conjugated points on the further segments belonging to $[\tau_i, \tau_{i+1}]$, $i = 1, \ldots, s-1$.

Now we have to study the index of the geodesic $\bar{\gamma}$ and compare it to that of γ.

The index form

$$I(V, W) = g(\sigma_{\bar{\gamma}'(0)}V(0), W(0)) + \int_0^1 \{g(V', W') - g(R(V, \bar{\gamma}')\bar{\gamma}', W)\}d\tau$$

is a symmetric bilinear form on the infinit dimensional vector space $\mathcal{V}^0(\bar{\gamma}; L)$. But as $\mathcal{V}^0(\bar{\gamma}; L)$ can be approximated by the finite dimensional vector space $\mathcal{A}^0(\bar{\gamma}, L)$ of the broken Jacobi fields so $\operatorname{Ind} I_{\mathcal{A}^0}(\bar{\gamma}) = \operatorname{Ind} I_{\mathcal{V}^0}(\bar{\gamma})$ holds. Notice that

$$\dim \mathcal{A}^0(\bar{\gamma}; L) = (s-1)(m-1),$$

where $m = \dim M$. On the other hand it is clear that the index and the augmented index are the same for $\bar{\gamma}$ since $\bar{v}$ is not a focal point and so

$$\dim \mathcal{J}^0(\bar{\gamma}; L) = 0.$$

Let now $r = \operatorname{Ind} I_{\mathcal{A}^0}(\gamma)$, where $r \le (s-1)(m-1)$. Then there are linearly independent broken Jacobi fields A_i for $i = 1, \ldots, (s-1)(m-1)$ for which $I_\gamma(A_i, A_i) < 0$ for $i = 1, \ldots, r$ and $I_\gamma(A_i, A_i) > 0$ for $i = r+1, \ldots, (s-1)(m-1)$ hold. It can be shown by continuity arguments that for each

$i = 1, \ldots, (s-1)(m-1)$ there exists a neighborhood V_i of ν such that along the geodesic $\bar{\gamma} = \varepsilon(\tau\bar{\nu})$ for $\bar{\nu} \in V_i$ one can choose a broken Jacobi field $\bar{A}_i$ sufficiently near to A_i in such a way that

$$I_{\bar{\gamma}}(\bar{A}_i, \bar{A}_i) < 0 \quad \text{for} \quad i = 1, \ldots, r$$

and

$$I_{\bar{\gamma}}(\bar{A}_i, \bar{A}_i) > 0 \quad \text{for} \quad i = r+1, \ldots, (s-1)(m-1)$$

still hold.

Let $V = \bigcap_{i=1}^{(m-1)(s-1)} V_i$. Evidently for $\bar{\nu} \in V$ the linear independence of the fields $\bar{A}_i$ can be easily achieved at the choice. Consequently,

$$\text{Ind } I(\bar{\gamma}) = \text{Ind } I(\gamma)$$

holds.

Proposition 6. Let $\nu \in F(L) \subset N(L)$ be a focal point of order r. Then ν has a normal neighborhood V in $N(L)$ such that for any ray $\bar{\varrho}$ of $N(L)$ intersecting V the number of focal points of L on $\bar{\varrho} \cap V$ counted with multiplicities is constant and equal to r.

Proof. Let $0 < \tau' < 1$ be such that $\xi\nu$ is not a focal point of L for $\tau' \leq \xi < 1$ and put $\nu' = \tau'\nu$. Consider now a neighborhood V' of ν' in $N(L)$ according to Prop. 5. Similarly let $\tau'' > 1$ be such that $\xi\nu$ is not a focal point of L for $1 < \xi \leq \tau''$. Consider a neighborhood V'' of ν'' in $N(L)$ according to Prop. 5.

Then for any $\bar{\varrho}$ of $N(L)$ intersecting V'' the number of focal points on the part of $\bar{\varrho}$ to the last common point with V'' is equal to the number of focal points of L on ϱ, $\tau \in [0, \tau'']$. On the other hand for any $\bar{\varrho}$ of $N(L)$ intersecting V' the number of focal points on the part of $\bar{\varrho}$ to the first common point with V' is equal to the number of focal points of L on ϱ, $\tau \in [0, \tau']$.

Consider now the set of rays $\bar{\varrho}$ of $N(L)$ which intersect both V' and V'' and let V be the union of segments of $\bar{\varrho}$ from the first common point with V' to the last common point with V''. The assertion of the proposition now follows obviously.

3. The ordinary focal locus

Let L be a submanifold of a Riemannian manifold M and $F(L)$ the focal locus of L in its normal boundle $N(L)$. A focal point $\nu \in F(L)$ is said to be ordinary point of $F(L)$ if there is a neighborhood U of ν in $N(L)$ such that any ray of $N(L)$ intersecting U has at most one point in common with $U \cap F(L)$; otherwise the point $\nu \in F(L)$ is said to be a branch point of $F(L)$.

The set $F^0(L)$ of the ordinary points of $F(L)$ is called the ordinary focal locus of L. The set $F^B(L)$ of the branch points of $F(L)$ will be studied below as well.

If $\nu \in F(L)$ is a focal point of order 1 of L, then $\nu \in F^0(L)$ is valid. In fact, there is a neighborhood U of ν in $N(L)$ such that for any ray ϱ of $N(L)$ intersecting U the number of focal points on $\varrho \cap U$ is one. (prop. 6.) Consequently ν is an ordinary point of $F(L)$.

The following theorem generalizes a result of F. W. WARNER on the conjugated locus [4] and it is the main result here.

THEOREM. *Let L be a submanifold of a Riemannian manifold M and F(L) its focal locus with the relative topology induced by the inclusion map*

$$F(L) \to N(L).$$

Then $F^0(L)$ the ordinary focal locus of L is an open and everywhere dense subset of $F(L)$; moreover $F^0(L)$ is a smooth submanifold of codimension 1 in $N(L)$. In fact, if $\nu \in F^0(L)$ and ϱ is the ray defined by ν then the decomposition

$$T_\nu NL = T_\nu F^0(L) \oplus T_\nu \varrho$$

is valid.

PROOF. The definition of $F_0(L)$ implies directly that $F^0(L)$ is an open subset of $F(L)$. In order to prove that $F^0(L)$ is everywhere dense in $F(L)$ it is sufficient to show that $F^B(L)$ is nowhere dense in $F(L)$ which means that the interior of $F^B(L)$ in $F(L)$ is empty. In order to argument by contradiction assume that ν is an interior point of $F^B(L)$ in $F(L)$, and let k be the order of the focal point ν.

Then by the proposition 6 there is a neighborhood U of ν in $N(L)$ such that $F(L) \cap U \subset F^B(L)$ holds and that any ray of $N(L)$ intersecting U intersects $F(L) \cap U$ in k points, where points of $F(L)$ are counted with their multiplicities. Since ν is a branch point of $F(L)$, there is a ray of $N(L)$ intersecting U which has at least two points in $F^B(L) \cap U$; the order of these two points cannot exceed $k-1$ by Prop. 6; let one of them ν_1. Now the repetition of the preceding argument yields a point $\nu_2 \in F^B(L) \cap U$ the order of which cannot exceed $k-2$. Further repetitions of the argument yield points $\nu_3, \dots, \nu_i \in F^B(L) \cap U$ such that the order of ν_i is equal to 1, where $1 \leq i \leq k-1$. But by a previous observation $\nu_i \in F^0(L)$ is valid. Thus a contradiction is obtained. Consequently, $F^0(L)$ is an everywhere dense subset of $F(L)$.

In order to show that $F^0(L)$ is a smooth submanifold of codimension 1 in $N(L)$ it is sufficient to prove that any $\nu \in F^0(L)$ has a neighborhood V in $N(L)$ such that the following are valid:

1. $V \cap F(L) = V \cap F^0(L)$,
2. $V \cap F^0(L)$ has an $(m-1)$ dimensional C^∞ manifold structure such that $V \cap F^0(L) \to N(L)$ is a smooth embedding.

Assume that the order of the focal point ν is k. Consider now the ray ϱ defined by ν and on account of Prop. 5. a coordinate system $(x^1, \dots, x^m)$ of $N(L)$ defined on a normal neighborhood V of ν and a coordinate system

$(y^1, \ldots, y^m)$ of M defined on a neighborhood V' of $\varepsilon(v)$ such that the following two conditions are satisfied:

1. $\varepsilon(V) \subset V'$,

2. $T_{\varrho(\tau)}\varepsilon\left(\frac{\partial}{\partial x^j}(\varrho(\tau))\right) = \varphi_j(\tau) \cdot \left(\frac{\partial}{\partial y^j}(\varepsilon(\varrho(\tau)))\right),$

$j = 1, \ldots, m$ for $\tau \in I = \varrho^{-1}(V)$,

where the φ_j, $\tau \in I$ are C^∞ functions such that φ_j are non vanishing for $j = 1, \ldots, m-k$ and φ_j vanish only at $\tau = 1$ where $\varphi_j' > 0$ for $j = m-k+1, \ldots, m$. Consider now the matrix representing $T\varepsilon \restriction V$ in the given coordinate systems. In details the relations

$$T_{\bar{v}}\varepsilon\left(\frac{\partial}{\partial x^j}(\bar{v})\right) = \sum_{i=1}^{m} \alpha_{ij} \cdot \frac{\partial}{\partial y^i}(\varepsilon(\bar{v})) \text{ for } j = 1, \ldots, m$$

yield the matrix $(\alpha_{ij})(\bar{v})$ at any $\bar{v} \in V$. The entries of this matrix are C^∞ functions on V. Moreover, along the ray ϱ the matrix (α_{ij}) is diagonal on account of Prop. 5 since it has the following form:

$$(\alpha_{ij})(\varrho(\tau)) = \begin{pmatrix} \varphi_1(\tau) & \ldots & 0 \\ 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \varphi_m(\tau) \end{pmatrix}.$$

Let now Δ_{k-1} be the $(k-1)$-th elementary symmetric functions of the eigenvalues of the matrix (α_{ij}). Then Δ_{k-1} is obviously real valued and of class C^∞ on V. In fact, from

$$\begin{vmatrix} \alpha_{11} - \lambda & & \ldots & \alpha_{1m} \\ & \alpha_{22} - \lambda & \ldots & \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & & \ldots & \alpha_{mm} - \lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_m - \lambda) = 0,$$

where $\lambda_i \in \mathbf{C}$, $i = 1, \ldots, m$ are the solutions of the Laplace-equation, it is easy to see that the coefficient Δ_{k-1} of $(-\lambda)^{k-1}$ is a real for any $\bar{v} \in V$. Now we have a C^∞ scalar-function

$$\Delta_{k-1}: V \to \mathbf{R}(V \subset N(L))$$

for which $\Delta_{k-1}(v) = 0$ holds. Moreover

$$\dot{\varrho}(1)\Delta_{k-1} = \frac{d}{d\tau}\Delta_{k-1} \restriction \varrho(\tau) = \left(\sum_{j=m-k+1}^{m} \varphi_j'(1)\right) \cdot \varphi_1(1) \ldots \varphi_{m-k}(1)$$

since $\Delta_{k-1}(\varrho(\tau))$ is the sum of products of $m-k+1$ factors which are among $\varphi_1(\tau), \ldots, \varphi_m(\tau)$. Consequently $\dot{\varrho}(1)\Delta_{k-1} \neq 0$ is valid, since the following holds:

$$\sum_{j=m-k+1}^{m} \varphi_j'(1) > 0$$

and $\varphi_1(1) \ldots \varphi_{m-k}(1) \neq 0$. But then there is a neighborhood $W \subset V$ of ν in $N(L)$ on which the radial derivative of Δ_{k-1} is not zero. Moreover W can be chosen to be normal and such that $F(L) \cap W \subset F^0(L) \cap W$ holds and for each ray ϱ of $N(L)$ intersecting W there is exactly one point of $W \cap F(L)$ on ϱ. Then on each ray ϱ of $N(L)$ intersecting W there is exactly one focal point of L in W of order k.

Now Δ_{k-1} is a function of class C^∞ whose differential is different from zero on W. Moreover

$$F(L) \cap W = \Delta_{k-1}^{-1}(0) \cap W$$

is valid. In fact, if $u \in F(L) \cap W$ then the rank of $T_u \varepsilon$ is $m-k$, consequently $\Delta_{k-1}(u) = 0$; assume conversely that $\Delta_{k-1}(w) = 0$ for $w \in W$, let w' be the unique focal point of L in W on the ray through w. Since W is a normal neighborhood there is a segment from w to w' in W, and, since $\Delta_{k-1}(w) = \Delta_{k-1}(w') = 0$ and the derivative of Δ_{k-1} along the segment is not zero, consequently $w = w'$.

Consequently $F(L) \cap W$ has a C^∞ manifold structure such that $F(L) \cap W \subset N(L)$ is a submanifold in the relative topology. Moreover $T_\nu N(L) = T_\nu F^0(L) \oplus T_\nu \varrho$ for $\nu \in F^0(L)$ since the radial derivative is not zero. So the theorem is proved.

Added in Proof: We have learnt recently that an assertion closely related to the statement of the Theorem of this paper was made by J. J. Hebda in The regular focal locus, *J. of Diff. Geom.*, **16** (1981), 421–429.

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ON THE CENTROIDS OF CONVEX SOLIDS OF REVOLUTION

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(Received April 2, 1984)

Suppose S is a convex solid of revolution in R^3 with axis of revolution AB. Let C be the centroid of S, let C' be the centroid of the (2-dimensional) domain D obtained by intersecting S with a plane through AB. A. MAIR raised the question how large the ratio $\frac{|AC|}{|AC'|}$ can be.

We prove that

$$\frac{1}{2} < \frac{|AC|}{|AC'|} < \frac{3}{2}. \tag{1}$$

REMARKS. 1. The ratio in question remains the same if one enlarges S around AB by a fixed factor.

2. Denote the points and their coordinates with the same symbols in a given coordinate system. It is well known that the coordinates C (and C' resp.) can be counted by the following integral

$$C = \frac{1}{\mathcal{V}(S)} \int_S x\,dS, \quad C' = \frac{1}{\mathcal{A}(D)} \int_D x\,dD$$

where x denotes the coordinates of a variable point in S (in D resp.) and $\mathcal{V}(S)$ denotes the volume of S and $\mathcal{A}(D)$ denotes the area of D.

3. A truncated cone exhibited on Fig. 1 shows that the best lower bound must be $\leqq 0.772\ldots$ and the best upper bound must be $\geqq 1.1368\ldots.$

PROOF. Consider the coordinate system (x, y) where the point A is the origin and the halfline AB is the positive part of the axis x. From above the domain D is bounded by the graph of a concave function f. In view of Re-

* Partially supported by Hungarian National Foundation for Scientific Research, No. 1238.

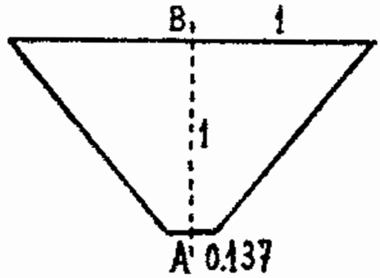

Fig. 1

mark 1 we may suppose that f is defined in the interval $[0, 1]$ such that $\sup_{x\in[0,1]} f(x) = 1$. Using Remark 2 we have

$$\frac{|AC|}{|AC'|} = \frac{\dfrac{\int_0^1 x f^2(x)\,dx}{\int_0^1 f^2(x)\,dx}}{\dfrac{\int_0^1 x f(x)\,dx}{\int_0^1 f(x)\,dx}}$$

(1) is an immediate consequence of the following sharp inequalities.

LEMMA. If $f(x)$ is a nonnegative concave function defined in the closed interval $[0, 1]$ and $\sup_{x\in[0,1]} f(x) = 1$, then

$$\frac{2}{3} \leqq \frac{\int_0^1 f^2(x)\,dx}{\int_0^1 f(x)\,dx} \leqq 1 \tag{2}$$

and

$$\frac{1}{2} \leqq \frac{\int_0^1 x f^2(x)}{\int_0^1 x f(x)} \leqq 1. \tag{3}$$

PROOF. Since $0 \leqq f(x) \leqq 1$ the upper bounds are obvious.
We prove the lower bound in (2).

Suppose that $f(x)$ attains 1 for x_0. We take the function g defined in the interval $[0,1]$ such that it is linear in the both intervals $[0,x_0]$, $[x_0, 1]$ further $g(x_0) = 1$ and finally

$$\int_0^{x_0} g(x)\,dx = \int_0^{x_0} f(x)\,dx; \quad \int_{x_0}^1 g(x)\,dx = \int_{x_0}^1 f(x)\,dx.$$

If $f(x) \not\equiv g(x)$ in the interval $(0, x_0)$ [in the interval $(x_0, 1)$ resp.], then the curve $f(x)$ intersects the curve $g(x)$ only for one $x_1 \in (0, x_0)$ (only for one $x_2 \in (x_0, 1)$ resp.). We have

$$\int_0^{x_0} [f^2(x) - g^2(x)]dx = \int_0^{x_1} [f^2(x) - g^2(x)]dx + \int_{x_1}^{x_0} [f^2(x) - g^2(x)]dx \geqq$$

$$\geqq 2f(x_1) \int_0^{x_1} [f(x) - g(x)]dx + 2f(x_1) \int_{x_1}^{x_0} [f(x) - g(x)]dx = 0.$$

Analogously

$$\int_{x_0}^1 [f^2(x) - g^2(x)]dx \geqq 0.$$

Thus

$$\frac{\int_0^1 f^2(x)\,dx}{\int_0^1 f(x)\,dx} \geqq \frac{\int_0^1 g^2(x)\,dx}{\int_0^1 g(x)\,dx} =$$

$$= \frac{\int_0^{x_0} \left[g(0) + (1 - g(0))\frac{x}{x_0}\right]^2 dx + \int_{x_0}^1 \left[g(1) + \frac{1-x}{1-x_0}(1 - g(1))\right]^2 dx}{\int_0^{x_0} \left[g(0) + (1 - g(0))\frac{x}{x_0}\right] dx + \int_{x_0}^1 \left[g(1) + \frac{1-x}{1-x_0}(1 - g(1))\right] dx} =$$

$$= \frac{\frac{x_0}{3}(g^2(0) + g(0) + 1) + \frac{1-x_0}{3}(2g(1) + 1)}{\frac{x_0}{2}(g(0) + 1) + \frac{1-x_0}{2}(g(1) + 1)} \geqq$$

$$\geqq \frac{\frac{x_0}{3}(g(0) + 1) + \frac{1-x_0}{3}(g(1) + 1)}{\frac{x_0}{2}(g(0) + 1) + \frac{1-x_0}{2}(g(1) + 1)} = \frac{2}{3}.$$

Note that equality holds for the function

$$f(x)=\begin{cases}\dfrac{x}{x_0} & \text{for } 0\leqq x\leqq x_0,\\[2mm] \dfrac{1-x}{1-x_0} & \text{for } x_0\leqq x\leqq 1,\end{cases}$$

where x_0 is arbitrary element of the interval $[0, 1]$.

We prove the lower bound in (3).

Let $f(x_0)=1$. We take the function g defined in the interval $[0, 1]$ such that it is linear in the both intervals $[0, x_0]$, $[x_0, 1]$ further $g(x_0)=1$ and

$$\int_0^{x_0} xg(x)\,dx=\int_0^{x_0} xf(x)\,dx$$

and

$$\int_{x_0}^{1} xg(x)\,dx=\int_{x_0}^{1} xf(x)\,dx.$$

If $f(x)\not\equiv g(x)$ in the interval $(0, x_0)$ (in the interval $(x_0, 1)$ resp.), then the curve $f(x)$ intersects the curve $g(x)$ only for one $x_1\in(0, x_0)$ (only for one $x_2\in(x_0, 1)$ resp.). We have

$$\int_0^{x_0}[xf^2(x)-xg^2(x)]\,dx=\int_0^{x_1}[xf^2(x)-xg^2(x)]\,dx+$$

$$+\int_{x_1}^{x_0}[xf^2(x)-xg^2(x)]dx\geqq 2f(x_1)\int_0^{x_1}[xf(x)-xg(x)]dx+$$

$$+2f(x_1)\int_{x_1}^{x_0}[xf(x)-xg(x)]dx=0.$$

Analogously

$$\int_{x_0}^{1}[xf^2(x)-xg^2(x)]dx\geqq 0.$$

Thus

$$\frac{\int_0^1 xf^2(x)\,dx}{\int_0^1 xf(x)\,dx}\geqq\frac{\int_0^1 xg^2(x)\,dx}{\int_0^1 xg(x)\,dx}=$$

$$=\frac{\int_0^{x_0} x\left[\frac{1-g(0)}{x_0}x+g(0)\right]^2dx+\int_{x_0}^{1} x\left[\frac{1-x_0g(1)}{1-x_0}-\frac{1-g(1)}{1-x_0}x\right]^2dx}{\int_0^{x_0}\left[\frac{1-g(0)}{x_0}x^2+g(0)x\right]dx+\int_{x_0}^{1}\left[\frac{1-x_0g(1)}{1-x_0}x-\frac{1-g(1)}{1-x_0}x^2\right]dx}=$$

$$=\frac{[3+2g(0)+g^2(0)]x_0^2+[(1-x_0)(1+3x_0)+2(1-x_0^2)g(1)+(1-x_0)(3+x_0)g^2(1)]}{2(2+g(0))x_0^2+2[(1-x_0)(1+2x_0)+(1-x_0)(2+x_0)g(1)]}\geqq\frac{1}{2}.$$

Note that equality holds for the function

$$f(x) = \begin{cases} \dfrac{x}{x_0} & \text{for } 0 \leqq x \leqq x_0, \\[2ex] \dfrac{1-x}{1-x_0} & \text{for } x_0 \leqq x \leqq 1, \end{cases}$$

where x_0 is an arbitrary element of $[0,1]$.

AN ALGEBRAIC PROOF FOR KRIPKE-STYLE COMPLETENESS OF A CLASS OF MULTIMODAL PROPOSITIONAL CALCULI

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In the following article I shall outline a sketchy proof for the Kripke-style completeness of a class multimodal propositional calculi — using the algebraic tools of Boolean algebras with distributive operators and the results and methods of HENKIN—MONK—TARSKI [1] and LEMMON [2]. In the first place the results of [1] on the embedding of Bo_α-s will be slightly generalised, then — after defining the needed concepts — a Lemmon-style representation theorem will be proved for getting a traditional possible world — semantics. In the first part the notational system of [1] (430—440 pp.) will be used, with the addition as follows. Let $\mathcal{B}_\alpha$ be the language of the description of Bo_α-s. A term of $\mathcal{B}_\alpha$ is called *positive* iff it contains only the operationsigns $+, \cdot, c_\varkappa$, the constants 0,1 and variables — i.e. it does not contain any occurences of the complementation-sign, $-$. Let $q_\varkappa$ denote for every $\varkappa < \alpha$ the '$-c_\varkappa-$' operation. A term of $\mathcal{B}_\alpha$ is *quasi-positive* iff all the occurences of '$-$' can be eliminated using substitutions of $q_\varkappa$-s for all occurences of '$-c_\varkappa-$'-s. Then we have the following

LEMMA 1. Let $\mathfrak{A}$ be a Bo_α and let $\mathcal{E}m\,\mathfrak{A}$ be the canonical embedding algebra of $\mathfrak{A}$. Furthermore, let $P_{\mathcal{B}_\alpha}$ and $Q_{\mathcal{B}_\alpha}$ be a positiv resp. quasi-positive term of the language $\mathcal{B}_\alpha$. Assume, that for the polynomials P, Q corresponding respectively to the terms $P_{\mathcal{B}_\alpha}$ and $Q_{\mathcal{B}_\alpha}$ $Px \leq Qx$ for every $x \in {}^\omega em^*\ A$. We then have $Px \leq Qx$ for all $x \in {}^\omega Em\,\mathfrak{A}$.

For the PROOF at first we need a slightly modificated version of the lemmas 2.7.11. and 2.7.12. of [1]. 2.7.11. must be modified to have

$$Qa \geq \Sigma\{Qx : x \in {}^\omega C,\ x_\varkappa \leq a_\varkappa,\ \varkappa \leq \omega\} \tag{1}$$

for every *quasi*-positive polynomial Q. The only necessary supplementary step for proving (1) is the proof of the hereditary character of it from Q to $q'_\varkappa Q$. But

$$(q'_\varkappa Q)a = q_\varkappa Qa = -c_\varkappa - Qa \geq -c_\varkappa - \sum_{x \in Ta} Qx = -c_\varkappa \prod_{x \in Ta} -Qx \geq$$

$$\geq -\prod_{x \in Ta} c_\varkappa(-Px) = \sum_{x \in Ta} -c_\varkappa - Px = \sum_{x \in Ta} (q'_\varkappa P)x.$$

The statement of the lemma 2.7.12. will remain true choosing Q quasi-positive, because here even the equality is hereditary from Q to $q'_\varkappa Q$, because

$$(q'_\varkappa Q)a = q_\varkappa Qa = -c_\varkappa - Qa = -c_\varkappa - \prod_{x\in Ta} Qx = -c_\varkappa \sum_{x\in Ta} -Qx =$$
$$= -\sum_{x\in Ta} c_\varkappa - Qx = \prod_{x\in Ta} -c_\varkappa - Qx = \prod_{x\in Ta} q_\varkappa x = \prod_{x\in Ta} (q'_\varkappa Q)x.$$

Thus for quasi-positive polynomials instead of 2.7.13. we have

$$Qa \geq \sum_{a\geq_\omega x\in{}^\omega C} \; \prod_{x\leq_\omega y\in{}^\omega em^* A} Qy.$$

For the positiv P applying the original 2.7.13. we have

$$Pa = \sum_{a\geq_\omega x\in{}^\omega C} \; \prod_{x\leq_\omega y\in{}^\omega em^* A} Py.$$

Using the condition $Py \leq Qy$ for $y \in em^*$ A now we have (for the sake of simplicity indices are omitted):

$$Pa = \Sigma\Pi Py \leq \Sigma\Pi Qy \leq Qa$$

which makes complete the proof of our lemma. (It must be remarked that in the proof we used the obvious monotonity and completely multiplicativity of $q_\varkappa$.)

Now it will be outlined how to use this result for the Kripke-style completeness proof of multimodal calculi.

First let us see some *definitions.*

$\mathcal{L}_\alpha$ is called *multimodal propositional language* iff it is an extension of the language of classical propositional logic with a number of modal possibility operator-sign M_i, $i \in \alpha$, where α is an ordinal. The definition of well formed formula is to be the natural generalization that of the usual unimodal case. An axiom-system in $\mathcal{L}_\alpha$ (given by schemes) is called quasi-positive iff

(i) it contains a system of asiom-schemes for classical propositional logic (including the rule modus ponens);

(ii) for all $i \in \alpha$ it contains the $M_i(A \vee B) \equiv M_i A \vee M_i B$ scheme;

(iii) it contains the rule $\vdash A \equiv B$ implies $\vdash M_i A \equiv M_i B$ for all $i \in \alpha$;

(iv) all the remaining axiom-schemes are of the form of a conditional with a positive antecedent and a quasi-positive consequent formula. (The meaning of the terms "positive resp. quasi-positive formula" can be defined on the analogy of the definition given in the language $\mathcal{B}_\alpha$ for Bo_α-s.)

LEMMA 2. Let $\mathcal{A}$ be a quasi-positive system of axioms in the multimodal language $\mathcal{L}_\alpha$. Then there exists an $\mathcal{M} \in \mathrm{Bo}_\alpha$ (which may be called by *multimodal algebra* in this context) *characteristic* for $\mathcal{A}$ in the sense of [2].

PROOF. The Lindenbaum-Tarski algebra of $\mathcal{A}$, i.e. the factoralgebra of the free formula-algebra over $\mathcal{L}_\alpha$ by the congruence-relation $\approx_{\mathcal{A}}$ will be suitable. (Here for every pair of formulae over $\mathcal{L}_\alpha$ A, B $A \approx_{\mathcal{A}} B :\equiv \mathcal{A} \vdash A \equiv B$. The needed evaluation characteristic for $\mathcal{A}$ may be the canonical homomorphism $A \to [A]_{\approx_{\mathcal{A}}}$. Notice that if $\mathcal{A}$ contains an axiom of form $A \supset B$, then for the elements of the above constructed characteristic algebra the $A^* \leq B^*$ inequality will be true, where A^* and B^* are the "natural" translations of A and B from the language $\mathcal{L}_\alpha$ to $\mathcal{B}_\alpha$, and A^* and B^* are positive resp. quasi-positive terms.

LEMMA 3. Let $\mathcal{A}$ be a quasi-positive system of axioms. Then there exists an $\mathcal{M}$ complete, atomic multimodal algebra characteristic for $\mathcal{A}$.

PROOF. Let us construct the Lindenbaum-Tarski algebra of $\mathcal{A}$, then its canonical embedding algebra as a Bo_α. Using Lemma 1. and 2. it may be proved that all the inequalities corresponding to our axioms will remain true on this algebra, which is characteristic fo $\mathcal{A}$ because the Lindenbaum-Tarski algebra is $\mathcal{A}$ is isomorphic with a sub-algebra of its embedding algebra.

DEFINITION. $\mathcal{K}_\alpha = \langle K, \{\langle Q_i, R_i\rangle : i \in \alpha\}\rangle$ is a multimodal Kripke-structure (mm. K. s.) iff $K \neq \emptyset$ is a set, α is an ordinal and for all $i \in \alpha$, $Q_i \times K$ and $R_i \times K \times K$. $\mathcal{M}_\alpha = \langle \mathcal{K}_\alpha, v\rangle$ is a multimodal Kripke-modell iff $\mathcal{K}_\alpha$ is an mm. K. s. and v is an evaluation of the primitive formulae of $\mathcal{L}_\alpha$ in every possible world i.e. $v \in K \times \mathcal{P} \to \{0,1\}$, where $\mathcal{P}$ is the set of the primitive formulae of $\mathcal{L}_\alpha$ The truthvalue of a multimodal formula may be defined inductively the usual way. The only interesting case is the evaluation of the possibility-operators: $M_i A$ is true in $w \in K$ iff $w \in Q_i$ or there exists a $w^* \in K$ such that $wR_i w^*$ and A is true in w^*. To have our completeness theorem in a usual form we only have to represent our complete, atomic characteristic multimodal algebra by some mm.K.s. We *define* the $\mathcal{K}^+$ multimodal algebra corresponding to the $\mathcal{K}$ mm.K.s. as follows:

$$\mathcal{K}^+ = \langle K, \cup, \cap, K\setminus, \emptyset, K, \mathbf{P}_i\rangle,$$

where $\mathbf{P}_i X = Q_i \cup R_i^{-1*} X$ for $X \subseteq K$.

Then we have the following

LEMMA 4. For every $\mathcal{M}$ complete, atomic multimodal algebra there exists a $\mathcal{K}$ mm.K.s. such that $\mathcal{K}^+ \cong \mathcal{M}$.

PROOF. We can construct the above mentioned $\mathcal{K}$. Let the basic set K of $\mathcal{K}$ be the set of all atoms of $\mathcal{M}$. Let Q_i be the set of atoms in $c.0$, and if a and b are atoms in $\mathcal{M}$ let $aS_i b$ be iff $b \leq c_i a$ in $\mathcal{M}$. Let us define $R_i = S_i^{-1}$. A very simple calculation shows that for such $\mathcal{K}$, $\mathcal{K}^+ \cong \mathcal{M}$ holds regarding the natural isomorphism.

Summarizing our result we may state the following general

COMPLETENESS THEOREM. *For every quasi-positive axiom-system of multimodal propositional logics there exists a characteristic class of mm.K. structures, where the properties of the relations of the mm.K.s. 's constituting the suitable class can be read off using the schemes of axioms.*

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ON SUPER PSEUDOPRIMES WHICH ARE PRODUCTS OF THREE PRIMES

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(Received March 1, 1985)

A composite integer m is called pseudoprime with respect to an integer $a>1$ if

$$a^m \equiv a \pmod{m}.$$

It is called super pseudoprime with respect to a if each divisor of it is a prime or a pseudoprime with respect to a. We simply say m is a pseudoprime (or a super pseudoprime) if it is one with respect to 2.

It is known that there are infinitely many super pseudoprime numbers. K. SZYMICZEK [7] proved that there are infinitely many super pseudoprimes which are products of exactly three distinct primes. This result was extended by J. FEHÉR and P. KISS [2] for super pseudoprimes with respect to a, where $4 \nmid a$. From a result of A. ROTKIEWICZ ([5], Theorem 2) it follows that for infinitely many primes p of the form $kx+h$, where $(k, h) = 1$, there exist primes q and r such that pqr is a super pseudoprime.

In this paper we extend the result of J. FEHÉR and P. KISS for any $a>1$ and partly also the result of A. ROTKIEWICZ for super pseudoprimes with respect to a. We shall prove:

THEOREM 1. *Let a and k be integers with conditions $a>1$ and $k>1$. Then there exist infinitely many triplets of distinct primes p, q and r of the form $kx+1$ such that pqr is a super speudoprime with respect to a.*

We also extend another result concerning the distribution of pseudoprimes. A. MAKOWSKI [4] proved that the series

$$\sum_{n=1}^{\infty} \frac{1}{\log p_n(a)}$$

is divergent, where $p_n(a)$ denotes the n-th pseudoprime with respect to a. We show:

THEOREM 2. *Let $a>1$ be an integer and let $\bar{a}$ be the squarefree kernel of a (i.e. a is of the form $a=\bar{a}\cdot u^2$, where $\bar{a}$ is square-free). If $\bar{a}\equiv\pm1 \pmod 4$, then the series*

$$\sum_{n=1}^{\infty}\frac{1}{\log P_n^{(3)}(a)}$$

is divergent, where $P_n^{(3)}(a)$ denotes the n-th super pseudoprime with respect to a which is product of exactly three distinct primes.

We shall use the following notation. Let $a>1$ and m be integers with $(a, m)=1$. We denote by $a(m)$ the least positive integer x such that $m|a^x-1$. $a(m)$ is called in general the exponent or order of a mod m. It is clear that if m is a composite number and $a(m)|m-1$ then m is pseudoprime number with respect to a, furthermore $a(p)|p-1$ for primes p with $(a, p)=1$.

We need some lemmas for the proof of our theorems.

LEMMA 1. For any integers $a>1$ and $n>6$ there is at least one prime p such that $a(p)=n$.

LEMMA 2. Let $a>1$ be an integer and let

$$\eta=\begin{cases}1 \text{ if } \bar{a}\equiv1 \pmod 4,\\ 2 \text{ if } \bar{a}\equiv2,3 \pmod 4,\end{cases}$$

where $\bar{a}$ is the square-free kernel of a. If $n>20$ and $n/\eta\cdot\bar{a}$ is an odd integer then there are at least two distinct primes p and q such that $a(p)=a(q)=n$.

LEMMA 3. Let a and k be integers with conditions $a>1$ and $k>1$. Then there exist infinitely many primes p of the form $kx+1$ such that $a(p)|(p-1)/2$ and $a(p)<(p-1)/2$.

LEMMA 4. Let p, q and r be distinct primes. The number pqr is a super pseudoprime with respect to a if

$$a(pqr)=[a(p), a(q), a(r)]|(p-1, q-1, r-1),$$

where $[x, y, \ldots]$ and $(x, y, \ldots)$ denote L.C.M. and G.C.D. of $x, y, \ldots$, respectively.

We have to prove only Lemma 3 and 4 since Lemma 1 is a known result of K. ZSIGMONDY [8] and G. D. BIRKHOFF—H. S. VANDIVER [1] and Lemma 2 follows from a theorem of A. SCHINZEL [6].

PROOF OF LEMMA 3. Let P denote the set of all primes p for which

$$a(p)\Big|\frac{p-1}{2} \text{ and } p\equiv1 \pmod k.$$

Let $a=2^\alpha\cdot a'$, where $\alpha\geq0$ and a' is an odd integer. By Dirichlet's theorem, there exist infinitely many primes of the form $8akx+1$ and if $p=8akx+1$ is a prime, then we have

$$\left(\frac{a}{p}\right)=\left(\frac{2^\alpha}{p}\right)\left(\frac{a'}{p}\right)=\left(\frac{a'}{p}\right)=\left(\frac{p}{a'}\right)=\left(\frac{1}{a'}\right)=1.$$

Thus the number a is a quadratic residue modulo p and from this it follow- that $a(p)|(p-1)/2$ for primes p of the form $8akx+1$. Thus P contains an infinite number of elements.

First we prove that the function g defined for primes by

$$g(p) = \frac{p-1}{a(p)} \tag{1}$$

is unbounded on the set P.

Let us suppose that $g(p)$ is a bounded function on the set P and so it has only a finite number of distinct values: $t_1, \ldots, t_s$. Let $x = 8ak$ and let us consider the integer

$$n = x(xt_1+1)\ldots(xt_s+1)+x.$$

Since $n>x>6$ by Lemma 1 there is a prime p for which $a(p) = n$ and so $g(p) = (p-1)/n$. From this

$$p = n\cdot g(p)+1 \tag{2}$$

follows. The prime number p has the form $8aky+1$ for some integer y and so, as above, p is an element of P. On the other hand by our assumption $g(p)= = t_i$ for some integer $i (1 \leq i \leq s)$, and so by (2)

$$p = nt_i+1 = x(xt_1+1)\ldots(xt_s+1)t_i+(xt_i+1)$$

is divisible by xt_i+1 $(\geq 8ak+1)$. This is a contradiction since p is a prime, thus the function $g(p)$ cannot be bounded.

Since the function $g(p)$ is unbounded on the set P, there exist infinitely many primes p in the set P for which $g(p)>2$, which by (1) proves Lemma 3.

We note that in [3], in our joint paper with P. Kiss, we proved by a similar argument that $g(p)$ is unbounded for the set of all primes.

Proof of Lemma 4. *Let p, q and r be distinct primes.* It is well known that $a(pqr) = [a(p), a(q), a(r)]$.

Suppose that $a(pqr)|(p-1, q-1, r-1)$. Let m be a divisor of the number pqr. Since $m|pqr$ we have $a(m)|a(pqr)$. By our assumption it follows that

$$p \equiv 1 \pmod{a(m)},\ q \equiv 1 \pmod{a(m)},\ r \equiv 1 \pmod{a(m)}$$

and so

$$m \equiv 1 \pmod{a(m)},$$

Hence

$$m \mid a^{a(m)}-1 \mid a^{m-1}-1 \mid a^m - a,$$

thus m really is a super pseudoprime with respect to a.

PROOF OF THEOREM 1. Let $p>12$ be a prime of the form $16akx+1$ and $a(p)\,|(p-1)/2$, $a(p)<(p-1)/2$. By Lemma 3 it follows that there exist infinitely many primes with such properties. Since $(p-1)/2>6$ by Lemma 1 there are distinct primes q and r for which

$$a(q) = (p-1)/2 \text{ and } a(r) = p-1.$$

Since $p = 16akx+1$ and $a(p)<(p-1)/2$ we have $p\neq q$, $p\neq r$ and

(3) $$q = 8akxy+1,\ r = 16akxz+1,$$

where y and z are some integers. Let $a = 2^{\alpha}\cdot a'$, where $\alpha\geq 0$ and a' is an odd integer. From (3) we have

$$\left(\frac{a}{q}\right)=\left(\frac{2^{a}}{q}\right)\left(\frac{a'}{q}\right)=\left(\frac{a'}{q}\right)=\left(\frac{q}{a'}\right)=\left(\frac{1}{a'}\right)=1$$

and so $a(q)|(q-1)/2$. From this it follows that $(p-1)|q-1$.

Hence

$$a(pqr) = [a(p), a(q), a(r)] = (p-1)|(p-1, q-1, r-1)$$

and so by Lemma 4 Theorem 1 is proved.

PROOF OF THEOREM 2. Since $\bar{a}\equiv\pm 1 \pmod 4$, by Lemma 2

$$\eta = \begin{cases} 1 \text{ if } \bar{a}\equiv 1 \pmod 4, \\ 2 \text{ if } \bar{a}\equiv -1 \pmod 4). \end{cases}$$

For every odd integer $x>20$, by Lemma 2 there exist two distinct primes p_x and q_x for which

$$a(p_x) = a(q_x) = x\bar{a}\eta.$$

Let r_x be a prime number for which

$$a(r_x) = \begin{cases} 2\bar{a}x\eta \text{ if } \eta = 1, \\ \bar{a}x\eta/2 \text{ if } \eta = 2. \end{cases}$$

Since $\bar{a}x\eta/2>6$, by Lemma 1 such a prime exists. Since $\bar{a}$ and x are odd integers, we have

$$a(p_x) = a(q_x) = \bar{a}x\eta|(p_x-1)/2,\ (q_x-1)/2$$

if $\eta = 1$ and

$$a(r_x) = \bar{a}x\eta/2|(r_x-1)/2$$

if $\eta = 2$. In both cases we have $a(p_xq_xr_x) = 2\bar{a}x$ and

$$a(p_xq_xr_x)|(p_x-1,\ q_x-1,\ r_x-1).$$

By Lemma 4 it follows that $p_xq_xr_x$ is a super pseudoprime with respect to a.

If $x, y > 20$ are odd integers and $x \neq y$ then $a(p_x q_x r_x) \neq a(p_y q_y r_y)$ and so $p_x q_x r_x \neq p_y q_y r_y$. Thus for every odd integer $x > 20$ there exists a number $p_x q_x r_x$ which is a super pseudoprime. On the other hand, we have seen that $a(p_x q_x r_x) = 2\bar{a}x$ and so

$$p_x q_x r_x \leq a^{2\bar{a}x} - 1 < a^{2\bar{a}x}.$$

From this it follows that

$$\sum_{n=1}^{\infty} \frac{1}{\log P_n^{(3)}(a)} \geq \sum_{\substack{x > 20 \\ x \text{ odd}}} \frac{1}{\log (p_x q_x r_x)} > \frac{1}{2\bar{a} \log (a)} \sum_{\substack{x > 20 \\ x \text{ odd}}} \frac{1}{x},$$

which proves Theorem 2.

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ÜBER DAS $(r, R)_k$-SYSTEM VON UNTERRÄUMEN IN E^n

Von

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(Eingegangen am 2. April 1985)

1. Einführung

Die Menge von d-dimensionalen Unterräumen $\{E_i^d\}$ $(0 \leq d \leq n-1)$ im n-dimensionalen euklidischen Raum E^n bildet ein $(r, R)_k$-System, mit anderem Wort, hat die $(r, R)_k$-Eigenschaft, wenn die reellen Zahlen $0 < r \leq R < \infty$ bei einem gegebenen Wert $k \geq 1$ derart existieren, daß $\{E_i^d\}$ die folgenden zwei Bedingungen erfüllt:

1.1. Jede offene n-dimensionale Kugel vom Radius r schneidet höchstens k Unterräume aus $\{E_i^d\}$;

1.2. Jede abgeschlossene n-dimensionale Kugel vom Radius R hat gemeinsame Punkte mit mindestens k Unterräumen aus $\{E_i^d\}$.

Bildet die Menge $\{E_i^d\}$ ein $(r, R)_k$-System und gelten weiterhin die Ungleichungen $r_1 \leq r$ und $R_1 \geq R$, dann bildet offenbar $\{E_i^d\}$ auch ein $(r_1, R_1)_k$-System. Wenn $r^* = \sup r$ und $R^* = \inf R$ solche Zahlen sind, für die die Menge $\{E_i^d\}$ noch die $(r^*, R^*)_k$-Eigenschaft hat, dann nennen wir den Quotient $\frac{R^*}{r^*}$ die k-Enge der Unterräume $\{E_i^d\}$. Die Grundaufgabe ist das Minimum von $\frac{R^*}{r^*}$ und die dem Minimum entsprechende Unterraummenge zu bestimmen.

Es ist klar, daß wir im Fall $k = 1$ und $d = 0$ im wesentlichen das von B. N. DELONE [2] definierte (r, R)-System und die von S. S. RYSKOV [10, 11] definierte Dichte des (r, R)-Punktsystems bekommen.

Von L. FEJES TÓTH [3] wurde der Begriff der engen k-fachen Kugelpackung und der lockeren k-fachen Kugelüberdeckung gegeben. Es ist leicht einzusehen, daß im Fall $k = 1$ die Bestimmung des Minimums von $\frac{R^*}{r^*}$, der engen k-fachen Kugelpackung und der lockeren k-fachen Kugelüberdeckung äquivalente Aufgaben sind.

Früher haben wir uns mit der k-Enge der $(r, R)_k$-Punktsysteme beschäftigt ($d = 0$). Wir haben einige untere Schranken für $\frac{R^*}{r^*}$ angegeben [7], weiterhin haben wir das Minimum von $\frac{R^*}{r^*}$ für gitterförmige Punktmengen in den Fällen $n = 2$, $k = 2, 3, 4, 5$ [6], [7]; $n = 3$, $k = 1, 2$ [4], [6] und $n = 4$, $k = 1$ [5] bestimmt. Im Fall $n = 3$, $k = 1$ hat K. BÖRÖCZKY [1] das Minimum von $\frac{R^*}{r^*}$ für nicht gitterförmige Punktmengen gefunden. In dieser Arbeit berichten wir über einige Anfangsergebnisse für den Fall $d > 0$.

2. Sätze in E^n

Zuerst definieren wir einige Anordnungen von der Unterräume $\{E_i^d\}$. Später beweisen wir, daß das Minimum der k-Enge von Unterräumen $\{E_i^d\}$ in dieser Anordnung auftritt.

Gelten $E_{i_1}^d = E_{i_2}^d = \ldots = E_{i_m}^d$ für die Unterräume $E_{i_1}^d, E_{i_2}^d \ldots E_{i_m}^d \in \{E_i^d\}$, dann ist der Unterraum E_i^d von m-facher Multiplizität.

2.1. Der Fall $k = 1$ und $d = n-1$. Es bezeichne $\{\bar{E}_i^{n-1}\}(-\infty < i < \infty)$ die Menge von parallelen Hyperebenen, in der der Abstand der benachbarten Hyperebenen $2c$ ist. Es ist offenbar, daß $\{\bar{E}_i^{n-1}\}$ für $r < c$ und $R \geq c$ ein $(r, R)_1$-System bildet und $\frac{R^*}{r^*} = 1$ gilt. ($c > 0$ ist eine reelle Zahl.)

2.2. Es seien $n, k \geq 2$ ganze Zahlen, $c > 0$ reelle Zahl und $d = n-1$. Wir nehmen an, daß $\{\bar{E}_{i_1}^{n-1}\}, \{\bar{E}_{i_2}^{n-1}\}, \ldots, \{\bar{E}_{i_k}^{n-1}\}$ in 2.1. definierte, nicht unbedingt verschiedene Hyperebenenmengen in E^n sind. Es sei

$$\{\bar{E}_{i_1 i_2 \ldots i_k}^{n-1}\} = \{\bar{E}_{i_1}^{n-1}\} \cup \{\bar{E}_{i_2}^{n-1}\} \cup \ldots \cup \{\bar{E}_{i_k}^{n-1}\}. \tag{1}$$

Es ist leicht einzusehen, daß die Hyperebenenmenge $\{\bar{E}_{i_1 i_2 \ldots i_k}^{n-1}\}$ im Fall $r \leq c$ und $R \geq c$ eine $(r, R)_k$-Menge bildet. Folglich gilt $\frac{R^*}{r^*} = 1$.

2.3. Der Fall $n \geq 2$ und $d = n-2$. Es sei ein reguläres Dreiecksgitter mit der Seitenlänge $2c$ in E^2 gegeben. Nehmen wir die Menge der $(n-2)$-dimensionalen Unterräumen $\{\bar{E}_i^{n-2}\}$, die in den Gitterpunkten zu E^2 total senkrecht sind $\left(\bar{E}_i^{n-2} \in \{\bar{E}_i^{n-2}\} \Rightarrow \bar{E}_i^{n-2} \perp_t E^2\right)$. Wenn $r \leq c$ und $R \geq \frac{2}{\sqrt{3}} c$ sind, dann bildet $\{\bar{E}_i^{n-2}\}$ ein $(r, R)_1$-System und gilt

$$\frac{R^*}{r^*} = \frac{2}{\sqrt{3}}.$$

2.4. Sätze.

SATZ 1. *Es seien $n \geq 1$, $k = 1$ und $0 \leq d \leq n-1$. Bildet die Menge der d-dimensionalen Unterräumen $\{E_i^d\}$ ein $(r, R)_1$-System, dann gilt*

(2) $$\frac{R^*}{r^*} \geq \sqrt{\frac{2(n-d)}{n-d+1}}.$$

Die Gleichheit tritt nur im Fall $d = n-1$ für $\{\bar{E}_i^{n-1}\}$ (s. in 2.1) und im Fall $d = n-2$ für $\{\bar{E}_i^{n-2}\}$ (s. in 2.3) auf.

SATZ 2. *Es seien $n \geq 2$, $k \geq 1$ und $1 \leq d \leq n-1$. Existiert eine Punktmenge mit der $(r, R)_k$-Eigenschaft und mit der k-Enge $\varkappa$ in E^{n-d}, dann gibt es eine Menge der d-dimensionalen Unterräumen $\{E_i^d\}$ mit der $(r, R)_k$-Eigenschaft und mit der k-Enge $\varkappa$ im Raum E^n.*

SATZ 3. *Es seien $n \geq 2$, $k \geq 1$ und $1 \leq d \leq n-1$, weiterhin $\varkappa_0$ sei das Infimum der k-Enge von Punktsystemen mit der $(r, R)_k$-Eigenschaft in E^{n-d} und $\varkappa_d$ sei das Infimum der k-Enge der Menge von d-dimensionalen Unterräumen $\{E_i^d\}$ mit der $(r, R)_k$-Eigenschaft in E^n. Dann gilt*

(3) $$\varkappa_d \leq \varkappa_0.$$

SATZ 4. *Es seien $n, k \geq 2$ ganze Zahlen. Hat die Menge von Hyperebenen $\{E_i^{n-1}\}$ die $(r, R)_k$-Eigenschaft, dann tritt die Gleichheit $\frac{R^*}{r^*} = 1$ nur für die in 2.2 definierte Hyperebenenmenge $\{\bar{E}_{i_1 i_2 \ldots i_k}^{n-1}\}$ auf.*

3. Die Beweise der Sätze

3.1. DER BEWEIS DES SATZES 1. Betrachten wir die Menge von Unterräumen $\{E_i^d\}$ in E^n, die die $(r, R)_1$-Eigenschaft hat. Es seien $r^* = \sup r$ und $R^* = \inf R$ solche reelle Zahlen ,für die die gegebene Menge $\{E_i^d\}$ noch die $(r^*, R^*)_1$-Eigenschaft hat. Wegen $k = 1$ ist der Abstand zweier beliebigen Unterräumen von $\{E_i^d\}$ mindestens $2r^*$. Nehmen wir eine n-dimensionale offene Kugel, die keinen gemeinsamen Punkt mit den Unterräumen von $\{E_i^d\}$ hat. (Die Existenz dieser Kugel folgt aus 1.1.) Wir vergößern diese Kugel bis der Lage, wo sie mindestens einen von den Unterräumen aus $\{E_i^d\}$ berührt. Es bezeichne E_1^d diesen Unterraum. Dann vergrößern wir diese Kugel neben der Beibehaltung der Berührung mit E_1^d weiter bis der Lage, in der unsere Kugel einen weiteren Unterraum $E_2^d \in \{E_i^d\}$ berührt. Wir setzen dieses Verfahren fort. Endlich bekommen wir eine Kugel, die bestimmte Unterräume aus $\{E_i^d\}$ berührt und deren weitere Vergrößerung nicht möglich ist.

Aus 1.1 folgt, daß die Anzahl der berührenden Unterräume aus $\{E_i^d\}$ endlich ist und aus 1.2 ergibt sich, daß auch der Radius der Kugel endlich ist.

Es sei G die Kugel in der Endlage, O bzw. $\overline{R}$ sei ihr Mittelpunkt bzw. Radius. Mit $E_1^d, E_2^d, \ldots, E^d$ bezeichnen wir die berührenden Unterräume, und die Berührungspunkte seien $P_1, P_2, \ldots, P_s$. $E^\nu (\nu \leq s-1)$ sei der Unterraum kleinster Dimension, der die Punkte $P_1, P_2, \ldots, P_s$ enthält. Die von den Punkten $P_1, P_2, \ldots, P_s$ bestimmte ν-dimensionale Kugel sei G_1. Mit O_1

bzw. R_1 bezeichnen wir den Mittelpunkt bzw. den Radius von G_1. Wegen der $(r^*, R^*)_1$-Eigenschaft gelten

$$P_jP_m \geq 2r^* \quad (j \neq m,\, j,\, m \in \{1, 2, \ldots, s\}) \tag{4}$$

und

$$\overline{R} \leq R^*. \tag{5}$$

Aus $G_1 \subset G$ folgt, daß

$$R_1 \leq \overline{R} \tag{6}$$

gilt und Gleichheit nur im Fall $O = O_1$ auftritt.

3.1.1. Es sei $\nu < n-d$. Wir zeigen, daß dieser Fall nicht vorkommt, weil wir die Kugel G derart weiter vergrößern können, daß kein Unterraum aus $\{E_i^d\}$ diese vergrößerte Kugel schneidet.

3.1.1.1. Nehmen wir an, daß $O \neq O_1$ ist. Es ist offenbar, daß $OO_1 \perp E^\nu$ und $OP_j \perp E_j^d$ $(j = 1, 2, \ldots, s)$ gelten. Betrachten wir die Hyperebene E_j^{n-1}, die die Kugel G im Punkt P_j berührt. Aus $OP_j \perp E_j^{n-1}$ folgen $E_j^d \subset E_j^{n-1}$ und $\sphericalangle (P_jO_1O) = 90°$. O_2 sei ein Punkt der Geraden O_1O in der Nähe von O, für den die Anordnung O_1OO_2 besteht. Dann gilt $\sphericalangle (P_jOO_2) > 90°$ und daraus folgt, daß der Abstand von O_2 und E_j^{n-1} größer als $\overline{R}$ ist. Deshalb ist auch der Abstand von O_2 und E_j^d größer als $\overline{R}$ für alle $j \in \{1, 2, \ldots, s\}$, folglich kann man die Kugel G vergrößern.

3.1.1.2. Es sei $O = O_1$, d.h. $O \in E^\nu$. Es seien $E^{n-\nu} \perp_l E^\nu$, $O \in E^{n-\nu}$, $O \in \overline{E}_j^d \| E_j^d$ $(j \in \{1, 2, \ldots, s\})$, weiterhin $T = E^{n-\nu} \setminus \left(\bigcup_{j=1}^{s} \overline{E}_j^d \cap E^{n-\nu} \right)$. Aus $n-\nu > d$ folgt $T \neq \emptyset$. Es liege $\overline{O} \in T$ in der Nähe von O. Wir zeigen, daß der Abstand zwischen $\overline{O}$ und E_j^d größer als $\overline{R}$ $(j \in \{1, 2, \ldots, s\})$ ist. Daraus folgt, daß man die Kugel G vergrößern kann.

Es sei Q_j der Punkt, für den $Q_j \in E_j^d$ und $\overline{O}Q_j \perp E_j^d$ gelten. So ergibt sich

$$O\overline{O} \perp OP_j \perp P_jQ_j \qquad (j \in \{1, 2, \ldots, s\}).$$

Anderseits gilt $O\overline{O} \nparallel P_jQ_j$ wegen $\overline{O} \notin \overline{E}_j^d$. Deshalb ist OP_j die Normaltransversale der Geraden $O\overline{O}$ und P_jQ_j, folglich gilt

$$\overline{O}Q_j > \overline{R}$$

für alle $j \in \{1, 2, \ldots, s\}$.

3.1.2. Es sei $\nu \geq n-d$. Es ist bekannt, wenn jede Kantenlänge eines ν-dimensionalen Simplexes mindestens $2r^*$ ist, dann ist der Umkugelradius mindestens $\sqrt{\frac{2\nu}{\nu+1}}\, r^*$ und Gleichheit tritt für das reguläre Simplex mit Kantenlänge $2r^*$ auf. Daraus folgt

$$R_1 \geq \sqrt{\frac{2\nu}{\nu+1}}\, r^*. \tag{7}$$

Aus (5), (6), (7) und $\nu \geq n-d$ ergibt sich (1), d.h., die Ungleichung

$$\frac{R^*}{r^*} \geq \sqrt{\frac{2(n-d)}{n-d+1}} \tag{8}$$

gilt.

Gleichheit tritt nur im Fall auf, wenn $O = O_1$, $\nu = n-d$, $s = n-d+1$ sind und das Simplex $P_1, P_2, \ldots, P_s$ ein $(n-d)$-dimensionales reguläres Simplex mit Kantenlänge $2r^*$ ist. Wir beweisen noch, daß $E_j^d \perp_t E^{n-d}$ $(j \in \{1, 2, \ldots, n-d+1\})$ im Fall der Gleichheit notwendig gilt. Weil E_j^d die Kugel G berührt, gilt $OP_j \perp E_j^d$. Nehmen wir indirekt an, daß es mindestens einen Unterraum unter $E_1^d, E_2^d, \ldots, E_{n-d+1}^d$, z. B. E_1^d gibt, für den E_1^d nicht $\perp_t E^{n-d}$ gilt. Es sei $O \in \overline{E^d}$ und $\overline{E^d} \perp_t E^{n-d}$. Es gibt eine Gerade g, die nicht parallel zu E_1^d ist und für die $O \in g$, $g \subset \overline{E^d}$ gelten. $\overline{O} \in g$ liege in der Nähe von O. Wie im Punkt 3.1.1 kann man zeigen, daß der Abstand von $\overline{O}$ und E_1^d größer als $\overline{R}$ ist. Deshalb gibt es eine Kugel $\overline{G}$ mit dem Mittelpunkt $\overline{O}$, deren Radius mindestens $\overline{R}$ ist und die höchstens $n-d$ Unterräume berührt. Folglich kann man die Kugel G auf Grund von 3.1.1. vergrößern.

Wir haben gesehen, daß die Unterräume von $\{E_j^d\}$ im Fall der Gleichheit in (8) parallel $(E_j \perp_t E^{n-d})$ sind und die Enge dieser Anordnung und die Enge des Punktsystems P_j in E^{n-d} gleich sind. Der Unterraum E^{n-d} hat eine Zerlegung in reguläre Simplexe nur in Fällen $n-d = 1$ und $n-d = 2$. Daraus ergibt sich, daß Gleichheit in (2) nur in den Fällen $d = n-1$ und $d = n-2$ auftreten kann. Anderseits ist es zur Erfüllung der Gleichheit notwendig, daß $P_1, P_2, \ldots, P_{n-d+1}$ ein $(n-d)$-dimensionales reguläres Simplex sei, d.h., P_1P_2 soll eine Strecke bzw. $P_1P_2P_3$ ein reguläres Dreieck sein. Diese zwei Eigenschaften gelten gleichzeitig nur für die in 2.1. bzw. 2.3. geschriebenen Anordnungen. □

3.2. Der Beweis des Satzes 2. Wir betrachten die Punktmenge $\{P_i\}$ mit der $(r, R)_k$-Eigenschaft und mit der k-Enge $\varkappa$ in E^{n-d}. Es seien $E^{n-d} \subset E^n$, $E_i^d \subset E^n$, $E_i^d \perp_t E^{n-d}$, $P_i \in E_i^d$. Wir nehmen an, daß die Multiplizität von E_i^d und die Multiplizität von P_i $(= E_i^0)$ gleich sind. Mit P' bezeichnen wir die orthogonale Projektion des Punktes $P \in E^n$ auf E^{n-d}. Weil der Abstand von P und E_i^d gleich $P'P_i$ ist und die Bedingungen 1.1. und 1.2. für die Punktmenge $\{P_i\} = \{E_i^0\}$ gelten, deshalb hat $\{E_i^d\}$ die $(r, R)_k$-Eigenschaft und ist $\varkappa$ ihre k-Enge. □

3.3. Der Satz 3. ist eine einfache Folgerung des Satzes 2. □

3.4. Vor dem Beweis des Satzes 4. sehen wir zwei Hilfssätze ein.

Hilfssatz 1. Wir nehmen an, daß die Menge von Hyperebenen $\{E_i^{n-1}\}$ die $(c, c)_k$-Eigenschaft in E^n hat, wo die Hyperebenen nicht unbedingt verschieden sind. Berührt eine n-dimensionale Kugel vom Radius c keine von den Hyperebenen aus $\{E_i^{n-1}\}$, dann ist die Zahl der diese Kugel schneidenden Hyperebenen aus $\{E_i^{n-1}\}$ genau k, wo die Schnittzahl einer Hyperebene und der Kugel gleich der Multiplizität dieser Hyperebene ist.

BEWEIS. Ist die Zahl der schneidenden Hyperebenen, auch die Multiplizität betrachtet, mehr als k, dann gilt die Eigenschaft 1.1. für $\{E_i^{n-1}\}$ nicht. Wenn diese Zahl kleiner als k ist, dann gilt aber die Eigenschaft 1.2. nicht.

HILFSSATZ 2. Wir nehmen an, daß die Menge von Hyperebenen $\{E_i^{n-1}\}$ die $(c, c)_k$-Eigenschaft in E^n hat. Berührt die Hyperebene $E_i^{n-1} \in \{E_i^{n-1}\}$ mit Multiplizität m in P eine n-dimensionale Kugel G vom Radius c, dann gibt es eine Hyperebene $'E_i^{d-1} \in \{E_i^{n-1}\}$ mit Multiplizität m, die die Kugel G in dem P gegenüberliegenden Punkt P' berührt, wo $1 \leq m \leq k$ ist.

BEWEIS. Mit O bezeichen wir den Mittelpunkt der Kugel G. Die Zahl der zu $\{E_i^{n-1}\}$ gehörigen und G schneidenen Hyperebenen ist höchstens $k-m$, auch die Multiplizität gerechnet. Im entgegengesetzten Fall könnten wir die Kugel in der Richtung $\overrightarrow{OP}$ ein bißchen derart verschieben, daß die Zahl der Translate von G schneidenden Hyperebenen aus $\{E_i^{n-1}\}$ mehr als k wäre. In diesem Fall gilt aber die Bedingung 1.1. für $\{E_i^{n-1}\}$ nicht.

Wir betrachten die Hyperebene $'E_i^{n-1}$, die die Kugel G in P' berührt ($\overrightarrow{PO} = \overrightarrow{OP'}$) und wir nehmen indirekt an, daß $'E_i^{n-1} \notin \{E_i^{n-1}\}$ gilt. Wir bewegen G parallel E_i^{n-1} derart, daß G weitere Hyperebenen aus $\{E_i^{n-1}\}$ nicht berührt. Wir verschieben G in der Richtung $\overrightarrow{PO}$ ein bißchen. Weil $'E_i^{n-1} \notin \{E_i^{n-1}\}$ ist, schneiden höchstens $k-m$ Hyperebenen aus $\{E_i^{n-1}\}$ die Translate von G, d.h., die Bedingung 1.2 gilt für $\{E_i^{n-1}\}$ nicht. Folglich gilt $'E_i^{n-1} \in \{E_i^{n-1}\}$.'

Es sei m' die Multiplizität von $'E_i^{n-1} \in \{E_i^{n-1}\}$. Wir zeigen, daß $m = m$ gilt. Wenn es $E_i^{n-1} \in \{E_i^{n-1}\}$, $'E_i^{n-1} \neq E_j^{n-1} \neq E_i^{n-1}$ gibt, die G berührt, dann bewegen wir G parallel E_i^{n-1} so, daß G und E_j^{n-1} sich nicht berühren werden. In diesem Fall ist die Zahl der Hyperebenen aus $\{E_i^{n-1}\}$, die die Translate von G schneiden, genau $k-m$. Im entgegengesetzten Fall gelte die Eigenschaft 1.2. für $\{E_i^{n-1}\}$ nicht.

Gilt $m > m'$ oder $m < m'$, dann verschieben wir G in der Richtung $\overrightarrow{OP'}$ ein bißchen und gelten die Bedingungen 1.2 bzw. 1.1 für $\{E_i^{n-1}\}$ nicht. Folglich ist $m = m'$. □

DER BEWEIS DES SATZES 4. $\frac{R^*}{r^*} \geq 1$ folgt aus den Punkten 1.1 und 1.2, sowie aus der Definition von r^* und R^*. Deshalb genügt es den Fall der Gleichheit zu untersuchen.

Wir nehmen an, daß die Menge von Hyperebenen E_i^{n-1} die $(r, R)_k$-Eigenschaft hat und $\frac{R^*}{r^*} = 1$ gilt, d.h.,

$$r^* = R^* = c \tag{9}$$

ist. Mit dieser Bezeichnung hat $\{E_i^{n-1}\}$ die $(c, c)_k$-Eigenschaft.

Es sei G eine n-dimensionale Kugel vom Radius c, die keine von den Hyperebenen aus $\{E_i^{n-1}\}$ berührt. (Die Existenz dieser Kugel folgt aus 1.1. und 1.2.). Es seien $\{E_{i_1}^{n-1}, E_{i_2}^{n-1}, \ldots, E_{i_s}^{n-1} \in \{E_i^{n-1}\}$ die die Kugel G schneidenden Hyperebenen, deren Multiplizität nacheinander $m_1, m_2, \ldots, m_s$ sind. Nach dem Hilfssatz 1. gilt

$$(10) \qquad m_1 + m_2 + \ldots + m_s = k.$$

Es seien $G_0 \neq G_{-1}$ zwei n-dimensionale Kugeln vom Radius c, die die Hyperebene $E_{i_1}^{n-1}$ im Punkt P_0 berühren. P_1 bzw. P_{-1} sei der gegenüberliegende Punkt von P_0 auf G_0 bzw. auf G_{-1}. Aus dem Hilfssatz 2. folgt, daß es $'E_{i_1}^{n-1}, {}^*E_{i_1}^{n-1} \in \{E_i^{n-1}\}$ geben, deren Multiplizität m_1 ist und die Kugel G_0 bzw. G_{-1} im Punkt P_1 bzw. P_{-1} berühren. Es seien $G_1 \neq G_0$ und $G_{-2} \neq G_{-1}$ n-dimensionale Kugeln vom Radius c, die die Hyperebene $'E_{i_1}^{n-1}$ bzw. ${}^*E_{i_1}^{n-1}$ im Punkt P_1 bzw. P_{-1} berühren. P_2 bzw. P_{-2} seien die gegenüberliegenden Punkte von P_1 bzw. P_{-1} auf G_1 bzw. auf G_{-2}. Wieder nach dem Hilfssatz 2. existieren Hyperebenen die $''E_{i_1}^{n-1}, {}^{**}E_{i_1}^{n-1} \in \{E_i^{n-1}\}$, deren Multiplizität m_1 ist und die G_1 bzw. G_{-2} im Punkt P_2 bzw. P_{-2} berühren. Mit der Fortsetzung dieses Verfahrens können wir eine Teilmenge $\{E_{i_{m_1}}^{n-1}\}$ von $\{E_i^{n-1}\}$ vorstellen, wo die Hyperebenen von $\{E_{i_{m_1}}^{n-1}\}$ parallel und von der Multiplizität m_1 sind und der Abstand der benachbarten Hyperebenen $2c$ ist.

Wir wiederholen das oben geschriebene Verfahren auch für die Hyperebenen $E_{i_2}^{n-1}, E_{i_3}^{n-1}, \ldots, E_{i_s}^{n-1}$. So existieren die Teilmengen $\{E_{i_{m_2}}^{n-1}\}$, $\{E_{i_{m_3}}^{n-1}\}, \ldots, \{E_{i_{m_s}}^{n-1}\}$ mit derselben Eigenschaften wie $\{E_{i_{m_1}}^{n-1}\}$.

Es sei

$$(11) \qquad \{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\} = \{E_{i_{m_1}}^{n-1}\} \cup \{E_{i_{m_2}}^{n-1}\} \cup \ldots \cup \{E_{i_{m_s}}^{n-1}\}.$$

Aus der Konstruktion folgt, daß $\{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\} \subseteq \{E_i^{n-1}\}$ gilt. Anderseits folgt (11) = (1) aus (10), d.h., $\{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\}$ bildet ein $(r, R)_k$-System und $\frac{R^*}{r^*} = 1$ ist.

Wir sehen ein, daß $\{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\} = \{E_i^{n-1}\}$ gilt. Wir nehmen an, daß eine Hyperebene E_j^{n-1} gilt, für die $E_j^{n-1} \notin \{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\}$ und $E_j^{n-1} \in \{E_i^{n-1}\}$ gelten. Es ist leicht einzusehen, daß es einen Punkt $Q \in E^n$ und Hyperebenen $E_{i_1}^{n-1} \in \{E_{i_{m_1}}^{n-1}\}, E_{i_{m_2}}^{n-1} \in \{E_{i_{m_2}}^{n-1}\}, \ldots, E_{i_s}^{n-1} \in \{E_{i_{m_s}}^{n-1}\}$ geben, wo der Abstand von Q und von $E_j^{n-1}, E_{i_1}^{n-1}, E_{i_2}^{n-1}, \ldots, E_{i_s}^{n-1}$ kleiner als c ist. Die n-dimensionale Kugel vom Mittelpunkt Q und vom Radius c schneidet die obigen Hyperebenen aus $\{E_i^{n-1}\}$. Wegen (10) schneidet diese Kugel mindestens $k+1$ Hyperebenen aus $\{E_i^{n-1}\}$. Das bedeutet aber, daß die Eigenschaft 1.1 für $\{E_i^{n-1}\}$ nicht gilt.

Daraus folgt

$$\{E_i^{n-1}\} = \{E_{i_{m_1} i_{m_2} \ldots i_{m_s}}^{n-1}\} = \{E_{i_1 i_2 \ldots i_k}^{n-1}\}. \ \square$$

4. Bemerkungen

4.1. Die Vermutung für die extremalen Figuren im Fall $d=n-3$ ist das folgende. Es ist ein raumzentriertes Würfelgitter in $E^3 \subset E^n$ gegeben. Nehmen wir die Menge der $(n-3)$-dimensionalen Unterräume $\{E_i^{n-3}\}$ in E^n, die in den Gitterpunkten zu E^3 total senkrecht sind. Es ist offenbar, daß die Enge von $\{E_i^{n-3}\}$ $\sqrt{\frac{5}{3}}$ ist. $\left(\sqrt{\frac{5}{3}} > \sqrt{\frac{3}{2}}\right.$ s. die Sätze 1. und 2.)

4.2. Die Vermutung ist, daß die Gleichheit in (3) auftritt.

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ON THE CUT LOCUS AND THE FOCAL LOCUS OF A SUBMANIFOLD IN A RIEMANNIAN MANIFOLD

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(Received April 10, 1985)

Focal loci of submanifolds in a Riemannian manifolds have been studied in detail in [1], [5], [7], [8], [9] and [10]. [1] and [9] contain some basic facts on cut loci of submanifolds (in [1] they are called minimum loci). In this paper section **1** contains the preliminaries where we establish our notational conventions, too. In section **2** we develop some of the matching and non-matching properties of two minimal geodesics from a cut point, which is not a first focal point, to the points of the sumanifold L. Section **3** deals with generalizations of some results of W. KLINGENBERG [6] from conjugate loci to focal loci. The results on the focal loci are obtained under some additional assumptions and with certain modifications.

1. Introduction

Let M be a compact connected n-dimensional Riemannian manifold of class c^∞ and consider a c^∞ compact connected m-dimensional submanifold $L\subset M$. Let $N(L)$ be the normal bundle of L, i.e., the set of all tangent vectors to M at points of L that are orthogonal to L. The exponential map of the Riemannian manifold M restricted to $N(L)$ is a map $\varepsilon : N(L)\to M$ of class c^∞. Consider now a unit vector w in the normal space $N_x(L)$ of L at x for $x\in L$ and the geodesic

$$c:R\to M$$

such that $c(0)=x$, $\dot c(0)=w$, $c(t)=z$ and $c\restriction[0,t]$ is the unique minimal geodesic. Let S_w be the supremum of these $t>0$ which is always well-defined, since M is compact. If S_w is finite, then $c\restriction[0,S_w]$ is a minimal geodesic. The point $\varepsilon(v)=c(S_w)$ is called a cut point of L in M and the point $v=S_w w$ is also said to be a cut point of L in $N(L)$. The set of such cut points $S_w w$ is called the cut locus of the submanifold L in the normal bundle while the set of cut points $c(S_w)$ is called the cut locus of L in M.

If the tangent linear map

$$T_\nu \varepsilon : T_\nu N(L) \to T_{\varepsilon(\nu)} M$$

of ε at $\nu \in N(L)$ is not injective, then ν is called a focal point of L in $N(L)$ and the point $\varepsilon(\nu)$ is also called a focal point of L in M. The set of such focal points ν is called the focal locus of L in $N(L)$ while the set of focal points $\varepsilon(\nu)$ is called the focal locus of L in M. The multiplicitiy k of a focal point is the dimension of the kernel of $T_\nu \varepsilon$. The sum of the multiplicities of the focal points gives the index of the geodesic by the Morse Index Theorem.

If $\nu = S_w w \in N(L)$ is a cut point of the submanifold L, then at least one of the following properties holds:

1. The point $\nu = S_w w \in N(L)$ is the first focal point of the submanifold L on the ray tw, $t \geqq 0$.

2. There are at least two different minimizing geodesics from the cut point $\varepsilon(S_w w) \in M$ to the submanifold L (cf. e.g. [9]).

The author thanks Prof. J. SZENTHE for many useful discussions during the preparation of this paper.

2. The closest point of the cut locus to the submanifold

Let $L \subset M$ be a submanifold and let c be a geodesic starting at a point of L which is orthogonal to L. Let z be a point on c. Then z is a cut point of L if c minimizes arc length up to z but not further. If c' is another geodesic from the cut point z to a different point of L such that c' does not match smoothly with c at z, then z is not a closest cut point of L. This fact is stated in the following lemma.

LEMMA 1. Let M be a c^∞ compact connected Riemannian manifold and let L be a c^∞ compact connected submanifold of M. Let $\varepsilon(\nu) = x \in M$ be a point of the cut locus $C(L)$ to L which is not a focal point of L. If c_1 and c_2 are two different minimizing geodesics from x to L such that they do not match smoothly at x, then there is a cut point of L in any neighborhood of x which is nearer to L than x.

PROOF. For the sake of convenience the zero section of the normal bundle $N(L)$ is identified with L and let $P_L : N(L) \to L$ be the projection map. Since $N(L)$ is a vector bundle, it is locally trivial, that is, every point y of the zero section of $N(L)$ has a neighborhood U' such that $P_L^{-1}(U')$ is isomorphic with $U' \times N_y L$. There is a Riemannian metric on $N(L)$ by basic results [4] and with respect to this metric $T_\nu \varepsilon$ preserves the length of vectors tangent to the rays of $N(L)$ by the generalized Gauss lemma [7]. Let $\nu \in N_y L$ be a non zero vector where $y \in U'$. Then locus of the end points of such ν with fixed length will be a sphere of dimension $n-m-1$. Consider with ν the family of vectors of the same length as ν in $P_L^{-1}(U')$, then corresponding to these vectors there is a union of the spheres which forms a piece of a hypersurface, say K, and hence a tangent space $T_\nu K$ at ν orthogonal to ν which is proved as follows: Let $\varrho:[0, \infty)\to N(L)$ be the ray of $N(L)$ defined by $\varrho(t) = tv$, $t\in[0, \infty)$, then $\dot\varrho(1) = v$. Consider a geodesic variation $W_t(s)$, $t\in[0, 1]$, $s\in[-\varepsilon, \varepsilon]$ of $\dot\varrho$ considered as a geodesic in $N(L)$ such that $W_t(o) = \varrho(t)$ and varied geodesics are of the same lengths and orthogonal to U'. Then the variation vector field $W = \left.\frac{\partial W_t(s)}{\partial s}\right|_{s=0}$ will be a Jacobi field tangent to the hypersurface K at v and is non zero field at v, since v is not a focal point of L in $N(L)$. This Jacobi field will be orthogonal to v at $t = 1$. This process can be applied for any element of T_vK. Since each variation vector filed which is an element of T_vK is orthogonal to v. Therefore T_vK will be orthogonal to v, i.e. for $X\in T_vK$, $\bar g(\dot\varrho(1), X) = 0$ holds, where $\bar g$ is induced metric of $N(L)$. Now we define geodesic $c_1:[0, 1]\to M$ such that $c_1(0) = z_1\in L$, $\dot c_1(0)\in N_{z_1}L$, $c_1(1) = x = \varepsilon(v)$. Consider for c_1 a family of neighboring geodesics each orthogonal to L, then under the restricted exponential map ε each member of this family is the image of non-zero vectors taken in $P_L^{-1}(U')$ corresponding to v. Since each vector in $P_L^{-1}(U')$ is of same length as that of v, Their images under map ε will be of the same length too. As $\varepsilon(v) = x$ is not a focal point of L in M, the image $\varepsilon(K)$ will be a piece of hypersurface containing x in M. Since T_vK is orthogonal to $\dot\varrho(1)$, therefore by the generalized Gauss lemma [7]

$$T_v\varepsilon(T_vK) = T_{\varepsilon(v)}\,\varepsilon(K)$$

will be orthogonal to $\dot c_1(1)$ i.e. for

$$Z = T_v\varepsilon X\in T_{\varepsilon(v)}\varepsilon(K),\ \dot c_1(1) = T_v\varepsilon(\dot\varrho(1));$$

$$g(\dot c_1(1), Z) = g(T_v\varepsilon\dot\varrho(1), T_v\varepsilon X) = \bar g(\dot\varrho(1), X) = 0,$$

where g is the Riemannian metric of M. Consequently the hypersurface $\varepsilon(K)$ is orthogonal to c_1. Similar result holds for the geodesic c_2 passing orthogonally through the point $z_2\in L$ to x. Since c_1 and c_2 do not match smoothly at x, the two tangent hyperplanes at x do intersect in the neighborhood of x. From this there exists a point x' close to x which is joined to L by two minimal geodesics c_1', and c_2' where c_1' is neighboring to c_1 and c_2' is neighboring to c_2 and each being shorter than c_1 and c_2. But this means that x' is on the cut locus $C(L)$ nearer to L than the point x; since it can easily be seen that geodesics of smaller length than c_1' minimizes its arc length uniquely. Now there are two possibilities:

(1) If $z_1\neq z_2$, then the minimizing geodesics from x' have different foot points, consequently they are different. But then x' is a cut point of L nearer to L than x.

(2) If $z_1 = z_2$, then since c_1 and c_2 are minimizing geodesics and not matching smoothly at x, there will be two different subspaces of codimension 1 in T_xM each orthogonal to c_1 and c_2 respectively at x and both the subspaces intersect in the neighborhood of x. Therefore there is a point x' in the neighborhood of x such that there are two different minimizing geodesics from x' to L and x' is nearer to L than x, consequently x' is a cut point of L nearer to L than x.

THEOREM 1. *Let M be a compact connected Riemannian manifold and let $L \subset M$ be a compact connected submanifold of M. Let the cut locus of L be non-empty and let $\varepsilon(v)$ be a closest point of the cut locus to L. If $\varepsilon(v)$ is not a focal point of L, then there are at most two different points of L which are at minimal distance from $\varepsilon(v)$.*

PROOF. Since $\varepsilon(v)$ is not a focal point of L, there are at least two different minimizing geodesics from L to the cut point $\varepsilon(v)$. We have to show that there are at most two different points of L which are foot points of minimizing geodesics from L to $\varepsilon(v)$. Let c_1 and c_2 be any two minimal geodesics through $\varepsilon(v)$ to two different points z_1 and z_2 of L. If c_1, c_2 do not match smoothly at $\varepsilon(v)$, then there is a point x' which is nearer to L than $\varepsilon(v)$. But then by lemma 1 the point x' will be a cut point of L, which contradicts the fact that $\varepsilon(v)$ is the closest cut point of the submanifold L. Thus if c_1 and c_2 are two minimizing geodesics from L to $\varepsilon(v)$ and $\varepsilon(v)$ is the closest cut point of L, then they match smoothly at $\varepsilon(v)$. It remains to prove that there are at most two different points of L which are at minimal distance from $\varepsilon(v)$. To prove this we assume that z_3 is a third foot point of a minimal geodesic c passing through $\varepsilon(v)$. But then neither the minimizing geodesics c_1, c nor c, c_2 match at $\varepsilon(v)$, since c_1 and c_2 match smoothly at $\varepsilon(v)$. Hence, the assertion of the theorem follows.

3. Focal points under some restrictions

The scalar product in T_pM, the tangent space of M at p is denoted by $\langle , \rangle$. The length of a path c in M is denoted by $\mathcal{L}(c)$.

PROPOSITION 1. *Let M be a complete connected n-dimensional Riemannian manifold of class C^∞ and let $L \subset M$ be a C^∞ compact connected submanifold such that the restricted exponential map has no focal points in $U(r)$, where $U(r)$ is the tube of radius r around the zero section in $N(L)$. Let $O' \in M$ and assume that c_0 and c_1 are different geodesic segments joining O' orthogonally to L and that there is a family h_t, $t \in [0, 1]$ of curves joining O' orthogonally to L such that*

$$h_0 = c_0, \quad h_1 = c_1,$$

and $\mathcal{L}(h_t) = \mathcal{L}(c_1)$ for all $t \in [0, 1]$. Then $\mathcal{L}(c_0) + \mathcal{L}(c_1) \geqq 2r$.

PROOF. We can assume that $\mathcal{L}(c_0) < r$. Since $U(r)$ does not contain focal points, the tangent linear map $T_v\varepsilon$ is everywhere non-singular in $U(r)$. Then the restricted exponential map ε is a covering map [5]. But every covering map has the curve lifting property ([2] pp. 25). Hence $c_0 = h_0$ can be lifted by the preimage ε^{-1} of ε restricted to $U(r)$ into $U(r)$ and this gives a straight segment $\bar{h}_0$ of length $\mathcal{L}(c_0)$ starting from the zero section of $N(L)$ to a point $\bar{O}' \in U(r)$. In this mammer we can lift the curves h_t for values of t suitably close to O to a family $\bar{h}_t$ going from the zero section to $\bar{O}'$. Since c_0 and c_1 are different, it will not be possible to lift h_t for all $t \in [0, 1]$ to $U(r)$. Hence for every $\varepsilon' > 0$, there exists a $t_0 \in [0, 1]$ such that h_{t_0} can be lifted to a curve $\bar{h}_{t_0}$ in $U(r)$ going from the zero section to O' and containing a point which has distance $<\varepsilon'$ from the boundary of $U(r)$. But then by the generalized Gauss lemma, $\mathcal{L}(c_0)+\mathcal{L}(h_{t_0})\geqq 2r-2\varepsilon'$. Since $\mathcal{L}(h_t)\leqq\mathcal{L}(c_1)$ for all $t\in[0, 1]$ and ε' is arbitrary, the result follows immediately.

THEOREM 2. *Let M be a comlete simply connected Riemannian manifold and $L\subset M$ be a compact connected submanifold such that there are no focal points in the neighborhood of radius r of the zero section in $N(L)$ but there are at least 2 focal points on each ray corresponding to an orthogonal geodesic in the neighborhood of radius $2r$ of the zero section in $N(L)$, Then the following hold:*

(1) *M is compact;*

(2) *For any O' sufficiently near to L which is not a focal point of L, there is precisely one geodesic of length $d(O', L)$ joining O' to L and all other geodesics joining O' to L orthogonally have length $\geqq 2r-d(O', L)$ and index $\geqq 2$;*

(3) *A non-trivial geodesic intersecting L orthogonally at its two end points has length $\geqq 2r$. The distance between L and its cut locus $C(L)$ is $\geqq r$.*

PROOF. To prove (1), it is sufficient to note that a geodesic segment of length $2r$, starting from L and being orthogonal to L contains focal points in its interior and, therefore, is not a curve of minimal length from its end point to L. Consequently a tube of radius $2r$ about L covers M. Since L is assumed to be compact, this implies that M is compact.

To prove (2): Since there are no focal points in the neighborhood of radius r of the zero section in $N(L)$ and at least 2 focal points on any ray corresponding to an orthogonal geodesic in the neighborhood of radius $2r$ of the zero section in $N(L)$, there is a $\delta>0$ such that each orthogonal geodesic segment of length $2r-\delta$ has index $\geqq 2$, as L is compact by the Morse Index Theorem. Let $O'\in M$ be a point which is not focal point of L such that $d(O', L)<\delta$. Let c_0 be a geodesic of minimal length $d(O', L)$ joining O' to L i. e. $\mathcal{L}(c_0) = d(O', L)$. Let c_1 be a geodesic $\neq c_0$ starting at O' and orthogonal to L. Since M is compact simply connected and L is connected, too, there exists a continuous family $\{h_t\}$, $t\in[0, 1]$ of curves h_t joining $c_0 = h_0$ and $c_1 = h_1$. In every such family consider a curve h_{t_0} which has maximal length in the family, where of course the value t_0 depends on the family. Consider now such a family that the length of $\bar{h}_{t_0}$ of this family is maximal among the lengths of h_{t_0} of all other families. Then this $\bar{h}_{t_0}$ is a geodesic $\neq h_0 = c_0$. In fact, if $\bar{h}_{t_0}$ is not a geodesic, then there is a variation of $\bar{h}_{t_0}$ where the lengths of curves in the variation are greater than that of $\bar{h}_{t_0}$. But then there would be families where the length of h_{t_0} is greater than that of $\bar{h}_{t_0}$ which is a contradiction. Since $\bar{h}_{t_0}$ is not of minimal length, so if $\bar{h}_{t_0}\neq c_1$, the construction made above allows us to assume that the index of $\bar{h}_{t_0}$ is 1 if $\bar{h}_{t_0}\neq c_1$. Applying proposition 1 to the homotopy $\{h_t\}$, $t\in[0, t_0]$, we obtain

$$\mathcal{L}(\bar{h}_{t_0})\geqq 2r-\mathcal{L}(c_0)\geqq 2r-\delta,$$

that is index $\bar{h}_{t_0}\geqq 2$. But then $\bar{h}_{t_0} = c_1$ and this yields the required result.

In order to prove (3) we assume that there is a non-trivial geodesic c intersecting L orthogonally at its two end points P_1 and P_2 and having length $\mathcal{L}(c) = 2r - 2a < 2r$. Then there exists an $O' \in M$, not focal point to L and so close to L that the family of paths joining O' to L contains a geodesic c' different from the uniquely determined geodesic c_0 of minimal length from L to O' and satisfying $\mathcal{L}(c') + \mathcal{L}(c_0) \leqq 2r - a < 2r$, which contradicts (2). This proves the first statement in (3). To prove the last statement of (3) we assume that p be a cut point of L, then, since the point P is not a focal point of L, there will be two minimizing geodesics c_0 and c_1 from P to L. If $\mathcal{L}(c_0) < r$ $\mathcal{L}(c_1) < r$, then

$$\mathcal{L}(c_0) + \mathcal{L}(c_1) < 2r,$$

which contradicts (2).

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ON THE ESTIMATE FOR THE DIFFERENCES OF THE DERIVATIVES OF PARTIAL SUMS ARISING BY THE EIGENFUNCTIONS OF HIGHER ORDER OF SCHRÖDINGER OPERATORS

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(Received May 8, 1985)

1. In the present work we prove an exact estimate for the difference of the first derivatives of the partial sums arising by the eigenfunctions of higher order of Sturm-Liouville operators. Our result extends that of [6] and [9] for the case when the eigenfunctions have an arbitrary order.

2. Let $G=(a, b)\subset \mathbf{R}$ be an arbitrary finite open interval, $q(x)\in L^p(G)$ $(p\geq 1)$ an arbitrary complex function and consider the formal Schrödinger operator

$$(1) \qquad Lu := -u''+qu.$$

Given a complex number λ, the function $u_{-1}: G\to \mathbf{C}$, $u_{-1}\equiv 0$ is called an eigenfunction on order -1 of the operator L with the eigenvalue λ. A function $u_k: G\to \mathbf{C}$, $u_k\not\equiv 0$ $(k=0, 1, 2, \dots)$ is said to be an eigenfunction of order k of the operator L with the eigenvalue λ if u_k together with its derivative is absolutely continuous on every compact subinterval of G and if for almost all $x\in G$ the equation $Lu_k(x)=\lambda u_k(x)-u_{k-1}(x)$ holds, where $u_{k-1}(x)$ is an eigenfunction of order $k-1$ with the same λ.

Let $\{u_k\}$ (resp. $\{\hat{u}_k\}$) be a Riesz basis in $L^2(G)$ consisting of eigenfunctions of the operator L(resp. $\hat{L}u=-u''+\hat{q}u$). Let λ_k and σ_k(resp. $\hat{\lambda}_k$ and $\hat{\sigma}_k$) denote the eigenvalue and the order of u_k(resp. $\hat{u}_k$) and having the following property:

$$(2) \qquad \sup \sigma_k<\infty, \ \sup \hat{\sigma}_k<\infty.$$

Let $f(x)$ be an absolutely continuous function on the closed interval $[a, b]$; further let μ be an arbitrary non-negative number. Introduce the notations

$$\sigma_\mu(f, x) := \sum_{|\operatorname{Re}\sqrt{\lambda_k}|<\mu} <f, v_k> u_k(x), \quad \hat{\sigma}_\mu(f, x) := \sum_{|\operatorname{Re}\sqrt{\hat{\lambda}_k}|<\mu} <f, \hat{v}_k> \hat{u}_k(x),$$

where $\{v_k\}$ (resp. $\{\hat{v}_k\}$) are the eigenfunctions of the adjoint operator $\bar{L}v := -v''+\bar{q}(x)v$ (resp. $\hat{\bar{L}}v := -v''+\hat{\bar{q}}(x)v$) with the eigenvalues $\{\bar{\lambda}_k\}$ (resp. $\{\hat{\bar{\lambda}}_k\}$). Our aim is to prove the following.

THEOREM. *Suppose K is an arbitrary compact subset of G, and let the potentials $\hat{q}(x)$, $\hat{q}(x)$ fulfil the condition*

$$q(x)\in L^p(G),\ \hat{q}(x)\in \hat{L}^p(G)\ (p,\ \hat{p}\in(1,\ \infty)).$$

For any $f\in W_1^1(G)$ the estimate

$$\left|\frac{d}{dx}\sigma_\mu(f,x)-\frac{d}{dx}\hat{\sigma}_\mu(f,x)\right|\le C(K)\|f\|_{W_1^1(G)}$$

holds uniformly in x on the compact K. The constant $C(K)$ depends only on K.

REMARK. This estimate can not be improved in the sense that we can't change the constant $C(K)$ into $\bar{o}(1)$ (cf. [6]).

3. Introduce in the investigation the spectral function $\Theta(x, y, \mu)$ of the operator L:

$$\Theta(x,y,\mu):=\sum_{|\operatorname{Re}\sqrt{\lambda_k}|<\mu}u_k(x)\overline{v_k(y)}.$$

Let K be an arbitrary compact subset of G and fix a number R_0 such that $0<R_0\le\frac{1}{4}$ dist $(K, \partial G)$; further suppose $R_0\le R\le 2R_0$. Denote S_{R_0} the average operator

$$S_{R_0}[f]:=\frac{1}{R_0}\int_{R_0}^{2R_0}f(R)dR.$$

Fixing $\mu>0$ and $x\in K$ arbitrarily, let us introduce the function $W:G\to\mathbf{R}$ by

$$W(x,y,\mu):=\begin{cases}\dfrac{1}{\pi}\dfrac{\sin\mu(y-x)}{y-x} & \text{if}\quad |x-y|\le R,\\ 0 & \text{if}\quad |x-y|>R.\end{cases}$$

For the proof of our theorem we shall need the following proposition which follows from some lemmas.

PROPOSITION. Suppose $q(x)\in L^p(G)$, $p>1$. Then the estimate

$$\left|\int_{y_1}^{y_2}\frac{\partial}{\partial x}[S_{R_0}W(x,y,\mu)-\Theta(x,y,\mu)]dy\right|\le D(K)\tag{3}$$

holds uniformly in x on the compact $K\subset G$ and in $y_1, y_2\in[a, b]$. The constant $D(K)$ depends only on K and $q(x)$.

For the sake of brevity we shall denote in the sequel by η_k an arbitrary square root of λ_k and we put $\varrho_k:=\operatorname{Re}\eta_k$, $\nu_k:=\operatorname{Im}\eta_k$. Throughout this paper $K=[c, d]\subset G$ will denote an arbitrary compact interval satisfying the following condition: $\exists R>0$ such that $K_R:=[c-R, d+R]\subset G$.

LEMMA 1. Consider the operator (1) on the interval $G=[0,1]$. Then the estimate

$$\left|\int_0^{2x} u_k(\xi)d\xi\right| \le \frac{C}{1+|\mu_k|}\left[\|u_k\|_{L^\infty(G)}\left(1+\frac{1}{2}|\nu_k|\right)^{\sigma_k}+\right.$$

$$\left.\cdot\|u_{k-1}\|_{L^\infty(G)}\left(1+\frac{1}{2}|\nu_k|\right)^{\sigma_{k-1}}\right] \tag{4}$$

holds uniformly in x on the interval $\left[0,\frac{1}{2}\right]$.

PROOF. We recall the generalized Titchmarsh formula of Joó I. [1]: for any $x+t\in G$, $x-t\in G$

$$u_k(x+t)+u_k(x-t)-2u_k(x)\cos\mu_k t =$$

$$=\int_{x-t}^{x+t}\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}[q(\xi)u_k(\xi)-u_{k-1}(\xi)]d\xi \quad \text{if} \quad \mu_k\ne 0 \tag{5}$$

and the proposition in [2] of KOMORNIK V.:

$$|u_k(x)|\exp\{|\nu_k|d_{a,b}(x)\}\le M_k\|u_k\|_{L^\infty(G)}(1+|\nu_k|d_{a,b}(x))^{\sigma_k}, \tag{6}$$

where $d_{a,b}(x):=\min\{|x-a|,|x-b|\}$ and the constant M_k does not depend on μ_k.

Let us assume $x\le\frac{1}{2}$. Integrating (5) in t from $-x$ to x and estimating we get easily

$$2\left|\int_0^{2x}u_k(\xi)d\xi\right|\le 4|u_k(x)|\left|\frac{\sin\mu_k x}{\mu_k}\right|+$$

$$+\int_{-x}^{x}\int_{x-t}^{x+t}\left|[u_k(\xi)q(\xi)-u_{k-1}(\xi)]\cdot\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\right|d\xi dt. \tag{7}$$

Using (6) and the inequality $|\sin\mu_k t|\le 2\exp|\nu_k|t$ ($t\in\mathbf{R}$, $\mu\in\mathbf{C}$) we obtain the estimate for the first term of (7):

$$4|u_k(x)|\left|\frac{\sin\mu_k t}{\mu_k}\right|\le \text{const}\,\|u_k\|_{L^\infty(G)}(1+|\nu_k|\cdot x)^{\sigma_k}\frac{1}{1+|\mu_k|}. \tag{8}$$

Taking into account that

$$t-|x-\xi|\le\min\{\xi,1-\xi\}:=d(\xi) \text{ if } t>0,$$

using (6) and the elementary inequality

(9) $2|\sin\mu_k(t-|x-\xi|)| \le \exp|\nu_k(t-|x-\xi|)|+1 \le 2\exp|\nu_k(t-|x-\xi|)|$

we get that

$$\left|u_k(\xi)\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\right| \le |u_k(\xi)|\frac{\exp\{|\nu_k|\cdot d(\xi)\}}{|\mu_k|} \le \tag{10}$$

$$\le \text{const}\,\|u_k\|_{L^\infty(G)}(1+|\nu_k|d(\xi))^{\sigma_k}\frac{1}{1+|\mu_k|} \quad \text{if} \quad t>0.$$

In case $t<0$, consider that $|t+|x-\xi|| \le \min\{\xi,\ 1-\xi\} := d(\xi)$, and the right hand side of (5) has the following form:

$$\int_{x-t}^{x+t}[u_k(\xi)q(\xi)-u_{k-1}(\xi)]\frac{\sin\mu_k(t+|x-\xi|)}{\mu_k}d\xi.$$

Using (6) and (9) we have, too:

$$\left|u_k(\xi)\frac{\sin\mu_k(t+|x-\xi|)}{\mu_k}\right| \le$$

$$\le \text{const}\,\|u_k\|_{L^\infty(G)}(1+|\nu_k|d(\xi))^{\sigma_k}\frac{1}{1+|\mu_k|} \quad \text{if} \quad t<0. \tag{11}$$

Similarly, we get that

$$\left|u_{k-1}(\xi)\frac{\sin\mu_k(t+|x-\xi)|}{\mu_k}\right| \le$$

$$\le \text{const}\,\|u_{k-1}\|_{L^\infty(G)}(1+|\nu_k|d(\xi))^{\sigma_k-1}\frac{1}{1+|\mu_k|}. \tag{12}$$

Taking into account $d(\xi)\le\frac{1}{2}$ from (7), (10)–(12) we obtain

$$\int_{-x}^{x}\int_{x-t}^{x+t}\left|[u_k(\xi)q(\xi)-u_{k-1}(\xi)]\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\right|d\xi\,dt \le$$

$$\le \text{const}\,\|u_{k-1}\|_{L^\infty(G)}\frac{\left(1+\frac{1}{2}|\nu_k|\right)^{\sigma_k-1}}{1+|\mu_k|}\int_{-x}^{x}|t|dt+ \tag{13}$$

$$+\text{const}\,\|p\|_{L^1(G)}\|u_k\|_{L^\infty(G)}\left(1+\frac{1}{2}|\nu_k|\right)^{\sigma_k}\int_{-x}^{x}dt.$$

The Lemma 1 follows from (7), (8) and (13). □

LEMMA 2. Given any compact $K \subset G$, there exists a constant $C_3(K, R)$ such that for $0 < 3R < 4R_0$; $q(x) \in L^p(G)(p > 1)$; $y_1, y_2 \in [a, b]$

$$(14) \qquad \sum_{|\mu - |\varrho_k|| \leq 1} \left| \int_{y_1}^{y_2} v_k(y) dy \operatorname{ch} \nu_k R \right|^2 < \frac{C^3(K, R)}{\mu^2}, \ \mu \geq 1.$$

PROOF. Using the Lemma 1 for $\{v_k\}$, the theorem 4 in [8] and (2) we have that

$$(15) \quad \left| \int_{y_1}^{y_2} v_k(x) dy \right| \leq \frac{C_1(K, R)}{1 + |\mu_k|} \operatorname{ch}^2 \nu_k R \left[\|v_k\|_{L^\infty(K)} + \|v_{k-1}\|_{L^\infty(K)} \right].$$

Hence using (61) in [4] and the inequality $\operatorname{ch} a \operatorname{ch} b \leq \operatorname{ch}(a+b)$ $(a, b > 0)$ we get

$$\sum_{|\mu - \varrho_k| \leq 1} \left| \int_{y_1}^{y_2} v_k(y) dy \operatorname{ch} \nu_k R \right|^2 \leq$$

$$\leq \frac{C_2(K, R)}{\mu^2} \left\{ \sum_{|\mu - |\varrho_k|| \leq 1} (\|v_k\|_{L^\infty(K)} \operatorname{ch} \nu_k 3R)^2 + \right.$$

$$\left. + \sum_{|\mu - |\varrho_k|| \leq 1} (\|v_{k-1}\|_{L^\infty(K)} \operatorname{ch} \nu_{k-1} 3R)^2 \right\} \leq \frac{4C_2(K, R) A}{\mu^2}, \ \mu \geq 1.$$

The Lemma 2 is proved.

LEMMA 3. Given any compact $K \subset G$, there exists a constant $C_3(K)$ such that for $q(x) \in L^p(G)$, $p > 1$

$$(16) \qquad \sum_{|\mu - |\varrho_k|| \leq 1} \|u_k'\|^2_{L(K)} \leq C_3(K) \cdot \mu^2, \ \mu \geq 1.$$

PROOF. This proof is made in the similar way of that of the Lemma 4 in [9]. □

Now we return to the proof of the proposition.

PROOF OF THE PROPOSITION. We count the Fourier coefficients of the function W according to the system $\{u_k\}$:

$$< u_k, W > = \int_{x-R}^{x+R} u_k(y) \frac{1}{\pi} \frac{\sin \mu(y-x)}{y-x} dy =$$

$$= \int_0^R \frac{1}{\pi} \frac{\sin \mu t}{t} [u_k(x+t) + u_k(x-t)] dt.$$

Applying the generalized Titchmarsh formula (5) we obtain for the Fourier coefficients of W the expansion:

$$<u_k, W> = u_k(x)\frac{2}{\pi}\int_0^\infty \frac{\sin \mu t \cos \varrho_k t}{t}\,dt - \tag{17}$$

$$-u_k(x)\frac{2}{\pi}\int_R^\infty \frac{\sin \mu t \cos \varrho_k t}{t}\,dt + u_k(x)\frac{2}{\pi}\int_0^R \sin \mu t \cot \varrho_k t\,\frac{\operatorname{ch} \nu_k t - 1}{t}\,dt -$$

$$-u_k(x)\frac{2i}{\pi}\int_0^R \sin \mu t \sin \varrho_k t \frac{\operatorname{sh} \nu_k t}{t}\,dt +$$

$$+\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin \mu t \sin \mu_k(t-|x-\xi|)}{\pi t \mu_k}\,dt\,d\xi +$$

$$+\int_{x-R}^{x+R} u_{k-1}(\xi)\int_{|x-\xi|}^{R}\frac{\sin \mu t \sin \mu_k(t-|x-\xi|)}{\pi t \mu_k}\,dt\,d\xi.$$

Similarly, we introduce the function

$$\omega(x, y, \mu) := \begin{cases} \dfrac{1}{\pi}\dfrac{\sin \mu R}{R} & \text{if} \quad |x-y| \le R, \\ 0 & \text{if} \quad |x-y| > R. \end{cases}$$

We obtain also

$$<u_k, \omega> = u_k(x)\frac{2\sin \mu R \sin \mu_k R}{\pi R \mu_k} +$$

$$+\frac{\sin \mu R}{R}\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin \mu_k(t-|x-\xi|)}{\pi \mu_k}\,dt\,d\xi + \tag{18}$$

$$+\frac{\sin \mu R}{R}\int_{x-R}^{x+R} u_{k-1}(\xi)\int_{|x-\xi|}^{R}\frac{\sin \mu_k(t-|x-\xi|)}{\pi \mu_k}\,dt\,d\xi.$$

We known that [4]:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \mu t \cos \varrho_k t}{t}\,dt = \delta(\mu, |\varrho_k|), \tag{19}$$

where

$$\delta(\mu, |\varrho_k|) := \begin{cases} 1 & \text{if } |\varrho_k| < \mu, \\ \dfrac{1}{2} & \text{if } |\varrho_k| = \mu, \\ 0 & \text{if } |\varrho_k| > \mu. \end{cases}$$

Introduce the notation

$$I^{\mu}_{|\varrho_k|}(R):=\frac{2}{\pi}\int_R^\infty \frac{\sin\mu t\cos\varrho_k t}{t}\,dt=$$

(20) $$=\frac{2}{\pi}\,\frac{\sin\mu R\sin|\varrho_k|R}{|\varrho_k|R}+\frac{\mu}{|\varrho_k|}\cdot K^{\mu}_{|\varrho_k|}(R),$$

where

$$K^{\mu}_{|\varrho_k|}(R):=-\frac{2}{\pi\mu}\int_R^\infty \frac{d}{dt}\left(\frac{\sin\mu t}{t}\right)\sin|\varrho_k|t\,dt.$$

On the other hand we have

(21) $$\frac{2}{\pi}\,\frac{\sin\mu R\sin\mu_k R}{R\mu_k}=\frac{2}{\pi}\,\frac{\sin\mu R\sin|\varrho_k|R}{R|\varrho_k|}+h(R,\mu_k,\mu),$$

where

$$h(R,\mu_k,\mu):=\frac{\sin\mu R\sin\varrho_k R(\operatorname{ch}\nu_k R-1)}{R\mu_k}+$$

$$+i\frac{\sin\mu R\cos\varrho_k R\operatorname{sh}\nu_k R-\nu_k\sin\mu R\sin\varrho_k R}{R\mu_k}$$

(cf. [9]).

From (17)–(21) in case $|\varrho_k|>1$ we have

$$\langle u_k,W\rangle-\langle u_k,\omega\rangle=u_k(x)\delta(\mu|\varrho_k|)-u_k(x)\cdot\frac{\mu}{|\varrho_k|}\cdot K^{\mu}_{|\varrho_k|}(R)+$$

$$+u_k(x)\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\,\frac{\operatorname{ch}\nu_k t-1}{t}\,dt-$$

$$-u_k(x)\frac{2i}{\pi}\int_0^R \sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt-u_k(x)h(R,\mu_k,\mu)+$$

$$+\int_{x-R}^{x+R} q(\xi)\,u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi+$$

$$+\int_{x-R}^{x+R} u_{k-1}(\xi)\int_{-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,u\xi-$$

$$-\frac{\sin\mu R}{R}\int_{-R}^{x+R} p(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,dt$$

(22) $$-\frac{\sin\mu R}{R}\int_{x+R}^{x-R} u_{k-1}(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi.$$

One can prove the convergence of the following eight series by similar method applied in [9]:

1. $$\sum_{|\varrho_k|\geq 1} \left| u_k(x) \cdot \frac{\mu}{|\varrho_k|} \cdot K''_{\varrho_k}(R) \right|^2,$$

2. $$\sum_{k=0}^{\infty} \left| u_k(x) \frac{2}{\pi} \int_0^R \sin \mu t \cos \varrho_k t \frac{\operatorname{ch} \nu_k t - 1}{t} dt \right|^2,$$

3. $$\sum_{k=0}^{\infty} \left| u_k(x) \frac{2i}{\pi} \int_0^R \sin \mu t \sin \varrho_k t \frac{\operatorname{sh} \nu_k t}{t} dt \right|^2,$$

4. $$\sum_{|\varrho_k|>1} |u_k(x) h(R, \mu_k \mu)|^2,$$

5. $$\sum_{k=0}^{\infty} \left| \int_{x-R}^{x+R} p(\xi) u_k(\xi) \int_{|x-\xi|}^{R} \frac{\sin \mu t \sin \mu_k(t-|x-\xi|)}{\pi t \mu_k} dt\, d\xi \right|^2,$$

6. $$\sum_{k=0}^{\infty} \left| \int_{x-R}^{x+R} p(\xi) u_k(\xi) \int_{x-R}^{R} \frac{\sin \mu_k(t-|x-\xi|)}{\pi t \mu_k} dt\, d\xi \right|^2,$$

7. $$\sum_{k=0}^{\infty} \left| \int_{x-R}^{x+R} u_{k-1}(\xi) \int_{x-\xi}^{R} \frac{\sin \mu t \sin \mu_k(t-|x-\xi|)}{\pi t \mu_k} dt\, d\xi \right|^2,$$

8. $$\sum_{k=0}^{\infty} \left| \int_{x-k}^{x+R} u_{k-1}(\xi) \int_{|x-\xi|}^{R} \frac{\sin \mu_k(t-|x-\xi|)}{\pi t \mu_k} dt\, d\xi \right|^2.$$

Multiplying (22) by $\overline{v_k(v)}$, summing in k for all $k \in \mathbf{N}$, applying the average operator to both sides and integrating in y from y_1 to y_2 ($a \leq y_1 \leq y_2 \leq b$) we get the following connection:

$$\int_{y_1}^{y_2} [S_{R_0} W(x, y, \mu) - \Theta(x, y, \mu)]\, dy = \int_{y_1}^{y_2} S_{R_0} \omega(x, y, \mu)\, dy +$$

$$+ \frac{1}{2} \sum_{|\varrho_k|=\mu} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy\, u_k(x) -$$

$$- \frac{2}{\pi} \sum_{|\varrho_k|\leq 1} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy\, u_k(x) S_{R_0} \frac{\sin \mu R \sin \mu_k R}{R \mu_k} -$$

$$- \sum_{|\varrho_k|\leq 1} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy\, u_k(x) S_{R_0} I''_{|\varrho_k|}(R) -$$

$$-\sum_{0\le|\varrho_k|>1}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,u_k(x)\frac{\mu}{|\varrho_k|}S_{R_0}K^{\mu}_{|\varrho_k|}(R)+$$

$$+\sum_{k=0}^{\infty}\int_{y_1}^{y_2}v_k\overline{(y)}\,dy\,u_k(x)S_{R_0}\frac{2}{\pi}\int_0^R\sin\mu t\cos\,\varrho_k t\frac{\operatorname{ch}v_k t-1}{t}\,dt-$$

$$-\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,u_k(x)S_{R_0}\frac{2i}{\pi}\int_0^R\sin\mu t\sin\varrho_k t\frac{\operatorname{sh}v_k t}{t}\,dt-$$

$$-\sum_{|\varrho_k|>1}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,u_k(x)S_{R_0}\,h(R,\mu_k,\mu)+$$

$$+\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\int_{x-R}^{x+R}q(\xi)\,u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi+$$

$$+\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\int_{x-R}^{x+R}u_{k-1}(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi-$$

$$-\sum_{k=0}^{\infty}\int_{K_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\frac{\sin\mu R}{R}\int_{x-R}^{x+R}p(\xi)u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi-$$

$$-\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\,dy\,S_{R_0}\frac{\sin\mu R}{R}\int_{x\ R}^{x+R}u_{k-1}(\xi)\int_{|x-\xi|}^{R}\cdot$$

$$\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi. \tag{23}$$

Denote $Q(x)$ $(P(x))$ the left (right) hand side of (23). Now our aim is to prove that the derivatives $\frac{d}{dx}P(x)$, $\frac{d}{dx}Q(x)$ exist and are bounded. By the above notations the following relation is true

$$\frac{dx}{d}Q(x)=\int_{y_1}^{y_2}\frac{\partial}{\partial x}\left[S_{R_0}W(x,y,\mu)-\Theta(x,y,\mu)\right]dy. \tag{24}$$

Let us denote $p_i(x)$ $(i=1,2,\ldots,12)$ the i^{th} member on the right hand side of (23). It is easy to see that on the compact K the derivatives $\frac{d}{dx}p_i(x)$ $(i=1,2,4)$ exist. The existence of the derivatives $\frac{d}{dx}p_i(x)$ $(i=3,6,7,8)$ si proved by the analogous method applied in [9] for the proof of the existence

$\frac{d}{dx} h_3(x)$ using also the lemmas 2 and 3 of the present paper. The existence of the derivative $\frac{d}{dx} p_5(x)$ can be proved similarly by repeating word for word the analogous proof used in [6] for that of (45).

Here we shall prove that the derivatives $\frac{d}{dx} p_i(x)$ $(i = 9, 10, 11, 12)$ exist.

Consider the function

$$p_9(x) = \sum_{k=0}^{\infty} \int_{y_1}^{y_2} v_k(\overline{y})\, dy\, S_{R_0} \int_{x-R}^{x+R} q(\xi)u_k(\xi) \int_{|x-\xi|}^{R} \frac{\sin\mu_t \sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\, dt\, d\xi.$$

We shall prove that on the compact K the following equality is true:

$$\frac{d}{dx} p_9(x) = \sum_{k=0}^{\infty} \int_{y_1}^{y_2} v_k(\overline{y})\, dy\cdot$$

$$\cdot S_{R_0}\left\{ -\frac{1}{\pi} \int_{x-R}^{x} q(\xi)u_k(\xi) \int_{x-\xi}^{R} \frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\, dt\, d\xi + \right.$$

$$\left. + \frac{1}{\pi} \int_{x}^{x+R} q(\xi)u_k(\xi) \int_{\xi-x}^{R} \frac{\sin\mu t\cos\mu_k(t+x-\xi)}{t}\, dt\, d\xi \right\} = P_{91}(x) + P_{92}(x),$$

where

$$P_{9\cdot}(x) := \sum_{k=0}^{\infty} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy\, S_{R_0}\left\{ -\frac{1}{\pi} \int_{x-R}^{x} q(\xi)u_k(\xi)\cdot \right.$$

$$\left. \cdot \int_{x-\xi}^{R} \frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\, dt\, d\xi \right\}. \tag{25}$$

In what follows we shall need the estimate

$$\int_{x-R}^{x} \frac{|q(\xi)|}{|x-\xi|^{\delta}}\, d\xi \le C_1(R_0) \quad \text{for} \quad q(\xi)\in L^p(G),\ p>1 \tag{26}$$

(cf. [9]).

Using (26) and the lemma 3 in [9] we obtain

$$P_{91}(x) \le C_5(R_0)C_1(R_0)\left\{ \sum_{0\le|q_k|\le 1} \left| \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \right| \|u_k\|_{L^\infty(K_{2R_0})} + \right.$$

$$+\sum_{1<|\varrho_k|\le\frac{\mu}{2}}\left|\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\right|\frac{\|u_k\|_{L^\infty(K_{2R_0})}}{|\mu-|\varrho_k||^{\delta/2}}+$$

$$\sum_{\frac{\mu}{2}<|\varrho_k|<\mu-1}\left|\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\right|\frac{\|u_k\|_{L^\infty(K_{2R_0})}}{|\mu-|\varrho_k||^{\delta/2}}+\sum_{|\mu-|\varrho_k||\le 1}\left|\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\right|\cdot$$

$$\cdot\|u_k\|_{L^\infty(K_{2R_0})}+\sum_{|\varrho_k|>\mu+1}\left|\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\right|\frac{\|u_k\|_{L^\infty(K_{2R_0})}}{|\mu-|\varrho_k||^{\delta/2}}\Bigg\}+$$

$$+C_5(R_0)\sum_{k=0}^{\infty}\left|\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\right|\frac{\|u_k\|_{L^\infty(K_{2R_0})}\operatorname{ch}\nu_k 2R_0}{1+|\mu-|\varrho_k||}. \tag{27}$$

1. Using the Lemma 2 and ((61) in [4]) we get

$$\sum_{0\le|\varrho_k|\le 1}\le C_3^{1/2}(K,R)A^{1/2}<\infty. \tag{28}$$

2. If $1\le|\varrho_k|\le\frac{\mu}{2}$ the relation $|\mu-|\varrho_k||\ge|\varrho_k|$ is true, too. Then using (15) and ((61) in [4]) we have

$$\sum_{1<|\varrho_k|\le\frac{\mu}{2}}\le 2C_1(K,R)A^{1/2}B^{1/2}\sum_{j=1}^{\infty}\frac{1}{j^{1+\delta/2}}<\infty, \tag{29}$$

here and from now on we use the proposition 1 in [4] for the system $\{v_k\}$: given any compact $K:=[c,d]\subset G$, there exists an $R>0$ such that $K_R:=$ $:=[c-R,d+R]\subset G$ satisfies

$$\sup_{\mu>0}\sum_{|\mu-|\varrho_k||\le 1}(\|v_k\|_{L^\infty(K)}\operatorname{ch}\nu_k R)^2<B<\infty. \tag{30}$$

3. If $\frac{\mu}{2}<|\varrho_k|\mu-1$ the relation $|\varrho_k|>|\mu-|\varrho_k||$ follows, too. Applying (15), (30) and ((61) in [4]) we obtain

$$\sum_{\frac{\mu}{2}<|\varrho_k|<\mu-1}\le 2C_1(K,R)A^{1/2}B^{1/2}\sum_{j=1}^{\left[\frac{\mu}{2}\right]}\frac{1}{j^{\delta/2}(\mu-j)}\le D(K,R,\delta)<\infty. \tag{31}$$

4. Using the Cauchy-Schwartz inequality, the Lemma 2 and ((61) in [4]) we have

$$\sum_{|\mu-|\varrho_k||\le 1}\le C_3^{1/2}(K,R)A^{1/2}<\infty. \tag{32}$$

5. From (15), (30) and ((61) in [4]) follows

(33) $$\sum_{|\varrho_k| > \mu+1} \leq 2C_1(K, R) A^{1/2} B^{1/2} \sum_{j=1}^{\infty} \frac{1}{j^{1+\alpha/2}} < \infty .$$

6. It is well-known that for any $q(x) \in L^1(G)$, there exists an $R > 0$ with

(34) $$\sup_{\mu > 0} \sum_{|\mu - |\varrho_k|| \leq 1} (\|v_k\|_{L^\infty(G)} \operatorname{ch} \nu_k R)^2 < V < \infty$$

(cf. [3]).

(4), (15), (34) and ((61) in [4]) imply

$$\sum_{k=0}^{\infty} \left| \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \right| \frac{\|u_k\|_{L^\infty(K_{2R_0})} \operatorname{ch} \nu_k 2R_0}{1 + |\mu - |\varrho_k||} \leq 8 \cdot C \cdot C_3(R) A^{1/2} V^{1/2} < \infty .$$

(35)

Now we substitute (28), (29), (31)–(33) and (35) into (27). We get for the quantity $P_{91}(x)$ the estimate

(36) $$|P_{91}(x)| < \infty .$$

We have the similar estimate for the quantity $P_{92}(x)$, too. Therefore the derivative $\frac{d}{dx} p_9(x)$ exists and (25) follows.

The existence of the derivatives $\frac{d}{dx} p_i(x)$ ($i = 10, 11, 12$) can be made in the quite similar way as before.

Summarising all of the above arguments we obtain that

$$\int_{y_1}^{y_2} \frac{\partial}{\partial x} [S_{R_0} W(x, y, \mu) - \Theta(x, y, \mu)]\, dy =$$

$$= \frac{d}{dx} \int_{y_1}^{y_2} S_{R_0} \omega(x, y, \mu)\, dy + \frac{1}{2} \sum_{|\varrho_k| = \mu} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \frac{d}{dx} u_k(x) -$$

$$- \frac{2}{\pi} \sum_{0 \leq |\varrho_k| \leq 1} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \frac{d}{dx} u_k(x) S_{R_0} \frac{\sin \mu R \sin \mu_k R}{R \mu_k} -$$

$$\sum_{0 \leq |\varrho_k| \leq 1} \int_{y_1}^{y_2} v_k(y)\, dy \frac{d}{dx} u_k(x) S_{R_0} I''_{|\varrho_k|}(R) -$$

$$- \sum_{|\varrho_k| > 1} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \frac{d}{dx} u_k(x) \cdot \frac{\mu}{|\varrho_k|} \cdot S_{R_0} K''_{|\varrho_k|}(R) +$$

$$+ \sum_{k=0}^{\infty} \int_{y_1}^{y_2} \overline{v_k(y)}\, dy \frac{d}{dx} u_k(x) S_{R_0} \left\{ \frac{2}{\pi} \int_0^R \sin \mu t \cos \varrho_k t \frac{\operatorname{ch} \nu_k t - 1}{t} dt \right\} -$$

$$-\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,\frac{d}{dx}\,u_k(x)S_{R_0}\left\{\frac{2i}{\pi}\int_0^R \sin\mu t\sin\varrho_k t\,\frac{\operatorname{sh}\nu_k t}{t}\,dt\right\}-$$

$$-\sum_{|\varrho_k|>1}\int_{y_1}^{y_2}v_k(y)\,dy\,\frac{d}{dx}\,u_k(x)S_{R_0}\{h(R,\mu_k,\mu)\}+$$

$$+\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\left\{-\frac{1}{\pi}\int_{x-R}^{x}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\,dt\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\int_x^{x+R}q(\xi)u_k(\xi)\int_{\xi-x}^{R}\frac{\sin\mu t\cos\mu_k(t+x-\xi)}{t}\,dt\,d\xi\right\}+$$

$$+\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\left\{-\frac{1}{\pi}\int_{x-R}^{x}u_{k-1}(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\,dt\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\int_x^{x+R}u_{k-1}(\xi)\int_{\xi-x}^{R}\frac{\sin\mu t\cos\mu_k(t+x-\xi)}{t}\,dt\,d\xi\right\}-$$

$$-\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\left\{\frac{1}{\pi}\,\frac{\sin\mu R}{R}\int_{x-R}^{x}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\cos\mu_k(t-x+\xi)\,dt\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\,\frac{\sin\mu R}{R}\int_x^{x+R}q(\xi)u_k(\xi)\int_{\xi-x}^{R}\cos\mu_k(t+x-\xi)\,dt\,d\xi\right\}-$$

$$-\sum_{k=0}^{\infty}\int_{y_1}^{y_2}\overline{v_k(y)}\,dy\,S_{R_0}\left\{\frac{1}{\pi}\,\frac{\sin\mu R}{R}\int_{x-R}^{x}u_{k-1}(\xi)\int_{x-\xi}^{R}\cos\mu_k(t-x+\xi)\,dt\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\,\frac{\sin\mu R}{R}\int_x^{x+R}u_{k-1}(\xi)\int_{\xi-x}^{R}\cos\mu_k(t+x-\xi)\,dt\,d\xi\right\}. \tag{37}$$

Above we had proved that all of series in the right hand side of (37) are convergent uniformly in x on the compact K and are bounded. Therefore

$$\left|\int_{y_1}^{y_2}\frac{\partial}{\partial x}\left[S_{R_0}W(x,y,\mu)-\Theta(x,y,\mu)\right]dy\right|\le D(K). \tag{38}$$

The proposition is proved. □

4. Proof of the theorem. The idea of the proof is the following. Introduce the notation

$$S_\mu(f,x):=\int_a^b\frac{\partial}{\partial x}S_{R_0}W(y)\cdot f(y)\,dy. \tag{39}$$

Applying (38) one can prove by the method used in [6] that

$$\frac{dx}{d}\sigma_n(f,x) - S_n(f,x) \le D_1(K)\|f\|W_1^1(G).$$

A similar estimate holds for the quantity $\frac{d}{dx}\hat{\sigma}_n(f,x)$, too. Consequently using the triangle inequality we obtain the required estimate

$$\left|\frac{d}{dx}\sigma_n(f,x) - \frac{d}{dx}\hat{\sigma}_n(f,x)\right| \le C(K)\|f\|W_1^1(G).$$

The theorem is proved. □

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SUR LE MINIMUM DE LA SOMME DES PUISSANCES DES DISTANCES

de

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(Reçu le 12. juin 1985)

Au 70-ème anniversaire du **L. Fejes Tóth**

C'est un problème de L. FEJES TÓTH (1967).

Dans l'espace euclidien on considere n points tels que la distance des deux points quelconques est tout au moins 1. Quelle est la configuration de points, quand la somme des distances réciproques est maximal?

Et voici un autre problème de L. FEJES TÓTH, qui est plus simple.

Dans le plan euclidien soient donnés les points $A_1, A_2, \ldots, A_n$ tels que la distance des deux points quelconques est tout au moins 1:

$$A_iA_k \leq 1, \quad 1 \leq i < k \leq n.$$

Quelle est la configuration plane de points, quand la somme des distances:

$$t(n) := \sum_{1 \leq i < k \leq n} A_i A_k$$

est minimal?

En 1967. J. HORVÁTH a determiné la configuration minimal dans le plan si $n \leq 7$ (Fig. 1.) [1].

Si $n = 7$, alors la configuration optimal c'est le hexagon régulier aux cotés-unité et son centre (Fig. 1/a). Si $n = 6$, alors il faut rayer (par exemple) le point A_7 (Fig. 1/b), si $n = 5$, alors il faut rayer le point A_6 (Fig. 1/c), si $n = 4$, alors il faut rayer le point A_5 (Fig. 1/d), et enfin si $n = 3$, alors le point A_4 (Fig. 1/e).

En 1974. Á. TEMESVÁRI a éxaminé la somme des puissances des distances [2]:

$$t_\nu(n) := \sum_{1 \leq i < k \leq n} A_i A_k^\nu$$

avec l'exposant $\nu \leq 2$, $\nu \neq 0$. Elle a démontré que la configuration optimal est la même que pour $\nu = 1$ si $n \leq 7$.

REMARQUE: Si $n = 4$ et $\nu = 2$, alors la configuration minimal n'est pas unique. Tous les rhombes unités sont minimaux quand les diagonales sont supérieures ou égales á 1.

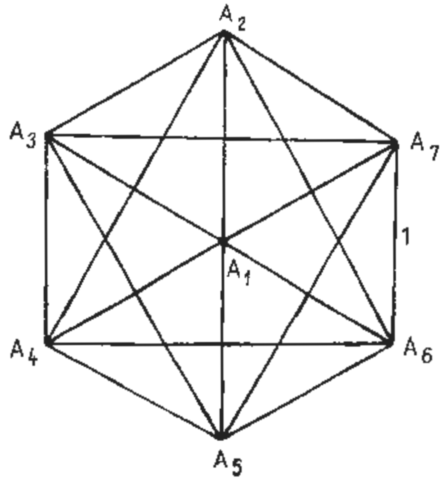

Fig. 1a.

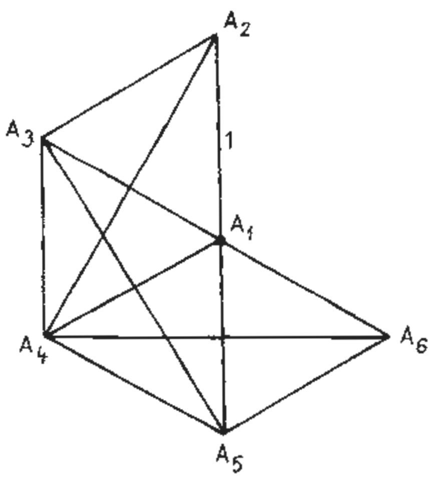

Fig. 1b.

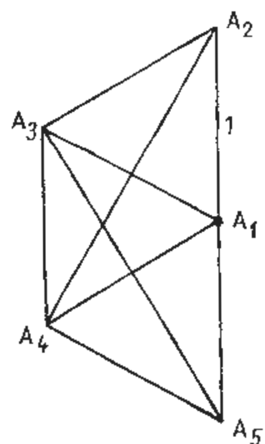

Fig. 1c.

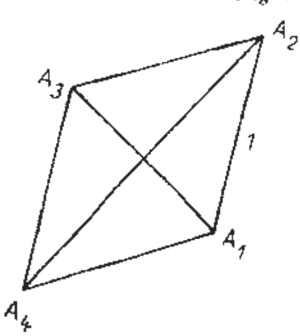

Fig. 1d.

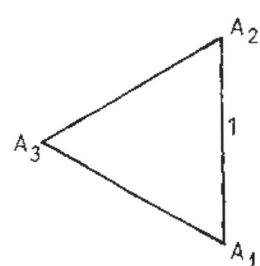

Fig. 1e.

C'est à dire, que si $n \leq 7$ et $\nu \leq 2$, $\nu \neq 0$, alors

$$t_\nu(7) \geq t_\nu^*(7) := 12 + 6 \cdot \sqrt{3}^\nu + 3 \cdot 2^\nu,$$

$$t_\nu(6) \geq t_\nu^*(6) := 9 + 4 \cdot \sqrt{3}^\nu + 2 \cdot 2^\nu,$$

$$t_\nu(5) \geq t_\nu^*(5) := 7 + 2 \cdot \sqrt{3}^\nu + 2^\nu,$$

$$t_\nu(4) \geq t_\nu^*(4) := 5 + \sqrt{3}^\nu$$

$$t_\nu(3) \geq t_\nu^*(3) := 3.$$

Nous démontrons le

THEORÈMÈ: *Dans le plan euclidien soient $A_1, A_2, \ldots, A_n$ n points tels que la distance des deux points quelconques est tout au moins 1:*

$$A_i A_k \geq 1, \quad 1 \leq i < k \leq n.$$

Si l'exposant $\nu > 2$, alors la somme des puissances des distances

$$t_\nu(n) = \sum_{1 \leq i < k \leq n} A_i A_k^\nu$$

est telle, que:

$$t_\nu(3) \geq t_\nu^*(3) := 3,$$

$$t_\nu(4) \geq t_\nu^*(4) := 4 + 2 \cdot \sqrt{2}^\nu,$$

$$t_\nu(5) \geq t_\nu^*(5) := 7 + 2 \cdot \sqrt{3}^\nu + 2^\nu, \qquad 2 < \nu \leq \nu_1,$$

$$t_\nu(5) \geq t_\nu^*(5) := 5 + 5(0{,}5 + 0{,}5 \cdot \sqrt{5})^\nu, \qquad \nu_1 \leq \nu,$$

où ν_1 est la racine de l équation:

$$2 + 2 \cdot \sqrt{3}^\nu + 2^\nu = 5(0{,}5 + 0{,}5 \cdot \sqrt{5})^\nu, \quad \nu_1 = 3{,}479 \ldots$$

COROLLAIRE: Sur la base du théorème la configuration minimal ($\nu > 2$) c'est

- le triangle régulier aux côtés-unité (Fig. 1/e) si $n = 3$,
- le carré-unité (Fig. 2.) si $n = 4$,
- le pentagone dégénéré aux côtés-unité (Fig. 1/c) si $n = 5$ et $2 < \nu \leq \nu_1$,
- le pentagone régulier aux côtés-unité (Fig. 3.) si $n = 5$ et $\nu_1 \leq \nu$.

On peut utiliser bien le remaniement du problème comme suit. Dans le plan euclidien soient $k_1, k_2, \ldots, k_n$ n cercles ouverts aux diamêtres-unité tels, que

$$k_i \cap k_j = 0 \quad 1 \leq i < j \leq n.$$

Quel est le placement des cercles, quand la somme de puissances des distances des centres est extrémale?

Nous utilisons trois lemmes.

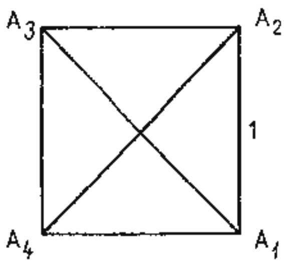

Fig. 2.

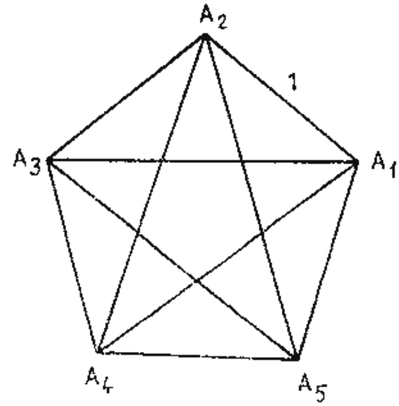

Fig. 3.

LEMME 1. Soit le polygone A l'enveloppe convexe des centres ($A_1, A_2, \dots$ $\dots A_n$) des cercles ouverts aux diamêtres-unité ($k_1, k_2, \dots, k_n$). Soient $A_{j-1}A_jA_{j+1}\sphericalangle \neq \pi$ où $A_{j-1}, A_j, A_{j+1} \in A$. Si $\nu > 2$ et si l'on peut déturner le cercle k_j du côté de $A_{j-1}A_{j+1}$ tellement que la distance $A_{j-1}A_j$, ou la distance A_jA_{j+1} est constante et $k_i \cap k_j = 0$, $1 \leq i < j \leq n$, alors la somme $t_\nu(n)$ décroîte.

La démonstration est la même que celle du lemme 1. dans [1].

LEMME 2. Soient A et B deux points sur le cercle-unité. Le centre du cercle soit O (Fig. 4.). Supposons que $F \in \widehat{AB}$, $AF = FB$, $AOF\sphericalangle := 2\alpha < \pi$, $C \in \widehat{AFB}$, $AC \geq CB$, $\widehat{AFC} \leq \pi$ et $2\delta := FOC\sphericalangle$. Alors la somme des puissances des distances AC et CB est donneé par

$$AC^\nu + CB^\nu = 2^\nu[\sin^\nu(\alpha+\delta)+\sin^\nu(\alpha-\delta)].$$

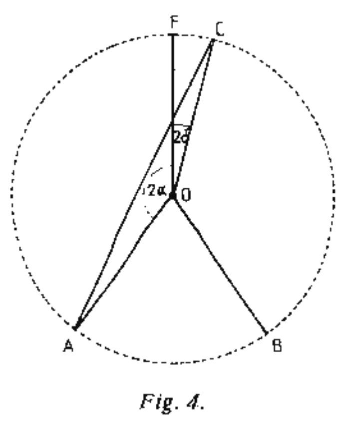

Fig. 4.

Soit i_ν le point d'inflexion de la fonction

$$[0, \pi/2] \to \mathbf{R}, \alpha \to \sin^\nu \alpha$$

(alors $i_\nu = \arccos(1/\sqrt{\nu})$. La fonction

$$f_\nu : [0, \alpha] \to \mathbf{R}, \delta \to 2^\nu [\sin^\nu(\alpha+\delta) + \sin^\nu(\alpha-\delta)]$$

si l'exposant $\nu > 2$, et

— si $0 < \alpha \leq \pi/4$, alors décroîte,
— si $\pi/4 < \alpha < i_\nu$, alors décroîte et puis croîte,
— si $i_\nu \leq \alpha < \pi/2$, alors croîte.

C'est la lemme 2. du [3].

Lemme 3. Soit $n \geq 3$, $\nu > 2$ et $A_1 A_2 \dots A_n$ un polygone convexe tel, que les points A_i $1 \leq i \leq n$ soient sur un cercle, et le centre du cercle soit dans ce polygone. Si la somme des puissances de côtés de ce polygone est minimale, et

— si $a_n \leq \nu$, alors le polygone est régulier,
— si $\nu \leq a_n$, alors le polygone est tel, que (par exemple) $A_1 A_3$ est le diamètre du cercle et $A_2 A_3 = A_3 A_4 = \dots = A_{n-1} A_n = A_n A_1$.

Le nombre a_n est la racin d'équation:

$$n \sin^{a_n}(\pi/4) = 1 + (n-1) \sin^{a_n}(\pi/2(n-1)),$$

$a_3 = 6.163\dots$, $a_4 = 3.169\dots$.
C'est le théorème dans le [3].

La démonstration du Théoréme.

1. Si $n = 3$ et $\nu > 2$, la démonstration est triviale.
2. Si $n = 4$ et $\nu > 2$, il faut éxaminer trois cas.
2.1. Si l'enveloppe convexe de 4 points est un segment, alors

$$t_\nu(4) \geq 3 + 2 \cdot 2^\nu + 3^\nu > 4 + 2^\nu = t_\nu^*(4).$$

2.2. Si l' enveloppe convexe de 4 points et un triangle, alors selon le lemme 1. il faut éxaminer seulement le cas, quand (par exemple) cercles k_2, k_3, k_4 touchent le cercle k_1 (Fig. 5.). Mais en ce cas sur la base du lemme 3.:

— si $2 < \nu \leq a_3$, alors $t_\nu(4) \geq 3 + 2 \cdot \sqrt{2}^\nu + 2^\nu > 4 + 2\sqrt{2}^\nu = t_\nu^*(4)$,

— si $a_3 \leq \nu$, alors $t_\nu(4) \geq 3 + 3 \cdot \sqrt{3}^\nu > 4 + 2\sqrt{2}^\nu = t_\nu^*(4)$.

2.3. Si l'enveloppe convexe de 4 points est un quadrilatère, alors sur la base du lemme 1. il faut considérer seulement les quadrilatères-unité. En ces cas si la somme

$$A_2 F^\nu + A_1 F^\nu$$

(Fig. 6.) est minimale, alors la somme $t_\nu(4)$ est aussi minimale. C'est pourquoi selon lemme 2. la configuration minimale c'est le carré-unité.

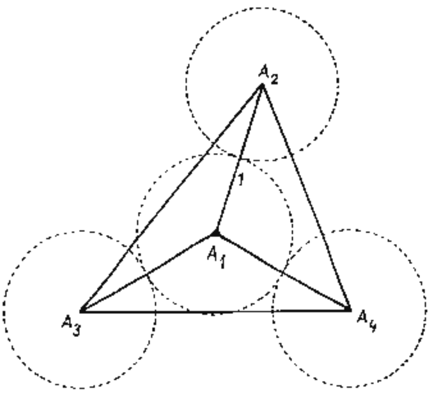

Fig. 5.

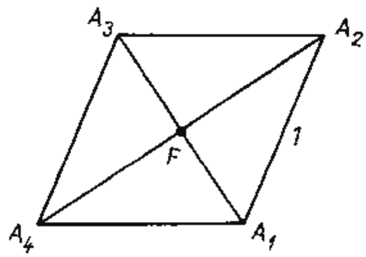

Fig. 6.

3. Si $n = 5$ et $\nu > 2$, alors il faut considérer quatre cas.

3.1. Si l'enveloppe convexe de 5 points est un segment, alors:

$$t_\nu(5) \ge 4+3\cdot 2^\nu+2\cdot 3^\nu+4^\nu > 7+2\sqrt{3}^\nu+2^\nu \ge t_\nu^*(5).$$

3.2. Si l'enveloppe convexe de 5 points $A_1, \ldots, A_5$ est un triangle (par exemple $A_3A_4A_5$, Fig. 7.), alors on peux voir, que il y a au moins un côté de ce triangle qui est supérieur où égal á 2. C'est pourquoi par conséquant le 2.2

— si $2 < \nu \le a_3$, alors $t_\nu(5) \ge 6+2\sqrt{2}^\nu+2\cdot 2^\nu > 7+2\sqrt{3}^\nu+2^\nu \ge t_\nu^*(5)$,

— si $a_3 < \nu$, alors $t_\nu(5) \ge 6+3\sqrt{3}^\nu+2^\nu > 7+2\sqrt{3}^\nu+2^\nu < t_\nu^*(5)$.

3.3. Si l'enveloppe convexe des points $A_1, \ldots, A_5$ est un quadrilatère (Fig. 8—9.), alors il faut considérer deux cas.

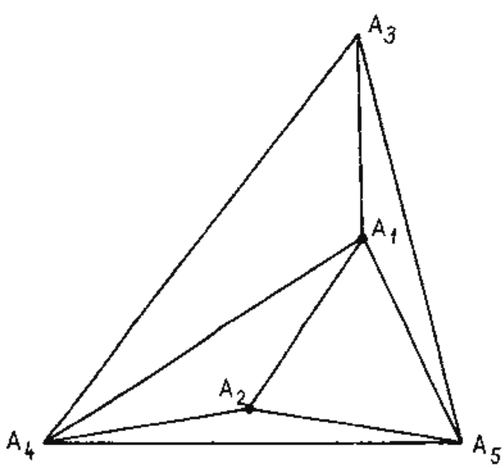

Fig. 7.

3.3.1. Le cercle k_1 touche aux cercle k_2, k_3, k_4 (Fig. 8.). Soit $A_i A_1 A_{i+1} \sphericalangle = \alpha_{i-1}$ ($i = 2, 3, 4, 5, 6 = 1$). Les points A_2, A_3, A_4, A_5 sont sur un cercle-unité, et le centre de ce cecrcle se trouve dans le quadrilatère $A_2A_3A_4A_5$.

On peut supposer, que:

$$2\pi/3 \leq \alpha_3 + \alpha_4 \leq \pi, \quad 2\pi/3 \leq \alpha_2 + \alpha_3 \leq \pi, \quad \alpha_2 \leq \alpha_4.$$

3.3.1.a Si $2 < \nu < a_4$ et $3\pi/4 \leq \alpha_3 + \alpha_4 \leq \pi$, alors

$$A_2A_4 \geq \sqrt{2+\sqrt{2}}, \; A_3A_5 \geq \sqrt{3}$$

et sur la base du lemme 3. (du quadrilatère $A_2A_3A_4A_5$):

$$t_\nu(5) \geq 7 + \left(\sqrt{2+\sqrt{2}}\right)^\nu + \sqrt{3}^\nu + 2^\nu > 7 + 2\sqrt{3}^\nu + 2^\nu = t_\nu^*(5).$$

3.3.1.b Si $a_4 \leq \nu \leq a_3$ et $3\pi/4 \leq \alpha_3 + \alpha_4 \leq \pi$, alors

$$A_2A_4 \geq \sqrt{2+\sqrt{2}}$$

et par conséquant le lemme 3. (du triangle $A_2A_3A_5$):

$$t_\nu(5) \geq 6 + 2\sqrt{2}^\nu + \left(\sqrt{2+\sqrt{2}}\right)^\nu + 2^\nu > 5 + 5(0{,}5 + 0{,}5\sqrt{5})^\nu \geq t_\nu^*(5).$$

3.3.1.c Si $2 < \nu \leq a_3$ et $2\pi/3 \leq \alpha_3 + \alpha_4 < 3\pi/4$, alors $\alpha_4 < 5\pi/12$ (parce que $\alpha_3 \geq \pi/3$), $\alpha_1 > 5\pi/6$ (parce que $\alpha_1 + \alpha_4 \geq \alpha_1 + \alpha_2 > 2\pi - 3\pi/4 = 5\pi/4$). Alors $\alpha_2/2 + \alpha_1 > \pi$ (parce que $\alpha_2 \geq \pi/3$) et $\alpha = (2\pi - \alpha_2)/4 \geq 19\pi/48 > i_\nu$ (parce que $\alpha_2 \leq \alpha_4$). C'est a dire que on peut utiliser le lemme 2. (Fig. 4. et Fig. 8.) On peut déturner le cercle k_2 du côté de k_5 sur le cercle k_1, et puis les cercles k_2 et k_5 ensemble du côté de k_4, et enfin le cercle k_3 du côté de k_4 sur le k_1:

$$t_\nu(5) \geq 7 + 2\sqrt{3}^\nu + 2^\nu \geq t_\nu^*(5).$$

3.3.1.d Si $a_3 \leq \nu$, alors $A_2A_4 \geq \sqrt{3}$ et sur la base du lemme 3. (du triangle $A_2A_3A_5$):

$$t_\nu(5) \geq 6 + 4\sqrt{3}^\nu > 5 + 5 \cdot (0{,}5 + 0{,}5\sqrt{5})^\nu = t_\nu^*(5).$$

3.3.2. Le cercle k_1 touche à k_3, k_4, mais il ne touche pas à k_2 (par exemple), (Fig. 9.). Soit

$$2\gamma := A_5A_1A_3 \sphericalangle, \; \beta := \pi - A_4A_1A_3 \sphericalangle.$$

Si $\gamma < \pi/4$, alors on peut déturner le cercle k_2 du côté de k_3 sur le k_1 et le cercle k_5 du côté de k_4 sur le k_1. La somme $t_\nu(5)$ décroite. Alors il faut èxaminer seulement le cas, quand $\pi/4 \leq \gamma \leq \pi/3$.

3.3.2.a Si $2 < \nu \leq a_3$ et $\beta \geq \pi/6$, alors $A_2A_4 \geq \sqrt{2+\sqrt{3}}$, et selon le lemme 3. (du triangle $A_3A_5A_4$):

$$t_\nu(5) \geq 6 + 2\sqrt{2}^\nu + \left(\sqrt{2+\sqrt{3}}\right)^\nu + 2^\nu > t_\nu^*(5).$$

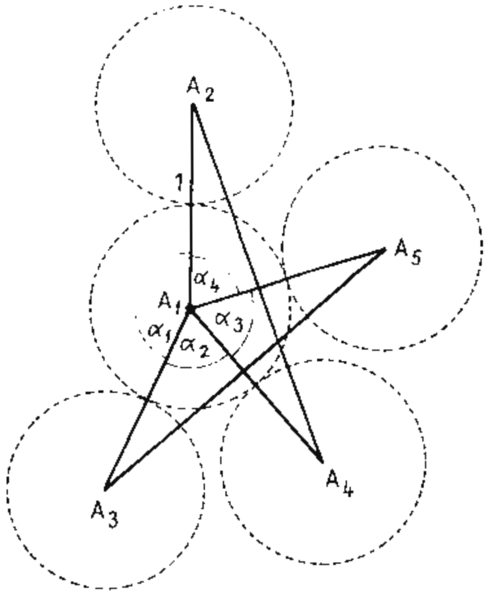

Fig. 8.

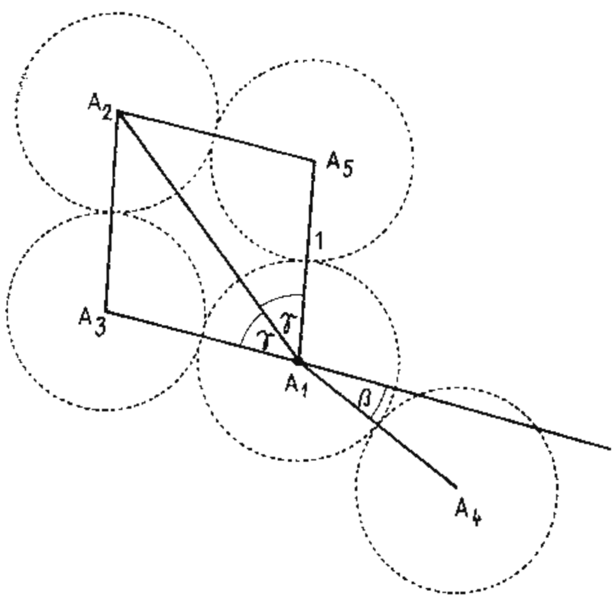

Fig. 9.

3.3.2.b Si $2<\nu\leq a_3$ et $\beta<\pi/6$, alors on peut faire décroître l'angle γ pour obtenir $A_5A_4' = 1$, où le point A_4' est l'image symétrique du point A_4 sur la droite A_1A_3. (Les distances $A_3A_2 = A_2A_5 = A_5A_1 = A_1A_3$ sont constantes.) On peut voir que

$$A_4A_5 \geq A_4'A_5 = 1, \text{ et } A_2A_4 \geq A_2A_4'$$

parce que $A_2A_1A_4\sphericalangle = \pi-\gamma+\beta > \pi-\gamma+\beta/2 = A_2A_5A_4'\sphericalangle$, et sur la base du lemme 2. la somme

$$A_3A_5^\nu + A_1A_2^\nu$$

aussi décroîte. C'est à dire, l'enveloppe convexe est un pentagone (v. 3.4).

3.3.2.c Si $a_3<\nu$, alors $A_2A_4\geq\sqrt{3}$, et sur la base du lemme 3. (du triangle $A_3A_5A_4$):

$$t_\nu(5) \geq 6+4\sqrt{3}^\nu > t_\nu^*(5).$$

3.4. Si l'enveloppe convexe des points A_1, A_2, A_3, A_4, A_5 est un pentagone, alors la configuration minimale doit être choisie d'entre le pentagone dégénéré aux côtés-unité (Fig, 1/c) et le pentagone régulier aux côtés-unité (Fig. 3.). Par conséquant le lemme 1. on peut considerer que

$$A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_1$$

(Fig. 10.). Soit $2\gamma_i = A_{i-1}A_iA_{i+1}\sphericalangle$ $(i = 2, 3, 4, 5, 6 = 1)$. alors la somme des puissances de distances:

$$t_\nu(5) = 5+\sum_{i=1}^{5} 2^\nu \sin^\nu\gamma_i,\ \pi/6\leq\gamma_i\leq\pi/2,\ (i = 1, 2, 3, 4, 5).$$

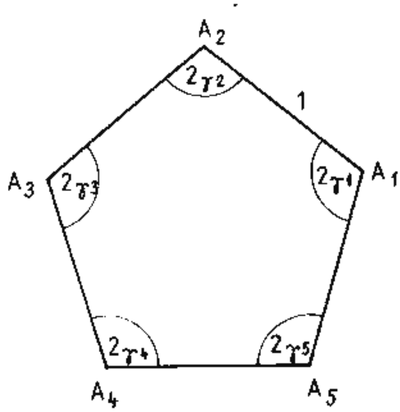

Fig. 10.

3.4.1. Si $\gamma_i < i_\nu$, $(i = 1, 2, 3, 4, 5)$, alors soit

$$5\alpha := \gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5,$$

et soit $A_{i-1}A_iA_{i+1} \sphericalangle := \alpha\,(i = 2, 3, 4, 5, 6 = 1)$. C'est le pentagone régulier et

$$t_\nu(5) \geq 5 + \sum_{i=1}^{5} 2^\nu \sin^\nu \alpha,$$

parce que la fonction

$$s_1 : [0, i_\nu] \to \mathbf{R},\ \alpha \to \sin^\nu\alpha,\ \nu > 2$$

est convexe (inégalité de Jensen).

3.4.2 S'il y a deux angles dans l'intervalle $[i_\nu, \pi/2]$ des angles γ_i $(i = 1, 2, 3, 4, 5)$ – par exemple γ_1 et $\gamma_2 \in [i_\nu, \pi/2]$ $\gamma_1 \leq \gamma_2$, – et l'angle γ_2 croîte, γ_1 décroîte tellement que la somme $\gamma_1+\gamma_2$ est constante, alors la somme des puissances ne croîte pas, parce que le fonction

$$s_2 : [i_\nu, \pi/2] \to \mathbf{R}, \quad \alpha \to \sin^\nu \alpha, \quad \nu > 2$$

est concave. En ce cas la configuration optimale sera le pentagone dégénéré, ou il faut éxaminer le cas suivant(3.4.3).

3.4.3. S'il y a seulement un angle (par exemple γ_1) dans l'intervalle $[i_\nu, \pi/2]$, alors soit

$$4\beta := \gamma_2+\gamma_3+\gamma_4+\gamma_5 \qquad (\beta < i_\nu).$$

Soit $\gamma = \alpha+\delta, \beta = \alpha-\delta$ et sur la base du lemme 2.

si $\alpha \geq i_\nu$, et γ croîte, β décroîte tellement que $\gamma+\beta$ est constant, alors $t_\nu(5)$ décroîte,

si $\pi/4 < \alpha < i_\nu$, et γ croîte, β décroîte ou γ décroîte et β croîte tellement que $\gamma+\beta$ est constant, alors $t_\nu(5)$ décroîte,

si $\alpha \leq \pi/2$, et γ décroît et β croîte tellement que $\gamma+\beta$ est constant, alors $t_\nu(5)$ décroîte. □

Et enfin quelques hypothéses:

1. $n = 6$ et $\nu > 2$.

1.a. Si $n = 6$ et $2 < \nu \leq \nu_2$ alors

$$t_\nu(6) \geq 9 + 4\sqrt{3}^\nu + 2 \cdot 2^\nu = : t_\nu^*(6),$$

et la configuration minimale est le même que pour $\nu \leq 2$ (Fig. 1/b).

1.b. Si $n = 6$ et $\nu_2 < \nu$ alors

$$t_\nu(6) \geq 5 + 5(2\sin(0,\ 4\pi))^\nu + 5(2\sin(0,\ 2\pi))^\nu =: t_\nu^*(6),$$

et la configuration minimale est le pentagone régulier avec son centre (Fig. 11.), où ν_2 est la racine de l'equation

$$4 + 4\sqrt{3}^\nu + 2\cdot 2^\nu = 5(2\sin(0,\ 4\pi))^\nu + 5(2\sin(0,\ 2\pi))^\nu, \qquad \nu_2 = 12{,}818\ldots.$$

2. Si $n = 7$ et $\nu > 2$, alors

$$t_\nu(7) \geq 12 + 6\sqrt{3}^\nu + 3\cdot 2^\nu =: t_\nu^*(7),$$

et la configuration minimale est la même que pour $\nu \leq 2$, (Fig. 1/a). □

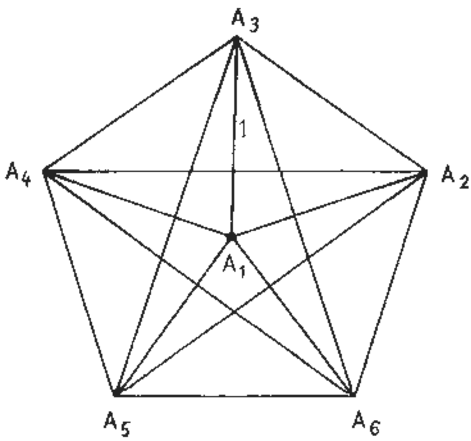

Fig. 11.

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НЕРАВЕНСТВО ТИПА БЕССЕЛЯ ДЛЯ СИСТЕМЫ СОБСТВЕННЫХ ФУНКЦИЙ НЕСАМОСОПРЯЖЕННОГО ОПЕРАТОРА ЛАПЛАСА

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(Поступило 22. 7. 1985)

В настоящей работе изучается вопрос о справедливости неравенства типа Бесселя для разложения функции, зависящей только от сферического радиуса в ряд по собственным функциям, вообще говоря, несамосопряженного оператора Лапласа в произвольной ограниченной трёхмерной области.

Пусть $\{u_n(x)\}$ — система собственных функций оператора Лапласа в области G. Под этим мы понимаем только следующее: каждая комплекснозначная функция $u_n(x)$ принадлежит классу $C^2(G)$ и для некоторого комплексного числа λ_n является в открытой области G решением уравнения $\Delta u_n+\lambda_n u_n=0$.

Определение. Будем говорить, что комплекснозначная функция $f(x)$ принадлежит в области G классу радиальных функций, если $f(x)$ зависит только от расстояния точки x от фиксированной точки y области G и обращается в нуль вне некоторого шара радиуса R лежащего целиком в области G.

Теорема. *Если функция $f(x)$ принадлежит в области G классу радиальныух функций и долпускает интеграл $\int_G |f(x)|^2\,dx$ и если справедливы два неравенства*

$$|\operatorname{Im}\sqrt{\lambda_n}|\le C_1, \tag{1}$$

$$\sum_{t\le|\sqrt{\lambda_n}|\le t+1} |u_n(y)|^2\le C_2(t+1)^2, \tag{2}$$

(во втором неравенстве y — любая внутренная точка области G, t — любое положительное вещественное число); то для функции $f(x)$ справедливо неравенство типа Бесселя

$$\sum_{n=1}^{\infty}\left|\int_G u_n(x)\overline{f(x)}\,dx\right|^2\le C_3\int_G |f(x)|^2\,dx. \tag{3}$$

Доказательство теоремы. Пусть $f(x)$ — произвольная функция, принадлежащая классу радиальных функций и допускающая в области G интеграл $\int\limits_G |f(x)|^2\,dx$. Тогда существует такая точка y внутри области G, что функция f зависит только от радиуса r сферической системы с центром в точке y и обращается в нуль вне шара радиуса R с центром в точке y, целиком лежащего в G.

Вычислим коэффициент Фурье f_n функции f по системе $\{u_n\}$, переходя к сферическим координатам. Получим

$$f_n = \int\limits_G u_n(x)\overline{f(x)}\,dx =$$

$$(4)\qquad = \int\limits_0^R \left[\overline{f(r)} \int\limits_0^{2\pi} \int\limits_0^{\pi} u_n(r,\Theta,\varphi)\sin\Theta d\Theta d\varphi\right] r^2\,dr.$$

Согласно известной формуле среднего значения Вебера (см, например, [1]) интеграл по углам, стоящий в (4) в квадратных скобках, равен

$$4\pi\cdot u_n(y)\,\frac{\sin r\sqrt{\lambda_n}}{r\sqrt{\lambda_n}}.$$

Таким образом,

$$(5)\qquad f_n = \frac{4\pi u_n(y)}{\sqrt{\lambda_n}} \int\limits_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})\,dr.$$

Для доказательства теоремы достаточно оценить сумму по всем номерам n

$$(6)\qquad \sum_{n=1}^{\infty} |f_n|^2 \equiv (4\pi)^2 \sum_{n=1}^{\infty} \frac{|u_n(y)|^2}{|\sqrt{\lambda_n}|^2} \left|\int\limits_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})\,dr\right|^2,$$

и доказать, что эта сумма мажорируется величиной, стоящей в правой части (3).

Представим сумму (6) в следующем виде

$$(7)\qquad \sum_{n=1}^{\infty} |f_n|^2 \equiv$$

$$\equiv (4\pi)^2 \sum_{k=1}^{\infty} \left\{ \sum_{\frac{2\pi}{R}(k-1)\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}k} \frac{|u_n(y)|^2}{|\sqrt{\lambda_n}|^2} \left|\int\limits_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})\,dr\right|^2\right\}.$$

Для каждого номера n, фигурирующего во внутренней (заключённой в фугурные скобки) сумме, справедливо представление

$$(8)\qquad \sqrt{\lambda_n} = \pm\frac{2\pi}{R}k + \delta_n,$$

где из двух знаков + и − следует взять какой − то один, а для велиины δ_n в силу неравенства (1) справедлива оценка

$$|\delta_n| = O(1). \tag{9}$$

Отдельно оценим каждую фигурную скобку в (7). Самую первую из указанных фигурных скобок, отвеючащую значению $k = 1$, оценим применяя к интегралу неравенство Коши − Буняковского и пользуясь заведомо справедливой при наличии неравенства (1) оценкой

$$\left|\frac{\sin r\sqrt{\lambda_n}}{r\sqrt{\lambda_n}}\right| \leq C_4 = \text{const}.$$

При этом получим, что*

$$\sum_{0\leq|\sqrt{\lambda_n}|<\frac{2\pi}{R}}' \frac{|u_n(y)|^2}{|\sqrt{\lambda_n}|^2}\left|\int_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})\,dr\right|^2 \leq$$

$$\leq \sum_{0\leq|\sqrt{\lambda_n}|\leq\frac{2\pi}{R}} |u_n(y)|^2 \int_0^R r^2|f(r)|^2dr \int_0^R r^2\left|\frac{\sin r\sqrt{\lambda_n}}{r\sqrt{\lambda_n}}\right|^2 dr \leq$$

$$\leq C_5 \int_G |f(x)|^2dx \sum_{0\leq|\sqrt{\lambda_n}|\leq\frac{2\pi}{R}} |u_n(y)|^2 \leq C_6 \int_G |f(x)|^2dx.$$

Оценим теперь фигурную скобку в (7) для любого номера $k \geq 2$. Для этого сначала оценим фигурирующий в этой фигурной скобке интеграл. Пользуясь представлением (8), мы перепишем этот интеграл в виде

$$\int_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})\,dr = \int_0^R r\overline{f(r)}\sin\left(\pm\frac{2\pi}{R}kr + \delta_n r\right)dr =$$

$$= \pm\int_0^R r\overline{f(r)}\sin\left(\frac{2\pi}{R}kr\right)\cos(\delta_n r)\,dr + \int_0^R r\overline{f(r)}\cos\left(\frac{2\pi}{R}kr\right)\sin(\delta_n r)\,dr.$$

*Мы учитываем, что $\int_G |f(x)|^2dx = 4\pi\int_0^R |f(r)|^2r^2dr$ и используем неравенство (1).

Интегрируя каждый из двух последних интегралов по частям, получим

$$\int_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})dr = \left[\mp\cos(\delta_n r)\int_r^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right]\Bigg|_{r=0}^{r=R} \mp$$

$$\mp\int_0^R \delta_n\sin(\delta_n r)\left(\int_r^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr +$$

$$+\left[-\sin(\delta_n r)\int_r^R \varrho\overline{f(\varrho)}\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right]\Bigg|_{r=0}^{r=R} +$$

$$+\int_0^R \delta_n\cos(\delta_n r)\left(\int_r^R \varrho\overline{f(\varrho)}\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr.$$

Окончательно, учитывая обращение в нуль трёх подстановок, получим

$$\int_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})dr = \pm\int_0^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho \mp$$

$$\mp\int_0^R \delta_n\sin(\delta_n r)\left(\int_r^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr + \tag{11}$$

$$+\int_0^R \delta_n\cos(\delta_n r)\left(\int_r^R \varrho\overline{f(\varrho)}\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr.$$

Используя оценку (9) и вытекающие из неё оценки $|\sin\delta_n r| = O(1)$ и $|\cos\delta_n r| = O(1)$, равномерные относительно r на сегменте $0 \leq r \leq R$, и применяя к интегралам в (11) неравенство Коши – Буняковского, мы получим из соотношения (11) следующее неравенство

$$\left|\int_0^R r\overline{f(r)}\sin(r\sqrt{\lambda_n})dr\right|^2 \leq 3\left|\int_0^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^2 + \tag{12}$$

$$+3C_7\int_0^R\left|\int_r^R \varrho\overline{f(\varrho)}\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^2 dr + 3C_8\int_0^R\left|\int_r^R \varrho\overline{f(\varrho)}\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^2 dr,$$

справедливое с одними и теми же постоянными C_7 и C_8 для всех номеров n, входящих в фигурную скобку в (7) с произвольным номером k.

Используя неравенства (10) и (12) в правой части (7), мы получим из соотношения (7):

$$\sum_{n=1}^{\infty} |f_n|^2 \le (4\pi)^2 C_6 \int_G |f(x)|^2 dx + \sum_{k=2}^{\infty} \left\{ \left[3 \left| \int_0^R \varrho \overline{f(\varrho)} \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 + \right.\right.$$

(13)
$$+ 3C_7 \int_0^R \left| \int_r^R \varrho \overline{f(\varrho)} \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 dr +$$

$$\left.\left. + 3C_8 \int_0^R \left| \int_r^R \varrho \overline{f(\varrho)} \cos\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 dr \right] \sum_{\frac{2\pi}{R}(k-1) \le |\sqrt{\lambda_n}| \le \frac{2\pi}{R} k} \frac{|u_n(y)|^2}{|\lambda_n|} \right\}.$$

Заметим теперь, что из оценки (2) вытекает, что для любого $k \ge 2$

$$\sum_{\frac{2\pi}{R}(k-1) \le |\sqrt{\lambda_n}| \le \frac{2k}{R} k} \frac{|u_n(y)|^2}{|\lambda_n|} \le \frac{R^2}{(2\pi)^2 (k-1)^2} \sum_{\frac{2\pi}{R}(k-1) \le |\sqrt{\lambda_n}| \le \frac{2\pi}{R} k} |u_n(y)|^2 \le C_9.$$

(14)

Учтем, наконец, что по условию теоремы функция $\varrho\, f(\varrho)$ одной переменной ϱ допускает интегрируемый квадрат модуля на сегменте $0 \le \varrho \le R$. Тем более допускает интегрируемый квадрат модуля по сегменту $0 \le \varrho \le R$ функция

$$F(\varrho) := \begin{cases} \varrho \overline{f(\varrho)} & \text{при } r \le \varrho \le R, \\ 0 & \text{при } 0 \le \varrho < r. \end{cases}$$

Поэтому в силу неравенства Бесселя для тригонометрической системы на сегменте $0 \le \varrho \le R$, мы можем утверждать, что

$$\sum_{k=2}^{\infty} \left| \int_0^R \varrho \overline{f(\varrho)} \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 \le 4\pi C_{10} \int_0^R \varrho^2 |f(\varrho)|^2 d\varrho \equiv C_{10} \int_G |f(x)|^2 dx,$$

(15)

$$\sum_{k=2}^{\infty} \int_0^R \left| \int_r^R \varrho \overline{f(\varrho)} \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 dr \le C_{11} \int_r^R \int_0^R \varrho^2 |f(\varrho)|^2 d\varrho\, dr \le C_{12} \int_G |f(x)|^2 dx,$$

(16)

(17)
$$\sum_{k=2}^{\infty} \int_0^R \left| \int_0^R \varrho f(\varrho) \cos\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^2 dr \le C_{13} \int_G |f(x)|^2 dx.$$

Вставляя оценки (14)—(17) в правую часть (13), мы окончательно получим, что существует постоянная C_{14} такая, что

$$\sum_{n=1}^{\infty} |f_n|^2 \le C_{14} \int_G |f(x)|^2 dx,$$

т. е. завершим доказательства требуемого неравенства Бесселя (3).

Замечание 1. Мы доказали неравенство Бесселя (3), не предполагая ни ортогональности, ни полносты, ни минимальности системы собственных функций $\{u_n(x)\}$. Единственными ограничениями при доказательстве этого неравенства явились оценки (1) и (2). Оценка (1), которая по существу эквивалентна принадлежности собственных значений так называемой карлемановской параболе (см. работу Т. Карлемана [2]), существенно использована нами в процессе доказательства.

Что же касается оценки (2), то легко убедиться в том, что это оценка является необходимым условием справедливости неравенства Бесселя (3) для функции из класса радиальных функций. В самом деле, если для системы $\{u_n(x)\}$ справедливо неравенство Бесселя (3), то применяя это неравенство к следующей конкретной функции из класса радиальных

$$f(r) := \begin{cases} \dfrac{t^2}{2\pi^2} \dfrac{\sin tr}{r} & \text{при } r \in [0, R], \\ 0 & \text{при } r > R \end{cases}$$

и повторяя технику, предложенную одним из авторов еще в статье [3] (стр. 83—87), мы установим, что для системы $\{u_n(x)\}$ справедлива оценка (2), равномерная относительно y на любом компакте области G.

Замечание 2. Конкретной моделью задачи, приводящей к системе $\{u_n(x)\}$ рассмотренного нами типа может служить система кратных экспонент, привлекающее в последние годы внимание многих авторов, см. например проблемы устойчивости турбулентной плазмы, где появится трехмерная задача А. А. Самарского — Н. И. Ионкина:

$$-\Delta u = \lambda u \text{ в кубе } \prod_{i=1}^{3} [0 \le x_i \le 1],$$

$$u|_{x_1=0} = 0, \quad \left.\frac{\partial u}{\partial x_1}\right|_{x_1=0} = \left.\left(\frac{\partial u}{\partial x_1} + \varepsilon u\right)\right|_{x_1=1},$$

$$u|_{x_2=0} = u|_{x_2=1} = u|_{x_3=0} = u|_{x_3=1} = 0.$$

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DENSEST PACKING OF SMALL NUMBER OF CONGRUENT SPHERES IN POLYHEDRA

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(*Received August 21, 1985*)

0. Introduction

The densest packing of n equal circles into a square was determined by SCHAER and MEIR [9] for $n \leqq 8$, by SCHAER [10] for $n = 9$ and by WENGERODT [16], [17] for $n = 16$ and $n = 25$. For some values of n greater than 9 GOLDBERG [3] SCHLÜTER [14] and SCHMITZ & KIRCHNER [15] constructed economical packings of n equal circles into a square. RUDA [8] investigated packings of equal circles in rectangles. The problem of densest packing of n equal circles into a circle was solved by PIRL [7] for $n \leqq 10$, who also gave constructions for some other values of n. Further examples of dense packings of equal circles in a circle are contained in the papers of KRAVITZ [6] and GOLDBERG [4].

In the present note we are going to investigate the analogous 3-dimensional problem. We can get to this problem-circle also in the following way. First we mention the famous question of L. FEJES TÓTH [1]: What is the minimum volume of the convex hull of k non-overlapping unit balls in Euclidean d-space. From this the next question can be derived. If P is a convex polyhedron in Euclidean 3-space, then we are looking for that packing of k non-overlapping unit balls which is contained by the smallest convex polyhedron homothetic to P. Or in a different manner: Let us find the densest packing of k non-overlapping congruent spheres in P. If P is a convex polyhedron circumscribed about a sphere, then the previous problem is equivalent to the following one: to distribute k points in P so that the least distance between the points is great as possible. $t(P, k)$ denotes this value in the 3-dimensional Euclidean space. In this paper we prove:

THEOREM. *If T, O, K denote the regular tetrahedron, octahedron and hexahedron of edge-length* 1, *then* $t(T, 2) = t(T, 3) = t(T, 4) = 1$,

$$t(T, 5) = \frac{\sqrt{6}}{4}, \quad t(T, 8) = t(T, 9) = t(T, 10) = \frac{1}{2}, \quad t(0, 2) = \sqrt{2},$$

$$t(0, 3) = \sqrt{6}-\sqrt{2}, \quad t(0, 4) = t(0, 5) = t(0, 6) = 1, \quad t(0, 7) = \frac{\sqrt{2}}{2},$$

* Partially supported by Hung. Nat. Found. for Sci. Research, No. 1238.

$$t(K,2)=\sqrt{3},\ t(K,3)=t(K,4)=\sqrt{2},\ t(K,8)=1,\ t(K,9)=\frac{\sqrt{3}}{2}$$

and locally (i.e. the points can move only in certain neighbourhoods (see the proofs)) $t(K,\ 13)=t(K,14)=\frac{\sqrt{2}}{2}$. *Furthermore Fig. 1. shows the corresponding extreme configurations.*

REMARK 1. Fig. 1 shows all incongruent extreme configurations of a given number of points.

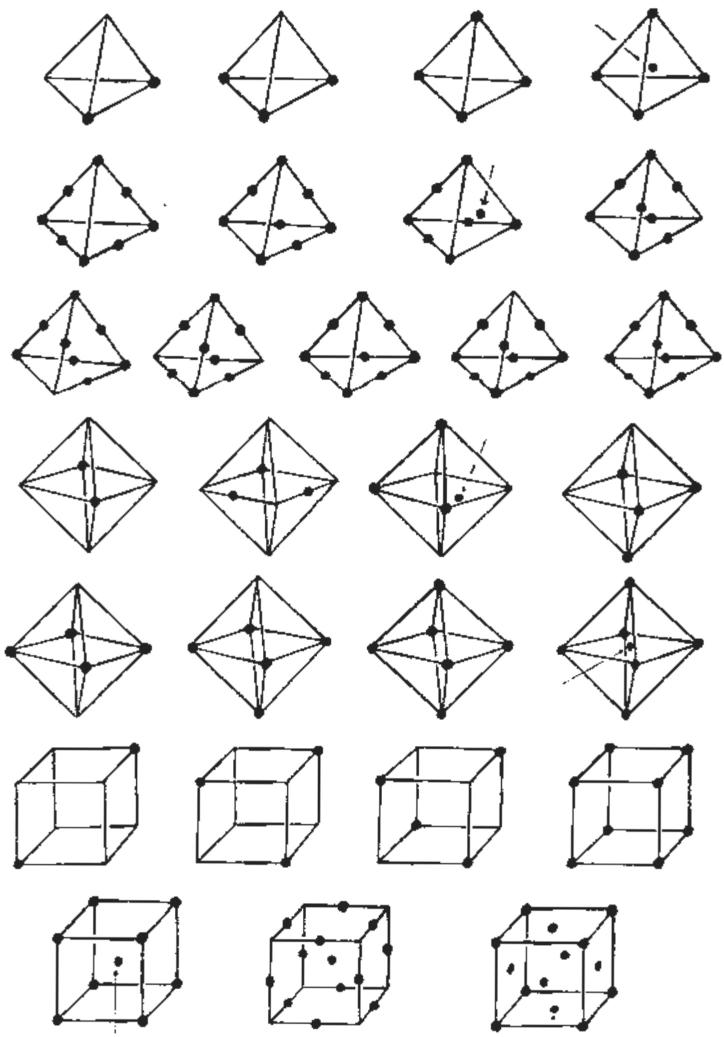

Fig. 1.

REMARK 2. One can conjecture that $t(K, 13) = t(K, 14) = \frac{\sqrt{2}}{2}$ not only locally.

REMARK 3. If I is the icosahedron of edge-length 1, then using the corresponding result of L. FEJES TÓTH [2] it is easy to show that $t(I, 12) = 1$ and the points must lie in the vertices of I.

REMARK 4. It is known that GOLSER [5] solved the problem in the case of octahedron for $k = 2, 3, 4, 5, 6$ and J. SCHAER [11], [12], [13] in the case of hexahedron for $k = 2, 3, 4, 5, 6, 8, 9$. Of course our proofs are different from the mentioned ones in the listed cases. On the other hand our aim was to find a geometric method which yields the corresponding extreme configurations in a lot of cases.

Now let us see the proof of the theorem.

1. Case of the octahedron

Let O be an octahedron of edge-length 1 with the vertices $O_1O_2O_3O_4O_5O_6$ (Fig. 2).

$P = O$, $k = 2$. In this case evidently $t(O, 2) = \sqrt{2}$ and the two points must lie in the opposite vertices of O.

$P = O$, $k = 3$. Choose the points H_1, H_2, H_3 of O so that $H_iH_j \geqq t(O, 3)$ $(i \neq j,\ i = 1, 2, 3,\ j = 1, 2, 3)$. Then the points H_1, H_2, H_3 determine a real triangle moreover these points lie on the edges of O. Without loss of generality we may suppose that either the pyramid $O_1O_2O_3O_4O_5$ contains the points H_1, H_2, H_3 (Fig. 2) or the points H_1, H_2, H_3 lie on such edges of O which are in pairs not neighbouring (Fig. 3).

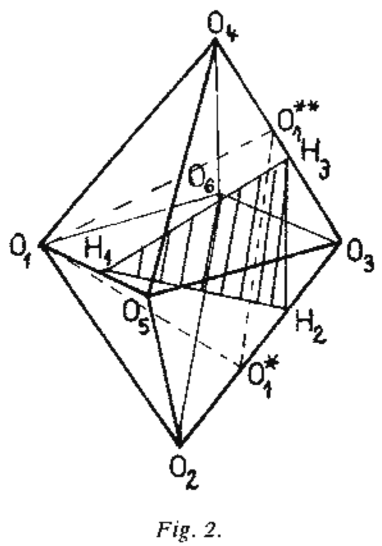

Fig. 2.

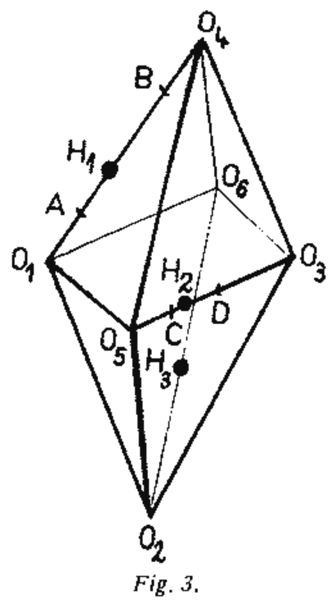

Fig. 3.

In the first case we can reason as follows. If we want to distribute 3 points in the unit square $O_1O_2O_3O_4$ so that the least distance between the points should be as great as possible, then these points have to form a regular triangle of side-length $\sqrt{6}-\sqrt{2}$ whose one vertex is also a vertex of the unit square. Let $O_1O_1^*O_1^{**}$ ($O_1^*\in O_2O_3$, $O_1^{**}\in O_3O_4$) be such a triangle. So in any case $t(O, 3)\geqq\sqrt{6}-\sqrt{2}>1$. But then at most one of the points H_1, H_2, H_3 can lie on the same triangle-facet of $O_1O_2O_3O_4O_5$ and on the union of edges O_5O_1, O_5O_2, O_5O_3, O_5O_4.

If $\{H_1, H_2, H_3\}\cap(O_5O_1\cup O_5O_2\cup O_5O_3\cup O_5O_4)=\emptyset$, then the points H_1, H_2, H_3 are interior points on the sides of the square $O_1O_2O_3O_4$ and so $t(O, 3)\leqq$ $\leqq\min\{H_iH_j|i\neq j,\ i=1, 2, 3,\ j=1, 2, 3\}<\sqrt{6}-\sqrt{2}$ which is a contradiction. Thus let H_1 be a point of the edge O_5O_1. Without loss of generality we may suppose that $H_2\in O_2O_3$ and $H_3\in O_3O_4$ (Fig. 2). If $H_1\neq O_1$, then replacing the point H_1 by O_1 we could remove the points H_2, H_3 so that

$$\min\{H_iH_j\ |i\neq j,\ i=1, 2, 3,\ j=1, 2, 3\}>t(O, 3)$$

which is a contradiction. Thus simply $H_1=O_1$ when $H_2=O_1^*$, $H_3=O_1^{**}$.

In the second case we may suppose that $H_1\in O_1O_4$, $H_2\in O_3O_5$, $H_3\in O_2O_6$ (Fig. 3). On account of symmetry we can suppose also that $H_2\in O_5D$ where D is the midpoint of the edge O_3O_5. After this we choose the points A, B, C on the edge O_1O_4 and O_3O_5, resp. so that $O_1A=O_4B=O_5C=2-\sqrt{3}$. In this way $H_1\in AB$ and $H_2\in CD$, respectively because otherwise either $H_1H_2<\sqrt{6}-\sqrt{2}$ or $H_1H_3<\sqrt{6}-\sqrt{2}$ although $H_1H_2\geqq t(O, 3)\geqq\sqrt{6}-\sqrt{2}$ and $H_1H_3\geqq t(O, 3)\geqq$ $\geqq\sqrt{6}-\sqrt{2}$, resp.. But then $H_1H_2\leqq AD<\sqrt{6}-\sqrt{2}$ and this is a contradiction.

$P=O$, $k=4$. A simple construction shows that $t(O, 4)\geqq 1$. Let H_1, H_2, H_3, H_4 be the points of O with the property $H_iH_j\geqq t(O, 4)$ ($\forall\, i, j=1, 2, 3, 4$, $i\neq j$). If it is necessary moving the points continuosly we can get the points H_1, H_2, H_3, H_5 on the boundary of O having not smaller distances between them than originally. Let us denote the midpoint of the edge O_iO_j ($i<j$, $i, j=1, 2, 3, 4, 5, 6$) by $F_{i,\,j}$ (Fig. 4). conv $\{F_{i,\,j}\}$ is a convex polyhedron with diameter 1 whose boundary consists of 6 squares and 8 equilateral triangles. Three cases are distinguish here:

(i) At last one of the points H_1, H_2, H_3, H_4 lies in the interior of one of the mentioned 8 equilateral tringles (for example $H_1\in\operatorname{int}(F_{1,4}\,F_{1,5}\,F_{4,5}\Delta)$).
(ii) The points H_1, H_2, H_3, H_4 lie on the boundary of the pyramids of edge-length 1/2 belonging to the vertices O_1, O_2, O_3, O_4. (The pyramid here has 4 triangles and a square as facets.)
(iii) The points H_1, H_2, H_3, H_4 lie on the boundary of the pyramids of edge-length 1/2 belonging to the vertices O_1, O_3, O_4, O_6.

It is easy to see that so we enumerated all the different cases.

(i) Since the unit spheres centred at the points O_2, O_3, O_6 intersect each other in the centre of the equilateral triangle $O_1O_4O_5$ therefore we distinguish two cases: Either is the point H_1 an interior point — for instance — of the

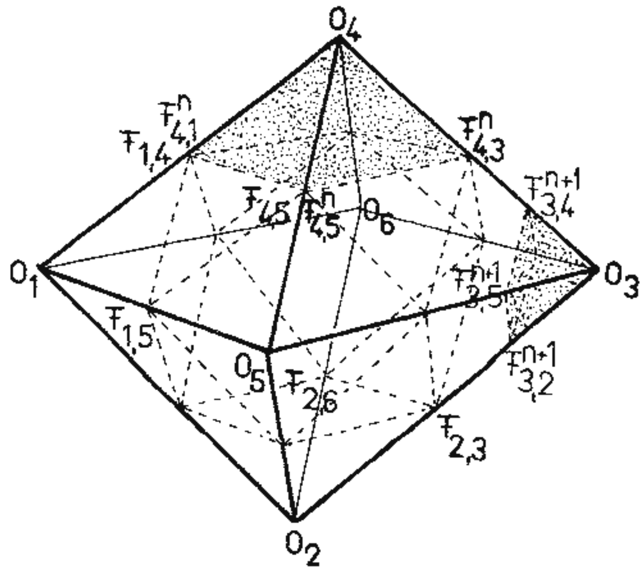

Fig. 4.

unit sphere centred at O_2 or H_1 is the center of the equilateral triangle $O_1O_4O_5$. In the first case the pyramids of edge-length 1/2 belonging to the vertices O_1, O_2, O_4, O_5 are contained by the interior of the unit sphere centred at H_1. So the points H_2, H_3, H_4 must belong to the pyramids of edge-length 1/2 lying at the vertices O_3, O_6. That is two points of H_2, H_3, H_4 belong to the same pyramid of edge-length 1/2 and so their distance is smaller than 1 which is a contradiction. In the second case the points H_2, H_3, H_4 must lie evidently in the vertices O_2. O_3, O_6.

(ii) Here we suppose that the point H_i $(1 \leq i \leq 4)$ belongs to the pyramid G_i^1 of edge-length 1/2 lying at the vertex O_i. After this G_i^n $(1 \leq i \leq 4)$ denotes the pyramid homothetic to G_i^1 of edge-length $1/2^n$ $(n = 1, 2, 3, \ldots)$ lying at the vertex O_i. We shall prove that $H_i \in G_i^{n+1}$ $(i = 1, 2, 3, 4)$ if $H_i \in G_i^n$ $(i = 1, 2, 3, 4)$. Let $F^n_{i,j}$ be such a point of the edge O_iO_j for which $O_iF^n_{i,j} = 1/2^n$. Using the symmetry it is enough to show that $H_3 \in G_3^{n+1}$ (Fig. 4). In order to prove this we have to show (using again the symmetry) that

$$H_3 \notin \operatorname{conv}\{F^n_{3,4}, F^n_{3,5}, F^n_{3,6}, F^{n+1}_{3,4}, F^{n+1}_{3,5}, F^{n+1}_{3,6}\} =: F.$$

Since the farthest points of the sets G_4^n and F are the points $F^n_{4,1}$ and $F^{n+1}_{3,5}$ (or $F^n_{4,1}$ and $F^{n+1}_{3,6}$) when

$$F^n_{4,1}F^{n+1}_{3,5} = F^n_{4,1}F^{n+1}_{3,6} = \sqrt{1 - \frac{2^{n+1}-3}{4^{n+1}}} < 1$$

therefore really $H_3 \notin F$. Thus necessarily $O_1 = H_1$, $O_2 = H_2$, $O_3 = H_3$, $O_4 = H_4$.

(iii) We know that the points H_1, H_2, H_3, H_4 lie one after another on the boundary of the pyramids G_1^1, G_3^1, G_4^1, G_6^1 of edge-length 1/2 lying at the vertices O_1, O_3, O_4, O_6. After this G_1^n, G_3^n, G_4^n, G^n denote the pyramids of edge-length $1/2^n$ belonging to the vertices O_1, O_3, O_4, O_6 ($n = 1, 2, 3, \ldots$). We prove that $H_1 \in G_1^{n+1}$, $H_2 \in G_3^{n+1}$, $H_3 \in G_4^{n+1}$, $H_4 \in G_6^{n+1}$ if $H_1 \in G_1^n$, $H_2 \in G_3^n$, $H_3 \in G_4^n$, $H_4 \in G_6^n$. For us it is enough to show that $H_2 \in G_3^{n+1}$ and $H_3 \in G_4^{n+1}$ (Fig. 5). The proof of $H_3 \in G_4^{n+1}$ is the same as the proof of $H_i \in G_i^{n+1}$ in the case (ii). So we have to deal only with the statement $H_2 \in G_3^{n+1}$. Let $F_{i,j}^n$ be such a point of the edge O_iO_j for which $O_iF_{i,j}^n = 1/2^n$. Moreover let K_n (L_n, respectively) be a point on the height of the triangle $O_3O_4O_6$ ($O_3O_2O_5$) belonging to the vertex O_3 for which

$$O_3K_n = \frac{\sqrt{3}}{2^{n+1}} \quad \left(O_3L_n = \frac{\sqrt{3}}{2^{n+1}}\right).$$

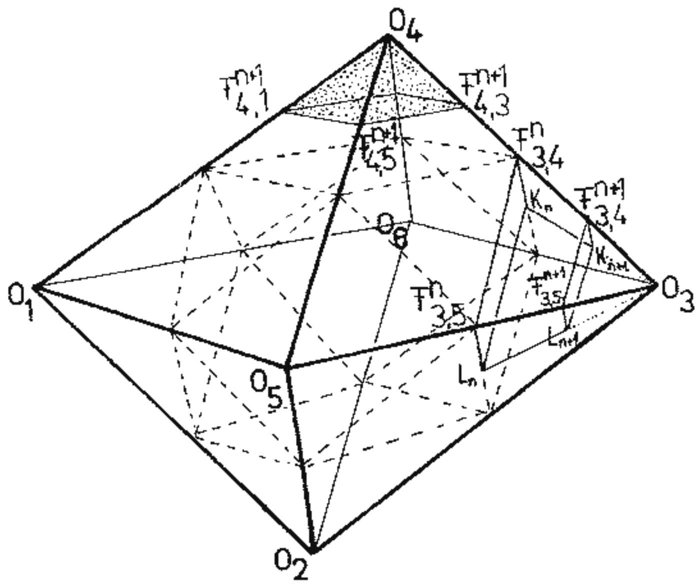

Fig. 5.

We have to show only that

$$H_2 \notin \operatorname{con}\{F_{3,4}^n,\ F_{3,5}^n,\ F_{3,4}^{n+1},\ F_{3,5}^{n+1},\ K_n,\ K_{n+1},\ L_n,\ L_{n+1}\} = : F.$$

We have proved that $H_3 \in G_4^{n+1}$. It is easy to see that the distance of any two points of the sets G_4^{n+1} and F cannot be greater than $O_4L_{n+1} < 1$ or $O_4L_n < 1$. So on account of $H_3H_2 \geq t(O, 4) \geq 1$ we get that $H_2 \notin F$. Thus necessarily $O_1 = H_1$, $O_3 = H_2$, $O_4 = H_3$, $O_6 = H_4$.

To sum it up the cases (i), (ii), (iii) yield that $t(O, 4) = 1$ and describe all the extreme configurations.

$P = O, k = 5$. This case together with the case $P = O, k = 6$ is trivial after the studying of the extreme configurations of $P = O, k = 4$.

$P = O, k = 7$. A simple construction shows that $t(O, 7) \geqq \frac{\sqrt{2}}{2}$. After this we choose the points $H_i (i = 1, 2, \ldots, 7)$ of O so that $H_i H_j \geqq t(O, 7) \geqq \frac{\sqrt{2}}{2}$ $(i \neq j,\ i, j = 1, 2, \ldots, 7)$. If we construct the Dirichlet cells of the vertices of O in O, then we get six congruent convex polyhedra of diameter $\frac{\sqrt{2}}{2}$. Thus one of these convex polyhedra contains at least two points from H_i consequently $t(O, 7) = \frac{\sqrt{2}}{2}$ and there exists only one possibility for the positions of the points H_i.

2. Case of the regular tetrahedron

Let T be a regular tetrahedron of edge-length 1 with the vertices $T_1T_2T_3T_4$ (Fig. 6).

$P = T, k = 2, 3, 4$. It is easy to see that $t(T, 2) = t(T, 3) = t(T, 4) = 1$ and the points must lie in the vertices of T.

$P = T, k = 5$. If we construct the Dirichlet cells of the vertices of T in T and follow the way of the proof in case $P = O, k = 7$, then we can get the corresponding result immediately.

$P = T, k = 8, 9, 10$. The Fig. 1 shows that $t(T, 8) \geqq 1/2$, $t(T, 9) \geqq 1/2$, $t(T, 10) \geqq 1/2$. At the same time let us denote the midpoint of the edge T_iT_j $(i < j,\ i, j = 1, 2, 3, 4)$ by $F_{i,j}$ (Fig. 6).

Since conv $\{F_{i,j}\}$ is an octahedron of edge-length 1/2 therefore we can divide T into 4 regular tetrahedron of edge-length 1/2 and an octahedron of edge-length 1/2. So if we consider at least 8 points of T, then either a tetrahedron

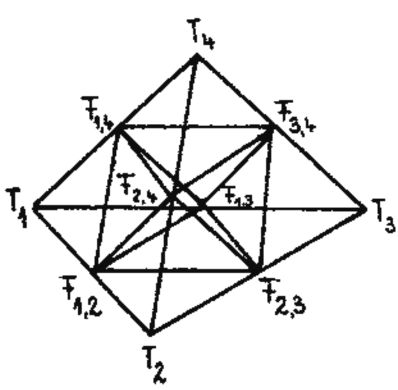

Fig. 6.

(of edge-length 1/2) contains at least two points or the octahedron of edge-length 1/2 contains at least 4 points. That is the minimal distance among the points is at most 1/2. Thus $t(T, 8) = t(T, 9) = t(T, 10) = 1/2$. Using the extreme configurations of the points in the octahedron we can get easily the corresponding extreme configurations in the regular tetrahedron.

3. Case of the hexahedron

Let K be the cube of edge-length 1 with the vertices $K_1K_2K_3K_4K_5K_6K_7$ K_8 (Fig. 7).

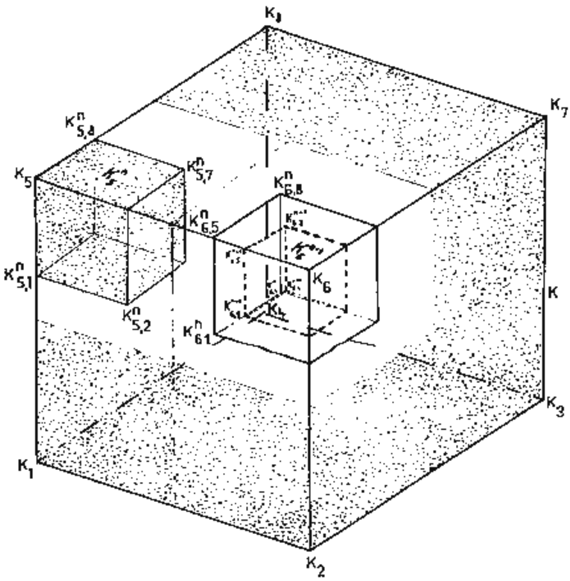

Fig. 7.

$P = K$, $k = 2$. This case is trivial.

$P = K$, $k = 3$. If we consider the vertices K_2, K_5, K_7, then we obtain that $t(K, 3) \geqq \sqrt{2}$. After this we choose the points H_1, H_2, H_3 of K so that $H_iH_j \geqq$ $\geqq t(K, 3)$ $(i \neq j,\ i, j = 1, 2, 3)$. It is clear that the points H_i form a real triangle moreover these points lie on the edges of K. If there exists a facet of K which contains two H_i points for example the points H_1, H_2 belonging to the square $K_1K_2K_5K_6$, then without loss of generality we may suppose that $H_1 = K_1$ and $H_2 = K_6$ after H_3 can be either in K_3 or K_8. If finally each facet of K contains at most one H_i point, then without loss of generality we may suppose that $H_1 \in K_1K_2$, $H_2 \in K_3K_7$, $H_3 \in K_5K_8$ (as interior points on the edges). So the points H_i divide the spatial brokenline $K_1K_2K_3K_7K_8K_5K_1$ into three parts when one of these parts for instance $H_1K_2K_3H_2$ has the length at most 2. But then $H_1H_2 < \sqrt{2}$ which is a contradiction.

$P = K, k = 4$. After the case $P = K, k = 3$ this is trivial.

$P = K, k = 8$. Considering the vertices of K we obtain that $t(K, 8) \geqq 1$. After this we choose the points H_i $(i = 1, 2, \ldots, 8)$ of K so that $H_iH_j \geqq t(K, 8)$ $(i \neq j, i, j = 1, 2, \ldots, 8)$. We define the following sequence: $\lambda_1 = 1/2$, $\lambda_{n+1} = 1 - \sqrt{1 - 2 \cdot \lambda_n^2}$ $(n = 1, 2, \ldots)$. Evidently $\lim_{n \to +\infty} \lambda_n = 0$. Now let us consider the cube of edge-length λ_n at the vertex K_i, which cube is homothetic to K. The corresponding vertex to K_j will be denoted by $K_{i,j}$ $(i, j = 1, 2, \ldots, 8, n = 1, 2, \ldots)$. At the same time let $K_i^n = \operatorname{conv}\{K_{i,j}^n \mid j = 1, 2, \ldots, 8\}$ $(i = 1, 2, \ldots, 8, n = 1, 2, \ldots)$. Of course $K_{i,i}^n = K_i$ if $i = 1, 2, \ldots, 8$ and $n = 1, 2, \ldots$, (Fig. 7).

Since the edge-length of K_i^1 $(i = 1, 2, \ldots, 8)$ equals 1/2 therefore we may suppose that $H_i \in K_i^1$ $(i = 1, 2, \ldots, 8)$. Further we prove that $H_i \in K_i^{n+1}$ $(i = 1, 2, \ldots, 8)$ if $H_i \in K_i^n$ $(i = 1, 2, \ldots, 8)$. Using the symmetry it is enough to show that $H_6 \in K_6^{n+1}$. In order to prove this we have to show only (using again the symmetry) that

$H_6 \notin (\operatorname{conv}\{K_{6,5}^n, K_{6,5}^{n+1}, K_{6,1}^n, K_{6,1}^{n+1}, K_{6,8}^n, K_{6,8}^{n+1}, K_{6,4}^n, K_{6,4}^{n+1}\}) \setminus \{K_{6,5}^{n+1}\} = \tilde{K}$. But this is a corollary of that fact that $H_5H_6 \geqq 1$ and if $H \in K_5^n$ and $\bar{H} \in \tilde{K}$, then $H\bar{H} < 1$ $(K_{5,4}^n\, K_{6,5}^{n+1} = 1)$. Thus necessarily $H_i = K_i$ $(i = 1, 2, \ldots, 8)$ and $t(K, 8) = 1$.

$P = K, k = 9$. This case is simple if we think over that a unit cube can be divided into 8 cubes of edge-length 1/2 and so at least one of these smaller cubes contains (at least) two points from the given 9 points of the unit cube.

$P = K, k = 13$. Let the directed lines $\overrightarrow{K_1K_2}$, $\overrightarrow{K_1K_4}$, $\overrightarrow{K_1K_5}$ of K be the axes of a system of co-ordinates. After this we define the points $H_{i,j,l} = \left(\frac{i}{3}, \frac{j}{3}, \frac{l}{3}\right)$ where $0 \leqq i, j, l \leqq 3$. The midpoint of the edge K_iK_j $(i < j)$ of the cube K is denoted by $F_{i,j}$ moreover let K_0 be the center of K (Fig. 8). And now let $0 < d \leqq 1/60$ be a real number. We construct a neighbourhood $\mathcal{F}_{i,j}^d$ for the midpoint $F_{i,j}$ $(i < j)$ as follows:

$$\mathcal{F}_{i,j}^d = \operatorname{conv}\{X_{i,j}^d, X_{j,i}^d, Y_{i,j}^d, Y_{j,i}^d, Z_{i,j}^d, Z_{j,i}^d\}$$

where

$$X_{i,j}^d \in K_iK_j,\ X_{j,i}^d \in K_iK_j \text{ and } X_{i,j}^dK_i = X_{j,i}^dK_j = \frac{1}{2} - d$$

furthermore

$$X_{i,j}^d Y_{i,j}^d Z_{i,j}^d \triangle \quad (X_{j,i}^d Y_{j,i}^d Z_{j,i}^d \triangle)$$

is an isosceles right-angled triangle with legs $3d$ whose vertices lie on the facets of K and whose plane is orthogonal to the edge K_iK_j and finally whose orientation from K_j is positive. In the end let

$$\mathcal{K}_0 = \operatorname{conv}\{H_{1,1,1}, H_{2,1,1}, H_{2,2,1}, H_{1,2,1}, H_{1,1,2}, H_{2,1,2}, H_{2,2,2}, H_{1,2,2}\}.$$

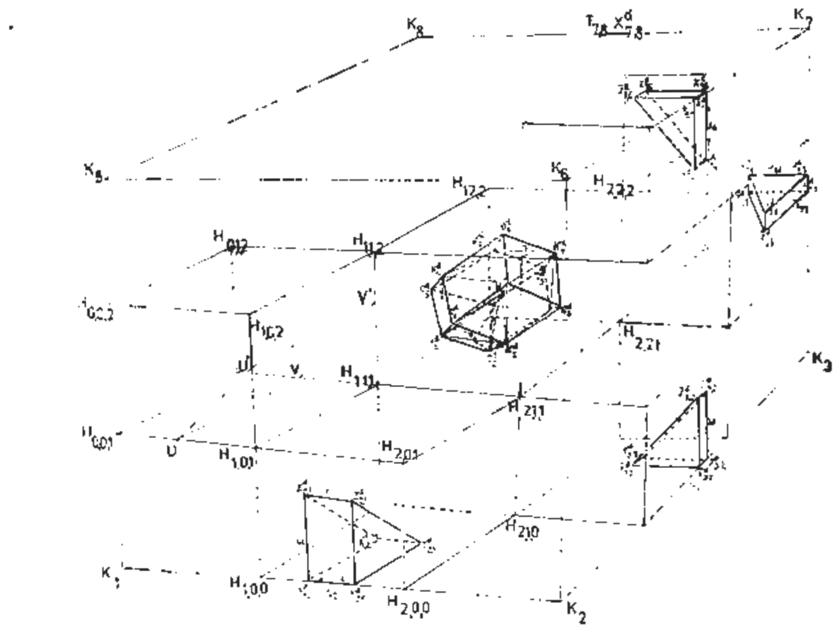

Fig. 8.

We show that if the points $F^*_{1,2}, F^*_{2,3}\ F^*_{3,4}, F^*_{1,4}, F^*_{1,5}, F^*_{2,6}, F^*_{3,7}, F^*_{4,8}, F^*_{5,6}, F^*_{6,7}, F^*_{7,8}, F^*_{5,8}\ K^*_0$ lie in K such that $F^*_{i,j}\in\mathcal{F}^d_{i,j}$ and $K^*_0\in\mathcal{K}_0$ moreover the distances among them are always at least $\frac{\sqrt{2}}{2}$, then necessarily $F^*_{i,j} = F_{i,j}\ K^*_0 = K_0$.

Suppose the opposite of this. Then we choose the smallest positive number d for which $F^*_{i,j}\in\mathcal{F}^d_{i,j}$.

First of all a simple remark. Let

$$\mathcal{H}_1 = \mathrm{conv}\{H_{1,0,0}, H_{2,0,0}, H_{2,1,0}, H_{1,1,0}, H_{1,0,1}, H_{2,0,1}, H_{2,1,1}, H_{1,1,1}\}$$

and

$$\mathcal{H}_2 = \mathrm{conv}\{H_{0,0,1}, H_{1,0,1}, H_{1,1,1}, H_{0,1,1}, H_{0,0,2}, H_{1,0,2}, H_{1,1,2}, H_{0,1,2}\}$$

moreover $H_1\in\mathcal{H}_1$, $H_2\in\mathcal{H}_2$ with the property $H_1H_2\geq\frac{\sqrt{2}}{2}$. A simple calculation shows that $H_2\notin \mathrm{conv}\,\{H_{1,0,1}, H_{1,1,1}, U, U', V, V'\}$ where the points U, U', V, V' lie on the edges $H_{0,0,1}\ H_{1,0,1}$; $H_{1,0,2}\ H_{1,0,1}$; $H_{0,1,1}\ H_{1,1,1}$; $H_{1,1,2}\ H_{1,1,1}$ of $\mathcal{H}_2$ and $UH_{1,0,1} = U'H_{1,0,1} = VH_{1,1,1} = V'H_{1,1,1} < \frac{\sqrt{10}-2}{6}$.

It yields directly that the point K_0^* belongs to the rhombic dodecahedron $\tilde{K}_1\tilde{K}_2\tilde{K}_3\tilde{K}_4\tilde{K}_5\tilde{K}_6\tilde{K}_7\tilde{K}_8\tilde{O}_1\tilde{O}_2\tilde{O}_3\tilde{O}_4\tilde{O}_5\tilde{O}_6$ whose cube $\tilde{K}_1\tilde{K}_2\tilde{K}_3\tilde{K}_4\tilde{K}_5\tilde{K}_6\tilde{K}_7\tilde{K}_8$ of edge-length $(4-\sqrt{10})/6$ with the center K_0 is homothetic to the unit cube K (the corresponding image of K_i is $\tilde{K}_i$ $i = 1, 2, \ldots, 8$) and whose octahedron $\tilde{O}_1\tilde{O}_2\tilde{O}_3\tilde{O}_4\tilde{O}_5\tilde{O}_6$ of edge-length $\sqrt{2}\cdot\frac{4-\sqrt{10}}{2}$ lies so that in the rhombic dodecahedron the vertex $\tilde{O}_1$ is connected with the vertices $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{K}_6$, $\tilde{K}_5$ by edges and similarly $\tilde{O}_2$ with $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{K}_3$, $\tilde{K}_4$; $\tilde{O}_3$ with $\tilde{K}_4$, $\tilde{K}_3$, $\tilde{K}_7$: $\tilde{K}_8$; $\tilde{O}_4$ with $\tilde{K}_5$, $\tilde{K}_6$, $\tilde{K}_7$, $\tilde{K}_8$; $\tilde{O}_5$ with $\tilde{K}_5$, $\tilde{K}_1$, $\tilde{K}_4$, $\tilde{K}_8$; $\tilde{O}_6$ with $\tilde{K}_6$, $\tilde{K}_2$, $\tilde{K}_3$, $\tilde{K}_7$ (Fig. 8 shows a homothetic example of this rhombic dodecahedron).

Then $F_{1,2}^*\tilde{K}_1 < \frac{\sqrt{2}}{2}$, $F_{1,2}^*\tilde{K}_2 < \frac{\sqrt{2}}{2}$, $F_{1,2}^*\tilde{O}_1 < \frac{\sqrt{2}}{2}$, $F_{1,2}^*\tilde{O}_2 < \frac{\sqrt{2}}{2}$ because $F_{1,2}^* \in \mathcal{F}_{1,2}^d \subset \mathcal{H}_1$ and we have some similar inequalities for the distances between the points $F_{i,j}^*$ $(i<j)$ and the corresponding vertices of the rhombic dodecahedorn. Since $\sphericalangle X_{2,1}^d\tilde{K}_1K_0 > \frac{\pi}{2}$ therefore we can find a unique point K_1^d in the interior of the segment $\tilde{K}_1K_0$ so that $X_{2,1}^d K_1^d = \frac{\sqrt{2}}{2}$. And now let us consider the scalar multiple of the rhombic dodecahedorn with center K_0 and scalar $K_1^dK_0/\tilde{K}_1K_0$. This maps the point $\tilde{K}_i$ $(\tilde{O}_j)$ onto K_i^d (O_j^d) $(i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 6)$. The image

$$\mathcal{K}_0^d = \operatorname{conv}\{K_1^dK_2^dK_3^dK_4^dK_5^dK_6^dK_7^dK_8^dO_1^dO_2^dO_3^dO_4^dO_5^dO_6^d\}$$

is again a rhombic dodecahedron. Using the construction it is easy to see that $K_0^* \in \mathcal{K}_0^d$.

Further first we prove that

$$F_{1,2}^* \notin \operatorname{conv}\{Y_{2,1}^d, Y_{1,2}^d, Z_{2,1}^d, Z_{1,2}^d\}$$

and in general

$$F_{i,j}^* \notin \operatorname{conv}\{Y_{j,i}^d, Y_{i,j}^d, Z_{j,i}^d, Z_{i,j}^d\}.$$

Using the symmetry for us it is enough to prove only the first case. But there it is sufficient to show that $Z_{2,1}^dK_0^* < \frac{\sqrt{2}}{2}$. Using the symmetries of $\mathcal{K}_0^d$ we can get this inequality as a corollary of the following inequalities:

$$Z_{2,1}^dK_8^d < \frac{\sqrt{2}}{2} \quad \text{and} \quad Z_{2,1}^dO_3^d < \frac{\sqrt{2}}{2}.$$

But we know that

$$X^d_{7,8}O^d_3 < X^d_{7,8}K^d_8 = \frac{\sqrt{2}}{2}$$

therefore we have only to prove that

$$Z^d_{2,1}K^d_8 < X^d_{7,8}K^d_8 \text{ and } Z^d_{2,1}O^d_3 < X^d_{7,8}O^d_3.$$

Denore F' (F'') the midpoint of the segment $K^d_8K^d_7$ ($Z^d_{2,1}\,Z^d_{1,2}$). Since $F''Z^d_{2,1} = F_{7,8}X^d_{7,8} = d$ therefore $Z^d_{2,1}\,K^d_8 < X^d_{7,8}\,K^d_8$ is equivalent to $F''F' < F'F_{7,8}$. We know that $K_0F' < d$ so indeed

$$(F''F')^2-(F'F_{7,8})^2 < \left[(3d)^2+\left(\frac{\sqrt{2}}{2}+d\right)^2-2(3d)\left(\frac{\sqrt{2}}{2}+d\right)\frac{1}{\sqrt{2}}\right]-$$

$$-\left[\frac{\sqrt{2}}{2}-d\right]^2=(9-3\sqrt{2})d\left(d-\frac{3-2\sqrt{2}}{9-3\sqrt{2}}\right)<0$$

$\left(\text{because } 0<d\leq\frac{1}{60}\right)$.

On the other hand

$$Z^d_{2,1}O^d_3 < X^d_{7,8}O^d_3$$

is equivalent to

$$F''O^d_3 < F_{7,8}O^d_3.$$

But on account of $K_0F' < d$ is $K_0O^d_3 < \sqrt{2}\cdot d$ so indeed

$$(F''O^d_3)^2-(F_{7,8}O^d_3)^2 < \left[\left(\frac{1}{2}-3d\right)^2+\left(\frac{1}{2}+\sqrt{2}\cdot d\right)^2\right]-\left[\left(\frac{1}{2}-\sqrt{2}\cdot d\right)^2+\frac{1}{4}\right]=$$

$$=9d\left(d-\frac{3-2\sqrt{2}}{9}\right)<0$$

$\left(\text{because } 0<d\leq\frac{1}{60}\right)$.

Finally we prove that $F^*_{i,j}\notin X^d_{j,i}Y^d_{j,i}Z^d_{j,i}\Delta$ and $F^*_{i,j}\notin X^d_{i,j}Y^d_{i,j}Z^d_{i,j}\Delta$ $(i<j)$. On account of the symmetry it is enough to show that $F^*_{1,2}\notin X^d_{2,1}\,Y^d_{2,1}\,Z^d_{2,1}\Delta$. Suppose that $F^*_{1,2}\in X^d_{2,1}\,Y^d_{2,1}\,Z^d_{2,1}\Delta$. Denote $\overline{X}^d_{3,2}$, $\overline{Y}^d_{3,2}$ $\overline{Z}^d_{3,2}$ such points on the segments $X^d_{3,2}\,X^d_{2,3}$, $Y^d_{3,2}\,Y^d_{2,3}$, $Z^d_{3,2}\,Z^d_{2,3}$ for which $\overline{X}^d_{3,2}\,X^d_{3,2}=\overline{Y}^d_{3,2}\,Y^d_{3,2}$ $=\overline{Z}^d_{3,2}\,Z^d_{3,2}=\frac{d}{2}$. Since $F^*_{2,3}\in\mathcal{T}^d_{2,3}$, $F^*_{1,2}F^*_{2,3}\geq\frac{\sqrt{2}}{2}$ and

$$(X^d_{2,1}\overline{Z}^d_{3,2})^2-\frac{1}{2}=(Z^d_{2,1}\overline{X}^d_{3,2})^2-\frac{1}{2}=\left(\frac{1}{2}-d\right)^2+\left(\frac{1}{2}+\frac{d}{2}\right)^2+(3d)^2-\frac{1}{2}=$$

$$=\frac{41}{4}d\left(d-\frac{2}{41}\right)<0\quad\left(0<d\leq\frac{1}{60}\right)$$

therefore

$$F^*_{2,3} \in \operatorname{conv}\{X^d_{3,2}, \overline{X}^d_{3,2}, Y^d_{3,2}, \overline{Y}^d_{3,2}, Z^d_{3,2}, \overline{Z}^d_{3,2}\}.$$

Denote moreover $\overline{X}^d_{7,3}, \overline{Y}^d_{7,3}, \overline{Z}^d_{7,3}$ that points on the segments $X^d_{7,3}X^d_{3,7}$, $Y^d_{7,3}Y^d_{3,7}$, $Z^d_{7,3}Z^d_{3,7}$ for which $X^d_{7,3}\overline{X}^d_{7,3} = Y^d_{7,3}\overline{Y}^d_{7,3} = Z^d_{7,3}\overline{Z}^d_{7,3} = \frac{3}{4}d$.

Since $F^*_{3,7} \in \mathcal{F}^d_{3,7}$, $F^*_{2,3}F^*_{3,7} \geqq \frac{\sqrt{2}}{2}$ and

$$(\overline{X}^d_{3,2}\overline{Y}^d_{7,3})^2 - \frac{1}{2} = (\overline{Y}^d_{3,2}\overline{X}^d_{7,3})^2 - \frac{1}{2} = \left(\frac{1}{2} - \frac{d}{2}\right)^2 +$$

$$+\left(\frac{1}{2} + \frac{d}{4}\right)^2 + (3d)^2 - \frac{1}{2} = \frac{149}{16}d\left(d - \frac{4}{149}\right) < 0 \quad \left(0 < d \leqq \frac{1}{60}\right)$$

therefore

$$F^*_{3,7} \in \operatorname{conv}\{X^d_{7,3}, \overline{X}^d_{7,3}, Y^d_{7,3}, \overline{Y}^d_{7,3}, Z^d_{7,3}, Z^d_{7,3}\}.$$

At last let $\overline{X}^d_{7,6}, \overline{Y}^d_{7,6}, \overline{Z}^d_{7,6}$ be points on the segments $X^d_{7,6}X^d_{6,7}$, $Y^d_{7,6}Y^d_{6,7}$, $Z^d_{7,6}Z^d_{6,7}$ with the property $X^d_{7,6}\overline{X}^d_{7,6} = Y^d_{7,6}\overline{Y}^d_{7,6} = Z^d_{7,6}\overline{Z}^d_{7,6} = \frac{3}{4}d$.

Similarly as before we can prove that

$$F^*_{6,7} \in \operatorname{conv}\{X^d_{7,6}, \overline{X}^d_{7,6}, Y^d_{7,6}, \overline{Y}^d_{7,6}, Z^d_{7,6}, Z^d_{7,6}\}.$$

But then

$$(F^*_{3,7}F^*_{6,7})^2 - \frac{1}{2} \leqq (\overline{X}^d_{7,3}\overline{Z}^d_{7,6})^2 - \frac{1}{2} - (Y^d_{7,3}\overline{X}^d_{7,6})^2 - \frac{1}{2} =$$

$$= 2\left(\frac{1}{2} - \frac{d}{4}\right)^2 + (3d)^2 - \frac{1}{2} = \frac{73}{8}d\left(d - \frac{4}{73}\right) < 0 \quad \left(0 < d \leqq \frac{1}{60}\right)$$

which is a contradiction. Thus indeed $F^*_{1,2} \notin X^d_{2,1}\,Y^d_{2,1}\,Z^d_{2,1}\,\Delta$. To sum it up we can find a positive number d_0 so that $0 < d_0 < d$ and $F_{i,j} \in \mathcal{F}^{d_0}_{i,j}$ $(i<j)$ which however contradicts the choosing of d. This means that necessarily $F^*_{i,j} = F_{i,j}$ $(i<j)$ and $K^*_0 = K_0$.

$P = K$, $k = 14$. Let us place the cube K in the system of co-ordinates in the following way:

$$K_1 = \left(-\frac{1}{2}, 0, -\frac{1}{2}\right), \; K_2 = \left(\frac{1}{2}, 0, -\frac{1}{2}\right), \; K_3 = \left(\frac{1}{2}, 1, -\frac{1}{2}\right),$$

$$K_4 = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right), K_5 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right), K_6 = \left(\frac{1}{2}, 0, \frac{1}{2}\right),$$

$$K_7 = \left(\frac{1}{2}, 1, \frac{1}{2}\right), K_8 = \left(-\frac{1}{2}, 1, \frac{1}{2}\right).$$

Then the centres of the facets of K are

$$O_1 = (0, 0, 0),\ O_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right),\ O_3 = (0, 1, 0),$$

$$O_4 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right),\ O_5 = \left(0, \frac{1}{2}, -\frac{1}{2}\right),\ O_6 = \left(0, \frac{1}{2}, \frac{1}{2}\right).$$

We construct the neighbourhood $\mathcal{K}_i$ ($\mathcal{O}_j$, respectively) for the point K_i ($i = 1, 2, \ldots, 8$) (O_j ($j = 1, 2, \ldots, 6$), respectively) as a scalar multiple of the unit cube K with center K_i (O_j, resp.) with scalar 1/3. So these neighbourhoods are cubes of edge-length 1/3. We shall prove that if we choose the points $K_i^* \in \mathcal{K}_i$ ($i = 1, 2, \ldots, 8$) and $O_j^* \in \mathcal{O}_j$ ($j = 1, 2, \ldots, 6$) so that the distances among them are at least $\frac{\sqrt{2}}{2}$, then necessarily $K_i^* = K_i$ ($i = 1, 2, \ldots, 8$) and $O_j^* = O_j$ ($j = 1, 2, \ldots, 6$) (Fig. 9).

First of all a simple remark. Since $O_1^* K_5^* \geq \frac{\sqrt{2}}{2}$ therefore

$$O_1^* \notin \operatorname{conv}\left\{\left(\frac{1}{48}, 0, \frac{1}{6}\right), \left(-\frac{1}{6}, 0, \frac{1}{6}\right), \left(-\frac{1}{6}, 0, -\frac{1}{48}\right), \left(\frac{1}{48}, \frac{1}{3}, \frac{1}{6}\right),\right.$$
$$\left.\left(-\frac{1}{6}, \frac{1}{3}, \frac{1}{6}\right), \left(-\frac{1}{6}, \frac{1}{3}, \frac{1}{48}\right)\right\}.$$

So using the symmetry if

$$K_{1,1} = \left(0, 0, -\frac{7}{48}\right),\ K_{1,2} = \left(\frac{7}{48}, 0, 0\right),\ K_{1,3} = \left(0, 0, \frac{7}{48}\right),$$

$$K_{1,4} = \left(-\frac{7}{48}, 0, 0\right),\ K_{1,5} = \left(0, \frac{1}{6}, -\frac{7}{48}\right),\ K_{1,6} = \left(\frac{7}{48}, \frac{1}{6}, 0\right),$$

$$K_{1,7} = \left(0, \frac{1}{6}, \frac{7}{48}\right),\ K_{1,8} = \left(-\frac{7}{48}, \frac{1}{6}, 0\right),\ K_{1,9} = \left(\frac{7}{96}, \frac{23}{96}, -\frac{7}{96}\right),$$

$$K_{1,10} = \left(\frac{7}{96}, \frac{23}{96}, \frac{7}{96}\right),\ K_{1,11} = \left(-\frac{7}{96}, \frac{23}{96}, \frac{7}{96}\right),\ K_{1,12} = \left(-\frac{7}{96}, \frac{23}{96}, -\frac{7}{96}\right),$$

$$K_{1,13} = \left(0, \frac{5}{16}, 0\right),$$

then $O_1^* \in \operatorname{conv}\{K_{1,i} \mid i = 1, 2, \ldots, 13\}$. Similarly $O_j^* \in \operatorname{conv}\{K_{j,i} \mid i = 1, 2, \ldots, 13\}$ $j = 2, 3, \ldots, 6$ where $K_{j,i}$ is the image of $K_{1,i}$ reflected in a corresponding central plane of K which reflection transforms the facet of K with center O_1 into the facet with center O_j. Now we prove that $O_j^* \in \operatorname{conv}\{K_{j,i} \mid i = 1, 2, \ldots, 8\}$ $j = 1, 2, \ldots, 6$. Using the symmetry it is enough to show

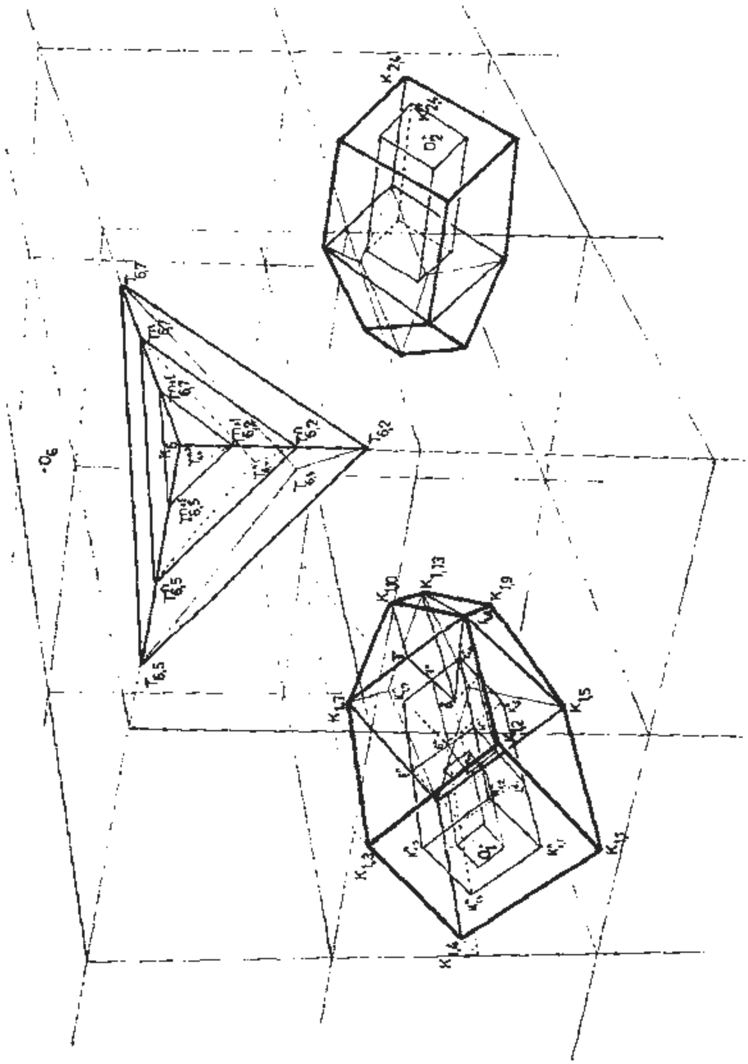

Fig. 9.

that $O_1^* \in \text{conv}\{K_{1,i} \mid i = 1, 2, \ldots, 8\}$. And for it we have to prove only that $O_1^* \notin \text{conv}\{K_{1,6}, K_{1,10}, K_{1,13}, F, \tilde{O}_1\}$ where $\tilde{O}_1$ is the centre of the square $K_{1,5}K_{1,6}K_{1,7}K_{1,8}$ and F is the midpoint of the segment $K_{1,6}K_{1,7}$. But this is true because the distance of any two points of the sets $\text{conv}\{K_{1,6}, K_{1,10}, K_{1,13}, F, \tilde{O}_1\}$ and $\text{conv}\{K_{2,i} \mid i = 1, 2, \ldots, 13\}$ is at most $\tilde{O}_1K_{2,4} = \sqrt{\frac{1105}{2304}} < \frac{\sqrt{2}}{2}$ though $O_1^*O_2^* \geq \frac{\sqrt{2}}{2}$.

After this let the vertex K_i $(i = 1, 2, \ldots, 8)$ of K be on the same edges of K with the vertices K_k, K_l, K_m and let us denote the opposite vertex to K_i of K by K_j. Let us consider the points $T_{i,k} \in K_iK_k$, $T_{i,l} \in K_iK_l$, $T_{i,m} \in K_iK_m$, $T_{i,j} \in K_iK_j$ for which $K_iT_{i,k} = K_iT_{i,l} = K_iT_{i,m} = \frac{4}{15}$ and $K_iT_{i,j} = \frac{4\sqrt{3}}{15}$. It is easy to see that if $\mathcal{T}_i = \text{conv}\{T_{i,k}, T_{i,l}, T_{i,m}, T_{i,j}, K_i\}$ $i = 1, 2, \ldots, 8$ then $K_i^* \in \mathcal{T}_i$. Now however let

$$K_{1,1}^n = \left(0,\ 0,\ -\frac{1}{12\cdot 2^{n-1}}\right),\quad K_{1,2}^n = \left(\frac{1}{12\cdot 2^{n-1}},\ 0,\ 0\right),$$

$$K_{1,3}^n = \left(0,\ 0,\ \frac{1}{12\cdot 2^{n-1}}\right),\quad K_{1,4}^n = \left(-\frac{1}{12\cdot 2^{n-1}},\ 0,\ 0\right),$$

$$K_{1,5}^n = \left(0,\ \frac{1}{6\cdot 2^{n-1}},\ -\frac{1}{12\cdot 2^{n-1}}\right),\quad K_{1,6}^n = \left(\frac{1}{12\cdot 2^{n-1}},\ \frac{1}{6\cdot 2^{n-1}},\ 0\right),$$

$$K_{1,7}^n = \left(0,\ \frac{1}{6\cdot 2^{n-1}},\ \frac{1}{12\cdot 2^{n-1}}\right),\quad K_{1,8}^n = \left(-\frac{1}{12\cdot 2^{n-1}},\ \frac{1}{6\cdot 2^{n-1}},\ 0\right)$$

and $O_j^n = \text{conv}\{K_{j,i}^n \mid i = 1, 2, \ldots, 8\}$ $n = 1, 2, \ldots, j = 1, 2, \ldots, 6$ where $K_{j,i}^n$ $(j = 2, 3, \ldots, 6)$ is the image of $K_{1,i}^n$ reflected in a corresponding central plane of K which reflection transforms the facet of K with center O_1 into the facet with center O_j. After this let the vertex K_i $(i = 1, 2, \ldots, 8)$ of K be on the same edges of K with the vertices K_k, K_l, K_m and let us denote the opposite vertex to K_i of K by K_j. Now let us consider the points $T_{i,k}^n \in K_iK_k$, $T_{i,l}^n \in K_iK_l$, $T_{i,m}^n \in K_iK_m$, $T_{i,j}^n \in K_iK_j$ for which

$$K_iT_{i,k}^n = K_iT_{i,l}^n = K_iT_{i,m}^n = \frac{1}{6\cdot 2^{n-1}}$$

and

$$K_iT_{i,j}^n = \frac{\sqrt{3}}{6\cdot 2^{n-1}}$$

and let

$$\mathcal{T}_i^n = \text{conv}\{T_{i,k}^n, T_{i,l}^n, T_{i,m}^n, T_{i,j}^n, K_i\}\quad n = 1, 2, \ldots,\ i = 1, 2, \ldots, 8.$$

It is easy to see that $O_j^* \in O_j^1$, $j = 1, 2, \ldots, 6$ and $K_i^* \in \mathcal{T}_i^1$, $i = 1, 2, \ldots, 8$. Finally we prove that $O_j^* \in O_j^{n+1}$, $j = 1, 2, \ldots, 6$ and $K_i^* \mathcal{T} \in_i^{n+1}$, $i = 1, 2, \ldots, 8$ if $O_j^* \in O_j^n$, $j = 1, 2, \ldots, 6$ and $K_i^* \in \mathcal{T}_i^n$, $i = 1, 2, \ldots, 8$.

First in order to prove $O_j^* \in O_j^{n+1}$ $j = 1, 2, \ldots, 6$ it is enough to show that $O_1^* \in O_1^{n+1}$. F^n, H^n, I^n, J^n, L^n denote the midpoints of the segments $K_{1,6}^n$ $K_{1,7}^n$ $K_{1,2}^n K_{1,6}^n$, $K_{1,3}^n K_{1,7}^n$, $K_{1,4}^n K_{1,8}^n$, $K_{1,1}^n K_{1,5}^n$ and G^n is the midpoint of $I^n H^n$ and $\tilde{O}_1^n$ ($\tilde{O}_1^{n+1}$, respectively) is the center of the square $K_{1,6}^n K_{1,7}^n K_{1,8}^n$ K_{15}^n ($H^n I^n J^n L^n$, respectively). Now we prove that

$$O_1^* \notin (\operatorname{conv}\{K_{1,5}^n, K_{1,6}^n, K_{1,7}^n, K_{1,8}^n, L^n, H^n, I^n, J^n\}) \setminus \{\tilde{O}_1^{n+1}\}.$$

This is a corollary of the next:

$$O_1^* \notin (\operatorname{conv}\{K_{1,6}^n, F^n, \tilde{O}_1^n, H^n, G^n, \tilde{O}_1^{n+1}\}) \setminus \{\tilde{O}_1^{n+1}\} = Z.$$

If $O_1^* \in Z$, then on account of $O_2^* \in O_2^n$ we get $O_1^* O_2^* < \tilde{O}_1^{n+1} K_{2,4}^n = \dfrac{\sqrt{2}}{2}$, a contradiction. So using the symmetry to get the result $O_1^* \in O_1^{n+1}$ it is enough to show that

$$O_1^* \notin \operatorname{conv}\{K_{1,2}^n, K_{1,2}^{n+1}, K_{1,3}^n, K_{1,3}^{n+1}, H^n, K_{1,6}^{n+1}\ I^n, K_{1,7}^{n+1}\} = U.$$

If here it were $O_1^* \in U$, then owing to $K_6^* \in \mathcal{T}_6^n$ we should get

$$O_1^* K_6^* \leq K_{1,3}^{n+1} T_{6,7}^n = \sqrt{\frac{1}{2} - \frac{3 \cdot 2^{n+2} - 17}{9 \cdot 2^{2n+4}}} < \frac{\sqrt{2}}{2}$$

which is again a contradiction.

For the second time we prove that $K_6^* \in \mathcal{T}_6^{n+1}$, (Similarly we can get $K_i^* \in \mathcal{T}_{in}^{+1}$, $i = 1, 2, \ldots, 8$.) We know that $K_6^* \in \mathcal{T}_6^n$ so we have to prove only that

$$K_6^* \notin \operatorname{conv}\{T_{6,2}^n\ T_{6,2}^{n+1}, T_{6,4}^n, T_{6,4}^{n+1}, T_{6,5}^n, T_{6,5}^{n+1}\}.$$

In the opposite case owing to $O_1^* \in O_1^{n+1}$

$$O_1^* K_6^* \leq K_{1,8}^{n+1} T_{6,2}^{n+1} = \sqrt{\frac{1}{2} - \frac{2^{n+2} - 3}{3 \cdot 2^{2n+4}}} < \frac{\sqrt{2}}{2}$$

is again a contradiction.

Thus indeed $O_j^* = O_j$, $j = 1, 2, \ldots, 6$ and $K_i^* = K_i$, $i = 1, 2, \ldots, 8$.

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A PROOF OF THE LEFT-HAND SIDE OF THE BURKHOLDER-DAVIS-GUNDY INEQUALITY FOR A LARGE CLASS OF ORLICZ SPACES

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(Received August 21, 1985)

1. The aim of the present note is to prove the left-hand side of the famous Burkholder-Davis-Gundy inequality (cf. [1] Theorem I 5.1,) at least for a large class of Orlicz spaces furnished with the Luxemburg norm. This class is generated by Young functions of the form $\Phi(x^2)$, where $\Phi(x)$ is itself a Young function having finite power p. This is defined by the formula

$$p = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)},$$

where $\varphi(x)$ denotes the right-hand side derivative of $\Phi(x)$.

Concerning the theory of the Young functions, Orlicz spaces and Luxemburg norms we refer to [2] and [3].

The class of the Young functions $\Phi(x^2)$ contains all the power functions x^p, where $2<q<+\infty$ but does not contain the power functions x^p with $1\leq q\leq 2$. Thus, unfortunately, our result does not cover all the possible Orlicz spaces which are generated by Young functions with finite power. However the proof we present merits to be published because of its simplicity. In the proof below we shall follow an idea of Garsia (cf. [4] Theorem II. 1. 1.).

There are different proofs of the Burkholder-Davis-Gundy inequality. In view of its importance it seems to be interesting to give more and more simpler proofs of it even for a restricted class of Young functions with finite power.

The right-hand side of the Burkholder-Davis-Gundy inequality can be deduced e.g. from the generalized Fefferman-Garsia inequality as it has been shown partially by Garsia [4] and totally by the author of the present paper (cf. [5]).

2. Let $\Phi(x)$ be a Young function. We also consider the conjugate Young function $\Psi(x)$. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X\in L_1(\Omega, \mathcal{A}, P)$ be a random variable. We consider a sequence of σ-fields $\mathcal{F}_0\subset\mathcal{F}_1\subset\mathcal{F}_2\subset\dots$ of events such that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right) = \mathcal{A}$. Consider the martingale

$$X_n = E(X|\mathcal{F}_n), \quad n = 0, 1, 2, \dots$$

where we suppose that $X_0 = 0$ a.s. The differences of the maringale $(X_n, \mathcal{F}_n)$ will be denoted by $d_0 = 0, d_1, d_2, \ldots$

Let $S_n(X) = S_n = \left(\sum_{i=1}^{n} d_i^2\right)^{1/2}$, $n = 1, 2, \ldots$, be the quadratic functions corresponding to the martingale $(X_n, \mathcal{F}_n)$. We define $S_0(X) = S_0 = 0$.

We say that $X \in \mathcal{H}_\Phi$, the Hardy space generated by the Young function Φ, if $\|S\|_\Phi < +\infty$, where

$$S = \left(\sum_{i=1}^{\infty} d_i^2\right)^{1/2}.$$

In this case we write

$$\|X\|_{\mathcal{H}_\Phi} = \|S\|_\Phi .$$

Here $\|Z\|_\Phi$ denotes the Luxemburg norm of the random variable Z. It can be easily seen that $\|\cdot\|_{\mathcal{H}_\Phi}$ is a seminorm.

Since

$$X_n^* = \max_{1 \leq k \leq n} |X_k| \leq \sum_{i=1}^{n} |d_i| \leq \sqrt{n}\, S_n$$

and since

$$|d_i| \leq 2X_i^* \leq 2X_n^*, \; i = 1, 2, \ldots, n,$$

which implies that

$$S_n = \left(\sum_{i=1}^{n} d_i^2\right)^{1/2} \leq 2\sqrt{n}\, X_n^*,$$

we see that $\|S_n\|_\Phi$ and $\|X^*\|_\Phi$ are at the same time finite or infinite.

3. In what follows let $\Phi(x)$ be a Young function with finite power p. Then the Young function $\Phi'(x) = \Phi(x^2)$ is also of finite power. In fact, we have

$$q = \sup_{x>0} \frac{x\varphi'(x)}{\Phi'(x)} = \sup_{x>0} \frac{2x^2\varphi(x^2)}{\Phi(x^2)} = 2 \sup_{x^2>0} \frac{x^2\varphi(x^2)}{\Phi(x^2)} = 2 \sup_{y>0} \frac{y\varphi(y)}{\Phi(y)} = 2p,$$

because together with $x > 0$ the values of x^2 run over the same set.

We shall prove the following

THEOREM 1. *We have*

$$\|X_n\|_{\mathcal{H}_\Phi} \leq 2\sqrt{p}\, \|X_n^*\|_{\Phi'}.$$

PROOF. First we prove that for any random variable Z the relation

$$\|Z\|_{\Phi'} = \sqrt{\|Z^2\|_\Phi}$$

holds. In fact, the case when $P(Z=0)=1$, is trivial. When $0<\|Z\|_{\Phi'}<+\infty$, then

$$E\left(\Phi'\left(\frac{|Z|}{\|Z\|_{\Phi'}}\right)\right)=1,$$

since together with Φ the Young function Φ' has also finite power. Therefore,

$$E\left(\Phi\left(\frac{Z^2}{\|Z\|_{\Phi'}^2}\right)\right)=1$$

and so $\|Z^2\|_\Phi \leqq \|Z\|_{\Phi'}^2$. On the other hand

$$1=E\left(\Phi\left(\frac{Z^2}{\|Z^2\|_\Phi}\right)\right)=E\left(\Phi'\left(\frac{|Z|}{\sqrt{\|Z^2\|_\Phi}}\right)\right).$$

Consequently, $\|Z\|_{\Phi'} \leqq \sqrt{\|Z^2\|_\Phi}$. These two inequalities together give the equality

$$\|Z\|_{\Phi'}=\sqrt{\|Z^2\|_\Phi}.$$

We also shall use the following elementary inequality: if $x\geqq y\geqq 0$ then

$$\Phi(x)-\Phi(y)=\int_y^x \varphi(t)dt\leqq\varphi(x)(x-y).$$

To prove our inequality we can supppose that $\|S_n\|_{\Phi'}$ is finite and positive which implies that $0<\|S_n^2\|_\Phi<+\infty$, too. This assumption can be made on the basis of what we have shown in section 2 about the norms of $\|S_n\|_{\Phi'}$ and $\|X_n^*\|_{\Phi'}$. Then since p is finite, we have

$$E\left(\Phi\left(\frac{S_n^2}{\|S_n^2\|_\Phi}\right)\right)=1.$$

From this, using the above elementary inequality

$$\|S_n^2\|_\Phi=\|S_n^2\|_\Phi E\left(\Phi\left(\frac{S_n^2}{\|S_n^2\|_\Phi}\right)\right)=\|S_n^2\|_\Phi\sum_{k=1}^n E\left(\Phi\left(\frac{S_k^2}{\|S_n^2\|_\Phi}\right)-\Phi\left(\frac{S_{k-1}^2}{\|S_n^2\|_\Phi}\right)\right)\leqq$$

$$\leqq\|S_n^2\|_\Phi\sum_{k=1}^n E\left(\varphi\left(\frac{S_k^2}{\|S_n^2\|_\Phi}\right)\left[\frac{S_k^2}{\|S_n^2\|_\Phi}-\frac{S_{k-1}^2}{\|S_n^2\|_\Phi}\right]\right).$$

Introduce the notation

$$\Theta_k=\varphi\left(\frac{S_k^2}{\|S_n^2\|_\Phi}\right)-\varphi\left(\frac{S_{k-1}^2}{\|S_n^2\|_\Phi}\right),\quad k=1,2,\dots,n.$$

Then from the preceding inequality we get

$$\|S_n^2\|_\Phi \leq \sum_{k=1}^{n}\sum_{l=1}^{k} E(\Theta_l(S_k^2-S_{k-1}^2)) = \sum_{l=1}^{n}\sum_{k=l}^{n} E(\Theta_l(S_k^2-S_{k-1}^2)) =$$

$$= \sum_{l=1}^{n} E(\Theta_l(S_n^2-S_{l-1}^2)) = \sum_{l=1}^{n} E(\Theta_l E(S_n^2-S_{l-1}^2|\mathcal{F}_l)) =$$

$$= \sum_{l=1}^{n} E(\Theta_l E((X_n-X_{l-1})^2|\mathcal{F}_l)) \leq 4\sum_{l=1}^{n} E(\Theta_l X_n^{*2}) = 4E\left(\varphi\left(\frac{S_n^2}{\|S_n^2\|_\Phi}\right)X_n^{*2}\right).$$

Let now $b>0$ be a constant to be determined later and use the Young inequality to the last expectation. Then we obtain

$$\|S_n^2\|_\Phi \leq 4b\left[E\left(\Psi\left(\varphi\left(\frac{S_n^2}{\|S_n^2\|_\Phi}\right)\right)\right)+E\left(\Phi\left(\frac{X_n^{*2}}{b}\right)\right)\right].$$

Since for all $x>0$ the elementary inequality

$$\Psi(\varphi(x)) = x\varphi(x)-\Phi(x) \leq (p-1)\Phi(x)$$

holds, we see that

$$\|S_n^2\|_\Phi \leq 4b\left[(p-1)E\left(\Phi\left(\frac{S_n^2}{\|S_n^2\|_\Phi}\right)\right)+E\left(\Phi\left(\frac{X_n^{*2}}{b}\right)\right)\right].$$

Choosing now $b = \|X_n^{*2}\|_\Phi$ we finally obtain

$$\|S_n^2\|_\Phi \leq 4p\|X_n^{*2}\|_\Phi.$$

Taking the square root of both sides we arrive at the announced inequality

4. Making use of the power q of Φ' instead of the power p of Φ the inequality of Theorem 1 can be written in the following form:

$$\|X_n\|_{\mathscr{H}_{\Phi'}} \leq \sqrt{2q}\,\|X_n^*\|_{\Phi'}.$$

This follows from the fact that $q/2 = p$.

Let $\Phi(x) = \frac{x^p}{p}$, where $1<p<+\infty$ is a power. Then $\Phi'(x) = \frac{x^q}{p}$, where $q = 2p$. In this case

$$\|X_n\|_{\mathscr{H}_{\Phi'}} = \frac{\|S_n\|_q}{(p)^{1/q}} = \frac{\|X_n\|_{\mathscr{H}_q}}{(p)^{1/q}}$$

and

$$\|X_n^*\|_{\Phi'} = \frac{\|X_n^*\|_q}{(p)^{1/q}},$$

so that

$$\|X_n\|_{\mathscr{H}_q} \leq \sqrt{2q}\,\|X_n^*\|_q,$$

which is the inequality II. 1. 3. of GARSIA [4]. So our inequality in Theorem 1 is a direct generalization of Garsia's inequality.

In [5] we proved, by using the generalized Fefferman-Garsia inequality, the right hand side of the Burkholder-Davis-Gundy inequality: if the Young function Φ has finite power p then we have

$$\|X_n^*\|_\Phi \leqq c_\Phi \|X_n\|_{\mathcal{H}_\Phi}.$$

Using this with $\Phi'(x) = \Phi(x^2)$ and comparing to the result of Theorem 1 we obtain the following

THEOREM 2. *We have*

$$\frac{1}{\sqrt{2q}} \|X_n\|_{\mathcal{H}_{\Phi'}} \leqq \|X_n^*\|_{\Phi'} \leqq C_{\Phi'} \|X_n\|_{\mathcal{H}_{\Phi'}},$$

where $C_{\Phi'}$ is a constant depending only on Φ', provided that the conditions of Theorem 1 are satisfied and Φ has finite power.

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MAXIMAL INEQUALITIES FOR NONNEGATIVE SUPERMARTINGALES

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(Received August 21, 1985)

1. Let $(X_n, \mathcal{F}_n)$, $n \geq 0$, be a nonnegative supermartingale with the Doob decomposition

$$X_n = M_n - A_n,\ n = 0, 1, 2, \ldots$$

where, as it is known, $(M_n, \mathcal{F}_n)$ is a nonnegative martingale and $0 = A_0 \leq A_1 \leq \leq A_2 \leq \ldots$ is a predictable increasing process (canonical process). It is proved that this decomposition is unique.

Using this decomposition we see that

$$\max_{0 \leq k \leq n} X_k = X_n^* \leq M_n^* = \max_{0 \leq k \leq n} M_k.$$

Let (Φ, Ψ) be a pair of conjugate Young functions. We shall consider the Orlicz space L^Φ furnished with the Luxemburg norm, which is a normed vector space. Concerning the notion of the Young functions, the Orlicz spaces generated by them and the Luxemburg norms we refer to [1] and [3].

We introduce the generalized power of the Young function Ψ, defined by the formula

$$\limsup_{x \to +\infty} \frac{x\psi(x)}{\Psi(x)}.$$

Here $\psi(x)$ is the right-hand side derivative of $\Psi(x)$.

The power q of the Young function Ψ is defined by the formula

$$q = \sup_{x > 0} \frac{x\psi(x)}{\Psi(x)}.$$

In their paper [6] J. MOGYORÓDI and T. F. MÓRI have proved the following assertion which we state as

LEMMA 1. Let (Φ, Ψ) be a pair of conjugate Young functions. In order that for Φ the maximal inequality

$$E\big(\Phi(X_n^*)\big) \leq a + E\big(\Phi(bX_n)\big)$$

be valid for arbitrary nonnegative submartingale $(X_n, \mathcal{F}_n)$, $n\geq 0$, with the maximum random variable $X_n^* = \max_{0\leq k\leq n} X_k$, where the constants a and b depend only on Φ, it is necessary and sufficient that the generalized power of the conjugate Young function be finite.

The necessity part is proved by helps of specially constructed nonnegative martingale. This shows that if the above maximal inequality holds then Φ must necessarily be of finite generalized power.

The purpose of the present note is to use this assertion to obtain maximal inequalities for nonnegative supermartingales.

In section **3** of the present paper we also give maximal inequalities for nonnegative supermartingales in case of concave Young functions.

2. We prove the following

THEOREM 1. *Let (Φ, Ψ) be a pair of conjugate Young functions. In order that for every nonnegative supermartingale $(X_n, \mathcal{F}_n)$ having the Doob decomposition $X_n = M_n - A_n$ the maximal inequality*

$$E(\Phi(X_n^*)) \leq a + E(\Phi(bM_n)), \quad n = 0, 1, 2, \ldots$$

hold with some constans $a\geq 0$ and $b>0$ depending only on Φ it is necessary and sufficient that the generalized power of Ψ be finite.

Moreover, it the power q of Ψ is finite and for every $k\geq 0$ we have

$$X_k \leq E(X|\mathcal{F}_k) \quad a.\,s.$$

with $X\in L^\Phi$, then

$$E(\Phi(X_n^*)) \leq E(\Phi(qX)).$$

PROOF. Since Φ increases we trivially have

$$E(\Phi(X_n^*)) \leq E(\Phi(M_n^*)).$$

Consequently, if the generalized power of Ψ is finite then with some constans $a\geq 0$ and $b>0$ depending only on Φ by Lemma 1 we have

$$E(\Phi(X_n^*)) \leq a + E(\Phi(bM_n)).$$

That the generalized power of Ψ must necessarily be finite follows at once if we consider nonnegative martingales (as special nonnegative supermartingales) and we take into account the necessity part of Lemma 1.

If q, the power of Ψ is finite then by Lemma 1 of [7] and by Jensen's inequality

$$E(\Phi(X_n^*)) \leq E(\Phi(\max_{0\leq k\leq n} E(X|\mathcal{G}_k))) \leq E(\Phi(qE(X|\mathcal{F}_n))) \leq E(\Phi(qX)).$$

This proves the assertion.

REMARKS. The inequalitites of this assertion are meant in the sense that both sides are at the same time infinite or finite and in this case the asserted inequalitites hold. This can be shown in a standard manner.

Let Φ be a Young function. For nonnegative submartingales $(X_n, \mathcal{F}_n)$ one can deduce the maximal inequality

$$E(\Phi(X_n^*)) \le a + E(\Phi(bX_n)),\ n = 1, 2, \dots$$

where $a \ge 0$ and $b > 0$ are constans depending only on Φ if the conditions of Lemma 1 are satisfied. If $(X_n, \mathcal{F}_n)$, $n \ge 1$, is a nonnegative supermartingale then we cannot expect the validity of the maximal inequality in this form even if Φ has finite (generalized) power. To show this let us give the following counterexample. We choose the sequence $0 < x_1 < x_2 < \dots$ according to the following rule: $\Phi(x_1) = 1$ and

$$\frac{\Phi(x_k)}{\Phi(x_{k+1})} = 1 - \frac{1}{k+1},\ k = 1, 2, \dots .$$

Let Ω be the set of the positive integers, $\mathcal{A}$ the σ-field of all subsets of Ω and define

$$P(\{\omega\}) = \frac{1}{\Phi(x_\omega)} - \frac{1}{\Phi(x_{\omega+1})},\quad \omega \in \Omega.$$

Then $(\Omega, \mathcal{A}, P)$ is a probability space. Let $\mathcal{F}_1 = (\varnothing, \Omega)$ be the trivial σ-field and for $n \ge 2$ define $\mathcal{F}_n$ to be the σ-field generated by the partition

$$\{1\}, \dots, \{n-1\}, \{n, n+1, \dots\}.$$

Then $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$. Consider the sequence

$$X_n = x_n\, \chi(\omega \ge n),\ n \ge 1,$$

of random variables. Then $(X_n, \mathcal{F}_n)$, $n \ge 1$, is a nonnegative supermartingale. In fact,

$$E(X_{n+1}|\mathcal{F}_n) = \chi(\omega \ge n) x_{n+1} \Phi(x_n) \frac{1}{\Phi(x_{n+1})} =$$

$$= x_n \chi(\omega \ge n) \frac{\Phi(x_n)}{x_n} \Bigg/ \frac{\Phi(x_{n+1})}{x_{n+1}} \le X_n \text{ a.s.},$$

since $\Phi(x)/x$ increases. Here χ_A is the indicator of the event A.

We have

$$X_n^*(\omega) = \begin{cases} x_\omega, & \text{if } \omega < n, \\ x_n, & \text{if } \omega \ge n, \end{cases}$$

and so

$$E(\Phi(X_n^*)) = \sum_{k=1}^{n-1} \Phi(x_k)\left(\frac{1}{\Phi(x_k)} - \frac{1}{\Phi(x_{k+1})}\right) + \frac{\Phi(x_n)}{\Phi(x_n)} =$$

$$= \sum_{k=1}^{n-1} \frac{1}{k+1} + 1 = \sum_{k=1}^{n} \frac{1}{k}.$$

This tends to $+\infty$ as $k\to+\infty$. At the same time for arbitrary $b>0$

$$E(\Phi(bX_n))=\frac{\Phi(bx_n)}{\Phi(x_n)}.$$

Now since Φ has finite power, say p, we have

$$E(\Phi(bX_n))\leq\begin{cases}1, & \text{if } b\leq 1.\\ b^p, & \text{if } b>1,\end{cases}$$

Consequently, there are no constans $a\geq 0$ and $b>0$ such that

$$E(\Phi(X_n^*))\leq a+E(\Phi(bX_n))$$

could hold for each n.

Let $X_n=E(A_\infty-A_n|\mathcal{G}_n)$, where $\{A_n\}$ is a canonical process such that $A_\infty=\lim\limits_{n\to+\infty}A_n\in L_1$. Then $(X_n,\mathcal{G}_n)$ is a potential and by the result of our theorem we have

$$E(\Phi(X_n^*))\leq E(\Phi(qA_\infty)),$$

since with $X=A_\infty$ the inequality

$$X_k\leq E(A_\infty|\mathcal{G}_k)$$

holds for every $k\geq 0$.

3. It seems to be interesting to prove maximal inequalities for concave Young functions, too. These are defined in the following way: let $\varphi(t)$ be a right continuous and non-increasing function that is integrable over any finite interval $(0,x)$. Then the function

$$\Phi(x)=\int_0^x\varphi(t)dt$$

is continuous, increasing and concave. We also suppose that

$$\lim_{x\to+\infty}\Phi(x)=+\infty.$$

The example of the nonnegative martingales (as special nonnegative supermartingales) shows that when the concave function is linear, or, "near" to the linear one, then by Doob's classical $L\log L$ inequality and Gundy's reverse inequality (cf. [4] and [1]) we cannot prove an inequality of the form

$$E(\Phi(X_n^*))\leq a+E(\Phi(bX_n)).$$

Consequently, we exclude from our considerations the concave Young functions which are "near" to the linear one. The exact meaning of this will be specified in the following assertion.

Before formulating this we state a known result about the nonnegative supermartingales in the form of the following

LEMMA 2. If $(X_n, \mathcal{F}_n)$, $n \geq 0$, is a nonnegative supermartingale, then the random variable

$$X^* = \sup_{n \geq 0} X_n$$

for every positive $\lambda > 0$ satisfies the inequality (cf. [1])

$$\lambda P(X^* \geq \lambda) \leq E(\min(X_0, \lambda)).$$

Now we are in the position to may prove the following

THEOREM 2. *If Φ is a concave Young function such that*

$$r = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)} < 1$$

and $(X_n, \mathcal{G}_n)$, $n \geq 0$, is a nonegative supermartingale, then

$$E(\Phi(X_n^*)) \leq \frac{1}{1-r} E(\Phi(X_0)).$$

PROOF. We use the maximal inequality

$$\lambda P(X_n^* \geq \lambda) \leq E(\min(X_0, \lambda))$$

stated in the preceding lemma. Let us integrate this on $(0, +\infty)$ with respect to the measure $(-d\varphi(\lambda))$ and remark that

$$\int_0^z \lambda(-d\varphi(\lambda)) = [-\lambda\varphi(\lambda)]_0^z + \int_0^z \varphi(\lambda) d\lambda = \Phi(z) - z\varphi(z),$$

since in the concave case

$$0 \leq \lambda\varphi(\lambda) \leq \Phi(\lambda)$$

and, consequently, $\lim_{\lambda \to 0} \lambda\varphi(\lambda) = 0$.

Therefore, by the Fubini theorem

$$\int_0^{+\infty} \lambda P(X_n^* \geq \lambda)(-d\varphi(\lambda)) = E\left(\int_0^{X_n^*} \lambda(-d\varphi(\lambda))\right) = E(\Phi(X_n^*)) - E(X_n^*\varphi(X_n^*)).$$

By our assumption

$$X_n^*\varphi(X_n^*) \leq r\Phi(X_n^*).$$

Therefore,

$$\int_0^{+\infty} \lambda P(X_n^* \geq \lambda)(-d\varphi(\lambda)) \geq (1-r)E(\Phi(X_n^*)).$$

On the other hand

$$\int_0^{+\infty} E(\min(X_0, \lambda))(-d\varphi(\lambda)) = E\left(\int_0^{X_0} \lambda(-d\varphi(\lambda)) + \int_{X_0}^{+\infty} X_0(-d\varphi(\lambda))\right) =$$
$$= E(\Phi(X_0)) - E(X_0\varphi(X_0)) + E(X_0\varphi(X_0)) - E(X_0\varphi(+\infty)) \leq E(\Phi(X_0)).$$

The comparison of these estimates gives

$$(1-r)E(\Phi(X_n^*)) \leq E(\Phi X_0)),$$

which proves the assertion.

Introduce the notation

$$\xi(x) = \Phi(x) - x\varphi(x).$$

A by-product of the preceding theorem says without any additional condition on Φ that for every nonnegative supermartingale we have

$$E(\xi(X_n^*)) \leq E(\Phi(X_0)).$$

We also see that

$$E(\xi(X_n^*) \leq E(\Phi(X_n^*)),$$

since

$$\xi(x) \leq \Phi(x).$$

The inequality of the preceding assertion can be obtained under less restrictive conditions, as well. Namely, we have

THEOREM 3. *In order that for some constants $a \geq 0$ and $0 < b < 1$ the inequality*

$$(1-b)E(\Phi(X_n^*)) \leq a + E(\xi(X_n^*))$$

hold, it is necessary and sufficient that the condition

$$r = \limsup_{x \to +\infty} \frac{x\varphi(x)}{\Phi(x)} < 1$$

be satisfied. In this case with some constant $K_\Phi > 0$ we have

$$E(\Phi(X_n^*)) \leq K_\Phi(1 + E(\Phi(X_0))).$$

PROOF. If $r < 1$ then with arbitrary b such that $r < b < 1$ and with appropriately chosen constant $a \geq 0$ we have for $x > 0$

$$x\varphi(x) \leq a + b\Phi(x).$$

Consequently,

$$E(\xi(X_n^*)) = E(\Phi(X_n^*)) - E(X_n^*\varphi(X_n^*)) \geq E(\Phi(X_n^*)) - bE(\Phi(X_n^*)) - a.$$

This shows the sufficiency of our condition. To prove the necessity suppose that the inequality

$$(1-b)E\big(\Phi(X_n^*)\big)\leq a+E\big(\xi(X_n^*)\big)$$

holds for any nonnegative supermartingale $(X_n, \mathcal{J}_n)$, $n\geq 0$, with some contants $0<b<1$ and $a\geq 0$. Apply this inequality to the special supermartingale $X_n\equiv x$, $n\geq 0$, where $x>0$ is arbitrary. Then

$$(1-b)\ \Phi(x)\leq a+\Phi(x)-x\varphi(x).$$

This means that

$$x\varphi(x)\leq a+b\Phi(x),$$

or, in other words,

$$\frac{x\varphi(x)}{\Phi(x)}\leq\frac{a}{\Phi(x)}+b.$$

Letting $x\to+\infty$ we get that

$$r=\limsup_{x\to+\infty}\frac{x\varphi(x)}{\Phi(x)}\leq b<1.$$

Finally, suppose that $r<1$ and take b such that $r<b<1$ be satisfied. Then comparing the inequality

$$(1-b)E\big(\Phi(X_n^*)\big)\leq a+E\big(\xi(X_n^*)\big)$$

to

$$E\big(\xi(X_n^*)\big)\leq E\big(\Phi(X_0)\big)$$

and taking

$$K_\Phi=\max\left(\frac{a}{1-b},\ \frac{1}{1-b}\right)$$

we obtain the second part of the assertion.

Consider the concave Young function $\Phi(x)=\log(x+1)$. Then

$$\frac{x\varphi(x)}{\Phi(x)}=\frac{x}{(x+1)\log(x+1)}.$$

By L'Hospital's rule

$$\lim_{x\to 0}\frac{x\varphi(x)}{\Phi(x)}=\lim_{x\to 0}\frac{1}{1+\log(x+1)}=1.$$

Consequently,

$$\sup_{x\to 0}\frac{x\varphi(x)}{\Phi(x)}=1.$$

At the same time

$$\lim_{x\to+\infty}\frac{x\varphi(x)}{\Phi(x)}=\lim_{x\to+\infty}\frac{1}{1+\log(x+1)}=0.$$

Thus our function $\Phi(x)=\log(x+1)$ does not satisfy the condition of Theorem 2 whilst the conditions of Theorem 3 are satisfied.

For the concave Young functions which are near to the linear one we can state only the result which we obtained in proving Theorem 2. We summarize it as

THEOREM 4. *Let $(X_n, \mathcal{J}_n), n\ge 0$, be a nonnegative supermartingale and let $\Phi(x)$ be a concave Young function. If $\xi(x)$ denotes the function $\Phi(x)-x\varphi(x)$ then*

$$E(\xi(X_n^*))\le E(\Phi(X_0)).$$

4. Theorem 1 from section 2 shows that in the case of (convex) Young functions $\Phi(x)$ which are "far" from the linear function (since $\Phi(x)/x\uparrow+\infty$ as $x\to+\infty$) any nonnegative supermartingale satisfies the maximal inequality of Theorem 1 provided the other conditions are fulfilled. The question arises whether we can give an inequality estimating $E(X_n^*)$ from above, when $(X_n, \mathcal{J}_n)$, $n\ge 0$, is a nonnegative supermartingale. This question is similar to Doob's classical problem: in case of nonnegative submartingales he proved that

$$E(X_n^*)\le\frac{e}{e-1}\left(1+E(X_n\log^+X_n)\right)$$

provided that $E(X_n\log^+X_n)<+\infty$. This inequality was generalized in [5] by J. MOGYORÓDI. Namely, he has shown that for arbitrary Young function Φ the maximal inequality

$$E(X_n^*)\le E(\Phi(X_n))+(1-t)^{-1}\ I$$

holds for any nonnegative submartingale provided that $0<t<1$ is such a parameter for which the Laplace transform

$$I=\int_0^{+\infty}e^{-t\lambda}\,d\Psi(\lambda)$$

converges. Here $\Psi(x)$ is the Young function which is conjugate to $\Phi(x)$.

For nonnegative supermartingales we state a similar maximal inequality in the next theorem. Before this, we formulate the following assertion from [5] which we state as

LEMMA 2. Let $Z_1, Z_2, \ldots$ be nonnegative random variables such that

$$\sum_{i=1}^{\infty}Z_i\le K<+\infty$$

and let $\Psi(x)$ be an arbitrary Young function. Let further $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ be an increasing sequence of σ-fields of events. If for some $t \in (0, K^{-1})$ the Laplace transform

$$I = \int_0^{+\infty} e^{-t\lambda}\, d\Psi(\lambda)$$

converges then $E(\Psi(A_\infty)) \leq (1-tK)^{-1} I$, where

$$A_\infty = \sum_{i=1}^{\infty} E(Z_i | \mathcal{G}_i).$$

Now we are able to formulate our

THEOREM 5. *Let $(X_n, \mathcal{F}_n)$, $n \geq 0$, be a nonnegative supermartingale with the Doob decomposition $X_n = M_n - A_n$. Let (Φ, Ψ) be a pair of conjugate Young functions. If for some $t \in (0, 1)$ the Laplace transform*

$$I = \int_0^{+\infty} e^{-t\lambda}\, d\Psi(\lambda)$$

converges then

$$E(X_n^*) \leq E(\Phi(M_n)) + (1-t)^{-1}\, I.$$

Especially, if $X_n = E(A_\infty - A_n | \mathcal{G}_n)$ is a potential, where $0 = A_0 \leq A_1 \leq A_2 \leq \ldots$ is a canonical increasing process such that $A_\infty \in L^\Phi$ then

$$E(X_n^*) \leq E(\Phi(A_\infty)) + (1-t)^{-1}\, I.$$

PROOF. Consider the events

$$A_1 = \{X_1 = X_n^*\},\ A_k = \{X_{k-1}^* < X_n^*,\ X_k = X_n^*\},\ k = 2, \ldots, n.$$

Then the random variables

$$Z_k = \chi_{A_k},\ k = 1, 2, \ldots, n$$

are such that

$$\sum_{k=1}^{n} Z_k = 1.$$

Here χ_{A_k} denotes the indicator of the event A_k, $k = 1, \ldots, n$. So the result of the preceding lemma can be applied to the Young function Ψ. Now by the fact that $X_k \leq M_k$ and that $(M_n, \mathcal{G}_n)$ is a nonnegative martingale we have

$$E(X_n^*) = \sum_{k=1}^{n} E(X_k \chi_{A_k}) \leq \sum_{k=1}^{n} E(M_k \chi_{A_k}) = \sum_{k=1}^{n} E(\chi_{A_k} E(M_n | \mathcal{G}_k)).$$

The conditional expectation being self-adjoint we get

$$E(X_n^*) \leq \sum_{k=1}^{n} E\big(M_n E(\chi_{A_k} | \mathcal{J}_k)\big) = E\left(M_n \sum_{k=1}^{n} E(\chi_{A_k} | \mathcal{J}_k)\right).$$

Apply now to the right-hand side the Young inequality

$$xy \leq \Phi(x) + \Psi(y);\ x \geq 0,\ y \geq 0$$

to get

$$E(X_n^*) \leq E\big(\Phi(M_n)\big) + E\left(\Psi\left(\sum_{k=1}^{n} E(\chi_{A_k} | \mathcal{J})\right)\right).$$

By the result of the preceding lemma applied to the second term on the right hand side we obtain

$$E(X_n^*) \leq E\big(\Phi(M_n)\big) + (1-t)^{-1}\ I.$$

This proves the first inequality. For the proof of the second one let us note that the Doob decomposition of the potential $X_n = E(A_\infty - A_n | \mathcal{J}_n)$ is $E(A_\infty | \mathcal{J}_n) - A_n$ and so $X_n \leq E(A_\infty | \mathcal{J}_n)$.

To get the second inequality use the Jensen inequality for

$$E\big(\Phi(E(A_\infty | \mathcal{J}_n))\big).$$

This proves the assertion.

Remark that our inequality is interesting in the case when Ψ has no finite power (because in this case Theorem 1 applies) or, when $\Phi(x)$ does not increase more quickly than x^p, where $p > 1$ is arbitrary. Such a function is for example

$$\Phi(x) = 4[(x+1)\log(x+1) - x].$$

In this case the conjugate Young function is equal to

$$\Psi(x) = 4(e^{\frac{x}{4}} - 1) - x.$$

Now if $t = \dfrac{1}{2}$ then

$$I = \int_0^{+\infty} e^{-\frac{\lambda}{2}} d\Psi(\lambda) = \int_0^{+\infty} e^{-\frac{\lambda}{2}} (e^{\frac{\lambda}{4}} - 1) d\lambda = 2.$$

Consequently, if $X_n = E(A_\infty - A_n | \mathcal{J}_n)$ is a potential then by the preceding inequality

$$E(X_n^*) \leq 4\big((A_\infty + 1)\log(A_\infty + 1) + 1\big),$$

provided that $A_\infty \in L \log L$.

This example shows that on the basis of the preceding theorem one can deduce for potentials an inequality which is similar to that of Doob for nonnegative submartingales belonging to $L \log L$.

It is interesting that for a special class of nonnegative supermartingales and for a large class of Young functions the preceding theorem can be reversed. This class is the family of the predictable nonnegative supermartingales.

DEFINITION 1. Let $(X_n, \mathcal{G}_n)$ be a nonnegative supermartingale, $n \geq 0$, where $\mathcal{G}_0 = (\emptyset, \Omega)$ is the trivial σ-field and suppose that there exists an adapted sequence $\lambda_0 \leq \lambda_1 \leq \dots$ of random variables such that for all $n \geq 1$ we have

$$X_n \leq \lambda_{n-1}$$

and $\lambda_\infty = \lim_{n \to +\infty} \lambda_n \in L_1$. Then $(X_n, \mathcal{G}_n)$ is said to be predictable in L_1.

Note that in this definition X_0 is constant a.s.

THEOREM 6. *If the nonnegative supermartingale $(X_n, \mathcal{G}_n)$, $n \geq 0$, is predictable in L_1 and*

$$\xi(x) = x\varphi(x) - \Phi(x) = O(x)$$

as $x \to +\infty$, then

$$E(\Phi(X_n)) \leq \lambda_0 \varphi(\lambda_0) + \xi(x_0) + KE(\lambda_n),$$

where $x_0 > 0$ is such a number for which $\xi(x) \leq Kx$ with some $K > 0$ if $x \geq x_0$.

PROOF. GARSIA (cf. [2], Theorem III. 3.4.) has shown that for nonnegative supermartingales which are predictable in L_1 the inequality

$$E(X_n\, \chi(\lambda_n \geq x)) \leq X_0 \chi(\lambda_0 \geq x) + xE(\chi(\lambda_n \geq x))$$

is satisfied. Here $x > 0$ is an arbitrary constant, χ_A denotes the indicator of the event A and

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

is a predicting sequence of $(X_n, \mathcal{G}_n)$ in L_1. Integrate this inequality on $(0, x)$ with respect to the measure generated by the nondecreasing and right continuous function $\varphi(x)$. Using the Fubini theorem we get

$$E(X_n \varphi(\lambda_n)) \leq X_0 \varphi(\lambda_0) + E(\xi(\lambda_n)),$$

since

$$\int_0^z x d\varphi(x) = \xi(z).$$

Note that $\varphi(x)$ increases and $\lambda_n \geq X_n$. Consequently,

$$E(X_n\varphi(\lambda_n)) \geq E(X_n\varphi(X_n)) \geq E(\Phi(X_n)).$$

From this

$$E(\Phi(X_n)) \leq \lambda_0\varphi(\lambda_0) + E(\xi(\lambda_n)).$$

λ_0 can be choosen such that $\lambda_0 \geq X_0$ hold.

Now if $\xi(x) = O(x)$ we can choose $x_0 \geq 0$ such that the inequality

$$\xi(x) \leq Kx$$

hold for $x \geq x_0$. Here $K > 0$ is a finite positive constant. Consequently,

$$E(\xi(\lambda_n)) \leq \xi(x_0) + KE(\lambda_n)$$

and so

$$E(\Phi(X_n)) \leq \lambda_0\varphi(\lambda_0) + \xi(x_0) + KE(\lambda_n).$$

This proves the assertion.

As an example consider the nonnegative supermartingale $(X_n, \mathcal{J}_n)$, for which $\mathcal{J}_0 = (\Phi, \Omega)$, $X_0 \leq c$ and for all $n \geq 0$ we have

$$X_{n+1} \leq cX_n \quad \text{a.s.}$$

Then for $n \geq 1$ we have $X_n \leq cX^*_{n-1} = \lambda_{n-1}$ a. s. If we suppose that $X^* \in L_1$ (e.g. $0 < c < 1$), then from the inequality of the preceding theorem we get

$$E(\Phi(X_n)) \leq c\varphi(c) + \xi(x_0) + KcE(X^*_n),$$

an inequality which is the reverse to that of Theorem 5.

Examples for nonnegative supermartingales with these properties can be found e.g. in [1].

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ON THE REPRESENTATION OF L^Φ-MEAN OSCILLATING RANDOM VARIABLES

By

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(Received September 17, 1985)

1. The aim of the present note is to generalize a representation theorem of GARSIA ([1] Theorem I.5.1. and Theorem I.5.2.) for the random variables belonging to the space $\mathcal{K}'_q$, where $2 \leqq q \leqq +\infty$. This space is defined as follows. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X \in L_1(\Omega, \mathcal{A}, P)$ be a random variable. We say that X belongs to $\mathcal{K}'_q$ if the set of the random variables γ defined by the formula

$$\Gamma_X'^{(q)} = \{\gamma : \gamma \in L_q,\ E(|X - X_{n-1}|^2 | \mathcal{F}_n) \leqq E(\gamma^2 | \mathcal{F}_n) \text{ a.s. } \forall n \geqq 1\}$$

is not empty. Here $\{\mathcal{F}_n\}$, $n \geqq 0$, is an increasing sequece of σ-fields of events such that $\mathcal{F}_\infty = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n) = \mathcal{A}$. Also, $X_n = E(X | \mathcal{F}_n)$, $n \geqq 0$, is the corresponding martingale. For the sake of commodity we suppose that $X_0 = 0$ a. s. If $\Gamma_X'^{(q)}$ is not empty then we let

$$\|X\|_{\mathcal{K}'_q} = \inf_{\gamma \in \Gamma_X'^{(q)}} \|\gamma\|_q.$$

It is easily seen that $\|\cdot\|_{\mathcal{K}_p'}$ is a seminorm on the family $\mathcal{K}'_q$. As it is well-known this space is the dual of the Hardy space $\mathcal{H}_p$, where $p^{-1} + q^{-1} = 1$. (In case $q = +\infty$ we put $p = 1$ and $\mathcal{K}'_\infty$ is the well-known BMO$_2$-space.) Concerning these notions and definitions we refer to the book by GARSIA [1].

Our definition of the $\mathcal{K}_q$-space, where $1 \leqq q \leqq +\infty$, is somewhat different from that of GARSIA (see e.g. [2]). We say that $X \in \mathcal{K}_q$, if the set

$$\Gamma_X^{(q)} = \{\gamma : \gamma \in L_q,\ E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leqq E(\gamma | \mathcal{F}_n) \text{ a.s. } \forall n \geqq 1\}$$

is not empty. In this case we let

$$\|X\|_{\mathcal{K}_q} = \inf_{\gamma \in \Gamma_X^{(q)}} \|\gamma\|_q.$$

It is easily seen that $\|\cdot\|_{\mathcal{K}_q}$ is a seminorm on the family $\mathcal{K}_q$. The space $\mathcal{K}_\infty$ is called also the BMO$_1$-space.

One can show that for $2 \leqq q \leqq +\infty$ the semi-norms $\|\cdot\|_{\mathcal{K}_q'}$ and $\|\cdot\|\mathcal{K}_q$ are equivalent (see e.g. [3]).

In this paper we characterize the random variables belonging to $\mathcal{K}_q$, where $1 < q \geqq +\infty$. In this manner we generalize the result of Garsia, by extending his result for the values of $1 < q < 2$. Our representation will also be applied to more general spaces than $\mathcal{K}_q$. For the definition of these we refer to section 2 of the present paper.

We shall be concerned with a Young function Φ and its conjugate (complementary) Young function Ψ. Concerning the theory of the Young functions we refer to [4] and [5]. The Young function is called of moderated growth and the quantity

$$p = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)}$$

is called its power, if this is finite. Here $\varphi(x)$ stands for the right-hand side derivative of Φ. We define similarly the power q of the conjugate Young function Ψ. When the power of a Young function is not finite then, of course, we say that it is not of moderated growth.

Every Young function Φ generates a space which we call Orlicz space. We say that the random variable X belongs to $L^{\Phi}(\Omega. \mathcal{A}, P)$ if there exists an $a > 0$ such that

$$E(\Phi(a^{-1}|X|)) \leq 1.$$

In this case we let

$$\|X\|_{\Phi} = \inf(a > 0 : E(\Phi(a^{-1}|X|)) \leqq 1).$$

It is easy to see that $\|\cdot\|_{\Phi}$ is a semi-norm on L^{Φ}. This is called the Luxemburg norm.([4])

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $X, Y \in L_1(\Omega, \mathcal{A}, P)$ be random variables and consider the sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ of σ-fields of events. For the sake of commodity we suppose that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right) = \mathcal{A}$. Then the martingales

$$X_n = E(X|\mathcal{F}_n),\ Y_n = E(Y|\mathcal{F}_n),\ n \geqq 0,$$

are such that their a.s. and L_1 limit is equal to X and Y, resp. We also suppose that $X_0 = Y_0 = 0$ a.s. The differences of the martingales $(X_n, \mathcal{F}_n)$, resp. $(Y_n, \mathcal{F}_n)$ will be denoted by $d_0 = 0, d_1, d_2, \ldots$ and $d_0' = 0, d_1', d_2', \ldots$, respectively.

Let (Φ, Ψ) be a pair of conjugate (complementary) Young functions. Throughout the present note we shall suppose that Ψ has finite power q.

In what follows we need some definitions.

DEFINITION 1. We say that X belongs to $\mathcal{K}_\Phi$ if the set $\Gamma_X^{(\Phi)}$ defined by the formula

$$\Gamma_X^{(\Phi)} = \{\gamma : \gamma \in L^\Phi,\ E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leq E(\gamma | \mathcal{F}_n) \text{ a.s. } \forall n \geq 1\}$$

is not empty. In this case we set

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Gamma_X^{(\Phi)}} \|\gamma\|_\Phi.$$

It is easy to verify that $\|\cdot\|_{\mathcal{K}_\Phi}$ defines a norm on the set $\mathcal{K}_\Phi$.

This definition covers the definition of the $\mathcal{K}_q$-spaces with $\Phi(x) = \dfrac{X^q}{q}$, $1 < q < +\infty$. Thus, exceptly the space $\mathcal{K}_1$, and $\mathcal{K}_\infty$ all the other $\mathcal{K}_q$-spaces are covered by this definition.

Note that if $X \in \mathcal{K}_\Phi$ then so is X_k for all $k = 1, 2, \ldots$, and we have $\|X_k\|_{\mathcal{K}_\Phi} \leq \|X\|_{\mathcal{K}_\Phi}$. In fact, it is easy to verify that the conditional expectations

$$E(|X_k - X_{n-1}| \,|\, \mathcal{F}_n),\ k = n,\ n+1,\ n+2, \ldots$$

are increasing in k a.s. Furthermore, $|X_k - X_{n-1}|$ converges in L_1 and a.s. to $|X - X_{n-1}|$ as $k \to +\infty$. Consequently, with arbitrary $\gamma \in \Gamma_X^{(\Phi)}$

$$E(|X_k - X_{n-1}| \,|\, \mathcal{F}_n) \leq E(|X - X_{n-1}| \,|\, \mathcal{F}_n) \leq E(\gamma | \mathcal{F}_n) \text{ a.s.}$$

Therefore, $X_k \in \mathcal{K}$ and

$$\|X_k\|_{\mathcal{K}_\Phi} \leq \|X\|_{\mathcal{K}_\Phi}.$$

DEFINITION 2. The random variable Y is said to belong to the Hardy-space $\mathcal{H}_\Psi$ if the random variable

$$S = S(Y) = \left(\sum_{i=1}^{\infty} d_i'^2\right)^{1/2}$$

belongs to L^Ψ. In this case we set

$$\|Y\|_{\mathcal{H}_\Psi} = \|S\|_\Psi.$$

It is easy to see that $\|\cdot\|\mathcal{H}_\Psi$ is a seminorm.

This definition covers the notion of the $\mathcal{H}_p$-spaces with $1 < p < +\infty$. Namely, we have to consider the Young function $\Psi(x) = \dfrac{x^p}{p}$. To define the Hardy space $\mathcal{H}_1$, we say that $Y \in \mathcal{H}_1$ if the random variable S belongs to L_1. In this case we define $\|X\|_{\mathcal{H}_1} = \|S\|_1$.

Note that together with Y the terms of the martingale $(Y_n, \mathcal{F}_n)$ also belong to $\mathcal{H}_\Psi$, $(\mathcal{H}_1)$. Indeed, we trivially have

$$S_n = S_n(Y) = \left(\sum_{i=1}^{n} d_i'^2\right)^{1/2} \leq S.$$

In their paper [6] S. ISHAK and J. MOGYORÓDI have proved the following generalization of the famous Fefferman inequality. We formulate it as

LEMMA 1. If $Y\in\mathcal{H}_{\Psi}$, where Ψ has finite power and $X\in\mathcal{K}_{\Phi}$ then for all n we have

$$|E(X_nY_n)|\leqq c_{\Phi}\|Y_n\|_{\mathcal{H}_{\Psi}}\|X_n\|_{\mathcal{K}_{\Phi}}<+\infty.$$

Moreover, the limit of $E(X_n\ Y_n)$ exists when $n\to+\infty$ and we have

$$|\lim_{n\to+\infty}E(X_nY_n)|\leqq c_{\Phi}\|Y\|_{\mathcal{H}_{\Psi}}\|X\|_{\mathcal{K}_{\Phi}}.$$

Here $c_{\Phi}>0$ is a constant depending on the Young functions (Φ,Ψ).

Also, if $X\in\mathrm{BMO}_1$ and $Y\in\mathcal{H}_1$ then with some constant c which is universal, we have

$$|E(Y_nX_n)|\leqq c\|Y_n\|_{\mathcal{H}_1}\|X_n\|_{\mathrm{BMO}_1}\ \text{and}\ |\lim_{n\to+\infty}E(Y_nX_n)|\leqq c\|Y\|_{\mathcal{H}_1}\|X\|_{\mathrm{BMO}_1}.$$

This last is another formulation of the famous Fefferman inequality.

3. We are now in the position to may prove the following

THEOREM 1. *If Ψ has finite power then every $X\in\mathcal{K}_{\Phi}$ can be represented in the form*

$$X=\sum_{i=1}^{\infty}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})],$$

where

$$\left\|\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{1/2}\right\|_{\Phi}E\left(\Phi\left(\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{1/2}\Big/\left\|\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{1/2}\right\|_{\Phi}\right)\right)\leqq c_{\Phi}\|X\|_{\mathcal{K}_{\Phi}},$$

and $c_{\Phi}>0$ is a constant depending only on (Φ,Ψ).

PROOF. Let $X\in\mathcal{K}_{\Phi}$ be fixed and consider any element Y from $\mathcal{H}_{\Psi}$ where Ψ is the conjugate Young function of Φ such that Ψ has finite power. We then have

$$|d_i'|=|Y_i-Y_{i-1}|\leqq S$$

and so $d_i'\in L_{\Psi}$. Also, as we have seen above in section 2,

$$|d_i|=|X_i-X_{i-1}|=E(|X_i-X_{i-1}||\mathcal{F}_i)\leqq E(\gamma|\mathcal{F}_i)\ \text{a.s.}$$

where $\gamma\in\Gamma_X^{(\Phi)}$ is arbitrary. Since $\gamma\in L^{\Phi}$ it follows by the Jensen inequality that $d_i\in L^{\Phi}$. Consequently, by the Hölder inequality

$$E(|d_i'd_j|)\leqq 2\|d_i'\|_{\Psi}\|d_j\|_{\Phi}.$$

This means that

$$E(Y_nX_n)=\sum_{i=1}^{n}E(d_i'd_i)$$

since for $i \neq j$ we have $E(d_i' d_j) = 0$. Therefore, for all $m \geqq n$ we have

$$E(Y_m X_m) - E(Y_n X_n) = \sum_{i=m+1}^{n} E(d_i' d_i) = E\big((Y_m - Y_n) X_m\big).$$

Then from Lemma 1 we get

$$|E(Y_m X_m) - E(Y_n X_n)| \leqq c_\Phi \|Y_m - Y_n\|_{\mathscr{H}_\Psi} \|X_m\|_{\mathcal{X}_\Phi} \leqq c_\Phi \|Y_m - Y_n\|_{\mathscr{H}_\Psi} \|X\|_{\mathcal{X}_\Phi}.$$

Since $\|Y_m - Y_n\|_{\mathscr{H}_\Psi} \to 0$ as $n, m \to +\infty$, we conclude that the sequence $E(X_n Y_n)$ has the Cauchy property.

Let then for all $Y \in \mathscr{H}_\Psi$

$$L(Y) = \lim_{n \to +\infty} E(Y_n X_n)$$

Lemma 1 again implies that

$$|L(Y)| \leqq c_\Phi \|Y\|_{\mathscr{H}_\Psi} \|X\|_{\mathcal{X}_\Phi}.$$

This shows that $L(Y)$ is a bounded linear functional on $\mathscr{H}_\Psi$.

Before continuing the proof of our assertion we state an auxiliary assertion which will be useful in the sequel and which can be found in [7] (Lemma 2.).

Let $\delta\mathscr{H}_\Psi$ denote the Banach space of sequences of random variables

$$\Theta = (\Theta_1, \Theta_2, \ldots)$$

with norm

$$\|\Theta\|_{\delta\mathscr{H}_\Psi} = \left\| \left(\sum_{i=1}^{\infty} \Theta_i^2 \right)^{1/2} \right\|_\Psi .$$

Lemma 2. If $\Lambda(\Theta)$ is a linear functional on $\delta\mathscr{H}_\Psi$ such that

$$|\Lambda(\Theta)| \leqq B \|\Theta\|_{\delta\mathscr{H}_\Psi},$$

then there is a $\sigma \in \delta\mathscr{H}_\Phi$ with

$$\|\sigma\|_{\delta\mathscr{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \Big/ \|\sigma\|_{\delta\mathscr{H}_\Phi} \right) \right) \leqq B$$

such that

$$\Lambda(\Theta) = \sum_{i=1}^{\infty} E(\sigma_i \Theta_i).$$

We continue the proof of our theorem.

Using this assertion we see that there exists a sequence of random variables $\{\sigma_n\}$ satisfying

$$\|\sigma\|_{\delta\mathscr{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \Big/ \|\sigma\|_{\delta\mathscr{H}_\Phi} \right) \right) \leqq c_\Phi \|X\|_{\mathcal{X}_\Phi}$$

such that

$$L(Y_n) = \sum_{i=1}^{n} E(d_i'\sigma_i) = E\left(Y_n\left[\sum_{i=1}^{n}\left(E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})\right)\right]\right).$$

But

$$L(Y_n) = E(Y_nX_n).$$

Consequently,

$$E(Y_nX_n) = E\left(Y_n\left[\sum_{i=1}^{n}\left(E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})\right)\right]\right)$$

for all $Y\in\mathcal{H}_\Psi$. In particular this holds for the random variable

$$Y = \operatorname{sign}\left(X_n-\sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]\right)-$$

$$-E\left(\operatorname{sign}\left(X_n-\sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]\right|\mathcal{F}_0\right).$$

It is easily verified that Y is a bounded random variable ($|Y|\le 2$) and it is $\mathcal{F}_n$-measurable. Consequently, $Y\in\mathcal{H}_\Psi$, since $Y = Y_n = Y_{n+1} = Y_{n+2} = \dots$ and so

$$S = \left(\sum_{i=1}^{n} d_i'^2\right)^{1/2}$$

which is bounded by $4\sqrt{n}$. Now with this random variable Y the preceding equality gives

$$E\left(\left|X_n-\sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]\right|\right) = 0,$$

since trivially

$$E\left(E\left(\operatorname{sign}\left(X_n-\sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]\right)\middle|\mathcal{F}_0\right)\cdot\right.$$

$$\left.\cdot\left(X_n-\sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]\right)\right) = 0.$$

These together give that

$$X_n = \sum_{i=1}^{n}[E(\sigma_i|\mathcal{F}_i)-E(\sigma_i|\mathcal{F}_{i-1})]$$

a.s. for all $n = 1, 2, \dots$.

This proves the assertion.

Remark. It should be mentioned that when Φ has also finite power then for arbitrary $X\in L^\Phi$ such that $P(X\neq 0)>0$ we have

$$E\left(\Phi\left(\frac{|X|}{\|X\|_\Phi}\right)\right) = 1.$$

From this and from the preceding assertion the following consequence can be deduced.

Corollary 1. Suppose that both Φ and Ψ have finite power. Then every $X \in \mathcal{X}_\Phi$ can be represented in the form

$$X = \sum_{i=1}^{\infty} [E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})] \text{ a.s. and in } \mathcal{X}_\Phi,$$

where

$$\left\| \left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \right\|_\Phi \leq c_\Phi \|X\|_{\mathcal{X}_\Phi},$$

and $c_\Phi > 0$ is a constant depending only on the pair (Φ, Ψ).

Proof. Use the result of Theorem 1 and the remark above. Under the assumptions of this corollary we have in Theorem 1

$$E\left\{\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \Big/ \|\sigma\|_{\delta\mathcal{X}_\Phi}\right)\right\} = 1.$$

This proves the corollary.

This assertion covers the case when the pair of the Young functions (Φ, Ψ) is the following: $\left(\frac{x^p}{p}, \frac{x^q}{q}\right)$ and $p > 1$, where $p^{-1} + q^{-1} = 1$. Thus we obtained the following special case of Theorem 1.

Corollary 2. Let $1 < p < +\infty$ and let $q = \frac{p}{p-1}$. Then every $X \in \mathcal{X}_p$ can be represented in the form

$$X = \sum_{i=1}^{\infty} [E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})],$$

where the random variables σ_i, $i = 1, 2, \ldots$ satisfy the inequality

$$\left(E\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{q/2}\right)\right)^{1/q} \leq c_p \|X\|_{\mathcal{X}_q}.$$

Here c_p is a constant depending only on (p, q).

This corollary generalizes Theorem 1.5.2. of Garsia [1] for the values of p such that $1 < p < +\infty$.

4. We can characterize the space $\mathcal{X}_\infty = \mathrm{BMO}_1$. This corresponds to the value 1 of p. Garsia has given a similar characterization for the random variables belonging to BMO_2. Since BMO_1 and BMO_2 have the same elements and their norms are equivalent, the characterization below seems to be superflous. However we give here the characterization for the sake of completeness.

THEOREM 2. *Let* $X \in \mathcal{K}_\infty = \mathrm{BMO}_1$. *Then* X *can be represented in the form*

$$X = \sum_{i=1}^{\infty} [E(\sigma_i | \mathcal{F}_i) - E(\sigma_i | \mathcal{F}_{i-1})],$$

where

$$\left\| \left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2} \right\|_\infty < +\infty.$$

PROOF. In many steps we can repeat the proof of Theorem 1. Let $X \in \mathrm{BMO}_1$ and consider any element Y of $\mathcal{H}_1$. We then have

$$|d_i'| = |Y_i - Y_{i-1}| \leqq S \in L_1.$$

Also,

$$|d_i| = |X_i - X_{i-1}| = E(|X_i - X_{i-1}| | \mathcal{F}_i) \leqq \|X\|_{\mathrm{BMO}_1} = B < +\infty.$$

This can be shown in the same manner as in the proof of Theorem 1. Consequently,

$$E(|d_i' d_j|) \leqq BE(|d_i'|).$$

This means that

$$E(Y_n X_n) = \sum_{i=1}^{n} E(d_i' d_i),$$

since for $i \neq j$ we have $E(d_i' \, d_j) = 0$. Therefore, for all $m \geqq n$ we have

$$E(Y_m X_m) - E(Y_n X_n) = E((Y_m - Y_n) X_m).$$

The second assertion of Lemma 1 then implies

$$|E(Y_m X_m) - E(Y_n X_n)| \leqq c \|Y_m - X_n\|_{\mathcal{H}_1} \|X_m\|_{\mathrm{BMO}_1} \leqq cB \|Y_m - X_n\|_{\mathcal{H}_1},$$

where $c > 0$ is the absolute constant of the Fefferman inequality of Lemma 1. Since $\|Y_m - Y_n\|_{\mathcal{H}_1} \to 0$ as $n, m \to +\infty$, we conclude that the sequence $\{E(Y_n X_n)\}$ has the Cauchy property.

Let then for all $Y \in \mathcal{H}_1$

$$L(Y) = \lim_{n \to +\infty} E(X_n Y_n).$$

The second part of Lemma 1 implies that

$$|L(Y)| \leqq c \|Y\|_{\mathcal{H}_1} \|X\|_{\mathrm{BMO}_1}.$$

This shows that $L(Y)$ is a bounded linear functional on $\mathcal{H}_1$.

To continue the proof we need an elementary assertion. Let $\delta\mathscr{H}_1$ denote the Banach space of sequences of random variables

$$\Theta = (\Theta_1, \Theta_2, \ldots)$$

with norm

$$\|\Theta\|_{\delta\mathscr{H}_1} = E\left(\left(\sum_{i=1}^{\infty} \Theta_i^2\right)^{1/2}\right).$$

We can state the following

LEMMA 3. *If $\Lambda(\Theta)$ is a linear functional on $\delta\mathscr{H}_1$ such that*

$$|\Lambda(\Theta)| \leqq B\|\Theta\|_{\delta\mathscr{H}_1}$$

then there is a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$ of random variables with

$$\left\|\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2}\right\|_{\infty} \leqq B$$

such that

$$\Lambda(\Theta) = \sum_{i=1}^{\infty} E(\sigma_i \Theta_i).$$

This assertion can be found e.g. in [1], Lemma 1.4.1.

We now continue the proof of our theorem.

Using the assertion of Lemma 3 we see that there exists a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$ of random variables satisfying

$$\left\|\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2}\right\|_{\infty} \leqq c\|X\|_{\mathrm{BMO}_1} = cB$$

such that

$$L(Y_n) = \sum_{i=1}^{n} E(d_i'\sigma_i) = E\left(Y_n\left[\sum_{i=1}^{n} \left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right)\right]\right).$$

At the same time

$$L(Y_n) = E(Y_n X_n).$$

Consequently,

$$E(Y_n X_n) = E\left(Y_n\left[\sum_{i=1}^{n} \left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right)\right]\right)$$

for all $Y \in \mathscr{H}_1$.

Beginning from this point the proof is the same as that of Theorem 1. This proves the assertion.

5. Concerning the result of Corollary 1 and Corollary 2 we can state more if we take into account a result of [8]. In this paper we proved that if both Φ and Ψ have finite power then the spacee $\mathcal{K}_\Phi$ contains the same elements as $\mathcal{H}_\Phi$. Moreover, the $\mathcal{K}_\Phi$-and the $\mathcal{H}_\Phi$-horms are equivalent. Consequently, the interesting is the result of Theorem 1 since we cannot deduce that $\mathcal{H}_\Phi$ coincides with $\mathcal{K}_\Phi$.

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SEPARATION AXIOMS FOR PROXIMITY AND CLOSURE SPACES

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(Received September 30, 1985)

Let X be a set, and denote by $\exp X$ the power set of X. A relation δ on $\exp X$ is said to be a *proximity* if it fulfils the following conditions:

P1. $A\delta B \Rightarrow B\delta A$,

P2. $\emptyset\bar{\delta}X$ ($\bar{\delta}$ means non $-\delta$),

P3. $A, B \subset X, A \cap B \neq \emptyset \Rightarrow A\delta B$,

P4. $A\delta B, A \subset A' \subset X, B \subset B' \subset X \Rightarrow A'\delta B'$,

P5. $A \cup B\delta C \Rightarrow A\delta C$ or $B\delta C$.

If δ satisfies P1 to P4, it is a *semi-proximity* on X (see [1]).

A *closure* on X is a mapping $c\colon \exp X \to \exp X$ satisfying

C1. $c(\emptyset) = \emptyset$,

C2. $A \subset X \Rightarrow A \subset c(A)$,

C3. $A \subset B \subset X \Rightarrow c(A) \subset c(B)$,

C4. $A, B \subset X \Rightarrow c(A \cup B) = c(A) \cup c(B)$.

If c fulfils C1 to C3, it is a *semi-closure* on X ([1]). A *topology* on X can be described as a closure satisfying

C5. $A \subset X \Rightarrow c(c(A)) = c(A)$.

If δ is a (semi-) proximity on X and we define, for $x \in X$, $A \subset X$,

(1) $$x \in c_\delta(A) \Leftrightarrow \{x\}\delta A,$$

then c_δ is a (semi-) closure on X ([1], (3.9), (3.10)). We say that δ *induces* c_δ or that δ is *compatible* with c_δ. For a given semi-closure c on X, there exists a semi-proximity compatible with c iff c satisfies

SC. $$x, y \in X,\ x \in c(\{y\}) \Rightarrow y \in c(\{x\}).$$

More precisely, if c fulfils SC, then a compatible semi-proximity is defined by

(2) $$A\delta_c^* B \Leftrightarrow A \cap c(B) \neq \emptyset \text{ or } c(A) \cap B \neq \emptyset;$$

if c is a closure, then δ_c^* is a proximity ([1], (3.12)). Another construction, namely

$$A\delta_c B \Leftrightarrow c(A)\cap c(B)\neq\emptyset, \tag{3}$$

furnishes a compatible semi-proximity iff c satisfies

SC′ $$x\in X,\ A\subset X,\ x\notin c(A)\Rightarrow c(\{x\})\cap c(A)=\emptyset;$$

if a closure c fulfils SC′, then δ_c is a proximity ([1], (3.15)).

Both SC and SC′ can be considered as separation axioms for the semi-closure c. SC′ clearly implies SC but the converse fails to be true ([1], (3.16)). They coincide of c is *separated*, i.e. if $c(\{x\})=\{x\}$ for $x\in X$, and also if c is a topology ([1], (3.17)).

Our first purpose is to introduce a further separation axiom for semi-closures, lying between SC and SC′. Later on, we shall consider some separation axioms for (semi-)proximities.

1. Axiom S_1C. For a semi-closure c on X, consider the condition

S_1C. $$x,\ y\in X,\ x\notin c(\{y\})\Rightarrow c(\{x\})\cap c(\{y\})=\emptyset.$$

Lemma 1. *SC′ implies* S_1C *and* S_1C *implies* SC. □

The converses hold if c is separated or if it is a topology. In general, these conditions are distinct.

Example 1. Let $X=\mathbf{R}^2$ and, for $z_0=(x_0, y_0)\in X$,

$$V(z_0)=\{(x_0, y): y\in\mathbf{R}\} \text{ if } y_0=0,$$
$$V(z_0)=\{(x_0, y): y\geq 0\} \text{ if } y_0>0,$$
$$V(z_0)=\{(x_0, y): y\leq 0\} \text{ if } y_0<0.$$

Let $z\in c(A)$ iff $V(z)\cap A\neq\emptyset$. Then clearly c is a closure satisfying SC (observe $c(\{z\})=V(z)$). However, for $z_i=(x_i, y_i)$, $x_1=x_2$, $y_1>0$, $y_2<0$, we have $z_1\notin c(\{z_2\})$ but $c(\{z_1\})\cap c(\{z_2\})\neq\emptyset$ so that c does not fulfil S_1C. □

Example 2. ([1], (3.16)). Let $X=\mathbf{R}^2$, and, for $z_0=(x_0, y_0)\in X$, $\varepsilon>0$, define

$$V_\varepsilon(z_0)=\{(x, y)\in X: x=x_0, \text{ or } y=y_0 \text{ and } |x-x_0|<\varepsilon\}.$$

Set $z\in c(A)$ iff $V_\varepsilon(z)\cap A\neq\emptyset$ for every $\varepsilon>0$. Then c is a closure on x and it is easily seen that it satisfies S_1C because $c(\{z_0\})=\{(x_0, y): y\in\mathbf{R}\}$. However, it is shown in [1] that c does not satisfy SC′ ($z=(0, 0)$, $A=\{(x, 1): x>0\}$). □

2. Axioms S_1P, S_1P', S_1P''. Let δ be a semi-proximity on X. It is clear ([1], (3.18)) that c_δ satisfies SC′ iff δ fulfils

SP′. $$\{x\}\bar{\delta}A\Rightarrow c_\delta(\{x\})\cap c_\delta(A)=\emptyset.$$

SP′ holds whenever δ is *separated*, i.e. if $\{x\}\delta\{y\}$ implies $x=y$. In [1], the following condition, obviously stronger than SP′, is introduced as well:

SP″. $$A,\ B\subset X,\ A\bar{\delta}B\Rightarrow c_\delta(A)\cap c_\delta(B)=\emptyset.$$

It is shown that SP″ is strictly stronger than SP′ ([1], (3.20)) and that separatedness and SP″ are independent of each other.

Now we consider three axioms for a semi-proximity δ on X:

S_1P. $$\{x\}\bar{\delta}\{y\} \Rightarrow c_\delta(\{x\}) \cap c_\delta(\{y\}) = \varnothing,$$

S_1P'. $$\{x\}\bar{\delta}\{y\} \Rightarrow \{x\}\bar{\delta}c_\delta(\{y\}),$$

S_1P''. $$\{x\}\bar{\delta}\{y\} \Rightarrow c_\delta(\{x\})\bar{\delta}c_\delta(\{y\}).$$

Lemma 2. *δ satisfies* S_1P *iff c_δ fulfils* S_1C. □

Lemma 3. SP′ *implies* S_1P. □

Lemma 4. S_1P'' *implies* S_1P' *and* S_1P' *implies* S_1P.

Proof. The first part is obvious. On the other hand, $\{x\}\bar{\delta}c_\delta(\{y\})$ and $z \in c_\delta(\{y\})$ imply $\{x\}\bar{\delta}\{z\}$, hence $z \notin c_\delta(\{x\})$, and the second part is proved. □

Lemma 5. *A separated semi-proximity fulfils* S_1P''. □

3. Axioms TP′, TP_p, LO_p. For a (semi-)proximity δ on X, further separation axioms are discussed in [1]. It is first shown that, for a proximity δ, c_δ is a topology iff δ satisfies

TP. $$\{x\}\bar{\delta}A \Rightarrow \{x\}\bar{\delta}c_\delta(A)$$

([1], (4.1)). It is easy to see that TP implies SP′, while TP and SP″ are independent of each other ([1], (4.2)).

Lemma 6. TP *implies* S_1P'. □

In [2], an axiom has been introduced that is stronger than TP and SP″:

LO. $$A\bar{\delta}B \Rightarrow c_\delta(A)\bar{\delta}c_\delta(B).$$

There are TP- and SP″-proximities that are not LO ([1], (4.3)).

We introduce further axioms similar to TP and LO:

TP′. $$\{x\}\bar{\delta}A \Rightarrow c_\delta(\{x\})\bar{\delta}A,$$

LO_p. $$A\bar{\delta}B \Rightarrow A\bar{\delta}\bigcup_{x\in B} c_\delta(\{x\}),$$

TP_p. $$\{x\}\bar{\delta}A \Rightarrow \{x\}\bar{\delta}\bigcap_{y\in A} c_\delta(\{y\}).$$

Lemma 7. *If δ is separated, it is* LO_p. □

Lemma 8. LO *implies* LO_p, LO_p *implies* TP′, *and* TP′ *implies* S_1P''.

Proof. The first two statements are obviuous. If δ it TP′ and $\{x\}\bar{\delta}\{y\}$, then $c_\delta(\{x\})\bar{\delta}\{y\} \Rightarrow c_\delta(\{x\})\bar{\delta}c_\delta(\{y\})$. □

Lemma 9. *Both* LO_p *and* TP *imply* TP_p. □

Lemma 10. TP_p *implies* S_1P'. □

4. Axiom S_2P. We introduce now an axiom for a semi-proximity δ on X that corresponds to the condition that c_δ satisfies a weak form of the Hausdorff separation axiom:

S_2P. $\{x\}\bar{\delta}\{y\}$ implies the existence of a $C\subset X$ such that $\{x\}\bar{\delta}X-C$, $C\bar{\delta}\{y\}$.

Lemma 11. *A* TP – *proximity δ satisfies* S_2P *iff the topology c_δ fulfils:*

$x\notin c_\delta(\{y\})\Rightarrow x$ and y have disjoint neighbourhoods. □

Lemma 12. S_2P *implies* S_1P'.

Proof. If $\{x\}\bar{\delta}\{y\}$ and C is chosen as in S_2P, then $z\in C$ implies $\{z\}\bar{\delta}\{y\}$, hence $z\notin c_\delta(\{y\})$, so that

$$c_\delta(\{y\})\subset X-C, \text{ and } \{x\}\bar{\delta}c_\delta(\{y\}). \quad \square$$

Let us now consider the axiom

R. $\{x\}\bar{\delta}A$ implies the existence of $C\subset X$ such that

$$\{x\}\bar{\delta}\, X-C,\ C\bar{\delta}A$$

introduced in [3].

Lemma 13. R *implies* S_2P. □

Lemma 14. R *implies* TP′ *and* TP.

Proof. Assume $\{x\}\bar{\delta}A$ and choose C according to R. Then, like in the proof of Lemma 12, $c_\delta(\{x\})\subset C$, and similarly $c_\delta(A)\subset X-C$, so that $c_\delta(\{x\})\bar{\delta}A$ and $\{x\}\bar{\delta}c_\delta(A)$. □

There is a separated proximity which is R and SP″ without being LO ([1], (4.7)).

5. Counterexamples. We show that the above more or less trivial implications cannot be reversed.

Example 3. Let $X=\mathbf{R}^2$, and $V(z)=\{x\}\times\mathbf{R}$ for $z=(x,y)\in X$, $V_\varepsilon(0,0)=$ $=(-\varepsilon,\varepsilon)\times\mathbf{R}$ for $\varepsilon>0$. Define c by

$$z\in c(A)\Leftrightarrow V(z)\cap A\neq\emptyset \text{ if } z\neq(0,0),$$

$$V_\varepsilon(z)\cap A\neq\emptyset \text{ for } \varepsilon>0 \text{ if } z=(0,0).$$

Then clearly c is a closure on X satisfying $c(\{z\})=V(z)$ for $z\in X$ so that SC is fulfilled and $\delta=\delta_c^*$ is a proximity compatible with c. δ is S_2P (because $\{z_1\}\bar{\delta}\{z_2\}$, $z_i=(x_i,y_i)$ implies, say, $x_1<x_2$, and

$$C=\{(x,y)\in X: x\leqq c\} \tag{4}$$

satisfies $\{z_1\}\bar{\delta}X-C$, $C\bar{\delta}\{z_2\}$ if $x_1<c<x_2$) and S_1P'' (because $c(V(z))=V(z)$). It is also TP_p since, for $A\subset X$, $A^*=\bigcup_{z\in A}c(\{z\})=\bigcup_{z\in A}V(z)$, $V(z_0)\cap A=\emptyset$ or $V_\varepsilon(0,0)\cap A=\emptyset$ implies $V(z_0)\cap A^*=\emptyset$ or $V_\varepsilon(0,0)\cap A^*=\emptyset$, respectively. However, if $A=(0,+\infty)\times\mathbf{R}$, $z=(0,1)$, then $\{z\}\bar{\delta}A$ and $(0,0)\in c(\{z\})\cap c(A)$: δ is not SP′. □

EXAMPLE 4. Let $X = \mathbf{R}^2$ and $p: X \to \mathbf{R}$ be defined by $p(x, y) = x$. Set
$A\bar{\delta}B \Leftrightarrow$ $p(A) \cap p(B) = \emptyset$ and either $A = \emptyset$ or $B = \emptyset$ or both A and B are finite.
It is easy to see that δ is a proximity on X. For $z_0 = (x_0, y_0) \in X$, we have $c_\delta(\{z_0\}) = \{x_0\} \times \mathbf{R}$, consequently, if $A \subset X$ is finite, then $c_\delta(A) = p(A) \times \mathbf{R}$. Hence δ fulfils SP″. However, for $z_0 = (0, 0)$, $z_1 = (1, 0)$, we have $\{z_0\}\bar{\delta}\{z_1\}$ and $\{z_0\}\delta c_\delta(\{z_1\})$ so that S_1P' fails to hold. □

EXAMPLE 5. Let X and p denote the same as in Example 4, and
$A\bar{\delta}B \Leftrightarrow p(A) \cap p(B) = \emptyset$ and either A or B is finite. Now again, δ is a proximity on X, and $c_\delta(\{z_0\}) = \{x_0\} \times \mathbf{R}$ for $z_0 = (x_0, y_0)$. moreover, $c_\delta(A) = p(A) \times \times \mathbf{R}$ for every $A \subset X$. Therefore δ fulfils SP″ and TP (in fact, c_δ is the product of the discrete topology on the x-axis and of the indiscrete one on the y-axis). δ also satisfies S_2P because, if $\{z_1\}\bar{\delta}\{z_2\}$ for $z_i = (x_i, y_i)$, and, say, $x_1 < c < < x_2$, then the set C in (4) can be chosen.

On the other hand, δ is not S_1P''; indeed, $z_0 = (0, 0)$, $z_1 = (1, 0)$ implies $\{z_0\}\bar{\delta}\{z_1\}$, $c_\delta(\{z_0\})\delta c_\delta(\{z_1\})$. □

EXAMPLE 6. ([1], (4.2)). Let $X = \mathbf{R}^2$ and, for $z_0 = (x_0, y_0) \in X$, $\varepsilon > 0$,
$V_\varepsilon(z_0) = \{(x, y) \in X : x = x_0 \text{ and } |y - y_0| < \varepsilon \text{ or } y = y_0 \text{ and } |x - x_0| < \varepsilon\}$.
If $z \in c(A)$ means that $V_\varepsilon(z) \cap A \neq \emptyset$ for every $\varepsilon > 0$, then c is a separated closure so that δ_c is a compatible proximity and satisfies by [1], (3.19), SP″. δ_c also fulfils S_2P: if $z_1 \neq z_2$, we choose for C a disc with centre z_1 and radius less than the Euclidean distance of z_1, $z_2 \in X$. However, it is clear that c is not a topology, hence δ_c fails to fulfil TP. □

EXAMPLE 7. Let

$$X = \{(x, y) \in \mathbf{R}^2 : x = 0 \text{ or } y = 0\},$$

and c be the topology induced by the pseudo-metric $\sigma(z_1, z_2) = |x_1 - x_2|$ for $z_i = (x_i, y_i)$. Set
$A\bar{\delta}B \Leftrightarrow c(A) \cap c(B) = \emptyset$ and either A or B is finite.

It is easy to see that δ is a proximity compatible with c. Hence δ is TP. It is also S_1P'' because $\{z_1\}\bar{\delta}\{z_2\}$ implies that either $c(\{z_1\})$ or $c(\{z_2\})$ is a singleton. Also, δ is obviously SP″. Finally δ is S_2P (one chooses the set C in (4) if $x_1 < c < x_2$).

However, for $z = (0, 0)$, $A = [1, 2] \times \{0\}$, we have $c(\{z\}) = \{0\} \times \mathbf{R}$, hence $\{z\}\bar{\delta}A$ but $c(\{z\})\delta A$. Therefore δ is not TP′. □

EXAMPLE 8. ([1], (3.20)). Let c be the usual topology on $X = \mathbf{R}$, and define $\delta = \delta_c^*$. Then δ is separated and R ([4], (3.2)), but fails to fulfil SP″: for $A = (0, 1)$, $B = (1, 2)$, we have $A\bar{\delta}B$ but $c(A) \cap c(B) \neq \emptyset$. □

EXAMPLE 9. Let c be a T_1-topology that is not T_2. Then $\delta = \delta_c$ is a separated, compatible proximity that is LO without being S_2P. □

EXAMPLE 10. Let c be a non-regular T_2-topology, and $\delta = \delta_c$. Then δ is compatible, LO, S_2P and separated, but fails to satisfy R (because then c would be regular [3]). □

Example 11. Let X and p be the same as in Example 4, $q(x, y) = y$, and set

$$A\bar{\delta}B \Leftrightarrow \quad p(A)\cap p(B) = \emptyset \text{ and either } A \text{ or } B \text{ has the form } P\cup Q \text{ where } p(P) \text{ and } q(Q) \text{ are finite.}$$

Then δ is a proximity on X. It is R because, if $\{z\}\bar{\delta}A$, we can put $C = \{x\}\times\mathbf{R}$, $z = (x, y)$ in order to obtain $\{z\}\bar{\delta}X-C$ and $C\bar{\delta}A$ (indeed, $p(C)\cap p(A) = \emptyset$ and $C = C\cup\emptyset$). δ is SP″ because $c_\delta(A) = p^{-1}(p(A))$ for $A\subset X$. However, δ is not LO_p: for $A = (0, 1)\times\mathbf{R}$, $B = (2, 3)\times\{0\}$, $A\bar{\delta}B$ and $A\,\delta \bigcup_{z\in B} c_\delta(\{z\})$. □

Example 12. Let X and p denote again the same as in Example 4, and define

$$A\bar{\delta}B \Leftrightarrow \quad p(A)\cap p(B) = \emptyset \text{ and either } p(A) \text{ is finite and there is } \varepsilon>0 \text{ such that } B\cap p^{-1}(V_\varepsilon(p(A))) \text{ is countable, or } p(B) \text{ is finite and there is } \varepsilon>0 \text{ such that } A\cap p^{-1}(V_\varepsilon(p(B))) \text{ is countable,}$$

where $V_\varepsilon(C) = \{x\in\mathbf{R}\colon |x-c|<\varepsilon \text{ for some } c\in C\}$. One easily sees that δ is a proximity on X. $\{z_1\}\bar{\delta}\{z_2\}$ iff $x_1\neq x_2$ for $z_i=(x_i,\ y_i)$, hence δ is $\mathrm{S_2P}$ (we use once more the set (4) if $x_1<x_2$). Further $c_\delta(\{z_1\}) = \{x_1\}\times\mathbf{R}$, whence δ is TP′. Finally if $A\bar{\delta}B$ and, say, $p(A)$ is finite and $B\cap p^{-1}(V_\varepsilon(p(A)))$ is countable, then $z_1\in c_\delta(A)$ implies $x_1\in p(A)$ and clearly $\{z_1\}\bar{\delta}B$, $z_1\notin c_\delta(B)$; thus δ is SP″.

However, δ is not TP_p: for $z_0 = (\pi, 0)$, $A = \mathbf{Q}\times\{0\}$, we have $\{z_0\}\bar{\delta}A$ and $\{z_0\}\delta \bigcup_{z\in A} c_\delta(\{z\})$. □

6. Main results. We can summarize our results in the following

Theorem 1. *The following implications are valid for a semi-proximity* δ (SP denotes that δ is separated):

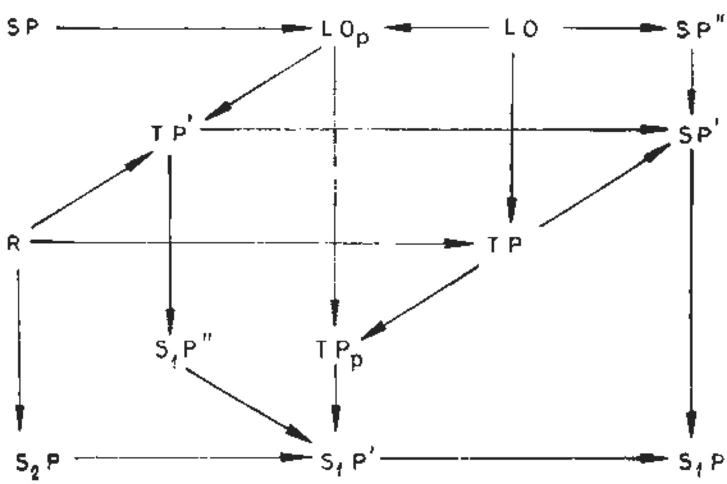

Fig. 1.

If $P_1, \ldots, P_n$, Q are axioms figuring in the diagram, and $P_i \to Q$ does not follow from the implications contained in the diagram ($i = 1, \ldots, n$), then there is a proximity that is P_i for $i = 1, \ldots, n$, but fails to be Q. □

According ti [1], if $f: X \to Y$ and δ is a proximity on Y, we define a proximity $f^{-1}(\delta)$ on X by

$$A\ f^{-1}(\delta)B \Leftrightarrow f(A)\delta f(B). \tag{5}$$

If δ_k is, for $k \in K$, a proximity on X, we define a proximity $\delta = \sup\{\delta_k\}$ on X by

$$A\bar{\delta}B \Leftrightarrow A = \bigcup_1^m A_i\ B = \bigcup_1^n B_j,\ m,\ n \in \mathbf{N}, \tag{6}$$

and there is, for each pair (i,j), a $k \in K$ such that $A_i \bar{\delta}_k B_j$.

An easy calculation furnishes, similarly to [1], (5.1), the following

THEOREM 2. *If Q denotes one of the symbols* S_1P, S_1P', S_1P'', S_2P, TP', LO_p, TP_p, *and δ satisfies Q, then $f^{-1}(\delta)$ satisfies Q as well. Similarly, if every δ_k fulfils Q, then the same holds for* $\delta = \sup\{\delta_k\}$. □

The author thanks Dr. J. DEÁK for many suggestions (e. g. for Examples 11 and 12.

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ON THE CONVERGENCE OF AMARTS IN ORLICZ SPACES

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(Received October 2, 1985)

1. During the recent years several papers have been pulished on the asymptotic martingales (amarts). The purpose of this note is to present a result concerning the convergence of amarts in Orlicz spaces. The martingale case was treated by J. MOGYORÓDI in [3]. Let Φ be a Young function having finite power and consider the corresponding Orlicz space $L^{\Phi}(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space. In [3] it is shown that if $(X_n, \mathcal{F}_n)$ is a martingale such that $\sup_{n\geq 1} \|X_n\|_{\Phi} < +\infty$ then X_n converges in L^{Φ} to its a.s. limit provided the above conditions are satisfied.

Concerning the theory of the Young functions and the Orlicz spaces we refer to [2] and [4]. The power of the Young function Φ is defined by the formula

$$\sup_{x>0} \frac{x\varphi(x)}{\Phi(x)}$$

where φ is the right hand side derivative of Φ.

In the present paper we prove the following assertion. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be an amart and let Φ be a Young function with finite power. If $\{X_n\}_{n\geq 1}$ satisfies the condition that the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ is uniformly integrable then X_n converges in L^{Φ} to its almost sure limit. The uniform integrability is also necessary provided Φ has finite power.

We show, by giving an example, that the finiteness of the power of Φ is also necessary. This example is due to T. F. MÓRI. Another example shows that the uniform integrability of the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ is a condition which implies $\sup_{n\geq 1} \|x_n\|_{\Phi} < +\infty$, but not conversely.

I wish to thank professor J. MOGYORÓDI for the valuable discussion and helpful criticism.

2. Throughout the present paper let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be an adapted sequence of real-valued random variables, where $\{\mathcal{F}_n\}_{n\geq 1}$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$.

We recall that a mapping $\tau:\Omega\to\bar{\mathbf{N}} = \{1, 2, \ldots, n, \ldots, +\infty\}$ is called a stopping time with respect to the above $\{\mathcal{F}_n\}_{n\geq 1}$ if $\{\omega:\tau(\omega) = n\}\in\mathcal{F}_n$ for each $n\in\mathbf{N} = \{1, 2, \ldots, n\ldots\}$.

We denote by T the set of all bounded stopping times with respect to $\{\mathcal{F}_n\}_{n\geq 1}$. We consider the natural ordering on T as follows: for all $\tau, \sigma\in T$ we say that $\tau\leq\sigma$ iff $\tau(\omega)\leq\sigma(\omega)$ for almost all $\omega\in\Omega$.

DEFINITION. Let $\{X_n\}_{n\geq 1}$ be a sequence of integrable random variables which is adapted to $\{\mathcal{F}_n\}_{n\geq 1}$. We call $(X_n, \mathcal{F}_n)$ an asymptotic martingale (amart), if the net $\{E(X_\tau)\}_{\tau\in T}$ converges to a finite limit

$$a = \lim_{\tau\in T} E(X_\tau)$$

(i. e. for arbitrary $\varepsilon>0$ there is $\tau_0\in T$ such that for all $\tau\in T$ with $\tau\geq\tau_0$ we have $|a-E(X_\tau)|<\varepsilon$).

Any amart has the so called Riesz decomposition. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be an amart. Then X_n can be uniquely written as $X_n = Y_n+Z_n$, where $(Y_n, \mathcal{F}_n)$ is a martingale, and $(Z_n, \mathcal{F}_n)$ is an amart with $Z_n\to 0$ in L_1. In addition, $\{Z_\tau\}_{\tau\in T}$ is uniformly integrable and $Z_n\to 0$ a. s.

We refer to [1] for a complete treatment of the asymptotic martingales, where the last assertion can also be found (see Theorem 3.2.).

3. Now we prove the following

THEOREM. *Let $(X_n, \mathcal{F}_n)$ be an amart in $L^\Phi(\Omega, \mathcal{F}, P)$, where the Young function Φ has finite power. Suppose that the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ is uniformly integrable. Then $\{X_n\}$ converges in L^Φ to its almost sure limit*

$$X_\infty = \lim_{n\to+\infty} \text{a.s.}\, X_n.$$

Conversely, if Φ has finite power and the amart $(X_n, \mathcal{F}_n)$ converges in L^Φ to a limit, then the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ is uniformly integrable.

PROOF. First assume that $\{\Phi(|X_n|)\}_{n\geq 1}$ is uniformly integrable. Since Φ is a Young function we have

$$\lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty.$$

Consequently, $\Phi(x)\geq x$ for sufficiently large values of x. From the uniform integrability of $\{\Phi(|X_n|)\}_{n\geq 1}$ it then follows that $\{X_n\}_{n\geq 1}$ is also uniformly integrable. According to the Riesz decomposition (see [1]) X_n can be written in the form

$$X_n = Y_n+Z_n,\ n\geq 1$$

where $(Y_n, \mathcal{F}_n)$ is a martingale and $(Z_n, \mathcal{F}_n)$ is an amart having the following properties: $Z_n\to 0$ a. s., $Z_n\to 0$ in L_1 and the net $\{Z_\tau\}_{\tau\in T}$ is uniformly integrable. By the second property $\{Z_n\}_{n\geq 1}$ is also uniformly integrable and this together with the uniform integrability of $\{X_n\}_{n\geq 1}$ implies that of $\{Y_n\}_{n\geq 1}$.

Consequently, $(Y_n, \mathcal{F}_n)$ is a regular martingale and so, there exists a random variable $X_\infty \in L_1(\Omega, \mathcal{F}, P)$ such that

$$Y_n = E(X_\infty | \mathcal{F}_n), \quad n = 1, 2, \dots.$$

and $Y_n \to X_\infty$ a.s. and in L_1. $\left[\text{Here, we can suppose without loss of the generality that } \mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right) = \mathcal{F}\right]$. From these we deduce that $X_n \to X_\infty$ a.s. and in L_1.

Now we prove that $X_\infty \in L^\Phi(\Omega, \mathcal{F}, P)$. For this purpose let

$$\sigma = \sup_{n \geq 1} \|X_n\|_\Phi$$

which by the uniform integrability of $\{\Phi(|X_n|)\}_{n \geq 1}$ is finite. Without loss of the generality we can suppose that $\sigma > 0$, since in the contrary case we would have $X_n = 0$ a. s., $n \geq 1$, which, from the point of view of our assertion is a trivial case. Then by the continuity of Φ we have

$$\lim_{n \to +\infty} \text{a. s. } \Phi\left(\frac{|X_n|}{\sigma}\right) = \Phi\left(\frac{|X_\infty|}{\sigma}\right),$$

which by the Fatou lemma implies that

$$E\left[\Phi\left(\frac{|X_\infty|}{\sigma}\right)\right] \leq \liminf_{n \to +\infty} E\left[\Phi\left(\frac{|X_n|}{\sigma}\right)\right] \leq 1.$$

From this it follows that $X_\infty \in L^\Phi(\Omega, \mathcal{F}, P)$ and $\|X_\infty\|_\Phi \leq \sigma < +\infty$. This means that $E[\Phi(|X_\infty|)] < +\infty$, since we have supposed that Φ has finite power. The above representation of $\{Y_n\}$ and the Jensen inequality together give

$$\Phi(|Y_n|) \leq \Phi[E(|X_\infty| | \mathcal{F}_n)] \leq E[\Phi(|X_\infty|) | \mathcal{F}_n] \text{ a. s.}$$

From this we deduce that $\{\Phi(|Y_n|)\}_{n \geq 1}$ is uniformly integrable, since the right hand side is uniformly integrable. From the same representation of $\{Y_n\}$ and from the Jensen inequality

$$\sup_{n \geq 1} \|Y_n\|_\Phi \leq \|X_\infty\|_\Phi \leq \sigma < +\infty.$$

This by the theorem of [3] implies that

$$\|Y_n - X_\infty\|_\Phi \to 0$$

as $n \to +\infty$.

Now we prove that $\{\Phi(|Z_n|)\}_{n \geq 1}$ is also uniformly integrable. This is an easy consequence of the elementary inequality

$$\Phi(|Z_n|) \leq \Phi(|X_n| + |Y_n|) \leq 2^{p-1}[\Phi(|X_n|) + \Phi(|Y_n|)]$$

where p denotes the power of Φ, and the inequality follows from the convexity and monotonity of Φ. Both sequences on the right hand side of this inequality being uniformly integrable we deduce that so is $\{\Phi(|Z_n|)\}_{n\geq 1}$. By the continuity of Φ we have $\Phi(|Z_n|)\to 0$ a. s. as $n\to+\infty$. This and the uniform integrability of $\{\Phi(|Z_n|)_{n\geq 1}$ together imply that $E[\Phi(|Z_n|)]\to 0$.

Let $\varepsilon>0$ be arbitrary. Then

$$E\left[\Phi\left(\frac{|Z_n|}{\varepsilon}\right)\right]\leq\frac{1}{\varepsilon^p}E[\Phi(|Z_n|)]\to 0,\ n\to+\infty,$$

if $0<\varepsilon<1$ and

$$E\left[\Phi\left(\frac{|Z_n|}{\varepsilon}\right)\right]\leq E[\Phi(|Z_n|)]\to 0,\ n\to+\infty,$$

if $\varepsilon\geq 1$. Consequently, $E\left[\Phi\left(\frac{|Z_n|}{\varepsilon}\right)\right]\leq 1$ if n is large enough, say $n\geq n_0(\varepsilon)$. This means that $\|Z_n\|_\Phi\leq\varepsilon$, if $n\geq n_0(\varepsilon)$. Consequently, $\|Z_n\|_\Phi\to 0$ as $n\to+\infty$ since $\varepsilon>0$ is arbitrary.

From these considerations we finally get

$$\|X_n-X_\infty\|_\Phi\leq\|Y_n-X_\infty\|_\Phi+\|Z_n\|_\Phi\to 0$$

if $n\to+\infty$.

Consequently, $\{X_n\}$ converges in L^Φ to its a. s. limit X_∞. This proves the first part of the assertion.

To prove the second part suppose that $(X_n, \mathcal{F}_n)_{n\geq 1}$ is an amart which converges in L^Φ to a random variable X_∞, where Φ has finite power. We prove that the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ is uniformly integrable.

First remark that from

$$\|X_n-X_\infty\|_\Phi\to 0,\ n\to+\infty$$

it follows that $X_\infty\in L^\Phi$ and that

$$\|X_n-X_\infty\|_1\to 0,\ n\to+\infty.$$

From this it follows that $\{X_n\}_{n\geq 1}$ is uniformly integrable. We again use the Riesz decomposition of X_n, according to which we have

$$X_n = Y_n+Z_n,\ n = 1, 2, \ldots.$$

The uniform integrability of $\{X_n\}$ and of $\{Z_n\}$ imply that of $\{Y_n\}$. Since $Z_n\to 0$ a. s. and in L_1 we have that the a. s. and L_1 limit of Y_n equals X_∞. Consequently,

$$Y_n = E(X_\infty|\mathcal{F}_n),\quad n = 1, 2, \ldots.$$

(Here we again auppose without loss of the generality that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty}\mathcal{F}_n\right) = \mathcal{F}$. Now we can show the uniform integrability of $\{\Phi(|Y_n|)\}_{n\geq 1}$. This follows from the inequality of Jensen, since

$$\Phi(|Y_n|)\leq\Phi[E(|X_\infty|\ |\mathcal{F}_n)]\leq E[\Phi(|X_\infty|)|\mathcal{F}_n]$$

and $E(\Phi(|X_\infty|)] < +\infty$ given that Φ has finite power. The uniform integrability of $\{\Phi(|Y_n|)\}_{n\geq 1}$ implies that

$$\sup_{n\geq 1} \|Y_n\|_\Phi < +\infty$$

and so by theorem of [3] we get

$$\|Y_n - X_\infty\|_\Phi \to 0.$$

Also

$$\|Z_n\|_\Phi = \|X_n - Y_n\|_\Phi \leq \|X_n - X_\infty\|_\Phi + \|Y_n - X_\infty\|_\Phi \to 0.$$

as $n \to +\infty$.

To prove the uniform integrability of $\{\Phi(|X_n|)\}$ it remains to show that $\{\Phi(|Z_n|)\}$ is also uniformly integrable.

For this purpose let us remark that if q denotes the (finite or infinite) power of the conjugate Young function Ψ then

$$\inf_{x>0} \frac{x\varphi(x)}{\Phi(x)} \geq \frac{q}{q-1}$$

where we set $\frac{q}{q-1} = 1$ if $q = +\infty$. From this it follows that if $c>1$ is arbitrary then

$$\ln \frac{\Phi(cx)}{\Phi(x)} = \int_x^{cx} \frac{\varphi(t)}{\Phi(t)} dt \geq \frac{q}{q-1} \int_x^{cx} \frac{dt}{t} = \frac{q}{q-1} \ln c$$

or, in other words

$$\frac{\Phi(cx)}{\Phi(x)} \geq c^{\frac{q}{q-1}}.$$

Therefore, for any ε such that $0<\varepsilon<1$, we have

$$\Phi(|Z_n|) \leq \varepsilon^{\frac{q}{q-1}} \Phi\left(\frac{|Z_n|}{\varepsilon}\right)$$

and integrating both sides of this inequality we get

$$E[\Phi(|Z_n|)] \leq \varepsilon^{\frac{q}{q-1}} E\left[\Phi\left(\frac{|Z_n|}{\varepsilon}\right)\right].$$

Since $\|Z_n\|_\Phi \to 0$ we see that if n is large enough then

$$E\left[\Phi\left(\frac{|Z_n|}{\varepsilon}\right)\right] \leq 1.$$

$0<\varepsilon<1$ being arbitrarily small, the two last inequalities imply that

$$E[\Phi(|Z_n|)]\to 0$$

as $n\to\infty$. From this it follows that $\{\Phi(|Z_n|)\}$ is uniformly integrable.

This completes the proof of our assertion.

REMARK. This theorem should be compared to the following easy assertion. Let $(X_n, \mathcal{F}_n)$ be an amart such that $\sup_{n\geq 1}\|X_n\|_\Phi<+\infty$, where Φ is an arbitrary Young function. Then X_n converges a. s. to a random variable X_∞ which belongs to L^Φ.

To see this, let us use the Riesz decomposition of $\{X_n\}$, i. e. let

$$X_n = Y_n+Z_n,\ n = 1, 2, \ldots$$

where $(Y_n, \mathcal{F}_n)$ is a martingale and $(Z_n, \mathcal{F}_n)$ is an amart such that $Z_n\to 0$ a. s. and in L_1. The condition $\sigma = \sup_{n\geq 1}\|X_n\|_\Phi<+\infty$, where we can suppose that $\sigma>0$, then implies that the sequence $\{X_n\}$ is uniformly integrable. This and the uniform integrability of $\{Z_n\}$ together imply that of $\{Y_n\}$. Consequently, there exists a random variable $X_\infty\in L_1(\Omega, \mathcal{F}, P)$ such that

$$Y_n = E(X_\infty|\mathcal{F}_n),\quad n = 1, 2, \ldots$$

and $Y_n\to X_\infty$ a.s. and in L_1. $\left[\text{Here we suppose, as usual, that } \mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty}\mathcal{F}_n\right)=\mathcal{F}\right]$. It remains only to prove that $X_\infty\in L^\Phi$. This can be made on the basis of $0<\sigma = \sup_{n\geq 1}\|X_n\|_\Phi<+\infty$ by the same way as in the proof of the first part of the theorem.

4. The theorem just proved says that whenever Φ has finite power then for the convergence of the amart $(X_n, \mathcal{F}_n)$ in L^Φ to a random variable it is necessary and sufficient that the sequence $\{\Phi(|X_n|)\}_{n\geq 1}$ be uniformly integrable.

Is the finiteness of the power of Φ also necessary in these considerations? The following example shows that we cannot omit this assumption in general.

Let Φ be a Young function and suppose that its power is equal to $+\infty$. Then we can construct such a martingale (which is always an amart) which converges a. s. to a limit but it does not converge to this limit in L^Φ. Namely, one can construct a sequence $x_i\uparrow+\infty$ such that

$$c_i=\frac{\Phi(2x_i)}{\Phi(x_i)}$$

tends increasingly to $+\infty$ as $i\to+\infty$. Also we can suppose that $c_i\geq 2^i$, since in the contrary case we pick out a subsequence $\{c_k'\}$ for which $c_k'\geq 2^k$. We choose the sequence $\{x_i\}$ in such a way that the series

$$c = \sum_{i=1}^{\infty} \Phi^{-1}(2x_i)$$

be finite.

Let us construct the probability space in the following way: Ω is the set of all positive integers, the σ-field $\mathcal{F}$ consists of all the subsets of Ω and let

$$P(\{k\}) = \frac{1}{c}\Phi^{-1}(2x_k),\ k = 1, 2, \ldots$$

Then for every $A \in \mathcal{F}$ the probability $P(A)$ is given by the formula

$$P(A) = \sum_{k \in A} P(\{k\})$$

The random variable $X = X(\omega)$ is defined by

$$X(\omega) = x_\omega,\ \omega \in \Omega$$

Then

$$E[\Phi(X)] = \sum_{i=1}^{\infty} \frac{\Phi(x_i)}{c\Phi(2x_i)} \leq \frac{1}{c} \sum_{i=1}^{\infty} \frac{1}{2^i} < +\infty .$$

This implies that there is a constant $K > 0$ such that

$$E\left[\Phi\left(\frac{X}{K}\right)\right] \leq 1$$

since if K increases then $\Phi\left(\frac{X}{K}\right)$ decreases and consequently, soon or later the last inequality will be satisfied by means of the monotone convergence theorem and by the finiteness of $E[\Phi(X)]$. From this we deduce that $\|X\|_\Phi < < +\infty$.

Let $\mathcal{F}_n$ be the σ-field generated by the partition

$$\Pi_n = (\{1\}, \{2\}, \ldots, \{n\}, \{n+1, n+2, \ldots\})$$

Since Π_{n+1} is finer than Π_n we see that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $n = 1, 2, \ldots$. Also, it is easily seen that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right) = \mathcal{F}$. Then the martingale $X_n = E(X|\mathcal{F}_n)$ $n = 1, 2, \ldots$ converges a. s. and in L_1 to X. By the Jensen inequality it follows that

$$\Phi(X_n) = \Phi[E(X|\mathcal{F}_n)] \leq E[\Phi(X)|\mathcal{F}_n]$$

and so $\{\Phi(X_n)\}$ is uniformly integrable. We show that at the same time $\{X_n\}$ cannot converge to X in L^Φ. This is shown by the fact that

$$E[\Phi(3|X - X_n|)] = +\infty, \quad n = 1, 2, \ldots$$

i. e. $\|X-X_n\|_\Phi > \frac{1}{3}$. To see this let us note that the value of X_n on the event $A_n = \{n+1, n+2, \dots\}$ is equal to

$$\frac{1}{P(A_n)} \int_{A_n} X_n dP = \frac{1}{P(A_n)} \int_{A_n} E(X \mid \mathcal{F}_n)\, dP = \frac{1}{P(A_n)} \int_{A_n} X dP =$$

$$= \frac{1}{\sum_{k=n+1}^{\infty} \frac{1}{c\Phi(2x_k)}} \cdot \sum_{k=n+1}^{\infty} \frac{x_k}{c\Phi(2x_k)}$$

which is a constant, say K_n, depending only on n. Let $k_0 \geq n+1$ be the index for which $3|X(k)-X_n(k)| \geq 2X(k)$ if $k \geq k_0$. This can be made since $X_n(k)$ is a constant for $k \geq n+1$ and $X(k) \uparrow +\infty$ as $k \to +\infty$. Thus there exists k_0 such that

$$3|X(k)-X_n(k)| = 3|x_k - K_n| \geq 2x_k$$

if $k \geq k_0$. Therefore,

$$E[\Phi(3|X-X_n|)] \geq \sum_{k \geq k_0} \Phi(|x_k - K_n|) \frac{1}{c\Phi(2x_k)} \geq \sum_{k \geq k_0} \frac{\Phi(2x_k)}{c\Phi(2x_k)} = +\infty$$

This shows that the assumption "Φ has finite power" cannot be omitted in general in the above theorem.

Finally we show by an example that the uniform integrability of the sequence $\{\Phi(|X_n|)\}$ is more than the condition $\sup_{n \geq 1} \|X_n\|_\Phi < +\infty$. Let $\Omega = [0, 1)$, $\mathcal{F}$ the σ-field of the Borel-sets of $[0, 1)$ and let P be the corresponding Lebesgue measure. We define the sequence $\{X_n\}$ of random variables as follows

$$X_n = \begin{cases} 2^{\frac{n}{2}} & \text{if } \omega \in [0, 2^{-n}), \\ 0 & \text{if } \omega \notin [0, 2^{-n}). \end{cases}$$

Let $\mathcal{F}_n$ be the σ-field generated by the dyadic intervals, i. e.

$$\mathcal{F}_n = \sigma(\{[k2^{-n}, (k+1)2^{-n})\}: 0 \leq k \leq 2^n - 1), \; n = 1, 2, \dots$$

The atoms of $\mathcal{F}_n$ are $[k2^{-n}, (k+1)2^{-n})$, $0 \leq k \leq 2^n - 1$, and it is easy to see that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right) = \mathcal{F}$. Also it is easy to show that $(X_n, \mathcal{F}_n)$ is an amart.

More, precisely, $(X_n, \mathcal{F}_n)$ is a nonnegative supermartingale tending obviously to 0.a. s. In fact,

$$E(X_{n+1}|\mathcal{F}_n) = \chi([0,2^{-n}))2^n \int_0^{2^{-(n+1)}} 2^{\frac{n+1}{2}}\, dP =$$

$$= \chi([0,2^{-n}))2^n . 2^{\frac{-(n+1)}{2}} \leq$$

$$\leq \chi([0,2^{-n}))2^{\frac{n}{2}} = X_n \text{ a. s.}$$

Now we have for all $n \geq 1$

$$\|X_n\|_2 = [E(X_n^2)]^{\frac{1}{2}} = 1$$

which shows that $\{X_n\}$ does not converge to its a. s. limit 0 in L_2. This also shows that $\{X_n\}$ is bounded in L_2. At the same time the sequence $\{X_n^2\}$ is not uniformly integrable, because for each $a > 0$ we have

$$\sup_{n \geq 1} E[X_n^2 \chi(X_n^2 \geq a)] = 1.$$

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ON A PROBLEM CONCERNING THE $1\frac{1}{2}$ BALL PROPERTY

By

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(Received November 19, 1985)

Introduction

In this paper we give an example of a Banach space which answers a question of G. GODINI [1].

In what follows, every normed linear space is real. Let $(X, \|\cdot\|)$ be a normed linear space, $M \subset X$ a subspace. We define the napping $P_M: X \to 2^M$ in the following way:

$$P_M(x) = \{m' \in M; \|x - m'\| = \inf_{m \in M} \|x - m\|\}.$$

Clearly, for an $x \in X$ if may happen that $P_M(x) = \emptyset$.

We define [1]

$$D_{P_M} = \{x \in X; P_M(x) \neq \emptyset\}.$$

We say that M has property * in X, if for all $x \in D_{P_M}$, $m \in M$

$$\inf_{m' \in P_M(x)} \|m - m'\| = \|x - m\| - \inf_{m'' \in M} \|x - m''\|.$$

We say that M has the $1\frac{1}{2}$ ball property, if the relations $m \in M$, $x \in X$, r_1, $r_2 \geq 0$, $M \cap B(x, r_2) \neq \emptyset$ and $\|x - m\| < r_1 + r_2$ imply

$$M \cap B(x, r_2) \cap B(m, r_1) \neq \emptyset.$$

GODINI raised the problem if there exists any normed linear space X with a closed subspace M such that

1. $D_{P_M} \neq M$.
2. M has the property * in X.
3. M does not have the $1\frac{1}{2}$ ball property in X.

In this paper we give such an example.

The example

Let $(X, \|.\|)$ be a locally uniformly convex, non-reflexive Banach space. Let us introduce the Banach space $(Y, \|\cdot\|_1)$ in the following way:

$$Y = X \oplus R,$$

$$\|(x, r)\| = \|x\| + |r|.$$

Because of the non-reflexivity of $(X, \|\cdot\|)$ there is an $f \in (X, \|\cdot\|)^*$, $\|f\| = 1$ such that f has no maximal value on $B_X(0, 1)$ (James's theorem, see, for example [2]). We set $Z = \operatorname{Ker} f$. It is well-known [3] that for all $x \in X \setminus Z$, $P_Z(x) = \emptyset$. We define

$$\tilde{Z} = \{(z, 0); z \in Z\},$$

and show that

$$P_{\tilde{Z}}(z_1, r) = \{(z_1, 0)\} \text{ for all } z_1 \in Z, r \in \mathbf{R}. \tag{1}$$

Clearly, $\|(z_1, r) - (z_1, 0)\|_1 = |r|$, and for all

$$(z, 0) \in \tilde{Z},$$

$$\|(z_1, r) - (z, 0)\|_1 = \|z_1 - z\| + |r| \geq |r|.$$

Here we have strict inequality if $z_1 \neq z$.

So, (1) is proved.

We show that for all $(x, r) \in Y$, $x \in X \setminus Z$,

$$P_{\tilde{Z}}(x, r) = \emptyset. \tag{2}$$

Let us assume to the contrary that $(z, 0) \in P_{\tilde{Z}}(x, r)$.

This means that

$$\inf_{z' \in Z} \|(x, r) - (z', 0)\|_1 = \inf_{z' \in Z} \|x - z'\| + |r| = \|(x, r) - (z, 0)\|_1 =$$
$$= \|x - z\| + |r|.$$

So, we have

$$\inf_{z' \in Z} \|x - z'\| = \|x - z\|,$$

and this contradicts $P_Z(x) = \emptyset$, (2) is proved.

(1) and (2) together imply

$$D_{P_{\tilde{Z}}} = \{(z, r); z \in Z, r \in R\}.$$

We prove now that $\tilde{Z}$ has property $*$ in Y.

For all $z \in Z$,

$$\|(z, 0) - P_{\tilde{Z}}(z_1, r_1)\|_1 = \|(z, 0) - (z_1, 0)\|_1 = \|z - z_1\|$$

and

$$\|(z_1, r)-(z, 0)\|_1 - \|(z_1, r) - P_{\tilde{Z}}(z_1, r)\|_1 = \|z_1 - z\| + |r| -$$
$$- \|(z_1, r) - (z_1, 0)\|_1 = \|z_1 - z\|.$$

Summing up, $\tilde{Z}$ has property $*$ in Y.

Finally, we show that $\tilde{Z}$ does not have the $1\frac{1}{2}$ ball property in Y.

Let $\bar{x} \in X \setminus Z$,

(3) $$\inf_{z \in Z} \|x - z\| = 1.$$

Let us introduce the element $(\bar{x}, 0) \in Y$. Fix a $\bar{z} \in \tilde{Z}$ with the property

(4) $$\|\bar{z} - \bar{x}\| = 2.$$

Let $\varepsilon > 0$ be arbitrary. Because of (3),

(5) $$B((\bar{x}, 0), 1+\varepsilon) \cap \tilde{Z} \neq \varnothing .$$

(4) implies

(6) $$\|(\bar{x}, 0) - (\bar{z}, 0)\| = 2 < 1 + 1 + \varepsilon.$$

On the other hand, we shall see that

(7) $$\tilde{Z} \cap B((\bar{x}, 0), 1+\varepsilon) \cap B((\bar{z}, 0), 1) = \varnothing$$

if $\varepsilon > 0$ is sufficiently small.

Clearly,

$$(x^*, 0) = \left(\frac{\bar{z}(1+\varepsilon)+\bar{x}}{2+\varepsilon}, 0\right) \in B((\bar{x}, 0), 1+\varepsilon) \cap B((\bar{z}, 0), 1).$$

Also,

$$\inf_{z \in Z} \|x^* - z\| = \frac{1}{2+\varepsilon} > \frac{1}{3}$$

in the case $\varepsilon < 1$.

Applying the local uniform convexity of $(X, \|\cdot\|)$, the diameter of the set

$$B((\bar{x}, 0), 1+\varepsilon) \cap B((\bar{z}, 0), 1)$$

becomes less than $\frac{1}{3}$ if ε is sufficiently small. Using (8), we obtain (7). Qu. e.d.

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THERE EXISTS A BASIS OF MINIMAL VECTORS IN EVERY $n \leq 7$ DIMENSIONAL PERFECT LATTICE

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(Received March 3, 1986)

Introduction

Ever since the end of the last century people have been examining those positive definite quadratic forms in n variables which, for a given (arithmetic) minimum (i.e. a minimum over the set of integral vectors of n variables, clearly excluding the zero vector) have the least determinant. The local solutions of this problem are called extreme forms [1].

The necessary condition for the extremality of a form, is that it is completely determined by the value of its arithmetic minimum and the vectors, for which this minimum occurs.

Positive forms satisfying the above condition are called perfect forms. Minkowski showed in [2], that the extreme forms correspond to locally densest lattice packings of congruent n-dimensional spheres.

All the extreme forms for $n \leq 5$ had been found already in 1877 [1], while in the course of the search for perfect forms, those for $n = 6$ (including extreme forms) were found only in 1957 [3]. For $n = 7$ there is no complete solution so far. The examination of perfect and extreme forms is important, because it may be a possible way to the solution of the problem of densest lattice packing of the space E^n, for greater values of n.

In [4] S. Ryškov retraced the search for the 7-dimensional perfect forms to the examination of a 21-dimensional polyhedron given by a limited number of faces. This was made possible by giving a positive answer for $n \leq 7$ to the following question.

For a given n, is there a basis of minimum vectors in every n-dimensional perfect lattice?

Let Π be the parallelepiped (and its volume), spanned by the minimal vectors of a lattice Γ. The index of Π with respect to Γ is defined by ind $\Pi = \Pi/V$, denoting by Π, resp. V, the volume of Π and that of the basic parallelepipedon of Γ. This index is an integer ≥ 1.

Let us examine the value

$$\Delta_n = \max \{\min \operatorname{ind} \Pi\}$$

where the minimum is taken on the set of parallelepipeds Π of Γ and the maximum on the set of all n-dimensional perfect lattices.

Now the above question will be:

For what value of n does $\Delta_n = 1$ hold?

The case of the lattice based on the 4-dimensional space centered cube shows that not every nimimum-parallelepiped will be a basic one; here the index of the cube is 2, while three of its edges and the half of one of its diagonal span a parallelepiped of index 1, that is, we obtain a basic parallelepiped.

It is known that

$\Delta_n = 1$	when $n \leq 6$,	RANKIN, 1964,
$\Delta_7 = 1$		WATSON, 1971,
$\Delta_n \geq 2$	when $n \geq 9$,	COXETER, 1951.

In this work we shall give a simpler proof for $\Delta_n = 1$, $n \leq 7$. The method of the proof can be applied directly for $n = 8$ as well, The proof $\Delta_8 = 1$ will appear in a forthcoming paper.

1. §. Important definitions and notations

1.1. The shortest vectors of a lattice are called minimal vectors. Generally, we regard these vectors as ones starting from the origo of the basis of the lattice. In enumerating them, we do not distinguish between a vector and its opposite. They will be taken as vectors of unit length. If the minimal vectors $\mathbf{e}_1 \dots \mathbf{e}_n$ in an n-dimensional lattice are linearly independent, then we say, that they form a parallelepiped of minima and we denote it by $\Pi = \Pi(\mathbf{e}_1, \dots, \mathbf{e}_n)$. From now on $\mathbf{e}$ and $\mathbf{m}$ will always indicate minimal vectors.

1.2. The frames and the positive quadratic forms.

In a space E^n we fix an orthonormal system of coordinates and consider in it an n-dimensional frame B with the origin of E^n as its starting point. Let the coordinates of the vectors of $\mathbf{b}_j$ be $\zeta_{1j} \dots \zeta_{nj}$. Thus with the frame B we can associate its coordinate matrix $B = (\zeta_{ij})$. The Gram matrix of B is the symmetric matrix $A = (a_{ij})$ with $a_{ij} = \mathbf{b}_i \mathbf{b}_j$. Obviously the relation $A = B^T B$ is valid and A is independent from the choice of the coordinate system. The necessary and sufficient condition for the congruence of two n-dimensional frames, is that after renumbering their vectors, if it is necessary, their Gram-matrices coincide.

The volume V of the parallelepiped spanned by the vectors of B satisfies

$$V^2 = (\det B)^2 = (\det B^T B) = \det A.$$

Corresponding to the matrix $A = (a_{ij})$ there is a quadratic form of n variables

$$f = f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \sum_{i,j}^{n} a_{ij} x_i x_j. \tag{1}$$

It is called the metric form of the frame B.

It is known that

1°: the quadratic form (1), is positive;

2°: for every positive quadratic form (1), there is a frame B in E^n, whose Gram-matrix coincides with the matrix of the form

3°: up to a congruence, this frame is unique.

On the basis of 1°–3° the correspondence $f \to \Gamma_f$ can be given, where Γ_f is the lattice based on the frame B.

1.3. In the rows $k = (k_1, \ldots, k_n) \neq (0, \ldots, 0)$ consisting of integer numbers, every positive form $f(x)$ attains its smallest value, $\min f > 0$, only a limited number of times; $\min f > 0$ is called its minimum. These rows are minimum representations, their elements are the coordinates of the minimal vectors in a certain B_f-basis of the lattice Γ_f. A form f is said to be perfect if its $\min f$ value and its minimum representations completely determine it. Thus Γ_f is a perfect lattice.

It is also known [8], that in perfect lattices there are at least $N = n(n+1)/2$ minimal vectors and among them n are linearly independent, these forming a p. m.

1.4. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a partially open parallelepiped of Γ, where $\mathbf{x} \in B$ if $\mathbf{x} = \sum_1^n x_i \mathbf{b}_i$ and $0 \leq x_i < 1$. Consider the n-lattice Γ^*, constructed on the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ as basic vectors. Γ will be given, in general, by a centering [4; 9] of Γ^*. In the case, when $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is a basic frame of Γ, we say that the centering is trivial. It is evident, that if the centering is not trivial, Γ has points in or on the boundary of the parallelepiped B other than its vertices. Therefore, we will often speak of Γ as the centering of the parallelepiped B. We consider only those centerings, which don't involves vectors shorter than the minimal vectors of the centered lattice. In order to stress this we often will speak of "admissible centerings".

Throughout this paper we consider only those lattices, which have a p.m. (denoted by Π in what follows).

1.5. Suppose Γ is a given n-lattice and $\Pi = \Pi(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ a p.m. in it with edges $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Consider the lattice Γ be a non-trivial centering of Γ^*, then Γ has points $\alpha_1, \ldots, \alpha_{p-1}$ in or on the boundary of Π, other than its vertices. Γ itself is the union of lattices Γ^*, $\alpha_1 + \Gamma^*, \ldots, \alpha_{p-1} + \Gamma^*$. The number p is the volume of Π, relative to the volume of a basic parallelepiped of Γ, it is called the index of Π (ind Π) or the index of centering [4].

We call vectors $\alpha_l (1 \leq l \leq p-1)$ defining of the centering, or, their coordinate rows, defining rows. The centering is defined when we know the vectors $\{\alpha_l\}$.

When we fix the order of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and their signs the coordinates of vectors $\{\alpha_l\}$ determine the type of centering. Consider all parallelepipeds $\Pi(\pm e_{t_1}, \ldots, \pm e_{t_n})$ which generate 2^n equivalent types (some of these types can simply coincide). These types form a class of centering.

Progressing upwards in the dimension, the first admissible centering of a p.m. is the spacecentering in E^4, when the parallelepiped is a cube, because its half-diagonals will be exactly of length 1.

A centering is called free centering when further new minimal vectors do not necessarily appear in it. The first example is the space centering in 5 dimensions, for if the minimum parallelepiped is a cube, then the shortest of the new vectors will be of length $\frac{1}{2}\sqrt{5} > 1$.

2. §. The application of the theory of centerings in the proof of lemmas

2.1. Since we are proving a statement concerning perfect lattices, on the basis of 1.3. we can restrict ourselves to lattices Γ, which include at least one minimum parallelepiped $\Pi_0 = \Pi_0(e_1, \ldots, e_n)$. Usually, this is not a basic parallelepiped in Γ. The lattice Γ^*, based on Π_0 is a sublattice of Γ, so Γ is a result of the centering of Γ^*. This centering is obviously "admissible", i.e., in Γ there is no shorter vector than $|e_i| = 1$. We give the lattice in the system $\{e_i\}$ so not all coordinates of all lattice-points are integers. All such admissible centerings for $n \leq 7$ can be found in the Table, which was taken from [4]. From each class of the centerings only one element is indicated, this is called "canonical". The centerings by means of which the lattices Γ are obtained from Π_0 $(e_1, \ldots e_n)$ were chosen to coincide with the canonical types from the given classes.

Obviously, $\det \Pi_0 = 1$ and since $\operatorname{ind} \Pi_0 = p$, for an arbitrarily chosen Π we have

$$\operatorname{ind} \Pi = p \cdot |\det \Pi|, \tag{2}$$

where Π indicates the coordinate matrix of the edges of the p.m. as well.

2.2. Table for the admissible centerings of the n-dimensional p.m.-s. In the Table it can be seen that the number of centering vectors in the last column in certain centerings is smaller than $p-1$ (see No. 5–6 and 8–14). Not all of them are there, but only those from which the others can be calculated (defining basis), so its number can be less than $p-1$. For example, in the case No. 6 for $p = 3$, we must have two defining vectors, but α_1 is on third of the diagonal of a 6-dimensional parallelepipedon, consequently $\alpha_2 = 2\alpha_1$.

2.3 Equivalent places. In the coordinate rows of a canonical centering, the places j and k (or the coordinates) are called equivalent if in all defining vectors α_i the coordinates α_{ij} and α_{ik} are equal. It can be easily shown, that in any latticepoint the equivalent coordinates are congruent modulo (1), though they are not necessarily equal. In the Table the equivalent coordinates are grouped; for better comprehension we will indicate these groups with ♀ from now on.