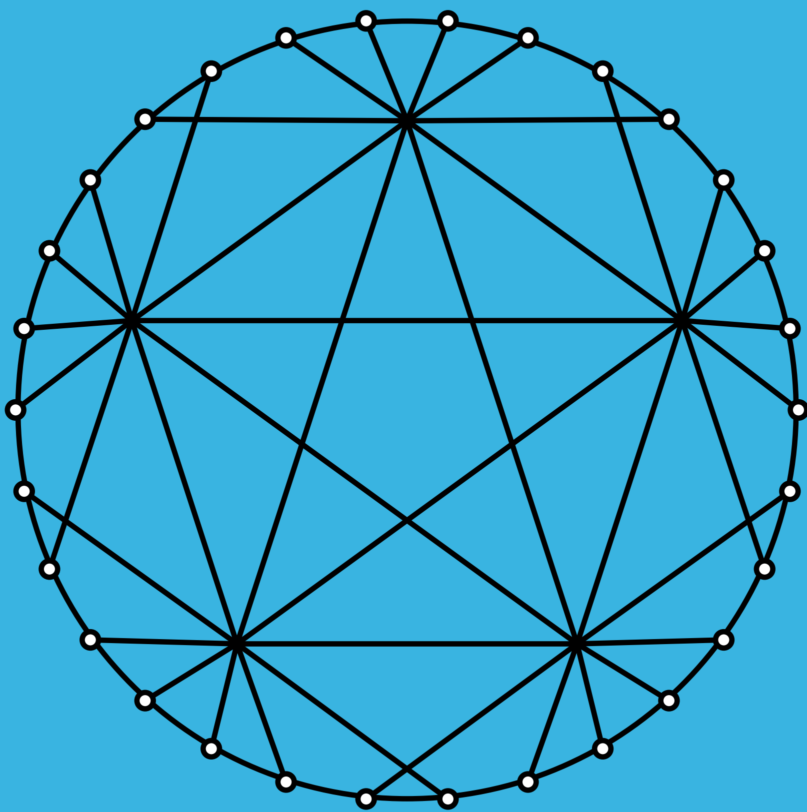
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 93
October 2021



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **93** (2021), 16–44

Anton Kotzig, the founder of graph theory in Slovakia

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Introduction

One of the most prominent Slovak mathematicians Anton Kotzig was born on October 22, 1919 in the village of Kočovce in northwestern Slovakia. He died on April 20, 1991 in Montreal. A brief account of his life and work appeared in [14].

Although a statistician and actuary by training, Kotzig was also well versed in economy and authored many works about applications of mathematics in economy. He also contributed to measurements, additive number theory and game theory. But the center of his scientific activity was graph theory and he is justifiably considered the founder of graph theory in Slovakia [113]. He came to graph theory by studying the book by Hungarian mathematician Dénes König [38] which for a long time was the only book on graph theory in existence. In the fifties and sixties of the last century, Kotzig made many fundamental contributions to graph theory. Unfortunately not all of these contributions attracted the attention that they deserved. They found their way into the well of foundations of graph theory only slowly, partly due to the fact that they were written mainly in Slovak. While Slovak is a wonderful language, it is not widely spoken. Many of Kotzig's fundamental results were rediscovered much later by others. (Some of his early papers were written in German but appeared in obscure journals; starting with early sixties, several of his papers were written in Russian: of course, all of his later papers are written in English, with a few in French.) Kotzig was a many-sided personality. Those who knew him appreciated him as an unending source of original ideas, problems and suggestions. He was very good at attracting young people to research in graph theory. Many have become his collaborators. He was not as well versed in the literature as some of his contemporaries but he made up for it by his inventiveness, often rediscovering on his own already existing results. In his later years he mastered both French and English.

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His scientific career spans two separate periods. He worked in Bratislava, Slovakia until 1969, the year he turned 50. In 1969 he emigrated to Canada where after one year in Calgary he stayed in Montreal until his death in 1991.

At the beginning of the sixties, he founded a very successful graph theory seminar in Bratislava which he himself led until his departure for Canada in 1969. This seminar is one of the longest existing graph theory seminars worldwide and continues to function till this day [113]. Many of its early participants became later successful and well known mathematicians. Let us mention here just a few: Juraj Bosák [112], [114], Štefan Znám [115], Ján Plesník, Pavol Glivjak, Jozef Širáň, Peter Horák, Martin Škoviera, Roman Nedela, Robert Jajcay. The Bratislava graph theory seminar is currently lead by Martin Škoviera.

It is virtually impossible to do justice in assessing Kotzig's graph theory contributions in a short survey article. The influence of his work is enormous and most of his contributions are of lasting value. The proper evaluation of Kotzig's life work awaits an effort of a professional biographer. The best I can do here is to attempt to highlight his most important contributions in graph theory as contained in his numerous publications.

Anton Kotzig

Dissertation “Connectivity and regular connectivity of finite graphs” and related topics

The beginnings of graph theory in Slovakia can safely be described as having their roots in the 1956 opus of Anton Kotzig entitled “Connectivity and regular connectivity of finite graphs” [44] which remains relevant to this day. One can say that in 1956 graph theory was in its infancy; at that time, only one book on graph theory was in existence. It was the book “Theorie der endlichen und unendlichen Graphen” by the Hungarian mathematician Dénes König [38] which appeared in 1936. Any conferences in graph theory still lay in the future. This was the time when graph theory was sometimes referred to as “the slums of topology”. While Kotzig's dissertation contains amazingly many fundamental results, it did not command such attention as it deserved. This is partly due to the fact that it was not distributed widely, and it was written in Slovak. Many results contained in Kotzig's dissertation were rediscovered by others much later. But it is also true that some of the results Kotzig discovered on his own have been known before. Lex Schrijver in his three-volume work “Combinatorial Optimization. Polyhedra and Efficiency” [116] gives a lot of credit to Kotzig. In his dissertation, Kotzig gives a proof of the following version of Menger's Theorem for undirected graphs.

Theorem 35. *Let G be an arbitrary graph containing vertices $u \neq v$ for which $\sigma_G(u, v) = k > 0$ then there exists a system of paths $\{C_1, C_2, \ldots, C_k\}$ such that each path connects vertices u, v and no two distinct paths have an edge in common. Such a system of paths in G exists only if $\sigma_G(u, v) \geq k$.*

Here $\sigma_G(u, v)$ is the minimum size of a $u - v$ cut in G.

Incidentally, in [55] Kotzig gives a proof of the directed arc-disjoint version of Menger's theorem (as Schrijver [116] remarks, without reference to Menger).

As an outgrowth of [44], Kotzig proves in [54] that there exists a unique minimum spanning tree T of a weighted graph G if and only if for each edge $e \notin T$, e is the longest edge in the cycle in $T \cup \{e\}$.

A very general method which has been used extensively in [44] is described in detail and analyzed in [57]. Let C be a class of graphs hereditarily closed for factors, and let T be a property of graphs from C satisfying two simple conditions: 1) no edgeless graph from C has property T, and 2) if a factor F of a graph G has property T then so does G.

If G has property T, a set with the smallest possible number of edges whose deletion causes G to lose property T is a *t-set* of G. The elements of a t-set are *t-edges* and the number of t-edges in a t-set is $\tau(G)$. An edge e of G is a t-edge if and only if its deletion decreases the value of $\tau(G)$ by one. A graph G is a T_k-graph if $\tau(G) = k$ and each edge of G is a t-edge. The following theorem is proved in [57]:

Theorem. *Let G be a graph such that $\tau(G) = k > 0$ and let j be a natural number, $j \leq k$. Then there exists at least one T_j-graph which is a factor of G. On the other hand, if G is an arbitrary T_k-graph then no proper factor of G is a T_k-graph.*

Seven concrete examples of properties T are discussed (where the class C is suitably chosen): 1) G is connected, 2) G is a factor of a given graph G^* and certain two vertices of G are connected, 3) G is a directed graph and for any two its vertices u, v there exists a directed path from u to v, 4) G has at least one 1-factor, 5) G possesses a Hamiltonian path, 6) G is nonplanar, and 7) the chromatic number of G is greater than n.

By [40], each edge of a connected regular graph G of degree n with a 1-factorization must be contained in a cycle of G. Equivalences on a vertex-set based on the degree of connectivity between vertices are considered and properties of partitions into equivalence classes of these relations with respect to the 1-factors of the 1-factorization are studied in [40]. In a connected regular graph of *odd* degree n with a 1-factorization the degree of connectivity between any two distinct vertices must be even or else equal to n.

In nine lemmas and four theorems of [45], relationships between spanning trees of a graph, its cycle bases and bases of cuts ("fundamental system of cuts") are explored. Most of the body of this paper as well as the papers mentioned above is devoted to rigorous proofs, some of them quite difficult.

In [59], interval graphs were called by Kotzig *Hajós graphs*. Kotzig proves that a bipartite graph is an interval graph if and only if each of its connected components is either an isolated vertex, a star, or a caterpillar. Interval graphs have been characterized by Lekkerkerker and Boland in [104].

Polyhedral graphs and Eulerian graphs

In his early often cited paper [41] Kotzig obtained important fundamental results on the weights of edges in polyhedral graphs (i.e. in 3-connected planar graphs). In such a graph, the *face weight* $\sigma_F(h)$ of an edge $h = \{u, v\}$ which is incident with faces S_1 and S_2 equals $\sigma_F(h) = m + n$ where S_1, S_2 is an m=gonal and n-gonal face, respectively. Similarly, the *vertex-weight* $\sigma_V(h) = x + y$ where x, y is the degree of the vertex u and v, respectively.

It is well known that every polyhedral graph contains at least one face with at most five vertices and at least one vertex whose degree is at most five. In [41], Kotzig proves:

Theorem. *In any polyhedral graph there exists an edge h such that its vertex-weight $\sigma_V(h) \leq 13$ and there exists an edge h' such that its face-weight $\sigma_F(h) \leq 13$.*

It is shown that these upper bounds cannot be lowered: if one subdivides each triangular face of an icosahedral graph by creating a new vertex in the center of the face joined by three new edges to the three original vertices of the face then each triangular face of the resulting polyhedral graph is incident with three vertices of which two are of degree 10 and one is of degree 3, thus for each edge h either $\sigma_V(h) = 13$ or $\sigma_V(h) = 20$. In the dual graph we have for each edge h either $\sigma_F(h) = 13$ or $\sigma_F(h) = 20$.

However, if the degree of each vertex is at least 4 then the upper bound for the edge weights can be lowered.

Theorem. *In any polyhedral graph with minimum vertex degree at least 4 there exists an edge h such that $\sigma_V(h) \leq 11$, and in any polyhedral graph whose all faces are polygons with at least 4 vertices there exists an edge h such that $\sigma_F(h) \leq 11$.*

Kotzig provides examples showing that this is best possible, and he also shows that further improvement is impossible even if one assumes that the minimum vertex degree is 5.

Some of Kotzig's results above have been extended by Ivančo, Jendroľ and Tuhársky (see, e.g., [37]) to non-orientable surfaces of genus g.

Almost a decade later, Kotzig returns to this theme in [58]. He concentrates mainly on those polyhedral graphs which are regular of degree d, where of course d must be one of $3, 4$, or 5. From among the many results in this paper, let us mention just two. In any polyhedral graph of degree 4 there exists an edge h such that $\sigma_F(h) \leq 8$, and this bound is best possible. In any polyhedral graph of degree 5 there are at least 30 edges incident with two triangles and thus for these 30 edges we have $\sigma_F(h) = 6$. The icosahedral graph provides such an example.

In [62] Kotzig provides yet another solution of an old problem by Eberhard, namely whether there exists a cubic polyhedral graph with an odd number of faces such that the number of vertices on each face is divisible by 3. A negative answer is obtained via a colouring theorem for the faces of such a polyhedral graph. Eberhard's problem was first solved by Motzkin [108] and then a different proof was given by Grünbaum [31].

In [67], Kotzig defines an n-regular polyhedron to be regularly variegated if there exists a sequence $u_1 < u_2 < \cdots < u_n$ such that for all $i \in \{1, 2, \ldots, n\}$, each vertex of the polyhedron is incident with exactly one u_i-gonal face. He proves that each regularly variegated polyhedral graph must be cubic, and either contains 12 quadrilaterals, 8 hexagons and 6 octagons, or else it contains 30 quadrilaterals, 20 hexagons and 12 decagons. Examples of regularly variegated polyhedra of both types are provided in which each face is a regular polygon.

A connected Eulerian graph G of degree $2n$ with an even number of edges can be partitioned into two factors F_1, F_2 as follows: the edges of any Eulerian trail $E = (e_1, e_2, e_3, \ldots)$ of G are assigned alternately to F_1 and F_2. In [42], Kotzig proves conversely, that any partition of G into two factors of degree n arises in this fashion from some Eulerian trail of G.

Let E_i be the set of $d_i = 2c_i$ edges incident with a vertex x_i in a connected Eulerian graph G $(i = 1, 2, \ldots, n)$ and let Q_i be an arbitrary partition of E_i into at least two classes. An Eulerian trail P of G is *admissible* with respect to the partitions $Q_1, Q_2, \ldots, Q_n$ when any two successive edges in P with a common vertex x_i belong to different classes Q_i. In [71], Kotzig proves that an Eulerian trail admissible with respect to the partition $Q_1, Q_2, \ldots, Q_n$ exists if and only if for $i = 1, 2, \ldots, n$ no class Q_i contains more than c_i edges.

The paper [2] deals with *planar* Eulerian multigraphs. It is shown that every planar Eulerian multigraph contains an Eulerian trail in which the transitions through any vertex never cross.

A very detailed and fundamental study of Eulerian trails ("lines") in 4-regular graphs is undertaken in [70]. It is not possible to reproduce here the content of this paper or even the statements, let alone proofs, of its seventeen theorems. Let us attempt, however, to give an indication of Kotzig's deep insight by presenting two of his theorems which do not require too many definitions. His Theorem 5 in [70] states that for any decomposition of a 4-regular graph G into two 2-factors Q_1, Q_2 there exists an Eulerian trail of G in which the edges regularly alternate between Q_1 and Q_2. Theorem 4 states that in a 4-regular *directed* graph there exists a directed Eulerian trail if and only if it is connected and the indegree and outdegree of each vertex are equal. This theorem holds even if one replaces "4-regular" with "Eulerian".

Finally, in [3] Kotzig's old result proved originally only for Eulerian multigraphs of degree 4 is extended to arbitrary Eulerian multigraphs: any Eulerian trail in such a multigraph can be obtained from any other Eulerian trail by a finite number of simple transformations (called κ-transformations).

Directed graphs and tournaments

This is another topic to which Kotzig contributed significantly. We have already mentioned his directed version of Menger's Theorem [55]. His first extensive paper on the topic [49] deals with the relationship between Eulerian graphs and balanced directed graphs. These are directed graphs in which for each vertex its indegree and outdegree are equal. These graphs are assumed to be without isolated vertices but multiple edges are allowed. Kotzig extends several theorem from [38], in particular, he is interested in the number $\rho(G)$ of different balanced orientations that can be obtained by assigning a direction to each edge of a connected Eulerian undirected graph G. He obtains a formula for $\rho(G)$ in terms of the number $\mu(G)$ which is the number of distinct decompositions of an Eulerian graph G into closed trails. For the number $\mu(G)$ itself a closed formula is obtained where $\mu(G)$ is expressed solely in terms of degrees of vertices of G. Kotzig also obtains a formula for the number of balanced subgraphs of a balanced directed graph. The last of the 13 theorems in this paper suggests a connection between the number of 1-factors in a bipartite graph G which contains at least one 1-factor, and the number of distinct balanced directed subgraphs of G.

In [66] two relations are considered on the set $\mathbf{G} = \{G_1, \ldots, G_k\}$ of directed graphs arising from an undirected graph G without loops or multiple edges by assigning a direction to each of its edges. Two graphs $G_i, G_j \in \mathbf{G}$ are in relation Ω if the outdegree is the same for each vertex $x \in V$ where V is the vertex-set of G. Two graphs $G_i, G_j \in \mathbf{G}$ are in relation *elta* if one can be obtained from the other by a sequence of transformations each consisting of reversing the orientation of the edges of a 3-cycle of G. The main result of the paper is that the two relations Ω and *elta* coincide if and only if G is a graph without bridges.

In [65], Kotzig proves several results on the existence of directed cycles in balanced tournaments. The most important of these is the following theorem.

Theorem. *Let V be any set of r vertices, $2 \leq r \leq 2n+1$, in a balanced tournament T on $2n+1$ vertices. Then T contains a directed k-cycle containing all r vertices of V either for all $k \in R$, or for all $k \in R \setminus \{r\}$, or for all $k \in R \setminus \{r+1\}$ where $R = \{r, r+1, \ldots, 2n+1\}$.*

In [84], the following question is studied: what is the minimum number of directed edges that must be removed from a tournament on n vertices so that the resulting directed graph is acyclic? Alternatively, what is the minimum number of directed edges that must have their direction reversed so that the resulting directed graph is acyclic? The two numbers in question are always equal. The maximum of these numbers taken over all tournaments on n vertices is denoted by $\mu^*(n)$. It is bounded below by the number $\kappa(n)$ of triples in a maximum partial triple system of order n which is well known to equal $\kappa(n) = \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor$ [25]. Kotzig shows that for infinitely many values of n, $\mu^*(n)$ is strictly greater than $\kappa(n)$ and conjectures that this is true for all $n > 10$.

In [16], a similar question is considered for bipartite tournaments (that is, directed graphs arising from orienting each edge of the complete bipartite graph $K_{m,n}$). These are studied through their connection to $(m \times n)$ $(0,1)$-matrices. The maximal order of cyclicity of such matrices and of (m,n)-bipartite tournaments is introduced and studied.

In [74] and [76], both written in French, Kotzig studies and proves many results on 3-cycles and 4-cycles in (primarily) balanced tournaments. Concerning the maximum number of 4-cycles in balanced tournaments, it turns out that he was unaware of earlier results (cf. [15]) establishing the maximum number of 4-cycles in a tournament, as he himself acknowledges in an erratum to [74].

In [75], Kotzig associates with each balanced directed graph a directed bipartite graph with twice as many vertices. Then he proves that a balanced tournament cannot be decomposed into Hamiltonian cycles if for every decomposition of this associated bipartite graph into 1-factors, at least one of the 2-factors obtained as a union of two distinct 1-factors contains an even number of cycles.

Cubic graphs

Kotzig devoted a whole series of papers to various properties of cubic graphs (in addition to papers on 1-factors and 1-factorizations of cubic graphs discussed in the following section): [68], [73], [41], [86], [6].

There are two ways to split an edge of a bipartite cubic graph G without loops or multiple edges (cf. [38]). The resulting graphs G' and G'' are also bipartite cubic but may contain multiple edges. If both G' and G'' contain multiple edges then the edge is *irreducible*. The graph G is irreducible if each its edge is irreducible. It is proved in [68] that a bipartite cubic graph is irreducible if and only if each of its components is isomorphic to the complete bipartite graph $K_{3,3}$.

Kotzig's paper [73] is a case of "independent discovery". In it Kotzig presents a construction for cubic graphs due to E.L. Johnson.

In [85] and [86], Kotzig studies the so-called change graphs of cubic graphs of class one. Given two distinct 1-factorizations F' and F'' of a cubic graph G, its change graph $CH(F', F'')$ is the subgraph of G induced by those edges which are of "different colour" in F' and F'', that is, belong to different 1-factors in F' and F''. Kotzig studies transformations on change graphs and their properties, in particular, for planar cubic graphs.

Finally, in [6], Abrham and Kotzig introduce and discuss a new type of a labelling for cubic graphs which they called ξ-labelling. This type of labelling is unlike the labellings discussed below in the section Labellings, and was motivated by the authors' investigations on additive sets of permutations (see below). A ξ-labelling of a cubic graph G is a $1-1$ mapping of its edge set $E(G)$ onto a set $X = \{x_1, x_2, \ldots, x_k\}$ (where $k = |E(G)|$) of integers such that (1) $\{\xi(e) + \xi(f) : e, f \in E(G),\ e, f \text{ adjacent}\} = X$, (2) if $e = \{u, v\} \in E(G)$, vertex u is incident also with edges p, q and vertex v is incident also with edges s, t then $\xi(p) + \xi(q) = \xi(s) + \xi(t)$. Several examples are given of cubic bipartite graphs with many ξ-labellings. Möbius ladders with $4q+2$ vertices have a ξ-labelling while those with $4q$ vertices do not. The prism P_{2n} with $2n$ faces has a ξ-labelling while another whole two-parameter class of cubic graphs is shown not to admit a ξ-labelling. A connected bipartite cubic graph with a ξ-labelling must be 3-edge-connected, but this need not be true for non-bipartite cubic graphs. Some transformations which increase the number of vertices and preserve the existence of ξ-labellings are also described in [6].

1-factors, 1-factorizations, perfect 1-factorizations

From the very beginning of his studies in graph theory, Kotzig was interested in the questions related to the existence of 1-factors (perfect matchings, or linear factors, as he called them in his early papers) in finite graphs. In particular, he was intrigued by the question of when does there exist a decomposition of a regular graph into 1-factors. When such a decomposition (1-factorization) exists, what properties does it have? Kotzig called graphs admitting 1-factorizations in which the union of any two 1-factors is a Hamiltonian cycle *strongly Hamiltonian*. Nowadays, such graphs are simply called graphs with a perfect 1-factorization.

Graphs with at least one 1-factor are subject of a deep and extensive three-part study [50], [51], [52]. This important fundamental work written in Slovak has unfortunately not been reviewed by Mathematical Reviews, only indexed, and even Zentralblatt für Mathematik reviewed only the first part [50]. There are 12 lemmas and 36(!) theorems in total, and it is not possible to describe these results in a concise way. A cycle (path) in a graph G with at least one 1-factor is called an α-cycle (an α-path) with respect to a certain 1-factor L if exactly one of any two its adjacent edges belongs to L. Theorem 2 states that the composition of an α-cycle with respect to L with L is again L. Theorem 4 says that if H_i is the set of all edges of all α-cycles with respect to L_i then $H_i = H$ for all i. To state any of the remaining theorems would require introducing many new concepts and definitions which is inappropriate for this kind of survey. The reader is referred to the original papers.

One-factors in lattice graphs, mainly two-dimensional, are subject of study in [61]. Kotzig shows, for example, that if L, L' are any two 1-factors of a two-dimensional lattice graph then L' can be obtained from L by a sequence of a finite number of simple transformations, each consisting in replacing two opposite edges of a basic quadrangle by the other pair of opposite edges,. Any 1-factor in a two-dimensional lattice graph necessarily contains a pair of opposite edges of some basic quadrangle.

A 1-factor L of a lattice graph is *significant* if there is at least one edge of L joining vertices of any two adjacent lines. Significant 1-factors in a lattice graph are in a $1-1$ correspondence with significant tilings of an $(m \times n)$ chessboard by dominoes (1×2 tiles). These tilings are shown to exist if and only if $m \geq 5, n \geq 5$ except when $m = n = 6$, a result also obtained by different means by Solomon Golomb.

Kotzig also shows that in any tiling of a 2-dimensional chessboard with dominoes, the dominoes can be coloured by four colours in such a way that no two dominoes sharing boundary have the same colour.

One-factorizations of Cartesian products are dealt with in [87]. Let H be the Cartesian product of regular graphs $G_1, G_2, \ldots, G_n$, that is, $H = G_1 \times G_2 \times \cdots \times G_n$. If for at least one $i \in \{1, 2, \ldots, n\}$ there exists a 1-factorization of G_i, or if there are at least two numbers i and j such that both G_i and G_j contain at least one 1-factor, then there exists a 1-factorization of H. Neither of these sufficient conditions is necessary.

Kotzig made many fundamental contributions to the theory of graphs of third degree (cubic graphs). In [46], he proves the following important theorem.

Theorem. *A connected cubic graph with an even number of edges has a 1-factorization if and only if its line graph has a 1-factorization.*

Kotzig's studies into the existence and construction of cubic graphs with perfect 1-factorizations are especially deep. They were initiated in [47] and continued in [53], [56], [39], [91] and finally in [92] which also contains a list of open problems. Of these, quite an important paper is [39]. In this paper, Kotzig establishes that the complete graph K_{2n} has a perfect 1-factorization (P1F) whenever $2n-1$ or n is a prime, and conjectures that a P1F of K_{2n} exists for all natural n. Actually, he states his conjecture in a very cautious way: Is there an n such that K_{2n} does not admit a perfect 1-factorization? Even in a much later paper [92] he only asks for which n does K_{2n} have a P1F. Remarkably, the two infinite series of orders for which Kotzig established the existence of a perfect 1-factorization are till today the only two such infinite series of orders, although another infinite class of P1Fs was obtained in [23] for those K_{2n} for which $2n-1$ is a prime. Other than that the existence of P1Fs of K_{2n} was established only for several sporadic orders. For more details on the problem of perfect 1-factorizations of the complete graph and its history, see [111]. Currently, the smallest order of a complete graph for which the existence of a P1F is in doubt is 64.

It is well known that any regular bipartite graph has a 1-factorization. Already in [47], Kotzig proves that a regular bipartite graph with n vertices can have a perfect 1-factorization only if $n \equiv 2 \pmod 4$. In [53] he proves that a *planar* bipartite graph with more than two vertices cannot have a perfect 1-factorization. In the deep study [56], it is shown, among other things, that starting with the graph with two vertices and three edges joining these two vertices and by applying repeatedly two types of two-vertex extensions, called ρ-extension and π-extension, respectively, one can construct any cubic graph with a perfect 1-factorization.

Graph decompositions

A classical theorem by Listing states that if a finite connected graph has exactly $2n$ vertices of odd degree then there exists a decomposition of its edge-set into n open trails. In [43], Kotzig proves several extensions of this theorem.

Theorem. *If a finite regular graph G of degree $2d+1$ with $2n$ vertices has at least one 1-factor, say L, then there exists an edge-decomposition of G into n open trails, each of length $2d+1$ and each containing exactly one edge of L.*

Theorem. *If G is a cubic graph with $2n$ vertices which has an edge-decomposition into n open trails of length 3 each, then G contains at least one 1-factor. An analogous statement for a regular graph of degree $d+1$ with $d > 1$ does not hold.*

Theorem. *A necessary condition for a regular graph G of degree $2d+1$ to possess an edge-decomposition into open trails of length $2d+1$ is that it does not contain a vertex incident with more than d bridges.*

This is augmented by the following theorem that we find in [48].

Theorem. *A regular graph G of degree $2d+1$ with $2n$ vertices can be* *decomposed into a factor of degree d and a factor of degree $d+1$ if and only if it can be decomposed into n open trails where each trail contains an odd number of edges.*

In [80] necessary and sufficient conditions are found for the existence of a decomposition of the complete graph K_n into regular bipartite *factors*. If the number of factors is k then $2^{k-1} < n \leq 2^k$. Curiously, due to an obviously unintended oversight, this paper appeared in Discrete Mathematics twice(!): it can be found in volume 2(1972), 383–387 and also in volume 4(1973), 65–69.

Decompositions of the 4-regular graph Q_n isomorphic to a circulant $S(n; \{1, 2\})$ into two 2-factors are considered in [26]. It is proved that in any decomposition of Q_n into two 2-factors, at least one of the factors must be a Hamiltonian cycle.

In one of his early papers [46], Kotzig proved that every connected graph with an even number of edges has an edge-decomposition into paths of length two; here, a path of length two is the simple graph with three vertices and two edges.

The edge-set of any tree T having $2n$ vertices can be decomposed into n paths, and n is the smallest number of paths into which the edge-set of T can be decomposed. In [69], a formula is given for the number of different decompositions of T into n paths in terms of d_i and g_i where d_i is the number of vertices of T of degree i and $g_i = i!!$ if i is odd, and $g_i = (i-1)!!$ if i is even.

Kotzig was the first to consider decompositions of complete graphs into cycles of fixed length in [64]. He proved, by a direct construction, that the complete graph K_n where $n \equiv 1 \pmod{8k}$ can be cyclically decomposed into copies of a $4k$-cycle. When k is a power of 2, $k = 2^a$, this sufficient condition for the existence of a decomposition of the complete graph into $4k$-cycles is also necessary. This paper marked the start of a series of papers dealing with decompositions of the complete graph into cycles of fixed length culminating in a complete solution of this problem many decades later. Credit is due Kotzig for his early work on the subject.

In [88], similar in spirit to [64], Kotzig considers decompositions of complete graphs into d-dimensional cubes. By establishing the existence of an α-labelling (see the section on labellings below) of the d-dimensional cube W_d, he proves that a decomposition of the complete graph K_n into copies of W_d exists for all $n \equiv 1 \pmod{d2^d}$. When d is even, this condition is also necessary. When d is odd, however, then another possibility is that $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$. He points out the existence of a decomposition of K_{16} into copies of W_3. Kotzig's results have subsequently been extended in [22] and [27], however, the existence of decompositions of K_n into copies of W_d when d is odd, $d \geq 5$, is still open.

The study of decompositions of the complete graph K_n into factors with given diameters was initiated in [21] where it is shown that K_n can be decomposed into m factors with diameters $d_1, d_2, \ldots, d_m$ if and only if $n \geq F(d_1, d_2, \ldots, d_m)$, the smallest number n for which such a decomposition exists. That paper and a large number of papers that followed studied the function $F(d_1, \ldots, d_m)$. If the diameter of each of the m factors equals d, one writes $F(d, \ldots, d) = F_m(d)$. The paper [98] studies the same problem with the additional requirement that the m factors be pairwise isomorphic. Values of n such that $m | \binom{n}{2}$ are *admissible.* Let $G_m(d)$ be the smallest cardinal number n such that there exists a decomposition of K_n into m pairwise isomorphic factors of diameter d, and let $H_m(d)$ be the smallest cardinal number n such that for all admissible $N \geq H_m(d)$ there exists a decomposition of K_n into m isomorphic factors of diameter d. Trivially, $F_m(d) \leq G_m(d) \leq H_m(d)$, and it is conjectured (but remains unproved in general) that $G_m(d) = H_m(d)$. For $m = 2$ one easily obtains $G_m(2) = H_m(2) = 4$ if $d = 3$, $= 5$ if $d = 2$ and $= \infty$ otherwise. The main body of [98] deals with the case of $m = 3$ isomorphic factors. In this case, the admissible values are $n \equiv 0$ or $1 \pmod{3}$.

Main result states:
Theorem 2. *Let $t > 1$ be an integer and let $d \in \{3, 4, \ldots, t+2\}$. The K_{3t} and K_{3t+1} can be decomposed into three isomorphic factors of diameter d. When $t = 2$ or $t = 3$, such a decomposition exists also with $d = t + 3$.*

As for diameter $d = 2$, combined with a much later result of an impossibility of a decomposition of K_{12} into three factors of diameter 2 [117], we get $G_3(2) = 13$ (actually, $F_3(2) = G_3(2) = H_3(2)$.

We find another pioneering contribution of Kotzig in his paper [77] where he introduces the notion of a P-groupoid and a P-quasigroup arising from a decomposition of a complete graph with an odd number of vertices into closed trails. The connection he establishes may be viewed as a generalization of the well-known relationship between Steiner triple systems and Steiner quasigroups [25]. A P-groupoid is a pair $(V, \times)$ where V is a finite set and $\times$ is a binary operation which is (1) idempotent, (2) such that if $a, b \in V, a \neq b$ then $a, b, a \times b$ are all distinct, and (3) $a \times b = c$ implies $c \times b = a$. It is shown that the set of P-groupoids with n elements is coextensive with the set of decompositions of the complete graph K_n into closed trails. When a P-groupoid satisfies one more condition (4) the equation $x \times a = b$ has a unique solution for all $a, b \in V$, then it is a P-quasigroup. The $(1, 3, 2)$-conjugate of a P-quasigroup is a commutative quasigroup which Kotzig calls K-quasigroup. In a somewhat later paper [102], Kotzig and Turgeon gave a constructive graph-theoretic proof that for every integer r such that $2r+1 \equiv 0 \pmod 7$ there exists a P-quasigroup of order $2r+1$ defining an Eulerian path in K_{2r+1}. Kotzig's paper [77] had as a consequence a furious activity (too broad and extensive to be fully surveyed here) leading to some deep and fairly definitive results. As a sample, let us just mention the paper [106] on Steiner pentagon systems, or many classification results for i-perfect cycle systems, surveyed, e.g., in [105].

Concerning the well-known Oberwolfach problem (OP) and its variants, Kotzig deals with them in [97], [32] and [35]. Specifically, in [35] a spouse-avoiding variant of OP (called therein NOP) is introduced for the first time, and many results, both general and for small orders are obtained. A special case of NOP, an analogue of Kirkman triple systems, the so called nearly Kirkman systems, was introduced earlier in [97]. All three above cited papers offer partial results on existence problems, and were thus later superseded by new results, in some cases settling the corresponding existence questions completely. So, for example, nearly Kirkman triple systems are now known to exist for all $n \equiv 0 \pmod 6$, $n \geq 18$; they do not exist for $n = 6$ or $n = 12$. The NOP problem is conjectured to have a solution in all cases except in the two cases mentioned but so far the conjecture has been proven only asymptotically.

Latin squares and quasigroups

Latin squares which contain no Latin subsquares of order 2, so-called N_2-Latin squares, are subject of [93] and [99]. Squares of this kind are applied in [93] to construct sets of pairwise disjoint Steiner triple systems. Both existence problems (for N_2-Latin squares and for large sets of disjoint Steiner triple systems) were subsequently completely settled: a Latin square of order n without a subsquare of order 2 exists for all natural n with the exception of $n = 2$ and $n = 4$, and a large set of disjoint Steiner triple systems of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$, $n \neq 7$.

The associativity index α of a finite groupoid is the number of ordered triples (a, b, c) such that $(ab)c = a(bc)$. It is known that for a quasigroup of order q which is not a group the associativity index is at least q and at most $q^3 - q$ if $q \geq 7$ but whether these bounds are best possible for all orders remains in doubt. In [94], Kotzig and Reischer derive results which improve these bounds for certain special classes of quasigroups. So, for example, if a quasigroup Q of order q has a one-sided identity then $\alpha \geq q^2$, and if Q is a loop then $\alpha \geq 3q^2 - 3q + 1$. Moreover, for every $q \not\equiv 2 \pmod 4$ there exists a quasigroup with left identity having $\alpha = q^2$ so that for these orders the bound is sharp. For a commutative quasigroup of order q, $\alpha \geq q^2$ and the bound is sharp whenever $q \not\equiv 2 \pmod 4$. When $q \equiv 2 \pmod 4$ there exists a commutative quasigroup with $\alpha = 2q^2$. The paper contains further partial results for idempotent quasigroups, Steiner quasigroups etc. but the general problem of determining the spectrum for the associativity index of finite quasigroups remains wide open. Several further contributions to this important and intriguing problem can be found in the paper by Grošek and Horák [30] as well as in several papers by Kepka, Drápal and their collaborators, too numerous to be mentioned here.

An interesting new problem is studied in [34]. A Latin square of order n *is h-homogeneous* if each its cell is contained in exactly h Latin subsquares of order 2. 0-homogeneous Latin squares are the N_2-Latin squares mentioned above. The $(n-1)$-homogeneous Latin squares exist if and only if n is a power of 2, while $(n-2)$-homogeneous Latin squares do not exist, and 1-homogeneous Latin squares of order n exist if and only if n is even. It is shown that $(n-3)$-homogeneous Latin squares of order n exist if and only if $n \in \{3, 4, 6, 8, 12, 16\}$. Several further constructions are given for various h-homogeneous Latin squares. This work is continued in [33] where a multiplicative construction is used to produce a 4-homogeneous Latin square of order 12, 6-homogeneous Latin squares of orders 20 and 24, and a 12-homogeneous Latin square of order 24. What is known about the existence of h-homogeneous Latin squares of order up to 26 is summarized here but many cases with $2 \leq h \leq 22$ and $9 \leq n \leq 26$ remain unresolved. To determine for which values of h does there exist an h-homogeneous Latin square of order n is a fascinating open problem.

Antipodal graphs

According to [72], a connected bipartite graph G is *centrally symmetric* if for each vertex x in G there is exactly one vertex $\bar{x}$ such that for all neighbours y of $\bar{x}$ the distance from x to y is smaller than the distance from x to $\bar{x}$. The vertices $x, \bar{x}$ are said to be antipodal pair. The distance between the two vertices of an antipodal pair equals the diameter $d(G)$, Each centrally symmetric graph has an even number of vertices and each has a fixed-point-free involution as an automorphism. Several structural properties of these graphs are derived here, and then in [29], [17] and [18], antipodal graphs are discussed: these are graphs where for each vertex x there is a unique vertex $\bar{x}$ at distance d where d is the diameter of G. Antipodal graphs of diameter 2 are isomorphic to the cocktail-party graph $K_n \setminus L$ where L is a 1-factor. Antipodal bipartite graphs of diameter 3 are isomorphic to $K_{n,n} \setminus L$. A construction is given in [18] producing all antipodal graphs of diameter 3. The authors use in the proof what is otherwise known in the theory of strongly regular graphs as Seidel switching, apparently discovered independently. A nonbipartite antipodal graph of diameter 3 must contain at least $2n - 12$ triangles, and for every even $n > 8$ there is an antipodal graph of diameter 3 with exactly $2n - 12$ triangles. Finally, in [19], antipodal graphs of diameter 4 and girth 5 are considered. Under an additional condition on the vertices equidistant from an antipodal pair $x, \bar{x}$ there is exactly one such graph in which vertices equidistant from x and $\bar{x}$ form a dodecahedron.

Labellings

In [95], Kotzig introduced the notion of a *magic labelling.* This is nowadays known as *edge-magic labelling* to distinguish it from vertex-magic labelling. For a graph G with m vertices and n edges, vertex set $V(G)$ and edge-set $E(G)$, a magic labelling f with a constant C is a one-to-one mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, m+n\}$ such that $f(x) + f(y) + f(\{x, y\}) = C$ for all edges $\{x, y\} \in E(G)$.

It is shown in [95] that all n-cycles for $n \geq 3$, all complete bipartite graphs, all caterpillars and matchings with an odd number of edges have an edge-magic labelling (called an M-valuation in [95]). The paper [95] has spurned a furious activity and the topic of magic labellings of all kinds has attracted for its simplicity and appeal many novices to graph theory as witnessed by Gallian in his survey [28]. There are four additional papers by Kotzig devoted wholly or partially to edge-magic labellings. In [96] it is shown that the complete graph K_n has an edge-magic labelling if and only if $n = 2, 3, 5,$ or 6. In [78], Kotzig proves the existence of a class of forests and of a class of regular graphs which cannot possess an edge-magic labelling. In [79] it is shown that if a 3-colourable graph G has an edge-magic labelling than the graph consisting of an odd number of components each isomorphic to

G also has an edge-magic labelling. Thus, for example, a graph consisting of an odd number of disjoint n-cycles has an edge-magic labelling. Finally, the paper [81] explores the relationship between α-labellings (see below) and edge-magic labellings of bipartite graphs. If a bipartite graph has an α-labelling then it also possesses an edge-magic labelling. It has been conjectured that all trees have an edge-magic labelling. Consequently, when looking for a counterexample, it suffices to look at those trees which do not admit an α-labelling. Several infinite classes of such trees are known. Despite the ongoing discussion about the importance of the topic, these papers confer another testimony to Kotzig's inventiveness. For an up-to-date status of research on edge-magic labellings, see [28].

After Gerhard Ringel posed in 1963 his famous problem on tree decompositions of the complete graph [109] and after an introduction of a hierarchy of graph labellings as a means to approach Ringel's problem [110], Anton Kotzig became very interested in this circle of problems. However, his first paper on this type of labellings appears only much later after he has moved to Canada. In order to describe most important contributions of Anton Kotzig to this topic, we need some definitions. A *labelling* ϕ of a graph G with n edges is a $1-1$ mapping of its vertex set $V(G)$ into the set $\{0, 1, \ldots, n\}$. The *value* of an edge $\{u, v\}$, $u, v \in V(G)$, in the labelling ϕ is $|\phi(u) - \phi(v)|$. If the set of edge-values of a labelling ϕ of a graph G with n edges is the set $\{1, 2, \ldots, n\}$ then ϕ is said to be a *graceful labelling* of G (an older name: β-labelling, or β-valuation). If, in addition, there exists a number x such that for every edge $\{u, v\} \in G$, one of the values $\phi(u), \phi(v)$ does not exceed x and the other is strictly greater than x, then the labelling ϕ is called an *α-labelling.*

Clearly, every α-labelling is also a graceful labelling but not conversely. The Graceful Tree Conjecture (GCT) states that every tree admits a graceful labelling. On the other hand, there exist trees which do not admit an α-labelling; the smallest such tree has 7 vertices.

The GCT is older than 50 years but in spite of an extensive effort, it remains open. In [82], Kotzig proves a result which in my opinion is among the strongest results toward settling the GTC. Given a tree T and an arbitrary edge $e = \{u, v\} \in E(T)$, let $T_i(e)$ be the tree obtained form T by replacing the edge e with a path of length i, $i = 1, 2, \ldots$ with end-vertices u, v. Kotzig proves in [82] that for any tree T and any its edge e, wouldn't "its any edge" be better? in the infinite set of trees $T(e) = \{T_i(e) : i = 1, 2, \ldots\}$ there is at most a finite number of trees without an α-labelling. Thus, in a certain sense, almost all trees admit an α-labelling even though there are several infinite classes known of trees without an α-labelling (cf. [36], [110]).

In his later years, Kotzig became interested in labellings of 2-regular graphs, an interesting topic although seemingly of lesser importance. In a 2-regular graph each component is a cycle. An Eulerian graph with n edges can have a graceful labelling only if $n \equiv 0$ or 3 (mod 4) [110]. For an n-cycle, this condition is also sufficient. It is stated in [90] and proved in [13] that in the case of a 2-regular graph with two components, this condition is also sufficient. However, the condition is not sufficient in general. Kotzig determines in [90] the smallest 2-regular graph satisfying the above necessary condition but not admitting a graceful labelling; it has three components, two of which are triangles and one is a pentagon. In the same paper [90], Kotzig derives another necessary condition for the existence of a graceful labelling of a 2-regular graph. If ω is the number of odd length cycles in a 2-regular graph G then for the number of its vertices one must have $|V(G)| \geq \omega \cdot (\omega + 2)$. It is also shown that for every natural ω there exists a 2-regular graceful graph with exactly $\omega \cdot (\omega + 2)$ vertices having exactly ω odd length cycles. While Kotzig proves in [90] that a 2-regular graph with three components, each of length $4k + 3$ (k natural) is graceful, he also convinces the reader that the existence problem for graceful labellings of 2-regular graphs is far from easy.

In [83], he treats the case of isomorphic components. In a series of papers [10], [11], joint with Jaromír Abrham culminating in [12], they settle completely the case when all cycles are of length 4: each such 2-regular graph admits even an α-labelling except in the case of three components when only a graceful labelling exists. In [101], it is shown that a d-regular graph with c components which are all complete (and thus isomorphic to K_{d+1}) has a graceful labelling if and only if $c = 1$ and $d < 4$. Exponential lower bounds on the number of graceful labellings of paths and of cycles are obtained in [8] and [7], respectively. See also [9].

Perfect systems of difference sets and additive permutations

A non-modular analogue of difference sets and supplementary difference sets well known in combinatorial design theory is considered in several papers starting with [100] and [20] although traces of the ideas involved one can find already in an early paper [60].

Let $A_i = (a_{i1} < a_{i2} < \cdots < a_{in})$, $i = 1, 2, \ldots, m$, be m sequences of integers, and let $D_i = \{a_{ij} - a_{ih} : 1 \leq h < j \leq n\}$ be their difference sets. The system $D = (D_1, \dots, D_m)$ is a perfect (m, n, c)-system if $D = D_1 \cup D_2 \cup \dots \cup D_m = \{c, c+1, \dots, c-1+m\binom{n}{2}\}$. The sets D_i are called the components. For example, the existence of a perfect $(m, n, 1)$-system corresponds to a graceful labelling of the graph consisting of m complete graphs K_n having one common vertex. The definition is quite restrictive as no (m, n, c)-system exists if $n \geq 6$, as is proved in [20].

Several further papers by Kotzig, most of them co-authored by Jaromír Abrham - nine, to be precise - deal with the properties of (m, n, c)-systems and also of more general perfect systems of difference sets (PSDS) where the components need not be of the same size. In these papers, various properties of these systems are derived and proved. Among other things, all (not too numerous) such systems with small number of components are exhibited and classified. For example, one such system with two components has $D_1 = \{1, 7, 8\}, D_2 = \{2, 3, 4, 5, 6, 9\}$. Somewhat surprisingly, this is the only perfect system of difference sets having exactly two components. There are only two such systems with one component, and altogether 28 such systems with three components [5].

It is difficult to judge the importance of these papers and the topic at this time. The truth is that the response to this set of papers regarding perfect systems of difference sets has been quite muted up to this time, reflected, among other things, by a negligible number of citations which these papers collectively elicited. Motivated by a real life problem of spacing antennas in radioastronomy, there have so far been very few applications. However, one can find some applications in a paper by Mathon [107] who uses PSDS to construct many new cyclic Steiner designs $S(2, 4, v)$. According to [24], PSDSs with n components of size 4 are known to exist for $n \in \{1, 4, 5, \dots, 36.41, 46\}$, and also for several infinite classes of n. Whether such PSDS exist for every number n of components of size 4 is still unanswered. As for PSDS with components of size 5, these can exist only when n is even, and for $n \leq 50$ are known to exist only when $n = 6, 8$ or 10 [107]. .

A related notion is that of additive sets of permutations. For an odd number $m = 2k+1$, let $X^1 = (-k, -k+1, \dots, -1, 0, 1, \dots, k-1, k)$ and let $X^2, \dots, X^n$ be permutations of X^1. Then $X^1, X^2, \dots, X^n$ is an *additive sequence of permutations* (ASP) of order m and length n if the vector sum of every subsequence of consecutive permutations is also a permutation of X^1. ASPs play a role in recursive constructions for PSDSs and vice versa (see, e.g, [1]). Twelve papers of Kotzig and his collaborators all published between 1979 and 1986 deal with the existence and various questions of additive sequences of permutations. Apart from the paper [4] about the relationship of additive sets of permutations and Skolem sequences, the response to the rest appears to be muted up until now.

Other contributions

Kotzig made important contributions to several other areas of graph theory.

In a paper [103] coauthored by Bohdan Zelinka we find the following interesting theorem:
For any positive integers r, s there exists a regular graph of degree $2r$ in which each edge belongs to exactly one cycle of length s. There are no regular graphs of odd degree with this property.

In [63], he proved three theorems on the existence of Hamiltonian cycles in lattice graphs.

Two of his papers deal with friendship graphs, in particular, with degrees of vertices in infinite friendship graphs. Obviously inspired by this, he considered the following generalization in [89]. For natural numbers r, k, let $P_r(k)$-graph (or r-regularly connected k-path graph) be a graph in which for each pair of vertices there exist exactly r paths of length k connecting these two vertices. The $P_1(1)$-graphs are complete graphs and $P_1(2)$-graphs are the friendship graphs. He conjectures that there exists no $P_1(k)$-graph for $k > 2$ and proves the conjecture for $k < 10$. Several examples of $P_r(k)$-graphs with $r > 1$ are presented in [89]. For example, the graph consisting of n K_5's having a common vertex is a $P_6(3)$-graph. The octahedron is a $P_8(5)$-graph. But the existence problem for $P_r(k)$-graphs remains wide open.

Two papers coauthored by Juraj Bosák and Štefan Znám deal with some metric problems in graph theory (see [114]). Kotzig also wrote about applications of graph theory to economic problems, permutations, number sequences and other topics.

Kotzig sometimes felt that his work was underappreciated. In the fifties and early sixties of last century when Kotzig did some of his best work, graph theory was still denigrated in certain circles. However, he did receive some of the highest honours granted to scientists in Czechoslovakia: in 1965 it was the Order for Outstanding Contributions to the Country, and the Czechoslovak State Prize in 1969. On the occasion of his 60th birthday, volume 12 of Annals of Discrete Mathematics edited by Alexander Rosa, Gert Sabidussi and Jean Turgeon was dedicated to him, with contributions from leading graph-theorists worldwide, such as Paul Erdös, Bill Tutte, Claude Berge, Branko Grünbaum and many others.

Kotzig's legacy is enormous, and his work and contributions are lasting.

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