Supplement For: Identification In Missing Data Models Represented By Directed Acyclic Graphs

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1 APPENDIX

A. Proofs

Proposition 1 *Given a DAG* $\mathcal{G}(\mathbf{X^{(1)}}, \mathbf{R}, \mathbf{O}, \mathbf{X})$*, the distribution* $p(R_i | \operatorname{pa}_{\mathcal{G}}(R_i))|_{\operatorname{pa}_{\mathcal{G}}(R_i) \cap \mathbf{R} = \mathbf{1}}$ *is identifiable from* $p(\mathbf{R}, \mathbf{O}, \mathbf{X})$ *if there exists*

(i) $\mathbf{Z} \subseteq \mathbf{X^{(1)}} \cup \mathbf{R} \cup \mathbf{O}$,

(ii) an equivalence relation $\sim$ *on* $\mathbf{Z}$ *such that* $\{R_i\} \in {}^{\mathbf{Z}}/_{\sim}$,

(iii) a set of elements $\mathbf{X}^{(1)}_{\widetilde{\mathbf{Z}}}$ *such that* $\mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}} \subseteq \mathbf{X}^{(1)}_{\widetilde{\mathbf{Z}}} \subseteq \mathbf{X}^{(1)}$ *for each* $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$,

(iv) $\mathbf{X^{(1)}} \cap \operatorname{pa}_{\mathcal{G}}(R_i) \subseteq (\mathbf{Z} \setminus \{R_i\}) \cup \mathbf{X}^{(1)}_{\{R_i\}}$,

(v) and a valid fixing schedule $\lhd$ *for* ${}^{\mathbf{Z}}/_{\sim}$ *in* $\mathcal{G}$ *such that for each* $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$, $\widetilde{\mathbf{Z}} \lhd \{R_i\}$.

Moreover, $p(R_i | \operatorname{pa}_{\mathcal{G}}(R_i))|_{\operatorname{pa}_{\mathcal{G}}(R_i) \cap \mathbf{R} = \mathbf{1}}$ *is equal to* $q_{\{R_i\}}$*, defined inductively as the denominator of (4) for* $\{R_i\}$*,* $\phi_{\lhd_{\{R_i\}}}(\mathcal{G})$ *and* $\phi_{\lhd_{\{R_i\}}}(p; \mathcal{G})$*, and evaluated at* $\operatorname{pa}_{\mathcal{G}}(R_i) \cap \mathbf{R} = \mathbf{1}$.

Proof. We first outline the essential argument made in this proof. We will reformulate the process of fixing according to a partial order in a missing data problem as a problem of ordinary fixing based on a total order in a causal inference problem where, previously missing variables are in fact observed. If we are able to show this, we can invoke results from [1], that guarantee that we obtain the desired conditional for each R_i.

Consider $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$, and define $\mathbf{X}^{(1)}_{\{\unlhd \widetilde{\mathbf{Z}}\}} \equiv \bigcup_{\mathbf{Z} \in \{\unlhd \widetilde{\mathbf{Z}}\}} \mathbf{X}^{(1)}_{\mathbf{Z}}$, and $\mathbf{R}_{\{\unlhd \widetilde{\mathbf{Z}}\}} \equiv \{R_k | X^{(1)}_k \in \mathbf{X}^{(1)}_{\{\unlhd \widetilde{\mathbf{Z}}\}}\}$, and similarly for $\mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}$ and $\mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}}$.

We first note that any total ordering $\prec$ on $\{\lhd \widetilde{\mathbf{Z}}\}$ consistent with $\lhd$ yields a valid fixing sequence on sets in $\{\lhd \widetilde{\mathbf{Z}}\}$ in $\mathcal{G}(\mathbf{R}, \mathbf{O}, \mathbf{X^{(1)}}, \mathbf{X}))$, where $\mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{R}, \mathbf{O}, \mathbf{X}$ are observed. The total ordering $\prec$ can be refined to operate on single variables where each set $\widetilde{\mathbf{Z}}$ is fixed as singletons following a topological total order where variables with no children in $\widetilde{\mathbf{Z}}$ would be fixed first. Such a total order is also valid and follows from the validity of $\lhd$ and the fact that at each step of the fixing operation in the total order, the Markov blanket of each Z contains only observed variables; hence no selection bias is induced on any singleton variables $\{\succ \widetilde{\mathbf{Z}}\}$.

We now show, by induction on the structure of the partial order $\lhd$, that for a particular $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$, $q_{\widetilde{\mathbf{Z}}}$ is equal to

$$\prod_{\mathbf{Z} \in \mathcal{Z}} \prod_{Z \in \mathbf{Z}} \tilde{q}(Z | \operatorname{mb}_{\tilde{\mathcal{G}}}(Z; \operatorname{an}_{\tilde{\mathcal{G}}}(\mathbf{D_Z}) \cap \prec_{\tilde{\mathcal{G}}} \{Z\}, \mathbf{R_Z})|_{(\mathbf{R} \cap \mathbf{Z}) \cup \mathbf{R_Z} = \mathbf{1}}, \tag{1}$$

obtained from a kernel

$$\tilde{q} \equiv \phi_{\{\lhd \widetilde{\mathbf{Z}}\}}(p(\mathbf{R}, \mathbf{O}, \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{X}); \mathcal{G}),$$

and CADMG

$$\tilde{\mathcal{G}} \equiv \phi_{\{\lhd \widetilde{\mathbf{Z}}\}}(\mathcal{G}(\mathbf{R}, \mathbf{O}, \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{X})),$$

where $\mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{R}, \mathbf{O}, \mathbf{X}$ are observed.

For any $\lhd$-smallest $\widetilde{\mathbf{Z}}$, $\widetilde{\mathbf{Z}}$ is independent of $\mathbf{R}_{\{\unlhd \widetilde{\mathbf{Z}}\}}$ given its Markov blanket; therefore treating $\mathbf{X}^{(1)}_{\{\unlhd \widetilde{\mathbf{Z}}\}}$ as observed results in the same kernel as $q_{\widetilde{\mathbf{Z}}}$.

We now show that the above is also true for any $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$. Assume the inductive hypothesis holds for all $\widetilde{\mathbf{Y}} \in \{\lhd \widetilde{\mathbf{Z}}\}$. Since $\lhd$ is valid, we obtain $q_{\widetilde{\mathbf{Z}}}$ by applying

$$\phi_{\lhd_{\widetilde{\mathbf{Z}}}}(q; \mathcal{G}) \equiv \phi_{\widetilde{\mathbf{Z}}}\Big(\frac{p(\mathbf{O}, \mathbf{X}, \mathbf{R} \setminus \mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}} = \mathbf{1})}{\prod_{\widetilde{\mathbf{Y}} \in \{\lhd \widetilde{\mathbf{Z}}\}} q_{\widetilde{\mathbf{Y}}}}; \phi_{\lhd_{\widetilde{\mathbf{Z}}}}(\mathcal{G})\Big), \tag{2}$$

where $q_{\widetilde{\mathbf{Y}}}$ are defined by the inductive hypothesis, and $\phi_{\widetilde{\mathbf{Z}}}$ is defined via

$$\frac{q(\mathbf{V} \setminus ((\mathbf{X^{(1)}} \setminus \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}) \cup \mathbf{R_Z}), \mathbf{R_Z} = \mathbf{1} | \mathbf{W})}{\prod_{\mathbf{Z} \in \mathcal{Z}} \prod_{Z \in \mathbf{Z}} q(Z | \operatorname{mb}_{\tilde{\mathcal{G}}}(Z; \operatorname{an}_{\tilde{\mathcal{G}}}(\mathbf{D_Z}) \cap \prec_{\tilde{\mathcal{G}}} (Z)), \mathbf{R_Z})|_{(\mathbf{R} \cap \mathbf{Z}) \cup \mathbf{R_Z} = \mathbf{1}}}, \quad (3)$$

where

$$q(\mathbf{V} \setminus (\mathbf{X^{(1)}} \setminus \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}) | \mathbf{W}) \equiv \frac{p(\mathbf{O}, \mathbf{X}, \mathbf{R} \setminus \mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}} = \mathbf{1})}{\prod_{\widetilde{\mathbf{Y}} \in \{\lhd \widetilde{\mathbf{Z}}\}} q_{\widetilde{\mathbf{Y}}}}.$$

Consider the equivalent functional in the model where we observe $\mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}$

$$\frac{q^{\dagger}(\mathbf{V} \setminus ((\mathbf{X^{(1)}} \setminus \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}) \cup \mathbf{R_Z}), \mathbf{R_Z} = \mathbf{1} | \mathbf{W})}{\prod_{\mathbf{Z} \in \mathcal{Z}} \prod_{Z \in \mathbf{Z}} q^{\dagger}(Z | \operatorname{mb}_{\tilde{\mathcal{G}}}(Z; \operatorname{an}_{\tilde{\mathcal{G}}}(\mathbf{D_Z}) \cap \prec_{\tilde{\mathcal{G}}} (Z)), \mathbf{R_Z})|_{(\mathbf{R} \cap \mathbf{Z}) \cup \mathbf{R_Z} = \mathbf{1}}}, \quad (4)$$

where

$$q^{\dagger}(\mathbf{V} \setminus (\mathbf{X^{(1)}} \setminus \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}) | \mathbf{W}) \equiv \frac{p(\mathbf{O}, \mathbf{X}, \mathbf{X}^{(1)}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \mathbf{R} \setminus \widetilde{\mathbf{R}}_{\{\lhd \widetilde{\mathbf{Z}}\}}, \widetilde{\mathbf{R}}_{\{\lhd \widetilde{\mathbf{Z}}\}} = \mathbf{1})}{\prod_{\widetilde{\mathbf{Y}} \in \{\lhd \widetilde{\mathbf{Z}}\}} q_{\widetilde{\mathbf{Y}}}},$$

and $\widetilde{\mathbf{R}}_{\{\lhd \widetilde{\mathbf{Z}}\}}$ is defined as the subset of $\mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}}$ that is fixed in $\{\lhd \widetilde{\mathbf{Z}}\}$.

The only difference between (3) and (4) for the purposes of the denominator is the variables in $\mathbf{R}_{\{\lhd \widetilde{\mathbf{Z}}\}} \setminus \widetilde{\mathbf{R}}_{\{\lhd \widetilde{\mathbf{Z}}\}}$. But the denominator is independent of these variables, by assumption. Thus, it follows that fixing on a valid partial order with missing data and fixing on a total order consistent with this partial order, as in causal inference, yield equivalent kernels.

The conclusion follows by Lemma 55 in [1].

□

Lemma 2 *Consider a DAG $\mathcal{G}(\mathbf{X^{(1)}}, \mathbf{R}, \mathbf{O}, \mathbf{X})$ such that for every $R_i \in \mathbf{R}$, $\{R_j | X^{(1)}_j \in \operatorname{pa}_{\mathcal{G}}(R_i)\} \cap \operatorname{an}_{\mathcal{G}}(R_i) = \emptyset$. Then for every $R_i \in \mathbf{R}$, a fixing schedule $\lhd$ for $\{\{R_j\} | R_j \in \mathcal{G}_{\mathbf{R} \cap \operatorname{de}_{\mathcal{G}}(R_i)}\}$ given by the partial order induced by the ancestrality relation on $\mathcal{G}_{\mathbf{R} \cap \operatorname{de}_{\mathcal{G}}(R_i)}$ is valid in $\mathcal{G}(\mathbf{X^{(1)}}, \mathbf{R}, \mathbf{O}, \mathbf{X})$, by taking each $\mathbf{X}^{(1)}_{\widetilde{\mathbf{Z}}} = \bigcup_{\mathbf{Z} \in \{\unlhd \widetilde{\mathbf{Z}}\}} \mathbf{X}^{(1)}_{\mathbf{Z}}$, for every $\widetilde{\mathbf{Z}} \in \{\unlhd \{R_i\}\}$. Thus the target law is identified.*

Proof. In order to prove that the target law is identified, we demonstrate that conditions (i-v) in Proposition 1 are satisfied for each R_i.

Conditions (i) and (ii) are trivially satisfied as we choose to fix $\mathbf{Z} \subseteq \mathbf{R}$, and we choose no equivalence relation, thus $^{\mathbf{Z}}/_{\sim}$ consists of singleton sets of Rs. Condition (iii) is also trivial as each $\mathbf{X}^{(1)}_{\widetilde{\mathbf{Z}}}$ is a union of the corresponding sets $\mathbf{X}^{(1)}_{\widetilde{\mathbf{Y}}}$, for $\widetilde{\mathbf{Y}}$ earlier in the partial order. In the proposed order we never fix elements in $\mathbf{X^{(1)}}$, and propose to keep elements in $\mathbf{X^{(1)}} \cap \operatorname{pa}_{\mathcal{G}}(R_j)$ for every $R_j \in \mathbf{Z}$. In particular, this also includes R_i, satisfying condition (iv).

Finally, we show that the proposed schedule $\lhd$ is valid by showing that each $\widetilde{\mathbf{Z}} \in {}^{\mathbf{Z}}/_{\sim}$ is fixable. There are 3 conditions for an element $\widetilde{\mathbf{Z}}$ to be fixable as mentioned in section 5. We go through each of these conditions and demonstrate each $\widetilde{\mathbf{Z}}$ in $^{\mathbf{Z}}/_{\sim}$ is a valid fixing in $\phi_{\lhd_{\widetilde{\mathbf{Z}}}}(\mathcal{G})$ where $\lhd$ is the proposed fixing schedule above.

In the proposed schedule each $\widetilde{\mathbf{Z}}$ is a singleton $R_j \in {}^{\mathbf{Z}}/_{\sim}$ that we are trying to fix in a graph $\phi_{\lhd_{R_j}}(\mathcal{G})$. Since $\mathbf{X}^{(1)}_{R_j} = \mathbf{X^{(1)}}$, $\phi_{\lhd_{R_j}}(\mathcal{G})$ is a CDAG. Thus, $\mathcal{D}(\phi_{\lhd_{R_j}}(\mathcal{G}))$ is just sets of singleton vertices. In particular, $\mathbf{D}_{R_j} = \{R_j\}$. Further, by definition of the schedule, it must be that $\operatorname{de}_{\phi_{\lhd_{R_j}}(\mathcal{G})}(R_j) = \{R_j\}$. This satisfies condition (i).

For condition (ii), we note that $\mathbf{S} \subseteq \operatorname{nd}_{\phi_{\lhd_{R_j}}(\mathcal{G})}(R_j)$ else, $\mathbf{S}$ contains some $R_k \in \operatorname{de}_{\mathcal{G}}(R_j)$ which should have been fixed prior to R_j by the proposed partial order. Thus, it follows that $\mathbf{S} \cap \{R_j\} = \emptyset$.

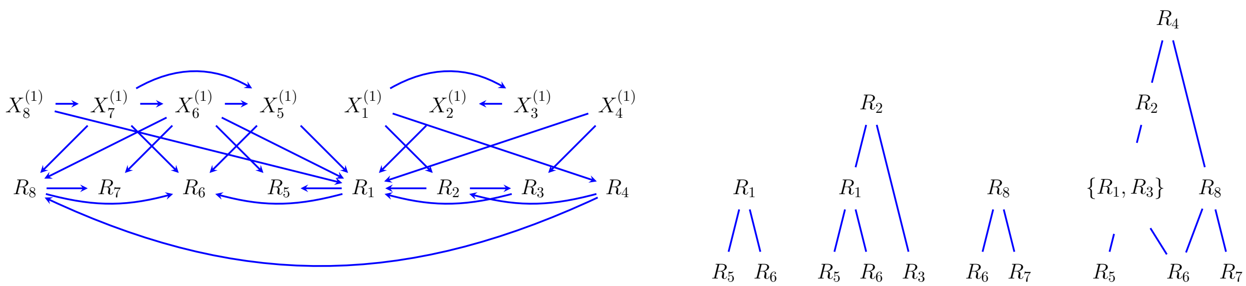
Finally, following the partial order, and under the assumption stated in the lemma, $\mathbf{R}_{\{R_j\}} \subseteq \{\lhd R_j\}$. We have also proved that $\mathbf{S} \subseteq \operatorname{nd}_{\phi_{\lhd_{R_j}}(\mathcal{G})}(R_j)$. Therefore, $R_j \perp\!\!\!\perp (\mathbf{S} \cup \mathbf{R}_{\{R_j\}}) \setminus \operatorname{mb}_{\phi_{\lhd_{R_j}}(\mathcal{G})}(R_j) | \operatorname{mb}_{\phi_{\lhd_{R_j}}(\mathcal{G})}(R_j)$.

Since each $\widetilde{\mathbf{Z}}$ is fixable, the proposed partial order $\lhd$ for each R_i is valid. Therefore, all five conditions in Proposition 1 are satisfied concluding the target law is ID.

□

B. An example to illustrate the algorithm

We walk the reader through identification of the target law for the missing data DAG shown in Fig. 1(a) in order to demonstrate the full generality of our missing ID algorithm. As a reminder, the target law is identified by (2) if we are able to identify $p(R_i | \operatorname{pa}_{\mathcal{G}}(R_i))|_{\mathbf{R}=\mathbf{1}}$ for each $R_i \in \mathbf{R}$. The identification of these conditional densities are shown in equations (i) through (viii). For a clearer presentation of this example, we switch to one-column format.



(a) $\mathcal{G}$ (b) (c) (d) (e)

Figure 1: (a) A complex missing data DAG used to illustrate the general techniques used in our algorithm (b-e) The corresponding fixing schedules of Rs.

We start with $\{R_3, R_5, R_6, R_7\}$. The fixing schedules for these are empty and we obtain the following immediately from the original distribution.

(i) $p(R_3|\,\text{pa}(R_3)) = p(R_3|R_2, X_4^{(1)}) = p(R_3|R_2, X_4, \mathbf{1}_{R_4})$,

(ii) $p(R_5|\,\text{pa}(R_5)) = p(R_5|R_1, X_6^{(1)}) = p(R_5|R_1, X_6, \mathbf{1}_{R_6})$,

(iii) $p(R_6|\,\text{pa}(R_6)) = p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)}) = p(R_6|R_1, R_8, X_5, X_7, \mathbf{1}_{R_5,R_7})$,

(iv) $p(R_7|\,\text{pa}(R_7)) = p(R_7|R_8, X_6^{(1)}) = p(R_7|R_8, X_6, \mathbf{1}_{R_6})$.

For R_1, we choose $\mathbf{Z} = \{R_1, R_5, R_6\}$, and no equivalence relations. Thus, ${}^{\mathbf{Z}}/_{\sim} = \{\{R_1\}, \{R_5\}, \{R_6\}\}$. The fixing schedule $\lhd$ is a partial order shown in Fig. 1(b) where R_5 and R_6 are incompatible, and $R_5 \prec R_1$, $R_6 \prec R_1$. Starting with the original $\mathcal{G}$ in Fig. 1(a), fixing R_5 and R_6 in parallel yields the following kernel.

$$q_{r_1}(\mathbf{X} \setminus \{X_5, X_6\}, X_5^{(1)}, X_6^{(1)}, \mathbf{R} \setminus \{R_5, R_6\}|\mathbf{1}_{R_5,R_6}) = \frac{p(\mathbf{X}, \mathbf{R} = \mathbf{1})}{p(R_5|R_1, X_6^{(1)})\; p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)})|_{\mathbf{R}=\mathbf{1}}}, \quad (5)$$

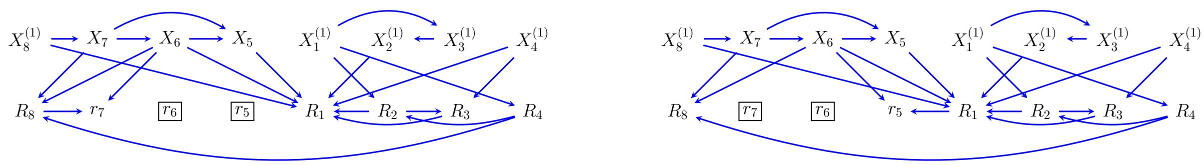
where the propensity scores in the denominator are identified using (ii) and (iii). The CADMG corresponding to this fixing operation is shown in Fig. 2(a).

$$\begin{aligned}
\text{(v)} \quad p(R_1|\,\text{pa}(R_1))|_{\mathbf{R}=\mathbf{1}} &= p(R_1|R_2, R_3, X_2^{(1)}, X_4^{(1)}, X_5^{(1)}, X_6^{(1)})|_{\mathbf{R}=\mathbf{1}} \\
&= q_{r_1}(R_1|R_2, R_3, X_2^{(1)}, X_4^{(1)}, X_5, X_6, \mathbf{1}_{R_5,R_6})|_{\mathbf{R}=\mathbf{1}} \\
&= q_{r_1}(R_1|R_3, X_2, X_4^{(1)}, X_5, X_6, \mathbf{1}_{R_2,R_5,R_6})|_{\mathbf{R}=\mathbf{1}} \\
&= q_{r_1}(R_1|R_3, X_2, X_4, X_5, X_6, \mathbf{1}_{R_2,R_4,R_5,R_6})|_{\mathbf{R}=\mathbf{1}} \quad \text{(by d-sep)}
\end{aligned}$$

where the last term can be obtained using kernel operations (conditioning+marginalization) on $q_{r_1}(.|.)$ defined in (5).

A similar procedure is applicable to R_8, where ${}^{\mathbf{Z}}/_{\sim} = \{\{R_8\}, \{R_7\}, \{R_6\}\}$; Fig. 1(d). Starting with the original $\mathcal{G}$ in Fig. 1(a), fixing R_6 and R_7 in parallel yields the following kernel.

$$q_{r_8}(\mathbf{X} \setminus \{X_6, X_7\}, X_6^{(1)}, X_7^{(1)}, \mathbf{R} \setminus \{R_6, R_7\}|\mathbf{1}_{R_6,R_7}) = \frac{p(\mathbf{X}, \mathbf{R} = \mathbf{1})}{p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)})\; p(R_7|R_8, X_6^{(1)})|_{\mathbf{R}=\mathbf{1}}}, \quad (6)$$



(a) $\phi_{\{R_5,R_6\}}(\mathcal{G})$ (b) $\phi_{\{R_6,R_7\}}(\mathcal{G})$

Figure 2: (a) Graph corresponding to the kernel obtained in (5) (b) Graph corresponding to the kernel obtained in (6).

where the propensity scores in the denominator are identified using (iii) and (iv). The CADMG corresponding to this fixing operation is shown in Fig. 2(b).

$$\begin{aligned}\text{(vi)} \quad p(R_8|\,\mathrm{pa}(R_8))|_{\mathbf{R}=\mathbf{1}} &= p(R_8|R_4, X_6^{(1)}, X_7^{(1)})|_{\mathbf{R}=\mathbf{1}} \\ &= q_{r_8}(R_8|R_4, X_6^{(1)}, X_7^{(1)}, \mathbf{1}_{R_6,R_7})|_{\mathbf{R}=\mathbf{1}} \\ &= q_{r_8}(R_8|R_4, X_6, X_7, \mathbf{1}_{R_6,R_7})|_{\mathbf{R}=\mathbf{1}}\end{aligned}$$

where the last term can be obtained using kernel operations (conditioning+marginalization) on $q_{r_8}(.|.)$ defined in (6).

For R_2, we choose $\mathbf{Z} = \{R_1, R_2, R_3, R_5, R_6\}$, and no equivalence relations. Thus, $^{\mathbf{Z}}/_{\sim} = \{\{R_1\}, \{R_2\}, \{R_3\}, \{R_5\}, \{R_6\}\}$. The fixing schedule $\lhd$ is a partial order where R_3, R_5, R_6 are incompatible and $R_5, R_6 \prec R_1 \prec R_2$ and $R_3 \prec R_2$ as shown in Fig. 1(c). In addition, the portion of the fixing schedule involving R_1, R_5, and R_6 is executed in a latent projection ADMG where we treat $X_2^{(1)}$ as being hidden as shown in Fig. 3(a), while the portion of the fixing schedule involving R_3 is executed in the original graph, Fig. 1(a).

$$\text{(vii)} \quad p(R_2|R_4, X_1^{(1)}) = q_{r_2}(R_2|R_4, X_1^{(1)}, \mathbf{1}_{R_1,R_3}), \tag{7}$$

where q_{r_2} corresponds to the kernel obtained by following the partial order of fixing R_3 and R_1, separately. That is,

$$q_{r_2}(.|\mathbf{1}_{R_1,R_3}) = \frac{p(\mathbf{X}, \mathbf{R}=\mathbf{1})}{q^1_{r_2}(R_1|R_2, R_3, X_2, X_5, X_6, X_3^{(1)}, X_8^{(1)}, \mathbf{1}_{R_5,R_6})\; p(R_3|R_2, X_4^{(1)})}. \tag{8}$$

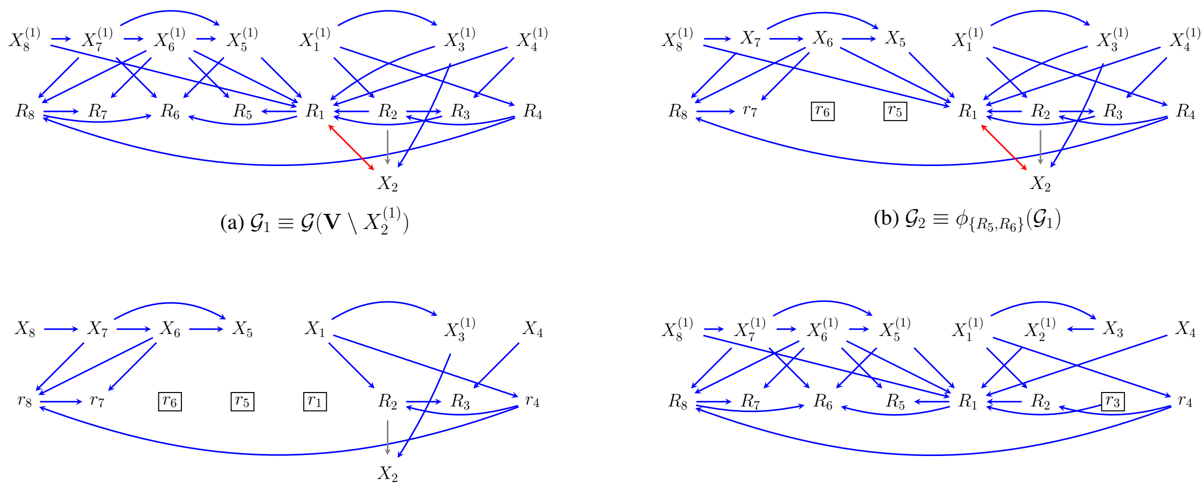
The propensity score for R_3 is obtained from (i) and $q^1_{r_2}$ is the kernel obtained by fixing R_5 and R_6 in parallel in a graph where $X_2^{(1)}$ is treated as hidden, as shown in Figures 3(a) and (b). That is,

$$q^1_{r_2}(\mathbf{X} \setminus \{X_5, X_6\}, X_5^{(1)}, X_6^{(1)}, \mathbf{R} \setminus \{R_5, R_6\}|\mathbf{1}_{R_5,R_6}) = \frac{p(\mathbf{X}, \mathbf{R}=\mathbf{1})}{p(R_5|R_1, X_6^{(1)})\; p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)})|_{\mathbf{R}=\mathbf{1}}}.$$

The propensity scores in the denominator above are identified using (ii) and (iii). For clarity, the CADMGs corresponding to fixing R_1 and R_3 are illustrated in Figures 3(c) and (d).

Finally, for R_4, we choose $\mathbf{Z} = \{\mathbf{R}\}$ and equivalence relation $R_1 \sim R_3$. Thus, $^{\mathbf{Z}}/_{\sim} = \{\{R_1, R_3\}, \{R_2\}, \{R_4\}, \{R_5\}, \{R_6\}, \{R_7\}, \{R_8\}\}$. The fixing schedule $\lhd$ is a partial order where $R_5, R_6 \prec \{R_1, R_3\} \prec R_2 \prec R_4$ and $R_6, R_7 \prec R_8 \prec R_4$ as shown in Fig. 1(e). In addition, the portion of the fixing schedule involving R_5, R_6, $\{R_1, R_3\}$, and R_2 is executed in a latent projection ADMG where we treat $X_2^{(1)}$ and $X_4^{(1)}$ as hidden variables, shown in Fig. 4(b), while the portion of the fixing schedule involving R_6, R_7, and R_8 is executed in the original graph, Fig. 1(a).

$$\text{(viii)} \quad p(R_4|X_1^{(1)}) = q_{r_4}(R_4|X_1^{(1)}, \mathbf{1}_{R_2,R_8}), \tag{9}$$



(a) $\mathcal{G}_1 \equiv \mathcal{G}(\mathbf{V} \setminus X_2^{(1)})$

(b) $\mathcal{G}_2 \equiv \phi_{\{R_5,R_6\}}(\mathcal{G}_1)$

(c) $\mathcal{G}_3 \equiv \phi_{R_1}(\mathcal{G}_2)$

(d) $\phi_{R_3}(\mathcal{G})$

Figure 3: Execution of the fixing schedule to obtain the propensity score for R_1 (a) Latent projection ADMG obtained by projecting out $X_2^{(1)}$ (b) Fixing R_5 and R_6 in $\mathcal{G}_1$ (c) Fixing R_1 in $\mathcal{G}_2$ (d) Fixing R_3 in the original graph.

where q_{r_4} corresponds to the kernel obtained by following the partial order of fixing R_2 and R_8, separately. That is,

$$q_{r_4}(.|\mathbf{1}_{R_2,R_8}) = \frac{p(\mathbf{X}, \mathbf{R} = \mathbf{1})}{q^1_{r_4}(R_2|R_4, X_2)\; q^2_{r_4}(R_8|R_4, X_6, X_7)}. \tag{10}$$

$q^1_{r_4}$ is the kernel obtained by fixing the set $\{R1, R3\}$ in graph $\mathcal{G}_2$ shown in Fig. 4(c). That is,

$$\begin{aligned} q^1_{r_4}(.|\mathbf{1}_{R_1,R_3,R_5,R_6}) &= \frac{q^3_{r_4}(.|\mathbf{1}_{R_5,R_6})}{q^3_{r_4}(R_1, R_3|R_2, R_4, X_2, X_3^{(1)}, X_4)} \\ &= \frac{q^3_{r_4}(.|\mathbf{1}_{R_5,R_6})}{q^3_{r_4}(R_1|R_2, R_4, X_2, X_3, X_4, \mathbf{1}_{R_3})\; q^3_{r_4}(R_3|R_2, R_4, X_2, X_4)} \end{aligned}$$

$q^3_{r_4}$ is the kernel obtained by fixing R_5 and R_6 in parallel in the graph $\mathcal{G}_1$ shown in Fig. 4(b). That is,

$$q^3_{r_4}(.|\mathbf{1}_{R_5,R_6}) = \frac{p(\mathbf{X}, \mathbf{R} = \mathbf{1})}{p(R_5|R_1, X_6^{(1)})\; p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)})|_{\mathbf{R}=\mathbf{1}}}.$$

The propensity scores in the denominator above are identified using (ii) and (iii).

Finally, $q^2_{r_4}$ is the kernel obtained by fixing R_6 and R_7 in parallel in the original graph $\mathcal{G}$, shown in Fig. 1(a). That is,

$$q^2_{r_4}(.|\mathbf{1}_{R_6,R_7}) = \frac{p(\mathbf{X}, \mathbf{R} = \mathbf{1})}{p(R_6|R_1, R_8, X_5^{(1)}, X_7^{(1)})\; p(R_7|R_8, X_6^{(1)})|_{\mathbf{R}=\mathbf{1}}}.$$

The propensity scores in the denominator above are identified using (iii) and (iv). For clarity, the CADMG corresponding to fixing R_8 is illustrated in Figures 4(a).

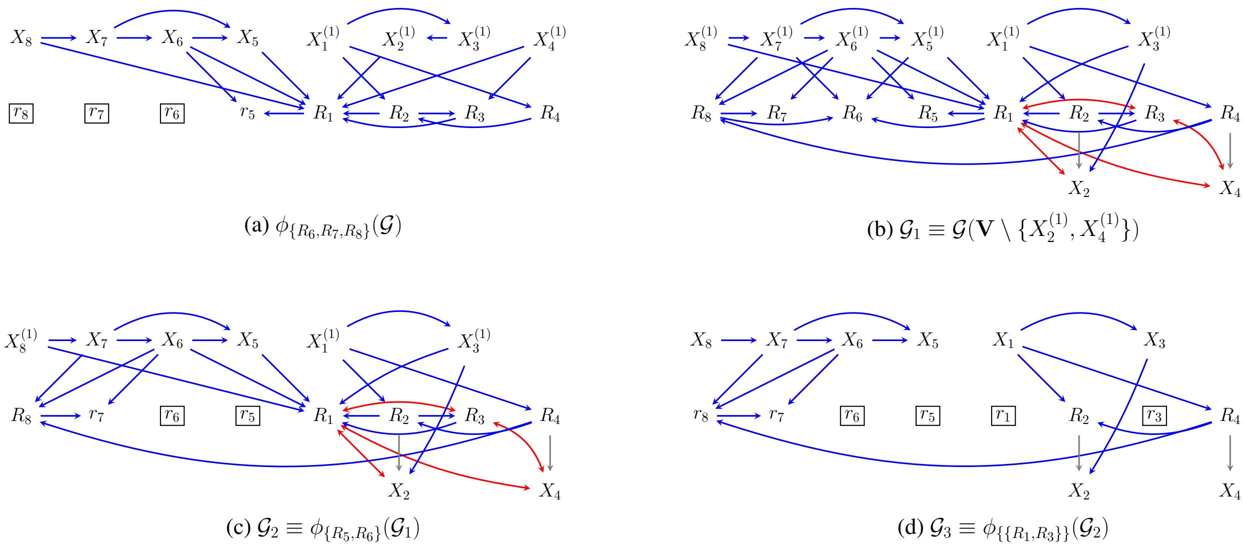
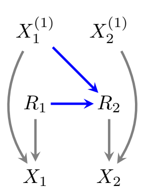


Figure 4: Execution of the fixing schedule to obtain the propensity score for R_4 (a) CADMG obtained by following the schedule to get the propensity score for R_8 (b) Latent projection ADMG obtained by projecting out $X_2^{(1)}$ and $X_4^{(1)}$ (c) Fixing R_5 and R_6 in $\mathcal{G}_1$ (d) Fixing R_1 in $\mathcal{G}_2$.

C. Table for Lemma 1



R_1	$p(R_1)$
0	a
1	$1-a$

$X_1^{(1)}$	$p(X_1^{(1)})$
0	b
1	$1-b$

$X_2^{(1)}$	$p(X_2^{(1)})$
0	c
1	$1-c$

R_2	R_1	$X_1^{(1)}$	$p(R_2 \mid R_1, X_1^{(1)})$
0	0	0	d
1	0	0	$1-d$
0	1	0	e
1	1	0	$1-e$
0	0	1	f
1	0	1	$1-f$
0	1	1	g
1	1	1	$1-g$

R_1	R_2	$X_1^{(1)}$	$X_2^{(1)}$	p(Full Law)	X_1	X_2	p(Observed Law)
0	0	0	0	$abcd$	?	?	$a\Big[db + f(1-b)\Big]$
		1	0	$a(1-b)cf$			
		0	1	$ab(1-c)d$			
		1	1	$a(1-b)(1-c)f$			
1	0	0	0	$(1-a)ebc$	0	?	$(1-a)eb$
		1	0	$(1-a)g(1-b)c$			
		0	1	$(1-a)eb(1-c)$	1		$(1-a)g(1-b)$
		1	1	$(1-a)g(1-b)(1-c)$			
0	1	0	0	$abc(1-d)$	?	0	$ac\Big[1-\Big(db+f(1-b)\Big)\Big]$
		1	0	$a(1-b)c(1-f)$			
		0	1	$ab(1-c)(1-d)$		1	$a(1-c)\Big[1-\Big(db+f(1-b)\Big)\Big]$
		1	1	$a(1-b)(1-c)(1-f)$			
1	1	0	0	$(1-a)(1-e)bc$	0	0	$(1-a)(1-e)bc$
		1	0	$(1-a)(1-g)(1-b)c$	1	0	$(1-a)(1-g)(1-b)c$
		0	1	$(1-a)(1-e)b(1-c)$	0	0	$(1-a)(1-e)b(1-c)$
		1	1	$(1-a)(1-g)(1-b)(1-c)$	1	1	$(1-a)(1-g)(1-b)(1-c)$

Any pair of $\{d, f\}$ would lead to different full laws. However, as long as $db + f(1-b)$ stays constant, the observe law would agree across all different full laws (which include infinitely many models). This is a general characterization of non-identifiable models with two binary random variables.

References

[1] Thomas S. Richardson, Robin J. Evans, James M. Robins, and Ilya Shpitser. Nested Markov properties for acyclic directed mixed graphs. *arXiv:1701.06686v2*, 2017. Working paper.