A Study of Conway's Thrackle Conjecture

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Abstract

A *thrackle* is a drawing of a simple graph on the plane, where each edge is drawn as a smooth arc with distinct end-points, and every two arcs have exactly one common point, at which they have distinct tangents. Conway, who coined the term thrackle, conjectured that there is no thrackle with more edges than vertices – a question which is still unsolved. A full thrackle is one with n vertices and n edges, and it is called non-extensible, if it cannot be a subthrackle of a counterexample to Conway's conjecture on n vertices. We define the notion of incidence type for a thrackle, which is the sequence of degrees of all vertices in increasing order. We introduce three reduction operations that can be applied to full subthrackles of thrackles. These reductions enable us to rule out the extensibility of many infinite series of incidence types of full thrackles. After defining the 1-2-3 group, we reduce Conway's conjecture to the problem of proving that thrackles from the 1-2-3 group are not extensible. Our result proves the hypothesis of Wehner, who predicted that a potential counterexample to Conway's conjecture would have certain graph-theoretic properties, which he described in [4].

1 Introduction

A *thrackle* is a plane drawing of an undirected simple graph (no loops and multiple edges) on n vertices by edges which are smooth curves (called lines) between vertices, with the condition that every two lines intersect at exactly one point, and have distinct tangents there [1]. More formally, we study the property of a graph to have a thrackle drawing. For the sake of brevity we do not make a notational distinction between a graph with the thrackle property and a thrackle drawing of this graph, referring to both of them as thrackle.

In the late 1960's, John H. Conway conjectured that the number of lines of a thrackle on n vertices cannot exceed n, which is known as the **Thrackle Conjecture**. After nearly forty years, despite many efforts by researchers, Conway's conjecture remains open. In [2], Lovasz, Pach, and Szegedy showed that every thrackle on n vertices has at most $2n-3$ lines. Five years later, Cairns and Nikolayevsky proved that the upper bound can be further lowered to $\frac{3}{2}(n-1)$ [3]. Wehner predicted that a minimal counterexample to Conway's thrackle conjecture, if it exists, would contain two cycles of one of the following types: *Figure-8* (two cycles share a vertex), *Theta* (two cycles share a path), *Dumb-bell* (two cycles connected by a path) [4]. We prove that Wehner's prediction is correct.

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2 Reduction Theory for Thrackles

2.1 Full thrackle and its incidence type

Definition 1 n-Cycle Thrackle: A thrackle that is an n-cycle.

Note that the 4-cycle cannot be a thrackle, nor a subgraph of a thrackle, since any drawing of a 4-cycle has two lines that do not intersect or intersect more than once.

Definition 2 Full Thrackle: A thrackle with n vertices and n lines.

It immediately follows that all the n-cycle thrackles are full thrackles.

Lemma 1 For a full thrackle on n vertices, the summation of the degrees of all vertices is $2n$.
Proof: It has n lines and each line starts and ends at a vertex. □

Lemma 2 If a full thrackle is not a cycle thrackle, then it has at least one vertex of degree 1.
Proof: By contradiction. Suppose there is no vertex of degree 1. Then by Lemma 1, every vertex must be of degree 2. This means we have a cycle thrackle, which is a contradiction. □

Definition 3 Incidence Type of a Thrackle: The *incidence type* of a thrackle on n vertices is a list of n integers sorted in increasing order, where each integer is the degree of a vertex.

For example, in Figure 1, the incidence type of the 3-Vertex thrackle and that of the 4-Vertex thrackle are $(2,2,2)$ and $(1,2,2,3)$, respectively.

2.2 Reductions of thrackles

Definition 4 α-Reduction: The removal of a vertex of degree 1 from a thrackle. The line associated with it is also removed. As a consequence, the degree of the vertex affected is decreased by 1.

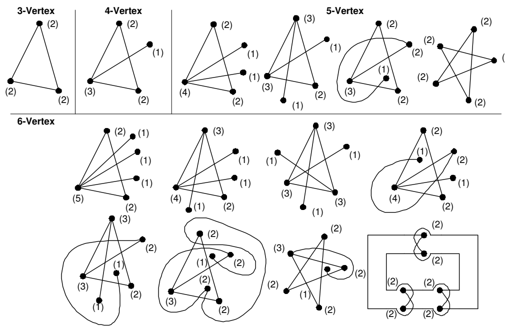

Figure 1: Sample full thrackles and vertex degrees

Lemma 3 The α-reduction of a full thrackle on n vertices is a full thrackle on $n-1$ vertices.
Proof: This follows from Definition 4 immediately. □

Wehner introduced a replacement of a 3-path by a 5-path method [4]. A 3-path or a 5-path refers to a graph fragment in a cycle, e.g. any three consecutive lines in a cycle will form a 3-path. We use Wehner's method reversely to remove two vertices in a cycle thrackle at a time. Because of the nature of the cycle thrackles, a graph fragment as in Figure 2 can always be found in a cycle thrackle on five or more vertices.

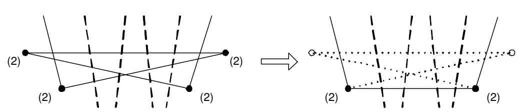

Figure 2: A cycle thrackle $(..., 2, 2, 2, 2, ...)$ is β-reduced to a cycle thrackle $(..., 2, 2, ...)$ [4]. Solid lines represent the five consecutive edges involved in the reduction, and dashed lines are the rest of the cycle thrackle's edges.

Definition 5 β**-Reduction:** The removal of two adjacent vertices and three lines associated with them from a five-consecutive-edge subgraph in a cycle thrackle (except the 6-cycle thrackle). In addition, the 'dangling' two vertices are connected with a new line (Figure 2).

Lemma 4 A β-reduction of an n-cycle thrackle ($n \geq 5$ and $n \neq 6$) is an $(n-2)$-cycle thrackle.
Proof: This follows from Definition 5 immediately. □

Note that for an n-cycle thrackle on $n \geq 5$ vertices, there are n ways to perform β-reduction. Also, β-reduction cannot be applied to the 6-cycle thrackle, because the 4-cycle is not a thrackle. To reduce the 6-cycle thrackle, we need a different reduction method. Before we show it, let us characterize the 6-cycle thrackles first. Wehner introduced notions called zero-, plus- and minus-configurations [4], which are used to specify how the lines in a directed 4-path ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$) intersect each other according to the order and the orientation with which the fourth line crosses the first and second line [4]. By using a computer program, he showed that the 6-cycle thrackles have no zero-configurations, but only plus-configurations. This means that all 6-cycle thrackles are identical in terms of the order and the orientation with which the six lines intersect each other.

Wehner demonstrated how to transform the 3-cycle thrackle into the 6-cycle thrackle [4]. Again, this can be used reversely to transform the 6-cycle thrackle back to the 3-cycle thrackle, as shown in Figure 3. This reduction proceeds as described below in Definition 6.

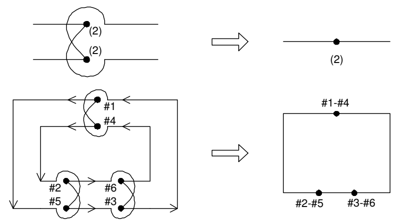

Figure 3: The 6-cycle thrackle (2, 2, 2, 2, 2, 2) is γ-reduced to the 3-cycle thrackle (2, 2, 2) [4].

Definition 6 γ**-Reduction:** The operation to reduce the 6-cycle thrackle to the 3-cycle thrackle, which involves the merge of three pairs of vertices and three pairs of lines. The operation proceeds as follows. Firstly, pick a vertex in the 6-cycle thrackle as the starting vertex, and travel along the lines continuously in one direction to visit each of the other vertices exactly once. Eventually, we will return to the vertex where we started. While visiting, number each vertex in ascending order. Secondly, for each vertex, identify its *in*-line and *out*-line, where an *in*-line is the line we use to 'enter' the vertex and an *out*-line is the one we use to 'leave' the vertex. Thirdly, merge vertices #1 with #4, #2 with #5, and #3 with #6. In addition, merge the two *in*-lines and the two *out*-lines of each pair of vertices.

Lemma 5 The γ-reduction of the 6-cycle thrackle is the 3-cycle thrackle.
Proof: As shown above, all 6-cycle thrackles are equivalent in terms of the order and the orientation with which the six lines intersect each other. Therefore, the reduction can be applied to all 6-cycle thrackles. Following the operations in Definition 6, the result is the 3-cycle thrackle. □

Next, we analyze the relationship between full thrackle incidence types.

2.3 Equivalence classes of full thrackles

Given a full thrackle on n vertices, there are only $2n$ degrees (Lemma 1) which have to be distributed over

n vertices with $1 \leq degree(v_i) \leq n-1$ for each vertex v_i; this limits the possible incidence types. We divide full thrackles into equivalence classes based solely on the number of vertices (see Figure 1).

Theorem 1 A full thrackle incidence type can be reduced to at least one full thrackle incidence type in a lower class, except for the incidence type (2, 2, 2).
Proof: By induction on number of vertices.
Induction basis: The 3-Vertex class. This class only contains one full thrackle incidence type (2, 2, 2).
Induction hypothesis: Suppose that all full thrackle incidence types that have no more than $n-1$ vertices can be reduced (via α, β, or γ reduction) to one or more full thrackle incidence types in a lower class.
Induction step: For an incidence type in n-Vertex class ($n > 3$), if it is the incidence type of a full thrackle, then the thrackle is either a cycle thrackle or it has at least one vertex of degree 1 (Lemma 2).
Case 1: If it is an n-cycle thrackle, it can be β-reduced ($n \geq 5$ and $n \neq 6$) or γ-reduced ($n = 6$). When it can be β-reduced, the reduced thrackle has $n-2$ vertices (Lemma 4), and it is in the $(n-2)$-Vertex class. When it can be γ-reduced, the result is the 3-cycle thrackle (2, 2, 2) (Lemma 5), which is in the 3-Vertex class.
Case 2: If it is not a cycle thrackle, then it can be α-reduced to a full thrackle on $n-1$ vertices (Lemma 3), which belongs to the $(n-1)$-Vertex class. □

2.4 Extensibility of full thrackles

Reductions are used as a tool to show that a full thrackle is not extensible if its reduced thrackle is not extensible. The non-extensibility of a subset of full thrackles called the *1-2-3 group* is not proved using reductions. Hence, we reduce Conway's thrackle conjecture to the problem of proving the non-extensibility of the 1-2-3 group.

Definition 7 Extensible Thrackle: A thrackle T is called extensible if there exists a new line between a pair of vertices of T such that the resulting geometric graph is still a thrackle.

Lemma 6 The full thrackle (2, 2, 2) (the 3-cycle thrackle) is not extensible.
Proof: Since the 3-cycle is a complete graph, no edges can be added. □

Next, we introduce a subset of full thrackles called the *1-2-3 group*. The significance of this group is that, unlike other full thrackles, a 1-2-3 group full thrackle has a body and a tail, which will also be defined.

Definition 8 1-2-3 Group: The set of full thrackles where the incidence type of each thrackle in the group is in the form of $(1, \underbrace{2, ..., 2}_{n-2}, 3)$ $(n \geq 4)$.

Now we characterize the structure of a full thrackle in the 1-2-3 group. We start with the simplest, which is the full thrackle (1, 2, 2, 3) as shown in Figure 1. As we can see, this full thrackle can be decomposed into two parts: a 3-cycle thrackle, and a line between the vertices of degree 1 and 3. We call this 3-cycle thrackle the *body* and the line the *tail* of this 1-2-3 group thrackle. If the tail contains just one vertex, it is said to have a length of 1. Next, we use induction to show that every full thrackle in the 1-2-3 group has a body and a tail.

Lemma 7 A full thrackle in the 1-2-3 group has a body and a tail, where the body is a cycle thrackle and the tail is a path between the vertices of degree 1 and 3.
Proof: By induction on number of vertices.
Induction basis: The simplest 1-2-3 group thrackle (1, 2, 2, 3), which has a body and a tail (attached to the vertex of degree 3).
Induction hypothesis: Suppose for a 1-2-3 group thrackle on $n-1$ vertices, it has a body and a tail which is a path between the vertices of degree 1 and 3.
Induction step: For a 1-2-3 group thrackle T on n vertices ($n \geq 5$), its incidence type is $(1, \underbrace{2, ..., 2}_{n-2}, 3)$ (by Definition 8). The vertex of degree 1 must be connected to either the vertex of degree 3 (case 1) or one of the vertices of degree 2 (case 2).
In case 1, apply α-reduction to T; the incidence type of the resultant thrackle is $(\underbrace{2, ..., 2}_{n-1})$, which is an $(n-1)$-cycle thrackle. Thus, the body of T is an $(n-1)$-cycle thrackle and the tail of T has a length of 1. Note that if the incidence type of T is (1, 2, 2, 2, 3) (when $n = 5$), the vertex of degree 1 must be connected to one of the vertices of degree 2 (since otherwise, T will contain a 4-cycle), which forces the α-reduction of T to fall into case 2 instead of case 1.
In case 2, again, we can apply α-reduction to T; the incidence type of the result is $(1, \underbrace{2, ..., 2}_{n-3}, 3)$, which is a 1-2-3 group thrackle T' on $n-1$ vertices. By the induction hypothesis, T' has a body and a tail which is a path between the vertices of degree 1 and 3. The line removed from T by α-reduction was therefore attached to the only vertex of degree 1 in T', which is the tail of T'. Thus, the body of T is the same as that of T', and the tail of T is of length 1 more than that of T'. □

For example, in Figure 1, there are two 1-2-3 group thrackles in the 6-Vertex class. One has a 3-cycle thrackle as its body and a tail of length 3, and the other has a 5-cycle thrackle as its body and a tail of length 1.

Conjecture 1 1-2-3 non-extensibility conjecture: A full thrackle in the 1-2-3 group is not extensible.

Theorem 2 Let T be a full thrackle and let T' be a full thrackle that T is reducible to. If the 1-2-3 non-extensibility conjecture is true and T' is not extensible, then T is not extensible.

Proof: By Lemma 3, 4 and 5, we know T' is a full thrackle too. By contradiction, assume T is extensible to a thrackle Y by adding a new line l. Applying to the sub-thrackle T in Y the same reduction that is used to reduce T to T', we can transform Y to Y'. Compare Y' to T'; the only difference between these two thrackles is line l, which indicates that T' can be extended to Y', contradicting the assumption that T' is not extensible. Note that the above proof works only when the two vertices associated with line l both still exist in Y' after the reduction operation is applied to Y. What if one or both of the vertices associated with line l are removed during the reduction? There are three cases:
Case 1: The reduction applied to T is a γ-reduction. In this case, T is the 6-cycle thrackle. Use the 6-cycle thrackle and the numbering scheme shown in Figure 3. Note that there are three groups of vertices. If line l is between two vertices that are not in the same group, the above proof by contradiction is still valid. If line l is between two vertices that are in the same group, for example, between #1 and #4, then vertices #1, #2, #3 and #4 will form a 4-cycle, which is a contradiction.
Case 2: The reduction applied to T is a β-reduction. In this case, T is a cycle thrackle on ≥ 5 vertices. Because β-reduction removes two adjacent vertices from a cycle thrackle, we can "carefully" pick for the reduction from T to T' two adjacent vertices neither of which are associated with line l. Since T has ≥ 5 vertices, such two adjacent vertices are always available. Therefore, the above proof by contradiction can still be applied.
Case 3: The reduction applied to T is an α-reduction. For those full thrackles that have ≥ 3 vertices of degree 1, we choose the vertex that is not associated with line l to apply α-reduction. Then, the above proof by contradiction is still valid. For those full thrackles that have exactly two vertices of degree 1, if line l is connected to only one of the two vertices, we can use the vertex that is not associated with line l to generate a contradiction. If line l is connected to both vertices of degree 1, we can show that the result is equivalent to adding a line to a 1-2-3 group thrackle. If the 1-2-3 non-extensibility conjecture is true, the result cannot be a thrackle. For those full thrackles that have exactly one vertex of degree 1, by Lemma 1, their thrackle incidence types must be in the form of $(1, 2, ..., 2, 3)$. All these full thrackles form precisely the 1-2-3 group. If the 1-2-3 non-extensibility conjecture is true, then, this completes case 3. □

Corollary 1 Suppose the 1-2-3 non-extensibility conjecture is true, then if a full thrackle T_1 can be reduced to T_2, T_2 can be reduced to T_3, …, T_{n-1} can be reduced to T_n, and T_n can be reduced to the full thrackle (2, 2, 2), then T_i is not extensible for all $1 \leq i \leq n$.

Proof: Using Theorem 2 and Lemma 6, we know T_n is not extensible. Applying Theorem 2 to T_n, shows T_{n-1} is not extensible. Using Theorem 2 recursively, we know T_i is not extensible for all $1 \leq i \leq n$. □

By Lemma 7, a full thrackle T in the 1-2-3 group has a body and a tail. Denote the vertex of degree 1 in T (which is the tip of its tail) by v, and that of degree 3 by u. If such a thrackle is extensible, then there exists a new line l that can be added. If l is added between two vertices neither of which is v, then a reduction argument leads to a contradiction (as shown in Theorem 2 above). Difficulty arises when vertex v is involved, which leads to the following three cases:
Case 1: Line l is between v and u.
Case 2: Line l is between v and one of the vertices in the body (except u).
Case 3: Line l is between v and one of the vertices on the tail (except u).
In any of the above cases, the result contains two cycles. In Case 1, two cycles share one vertex in common. In Case 2, they share a path in common. In Case 3, two cycles are connected by a single path. Thus, we have just proved Wehner's prediction of what a minimal counterexample to Conway's thrackle conjecture may look like [4]. If the 1-2-3 non-extensibility conjecture is true, then none of these three cases exist.

Corollary 1 tells us that if the 1-2-3 non-extensibility conjecture is true, then in the full thrackle incidence type hierarchy, every incidence type is not extensible, because, by Theorem 1, from any incidence type, there is a reduction sequence which leads to the incidence type (2, 2, 2). Thus, we can state the following theorem:

Theorem 3 If the 1-2-3 non-extensibility conjecture is true, then any full thrackle is not extensible and Conway's conjecture is true.

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References

[1] J. S. B. Mitchell and J. O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 2001. Also in *SIGACT News* 32(3):63-72 (2001).

[2] L. Lovasz, J. Pach, and M. Szegedy. On conway's thrackle conjecture. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 147-151, 1995.

[3] G. Cairns and Y. Nikolayevsky. Bounds for generalized thrackles. *Discrete Comput. Geom.*, 23(2):191-206, 2000.

[4] S. Wehner. On the thrackle problem. `http://www.thrackle.org/thrackle.html` .