On (Pointed) Minimum Weight Pseudo-Triangulations

Oswin Aichholzer*‡ Franz Aurenhammer†‡ Thomas Hackl*‡ Bettina Speckmann§

Abstract

In this note we discuss some structural properties of minimum weight (pointed) pseudo-triangulations.

1 Introduction

Optimal triangulations for a set of points in the plane have been, and still are, extensively studied within Computational Geometry. There are many possible optimality criteria, often based on edge weights or angles. One of the most prominent criteria is the *weight* of a triangulation, that is, the total Euclidean edge length. Computing a *minimum weight triangulation* (MWT) for a point set has been a challenging open problem for many years [4] and various approximation algorithms were proposed over time; see e.g. [3] for a short survey. Mulzer and Rote [7] showed only very recently that the MWT problem is NP-hard.

Pseudo-triangulations are related to triangulations and use *pseudo-triangles* in addition to triangles. A pseudo-triangle is a simple polygon with exactly three interior angles smaller than π. Also for pseudo-triangulations several optimality criteria have been studied, for example, concerning the maximum face or vertex degree [5]. Optimal pseudo-triangulations can also be found via certain polytope representations [8] or via a realization as locally convex surfaces in three-space [1]. Not all of these optimality criteria have natural counterparts for triangulations. Here we consider the classic minimum weight criterion for pseudo-triangulations.

Rote et al. [9] were the first to ask for an algorithm to compute a minimum weight pseudo-triangulation (MWPT). The complexity of the MWPT problem is unkown, but Levcopoulos and Gudmundsson [6] show that a 12-approximation of an MWPT can be computed in $\mathcal{O}(n^3)$ time. Moreover, they give an $\mathcal{O}(\log n \cdot w(\mathrm{MST}))$ approximation of an MWPT, in $\mathcal{O}(n \log n)$ time. Here $w(\mathrm{MST})$ is the weight of the minimum Euclidean spanning tree, which is a subset of the obtained structure.

*Institute for Software Technology, Graz University of Technology, [oaich|thackl]@ist.tugraz.at.

†Institute for Theoretical Computer Science, Graz University of Technology, auren@igi.tugraz.at.

‡Supported by the Austrian FWF Joint Research Project 'Industrial Geometry' S9205-N12.

§Department of Mathematics and Computer Science, TU Eindhoven, speckman@win.tue.nl

A pseudo-triangulation is called *pointed* (or minimum) if every vertex p has one incident region (either a pseudo-triangle or the exterior face) whose angle at p is greater than π. A pointed pseudo-triangulation minimizes the number of edges among all pseudo-triangulations of a given point set. Since a spanning tree is not necessarily pointed (see [2]) the pseudo-triangulation constructed by the approximation algorithm of [6] is also not necessarily pointed. It is logical to conjecture that the MWPT should be pointed. However, we show that this does not need to be the case. As a consequence, the MWPT and the minimum weight pointed pseudo-triangulation (MWPPT) of a point set are different concepts. We also discuss the relation of MWP(P)Ts to greedy pseudo-triangulations and we give conditions on point sets under which the MWPT is lighter than the MWT.

2 Does an MWPT have to be pointed?

This question is answered in the negative below. The first example (Figure 1 (left)) shows the MWPT of a non-simple polygon with an interior point. This pseudo-triangulation is not pointed, whereas the MWPT of the underlying point set (not shown here) happens to be pointed. In general, however, we have:

Lemma 1 *The minimum weight pseudo-triangulation of a point set is not necessarily pointed.*

Proof. See Figure 1 (right). This pseudo-triangulation, $\mathcal{PT}$, contains exactly one non-pointed vertex. To make $\mathcal{PT}$ pointed, we have to reduce the number of its edges by exactly one. However, no single edge can be removed to achieve this. Observe further that the shortest non-used edge is longer than the longest used edge (apart

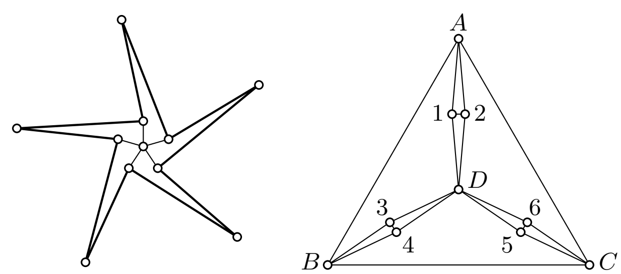

Figure 1: Non-pointed MWPT: inside a polygon (left) and of a point set (right).

from convex hull edges, of course). Therefore, it suffices to show that no two edges of $\mathcal{PT}$ can be replaced by a single new edge without increasing the weight of the pseudo-triangulation.

We now exclude the unused edges one by one. Edge $\overline{A3}$ is longer than the edges $\overline{A1}$ and $\overline{1D}$ together, which are the longest interior edges of $\mathcal{PT}$. Therefore, $\overline{A3}$ may not be inserted instead of other edges because this would raise the weight of $\mathcal{PT}$. Edge $\overline{A4}$ is inapplicable because it is even longer than edge $\overline{A3}$. If we insert edge $\overline{13}$ then we also have to insert edge $\overline{A3}$ (or edge $\overline{B1}$, which is of the same length) to maintain a pseudo-triangulation. But we already argued that the insertion of edge $\overline{A3}$ is not allowed, and therefore edge $\overline{13}$ cannot be inserted, either. Edges $\overline{14}$ and $\overline{24}$ are inapplicable for similar reasons: Insertion of edge $\overline{14}$ forces either edge $\overline{B1}$ or edge $\overline{A4}$, and inserting edge $\overline{24}$ makes it necessary to add edge $\overline{A4}$ or two of the previously mentioned edges to maintain a pseudo-triangulation. The last possible edge is $\overline{AD}$, which either involves the insertion of an already excluded edge, or can be exchanged with edge $\overline{12}$, which is the shortest edge of $\mathcal{PT}$. $\square$

3 Vertex degrees in a MW(P)PT

Lemma 2 *A minimum weight (pointed) pseudo-triangulation can have vertices with arbitrarily high vertex degree.*

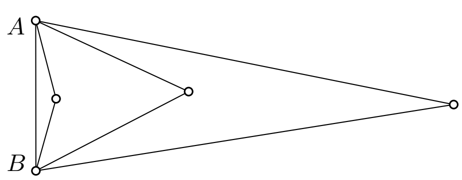

Figure 2: An MWPPT with linear vertex degree.

Proof. See Figure 2. For each two consecutive triangles based on $\overline{AB}$, the distance between their tips is larger than the longest edge of the smaller triangle. This implies that the shown pseudo-triangulation is indeed minimum weight. The degree of the vertices A and B is $n-1$ if the example is drawn on n points. $\square$

4 Greedy (pointed) pseudo-triangulations

The greedy pseudo-triangulation of a point set S is obtained by inserting edges spanned by S in increasing length order, such that no crossings are caused and until a pseudo-triangulation of S is obtained. Though such a greedy pseudo-triangulation clearly exists, the concept is not meaningful, as we are going to show below.

Lemma 3 *Let ∇ be any pseudo-triangle that is not a triangle. Then ∇ contains some diagonal that is shorter than the longest edge of ∇.*

Proof. As the sum of angles in a triangle is π, it is immediate that the three (interior) angles at the corners of ∇ sum up to less than π. Hence there exists a corner c of ∇ where the interior angle is less than $\frac{\pi}{3}$. Let s be the line segment connecting the two vertices of ∇ neighbored to c. Moreover, denote with Δ the triangle spanned by s and c. Clearly, the longest edge of Δ is not s but rather an edge of ∇, say e. So, if s is a diagonal of ∇ then we are done. Otherwise, there have to exist vertices in the interior of Δ. Corner c sees at least one of them, u, and $\overline{cu}$ is a diagonal of ∇ that is shorter than e. The lemma follows. $\square$

Corollary 4 *For every point set S, the greedy pseudo-triangulation equals the greedy triangulation.*

Proof. Assume that the greedy pseudo-triangulation of S contains a pseudo-triangle, ∇, that is not a triangle. Then, by Lemma 3, ∇ contains some diagonal, d, being shorter than its longest edge. So, during the greedy process of constructing the pseudo-triangulation, d would have been inserted before completing the insertion of the edges that form ∇ – a contradiction. $\square$

Requiring pointedness of a greedy pseudo-triangulation changes the situation. This concept is well defined, too, as each face of the pointed graph produced so far – if not a pseudo-triangle – can be split into two faces using any geodesic and without violating pointedness. Not surprisingly, the greedy pointed pseudo-triangulation can differ from the MWPPT. Figure 3 gives a simple example. This raises the question how well the greedy pointed pseudo-triangulation approximates the MWPPT.

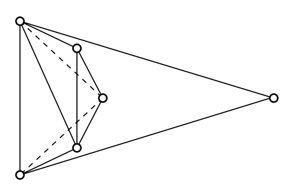

Figure 3: The greedy pointed pseudo-triangulation (solid) differs from the MWPPT (dashed).

5 Comparing MWPT and MWT

We now compare the minimum weight pseudo-triangulation to the minimum weight triangulation of a point set. A useful structure for this comparison is the so-called *wheel*. A wheel is the star-like triangulation of a convex polygon with exactly one interior vertex, the *hub* of the wheel. We call the vertex degree of the hub (i.e., the size of the convex polygon) the *degree* of a wheel. The *spokes* of a wheel are the edges of the wheel incident to the hub. Let us call a *big angle* an angle that is larger than π.

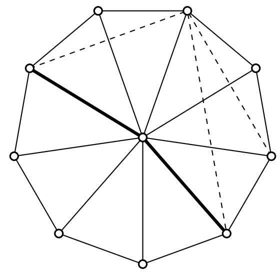

Figure 4: This regular wheel of degree 9 is both the MWT and the MWPT of the underlying point set.

Theorem 5 *There are point sets for which the minimum weight pseudo-triangulation is a triangulation.*

Proof. Consider the regular wheel in Figure 4. It is easy to see that this wheel is the MWT of the underlying point set. To construct a pseudo-triangulation that is not a triangulation we have to make the hub pointed, as it is the only interior vertex. This involves removing at least 4 spokes, and inserting 3 new edges afterwards (dashed edges in Figure 4). Let δ be the length of a short new edge, and let κ be the length of the long new edge. Further, let R be the length of a spoke. Applying the law of cosine we get $\delta = \sqrt{2R^2 \cdot \left(1 - \cos\frac{4\pi}{9}\right)}$ and $\kappa = \sqrt{2R^2 \cdot \left(1 - \cos\frac{2\pi}{3}\right)}$. As $2 \cdot \delta + \kappa > 4 \cdot R$, the constructed pseudo-triangulation is longer then the MWT. The assertion follows, as removing more than 4 spokes in the first place results in an even larger discrepancy. □

The point set used in the proof of Theorem 5 contains only one vertex in the interior. Hence the question arises whether requiring a certain number of interior vertices in a point set always ensures the existence of pointed vertices in its MWPT. We settle this question in the affirmative in the remainder of this section.

Observation 1 *If the MWPT of a point set is a triangulation then, for each interior vertex, its incident triangles form a wheel.*

This holds because, otherwise, some edge incident to such a vertex could be removed, creating a proper pseudo-triangle. We continue with a series of properties of wheels that imply the property MWPT $\neq$ MWT.

Observation 2 *Consider any two spokes in a wheel, T, and let α be the big angle between them. If, within α, there is only one spoke then removing it gives a pseudo-triangulation that is lighter than T.*

The next assertion easily follows from Observation 2.

Observation 3 *If the minimum weight triangulation of a point set is a wheel of degree 3 or 4 then there exists a pseudo-triangulation on the same (four or five) points that is lighter.*

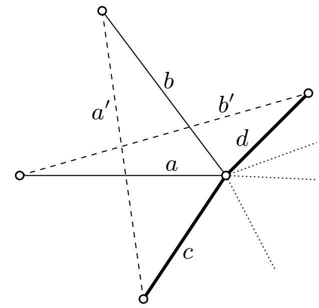

Figure 5: A wheel with two spokes in a big angle.

Whenever in a wheel there exist exactly two spokes, a and b, within the big angle of two other spokes, c and d, such that $|a| > |c|$ or $|b| > |d|$, then we can remove a and b and construct a shorter pseudo-triangulation by adding an edge between the non-hub endpoints of either b and c or a and d; see Figure 5. More formally we get:

Observation 4 *Let c and d be two spokes of a wheel whose big angle contains exactly two other spokes, a and b. Further, let a' be the edge between the non-hub endpoints of b and c, and let b' be the edge between the non-hub endpoints of a and d. If $|a| > |c|$ then $|a| + |b| > |a'|$. If $|b| > |d|$ then $|a| + |b| > |b'|$.*

Lemma 6 *If the minimum weight triangulation of a point set is a wheel of degree 5 then there exists a pseudo-triangulation on these six points that is lighter.*

Proof. See Figure 6. Let a be the longest spoke. Let c and d be the spokes that span the smallest big angle, α, that spoke a lies in. If a is the only spoke in α then removing a results in a lighter pseudo-triangulation, by Observation 2. If there exist more than 2 spokes within α then, again by Observation 2, removing c or d results in a pseudo-triangulation lighter than the wheel, as there exist only 5 spokes in total. It remains the case where there exists one other spoke, b, besides a within α. As a is the longest spoke we have $|a| > |c|$ and thus we can apply Observation 4 and get a lighter pseudo-triangulation again. □

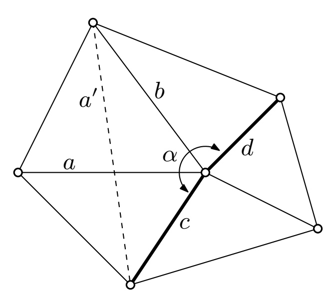

Figure 6: A non-regular wheel of degree 5.

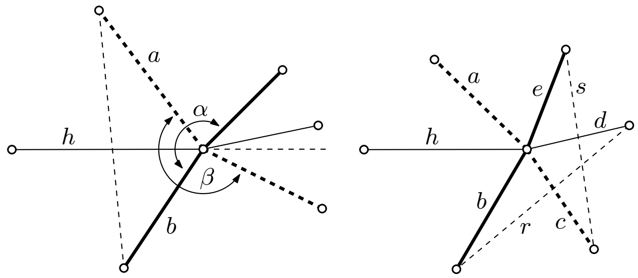

Figure 7: Two non-regular wheels of degree 6.

Lemma 7 *If the minimum weight triangulation of a point set is a wheel of degree* 6 *then there exists a pseudo-triangulation on these seven points that is lighter.*

Proof. Let h be the longest spoke. Bearing in mind Observation 2 and general position, let there be at least 2 spokes on either side of the line supporting h. If $\alpha > \pi$ or $\beta > \pi$ in Figure 7 (left) then Observation 4 implies the assertion. So assume $\alpha < \pi$ and $\beta < \pi$ (Figure 7 (right)). If $|c| > |d|$ then we can remove spokes b and c and add edge r to get a pseudo-triangulation lighter than the MWT. If $|c| < |d|$ then we can remove spokes d and e and add edge s to get a pseudo-triangulation lighter than the MWT. □

Lemma 8 *Let S be a point set with h points on the convex hull and at least* 3 *interior points. For every triangulation of S the sum of degrees of the convex hull vertices, ρ_h, is at least* $3 \cdot h + 3$.

Proof. It is easy to see that ρ_h is minimized if the interior vertices have a triangular convex hull. Thus we only have to consider h points in convex position and a triangle, Δ, inside. The number of triangulation edges then is $2 \cdot h + 6$, exactly $h + 3$ of which are interior to the belt $conv(S) \setminus \Delta$. But each of these interior edges has to be incident to some vertex of $conv(S)$. We conclude $\rho_h \geq (h + 3) + 2 \cdot h = 3 \cdot h + 3$. □

Theorem 9 *If a set S of $n \geq 15$ points contains more than $\frac{n-9}{2}$ interior points then its minimum weight pseudo-triangulation contains pointed interior vertices.*

Proof. Any triangulation with average *interior* vertex degree $\overline{\rho}_i < 7$ has at least one interior vertex of degree at most 6. From Observations 1 and 3 and Lemmas 6 and 7 we know that, in such a case, we can construct a corresponding pseudo-triangulation which is lighter than this triangulation.

The sum of all vertex degrees in a triangulation of S is exactly $6 \cdot n - 2 \cdot h - 6$ if h points of S are extreme. By Lemma 8, the sum of interior vertex degrees is at most $6 \cdot n - 5 \cdot h - 9$, which gives $n + 5 \cdot i - 9$ if there are $i = n - h \geq 3$ interior points. The average interior vertex degree thus is $\overline{\rho}_i \leq \frac{n-9}{i} + 5$. If we want $\overline{\rho}_i < 7$ then $i > \frac{n-9}{2}$. □

We remark that the bound in Theorem 9 improves if one can show that for wheels of degree 7 or higher there exist pseudo-triangulations with less weight.

6 Conclusion

We have given some properties of minimum weight pseudo-triangulations. A main open question is how much weight can be saved when relaxing from triangulations to pseudo-triangulations. Theorem 5 shows that there might be no gain at all and, even worse, the MW-PPT may be longer than the minimum weight triangulation. On the other hand, Theorem 9 suggests that the gain might be linear in the number of interior vertices.

References

[1] O. Aichholzer, F. Aurenhammer, P. Brass, and H. Krasser. Pseudo-triangulations from surfaces and a novel type of edge flip. *SIAM Journal on Computing*, 32:1621–1653, 2003.

[2] O. Aichholzer, C. Huemer, and H. Krasser. Triangulations without pointed spanning trees. *Computational Geometry: Theory and Applications*, 2007. To appear.

[3] F. Aurenhammer and Y.-F. Xu. Optimal triangulations. In P. Pardalos and C. Floudas, editors, *Encyclopedia of Optimization*, volume 4, pages 160–166. Kluwer, 2001.

[4] M. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[5] L. Kettner, D. Kirkpatrick, A. Mantler, J. Snoeyink, B. Speckmann, and F. Takeuchi. Tight degree bounds for pseudo-triangulations of points. *Computational Geometry: Theory and Applications*, 12(1&2):1–12, 2003.

[6] C. Levcopoulos and J. Gudmundsson. Minimum weight pseudo-triangulations. In *Proc. 24th International Conference on Foundations of Software Technology and Theoretical Computer Science*, pages 299–310, 2004.

[7] W. Mulzer and G. Rote. Minimum weight triangulation is np-hard. In *Proc. 22nd Symposium on Computational Geometry*, pages 1–10, 2006.

[8] G. Rote, F. Santos, and I. Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry - The Goodman-Pollack Festschrift*, volume 25 of *Algorithms and Combinatorics*, pages 699–736. Springer, 2003.

[9] G. Rote, C. Wang, L. Wang, and Y. Xu. On constrained minimum pseudotriangulations. In *Proc. 9th Symposium on Computing and Combinatorics*, volume 2697 of *LNCS*, pages 445–454, 2003.