Finding segments and triangles spanned by points in $\mathbb{R}^3$

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Abstract

Given a set S of n points in $\mathbb{R}^3$ we consider finding the farthest line segment spanned by S from a query point q given as part of the input, and finding the minimum and maximum area triangles spanned by S. For the farthest line segment problem we give an $O(n \log n)$ time, $O(n)$ space algorithm, matching the time and space complexities of the planar version. The algorithm is optimal in the algebraic decision tree model. We further prove that the minimum area triangle spanned by S can be found in $O(n^{2.4} \log^{O(1)} n)$ time and space, and the maximum area triangle spanned by S can be found in $O(h^{2.4} \log^{O(1)} h + n \log n)$ time and $O(h^{2.4} \log^{O(1)} h + n)$ space, where h is the number of vertices of the convex hull of S ($h = n$ in the worst case).

1 Introduction

Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^3$, we consider finding the farthest line segment spanned by S from a query point q given as part of the input, and finding the minimum and maximum area triangles spanned by S. For each problem we give efficient algorithms for finding the corresponding geometric structure.

The problem of finding the farthest line segment spanned by S from a query point q given as part of the input, in its 2-dimensional version, was introduced in [2] and has sparked the development of fundamental data structures [1, 7] that surprisingly enough were not addressed by previous work. In [2], they give an optimal, $O(n \log n)$ time, $O(n)$ space algorithm for solving the problem. In [7], they address the planar all-farthest-segments problem, that asks to compute the farthest line segment for each of the points in S, and give an optimal, $O(n \log n)$ time, $O(n)$ space algorithm, improving a previous result on the same problem [6].

In [5], they investigate the number of minimum (nonzero) volume tetrahedra spanned by n points in $\mathbb{R}^3$ and give an $O(n^3)$ time algorithm for reporting all tetrahedra of minimum nonzero volume. In [4], they show that a set S of n points in $\mathbb{R}^3$ can define $O(n^2)$ minimum area triangles, which is asymptotically tight, and that there exist n-element point sets that span $\Omega(n^{4/3})$ triangles of maximum area.

Results. We present the following results in $\mathbb{R}^3$. (i) For computing the farthest line segment spanned by S from a query point q that is part of the input we prove that a key property in [6] can be extended to $\mathbb{R}^3$ and give an $O(n \log n)$ time, $O(n)$ space algorithm for the 3-dimensional version of the problem, matching the time and space complexities of the planar version [2]. The algorithm is optimal in the algebraic decision tree model. (ii) We prove that the minimum area triangle spanned by S can be found in $O(n^{2.4} \log^{O(1)} n)$ time and space, and the maximum area triangle spanned by S can be found in $O(h^{2.4} \log^{O(1)} h + n \log n)$ time and $O(h^{2.4} \log^{O(1)} h + n)$ space, where h is the number of vertices of the convex hull of S.

Definitions and terminology. For a set of points S, we use $CH(S)$ to denote the convex hull of S. We define the distance between a point q and a line segment s to be the minimum distance between q and any point on s. The triangle defined by the points p_i, p_j, and p_k is denoted as $\triangle_{ijk}$.

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Given a line segment $\overline{qp}$ and a plane Π orthogonal to $\overline{qp}$ at q, the proper halfspace for p is that halfspace bounded by Π that does not contain p. We use the notation H_i to denote a halfspace bounded by a plane that contains the point p_i.

2 Finding the Farthest Line Segment

Lemma 1 *The farthest line segment from q spanned by S has at least one endpoint at a vertex of $CH(S)$. Moreover, (i) if both endpoints are vertices of $CH(S)$ then the line segment is an edge of $CH(S)$ and (ii) if only one endpoint is a vertex of $CH(S)$ then the other endpoint p is the farthest point from q among those points in S that are not vertices of $CH(S)$; in this case the distance from q to p is also the distance from q to the line segment.*

Proof. We make the proof by contradiction on various cases that do not satisfy the conditions in the lemma. Let $\overline{p_ip_j}$ be the farthest line segment from q, for some $i, j \in \{1, 2, \ldots, n\}, i \neq j$. Without loss of generality, to simplify the exposition, we assume that one endpoint, say p_j, of $\overline{p_ip_j}$ is further from q than the other one. Assume the endpoints of the farthest line segment do not satisfy the conditions in the lemma. There are a few choices for p_i and p_j: (1) both are interior to $CH(S)$; (2) one is interior and one is on a face of the convex hull; (3) one is interior and one is on an edge of the convex hull; (4) each one is on a face of the convex hull (possibly the same); (5) each one is on an edge of the convex hull (possibly the same); (6) one is on an edge of the convex hull and one is on a face of the convex hull; (7) each one is a vertex of the convex hull, but the two vertices do not define an edge of $CH(S)$. All these cases can be proved false following the same strategy, which we only illustrate for case (1) and case (7). For case (1), refer to Figure 1.

Consider the plane orthogonal to $\overline{qp_j}$ at p_j. By convexity of $CH(S)$, the proper halfspace for q defined by this plane must contain a vertex p_k of $CH(S)$ and thus $\overline{p_jp_k}$ is further from q than $\overline{p_ip_j}$, a contradiction.

Consider now case (7). Assume first that $\overline{p_ip_j}$ intersects the interior of the convex hull. If one of p_i or p_j, say p_i, gives the distance from q to $\overline{p_ip_j}$ then we take the plane Π orthogonal to $\overline{qp_i}$ at p_i and notice that the proper halfspace for q defined by this plane contains $\overline{p_ip_j}$. The proper halfspace for q defined by Π must contain another vertex p_k of $CH(S)$ (else, $\overline{p_ip_j}$ is an edge of $CH(S)$). Then, p_k and p_j define a line segment $\overline{p_kp_j}$ that is further from q than $\overline{p_ip_j}$, a contradiction. If the distance from q to $\overline{p_ip_j}$ is given by a point p interior to $\overline{p_ip_j}$ then we take the plane Π orthogonal to $\overline{qp}$ at p and notice that $\overline{p_ip_j} \in \Pi$. By convexity of $CH(S)$, the proper halfspace for q defined by this plane must contain a vertex p_k of $CH(S)$ and thus $\overline{p_jp_k}$ and $\overline{p_ip_k}$ are further from q than $\overline{p_ip_j}$, a contradiction.

Assume now that $\overline{p_ip_j}$ is on a face of the convex hull. Again, we have two possibilities, as above. In the second case however, it may be possible that the face containing $\overline{p_ip_j}$ is included in Π. If this is the case, it is easy to see that any vertex p_k of $CH(S)$ on that face, that forms and edge of $CH(S)$ with p_i (or p_j), defines a line segment $\overline{p_ip_k}$ (resp., $\overline{p_jp_k}$) that is farther from q than $\overline{p_ip_j}$, leading again to a contradiction.

Thus, we are in one of the two cases in the lemma: either $\overline{p_ip_j}$ is an edge of $CH(S)$, or one of p_i, p_j is a vertex of $CH(S)$ and the other one is not. To finish the proof of the lemma we need to consider the second situation. Recall that we assumed p_j is farther from q than p_i. If p_j is the endpoint that is not a vertex of $CH(S)$ then from the proof by contradiction for case (1) it follows $\overline{p_ip_j}$ is not the farthest segment. Thus, p_j must be the vertex of $CH(S)$. On the other hand, the distance from q to $\overline{p_ip_j}$ must be given by the distance from q to p_i, or otherwise we can apply the proof for case (1) with p_j replaced by p_i, obtaining that $\overline{p_ip_j}$ is not the farthest line segment. Finally, if p_i is not the

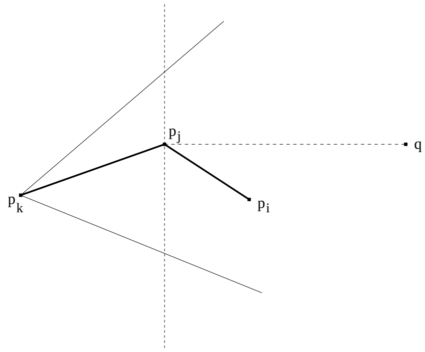

Figure 1: Violation of farthest segment.

farthest point from q among those points in S that are not vertices of $CH(S)$, then let p_k be the farthest point. We can apply the proof for case (1) with p_j replaced by p_k and obtain that $\overline{p_i p_j}$ is not the farthest line segment, again a contradiction. □

Using Lemma 1, we have the following simple algorithm. Start by computing the convex hull $CH(S)$ of S, in $O(n \log n)$ time and $O(n)$ space. From the edges of $CH(S)$, select the farthest one, e_1. This is one of the two possible candidates for the farthest line segment, according to Lemma 1, and can be found in $O(n)$ time. Let V denote the set of points that are vertices of $CH(S)$ and let $S' = S \setminus V$. Find the farthest point from q in S', which takes $O(n)$ time. Let this point be p_i, where $1 \leq i \leq n$. If p_i is closer to q than e_1 then report e_1 as the farthest line segment. Else, find the farthest line segment from q with an endpoint at p_i and the other endpoint in V, which can be done in $O(n)$ time, and report this segment, e_2, as the farthest line segment. The optimality follows from [2].

Theorem 2 *Given a set S of n points in $\mathbb{R}^3$, and a query point $q \in \mathbb{R}^3$, the farthest line segment from q spanned by S can be found in $O(n \log n)$ time and $O(n)$ space, which is optimal.*

3 Minimum and maximum area triangles

Our results for finding the minimum and maximum area triangles spanned by a set S of n points in $\mathbb{R}^3$ make use of a data structure in [3], that uses the fact that in $\mathbb{R}^3$ the Euclidean distance between a point and a line, as a function of the line, admits a linearization into a space of dimension 9. With s a parameter that controls the trade-off between the query time and the space and preprocessing time, $n \leq s \leq n^{9/2}$, they [3] show that S can be preprocessed with $O(s \cdot \log^{O(1)} n)$ space and time such that given a query line L the farthest (or closest) point of S from L can be found in $O(n \log n / s^{1/\lfloor 9/2 \rfloor})$ time. We use this data structure in the theorem below.

Theorem 3 *Given a set S of n points in $\mathbb{R}^3$, a minimum area triangle spanned by S can be computed in $O(n^{2.4} \log^{O(1)} n)$ time and space.*

Proof. For each pair of points $p_i, p_j \in S$, $i \neq j$, the minimum area triangle defined by the line segment $\overline{p_i p_j}$ with the points in $S \setminus \{p_i, p_j\}$ can be found by finding the point with minimum distance to the line supporting $\overline{p_i p_j}$. Thus, over all pairs $p_i, p_j \in S$ we have $O(n^2)$ such queries. Balancing the preprocessing time with the query time leads to the claimed bounds. □

For the maximum area triangle we have:

Lemma 4 *The vertices of the maximum area triangle spanned by S are among the vertices of $CH(S)$.*

Proof. Let p_i, p_j, and p_k be the vertices of the maximum area triangle. Assume one or more vertices of $\triangle_{ijk}$ are not among the vertices of $CH(S)$. Let p_k be one of the vertices of $\triangle_{ijk}$ that is not a vertex of $CH(S)$, and assume p_k is interior to $CH(S)$. Let p be the point on the line supporting $\overline{p_i p_j}$ that defines the distance $\delta(p_k, \overline{p_i p_j})$, from p_k to that line. Let Π be the plane through p_k, that is orthogonal to $\overline{p p_k}$ (see Figure 2). Notice that a vertex p_k' of $CH(S)$ must lie in the proper halfspace for p defined by Π. Moreover, the points p_i and p_j are not contained in that halfspace.

Notice also that the distance from p_k' to the line supporting $\overline{p_i p_j}$ is greater than the distance from p_k to that

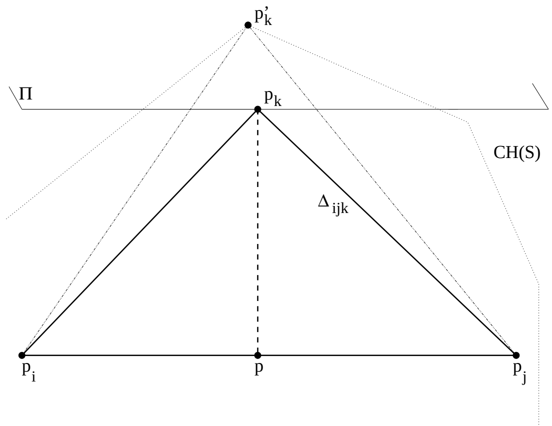

Figure 2: The area of $\triangle_{ijk}$ can be increased by extending the height away from the base

line. Since we did not change the length of the base $\overline{p_ip_j}$ of the resulting triangle, but only increased the height by finding a point p'_k of S that is farther from the line supporting $\overline{p_ip_j}$, the area of this new triangle must be greater than that of $\triangle_{ijk}$, thus a contradiction.

Assume now that p_k is interior to a face of $CH(S)$. If that face is not in Π then the proof above applies resulting in a triangle of larger area. Let the face containing p_k be in Π. Take the line L through p_k and parallel to $\overline{p_ip_j}$. This line is orthogonal to $\overline{pp_k}$ at p_k. The line L intersects the face of the convex hull at two points. If one of the intersection points is on an edge e of $CH(S)$, then one of the end vertices p'_k of e, together with p_i and p_j, defines a triangle of larger area than $\triangle_{ijk}$. If both intersection points are vertices of the convex hull, then any other convex hull vertex p'_k on that face, together with p_i and p_j, defines a triangle of larger area than $\triangle_{ijk}$. □

Theorem 5 *Given a set S of n points in $\mathbb{R}^3$, a maximum area triangle spanned by S can be found in $O(h^{2.4}\log^{O(1)} n + n\log n)$ time and $O(h^{2.4}\log^{O(1)} n + n)$ space, where h is the number of vertices of $CH(S)$.*

Proof. Find $CH(S)$ in $O(n\log n)$ time and use the approach in Theorem 3 on the vertices of $CH(S)$. □

4 Conclusion

In this paper we discussed finding the farthest line segment spanned by S from a query point q given as part of the input, and finding the minimum and maximum area triangles spanned by S. For each of these problems we described efficient, exact algorithms for finding the corresponding geometric structure.

A number of open problems remain with respect to computing closely related geometric structures. One interesting problem is to answer whether it is possible to find the farthest line spanned by S from a query point q, given as part of the input, in subquadratic time. While we can prove interesting properties for this problem, we have not been able to find a subquadratic time algorithm for it. We notice that in some sense the problem seems harder than the problem of finding the farthest plane spanned by S.

Finally, there are the problems of finding the closest line segment and the closest line spanned by S from q. It would be interesting to see whether either of these two problems can be solved in subquadratic time.

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