Improved Layouts of the Multigrid Network

Shabnam Aziza* Therese Biedl††

Abstract

In a previous paper, Calamoneri and Massini studied the problem of drawing the multigrid network in "a grid of minimum area". In this paper we show that we can draw the multigrid network in a smaller grid, and can reduce the number of bends and the number of crossings as well.

1 Introduction

The multigrid network, M_N is a graph consisting of a set of square grids (or *arrays*), connected to each other in a particular way (see Sect. 2 for precise definitions). For parallel algorithms M_N is one possible architecture, and in particular proves useful for speeding up convergence for finite difference problems (see for example [6, p.99].)

Calamoneri and Massini studied how to embed M_N in the 2-dimensional grid, and achieve what they call "a grid of minimum area", which is a grid of size $(\frac{5}{2}N - 3) \times (3N - 4)$ [1]. This is clearly asymptotically optimal, since M_N has $\theta(N^2)$ vertices, but the factors can be improved. This is the topic of this paper. We use two main changes. First, we relocated nodes slightly; this alone removes $\theta(N)$ crossings. Then we replace nodes with other shapes. Calamoneri and Massini represented nodes either as points or as horizontal segments of length 1. The more freedom we allow ourselves in node representations, the more we can improve their bounds. By using vertical segments, the number of bends can be cut in half and the number of crossings can be reduced. If we allow horizontal segments of length 2, or small boxes, we can eliminate all bends. All of our constructions have smaller grid size than the one by Calamoneri and Massini. Table 1 gives a summary of our results.

2 Definitions

The *N-multigrid network* M_N, which exists for N a power of 2, is defined as follows: there are $\log N + 1$ 2-dimensional grids or *arrays*. For $0 \le k \le \log N$, each array is of size $N/2^k \times N/2^k$.[1] For $1 \le k \le \log N$, node (i, j) of the $N/2^k \times N/2^k$ array is connected to node $(2i, 2j)$ of the $N/2^{k-1} \times N/2^{k-1}$ array for all $0 \le i, j \le N/2^k - 1$. See also Fig. 1.

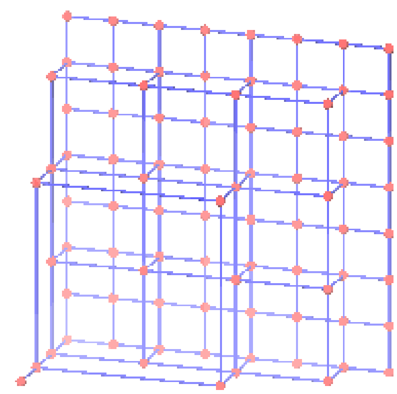


Figure 1: A 3D view of the multigrid M_8 (Viewpoint computed by R. Webber; see also [5] and [7]).

M_N has $\frac{4}{3}N^2 - \frac{1}{3}$ nodes. We use the notation $(i, j)_k$ for node (i, j) of the $N/2^k \times N/2^k$ array, $0 \le i, j \le N/2^k - 1$, $0 \le k \le \log N$. M_N has two types of edges: *intra-edges* connect nodes of the same array, *inter-edges* connect nodes in different arrays.

A *layout* of a multigrid network is an embedding of the graph in the 2-dimensional grid. The *grid size* of the layout is denoted by $(a + 1) \times (b + 1)$ where $a \times b$ is the dimension of the smallest axis-aligned bounding box of the layout.[2]

Since every node of the grid has only 4 incident edges, but some nodes of the multigrid network have 5 or 6 incident edges, we sometimes must use segments or boxes for nodes. We use the term *dot node* for a node that is represented by a dot. *Horizontal/vertical nodes* are nodes that are represented by a line segment of that orientation and length 1. *Long nodes* are represented by a horizontal segment of length 2. Finally *box nodes* are nodes that are represented by a box intersecting two rows and two columns. See also Fig. 2. We will use no other types of segments or boxes for nodes.

M_N is not planar, and hence any 2D embedding necessarily has crossings. We distinguish two different types of crossings. When an intra-edge of one array crosses an intra-edge of another array, we call this an

*saziza@yahoo.com

†biedl@uwaterloo.ca. Research supported by NSERC.

[1] It would sometimes simplify notation to use $\log N$ grids instead and denote them as $2^k \times 2^k$ arrays. However, to keep our results comparable to those in [1], we will follow their notation.

[2] In our pictures, we often insert empty rows and columns to keep the symmetry of the arrays visible. These are not counted for the grid size.

Approach	Grid size	Bends	Intra-crossings	Inter-crossings
[1]	$(\frac{5}{2}N-3) \times (3N-4)$	$\frac{1}{6}N^2-\frac{8}{3}$	$\frac{8}{3}N^2-4N\log N-\frac{8}{3}$	$\frac{5}{12}N^2-\frac{3}{2}N+\frac{1}{3}$
Sect. 4.2	$(\frac{7}{3}N-\frac{7}{3}) \times (\frac{8}{3}N-\frac{7}{3})$	$\frac{1}{12}N^2-\frac{4}{3}$	$\frac{8}{3}N^2-8N\log N$ $+12N-4\log N-\frac{44}{3}$	$\frac{1}{12}N^2-N-\frac{10}{3}$
Sect. 4.3	$(2N-1) \times (3N-3)$	0	$\frac{8}{3}N^2-8N\log N$ $+12N-4\log N-\frac{44}{3}$	0
Sect. 4.4	$(\frac{7}{3}N-\frac{7}{3}) \times (\frac{8}{3}N-\frac{7}{3})$	0	$\frac{8}{3}N^2-8N\log N$ $+12N-4\log N-\frac{44}{3}$	0

Table 1: Overview of our results. Sect. 3 explains intra-crossings and inter-crossings. Calamoneri and Massini also bound the maximum edge length by $\frac{3}{2}N-3$; our layouts have the same bound.

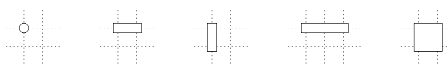


Figure 2: Different types of nodes: dot, horizontal, vertical, long and box.

intra-crossing. When an inter-edge crosses an intra-edge, we call this an *inter-crossing*. No other types of crossings occur in the layouts in this paper (they are not principally forbidden, but do not seem to help in improving the bounds.)

3 The layout by Calamoneri and Massini

Calamoneri and Massini [1] gave a layout of M_N in a $(\frac{5}{2}N-3) \times (3N-4)$-grid with $\frac{1}{6}N^2-\frac{8}{3}$ bends and a maximum edge length of $\frac{3}{2}N-3$. They use dot-nodes for some of the $N \times N$-array nodes and for the only node in the 1×1-array, and horizontal nodes otherwise. They generally place node $(i,j)_k$, $1 \le k \le \log N$, in the middle of the square generated by the node $(2i,2j)_{k-1}$, $(2i+1,2j)_{k-1}$, $(2i,2j+1)_{k-1}$, $(2i+1,2j+1)_{k-1}$. See Figure 3.

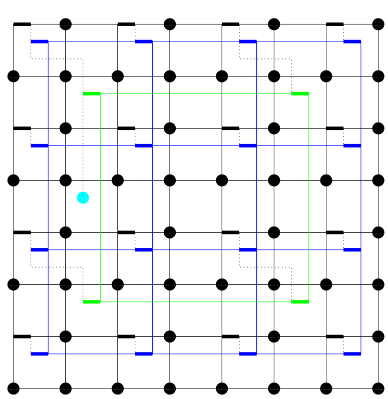


Figure 3: The layout of M_8 according to [1].

They did not analyse the number of crossings in their layout; we give a bound here. It is quite easy to show (and similar to the analysis in Sect. 4) that their layout has

$$\sum_{k=1}^{\log N-1} 2\left(N/2^k-1\right)\left(N/2^k\right)\sum_{\ell=0}^{k-1}2^{k-\ell} = \frac{8}{3}N^2-4N\log N-\frac{8}{3}$$

intra-crossings. Every inter-edge between the $N/2^{k-1} \times N/2^{k-1}$ array and the $N/2^k \times N/2^k$ array has 2^k-1 inter-crossings, for $2 \le k \le \log N-1$, and the inter-edge from the 1×1-grid to the 2×2-grid has $N/2-1$ inter-crossings, yielding

$$N/2-1+\sum_{k=2}^{\log N-1}\left(N/2^k\right)^2 \cdot (2^k-1) = \frac{5}{12}N^2-\frac{3}{2}N+\frac{1}{3}$$

inter-crossings.

4 Improved Layouts

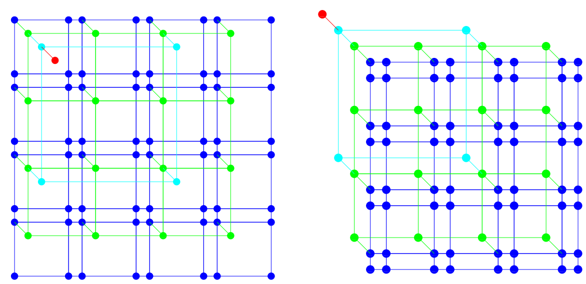
Now we will show some improvement over the layout of Calamoneri and Massini [1]. We use dot notes as they did (i.e., for some nodes in the $N \times N$-grid and the node in the 1×1-grid), but change the type of representation and location of the other nodes.

Obviously allowing other representations of nodes changes the model. On the other hand, no particular reason was given in [1] for using only dot and horizontal nodes, and allowing e.g. vertical nodes should have little impact on the applications, yet allows much improvement in the number of bends.

4.1 Reducing crossings

We remove many of the crossings by simply relocating the nodes. We do the following two changes: (i) Rather than placing each smaller array "at the center" of the previous array, we place the smaller array in such a way that inter-edges are very short. This eliminates many, and sometimes all, inter-crossings, (ii) Instead of placing each smaller array "inside" the previous array, we place the smaller array such that two of its sides are outside the previous array. This eliminates some intra-crossings, and some more inter-crossings.

We illustrate these changes in Fig. 4. We draw nodes here as dots, and inter-edges as diagonals, as to not fix yet the type of representation of nodes. The grid size of these drawings is easily computed to be $(2N-1)\times(2N-1)$. Note that if we allow diagonal edges, this would give an extremely compact drawing with no bends, and no inter-crossing. Also note that one more column can be eliminated by moving the only node of the 1×1 array to the column of its neighbours.



(a) (b)

Figure 4: (a) Making inter-edges shorter. (b) Moving arrays partially to the outside.

The number of intra-crossings can be computed as follows. Observe that the $N/2^k \times N/2^k$-array has $2 \cdot N/2^k \cdot (N/2^k - 1)$ edges. Of those, the top $N/2^k - 1$ edges and the left $N/2^k - 1$ edges don't have any crossing with the $N/2^\ell \times N/2^\ell$-array, for $\ell < k$. The remaining $2\left(N/2^k - 1\right)^2$ edges have $2^{k-\ell}$ crossings each with edges of the $N/2^\ell \times N/2^\ell$-array, for all $\ell < k$. The number of intra-crossings hence is

$$\begin{aligned} & \sum_{k=1}^{\log N-1} 2\cdot(N/2^k-1)^2 \sum_{\ell=0}^{k-1} 2^{k-\ell} \\ = & \sum_{k=1}^{\log N-1} 2\cdot(N/2^k-1)^2(2^{k+1}-2) \\ = & \frac{8}{3}N^2 - 8N\log N + 12N - 4\log N - \frac{44}{3}. \end{aligned}$$

4.2 Using vertical nodes

Calamoneri and Massini used only dot-nodes and horizontal nodes for their layouts. In this section, we show that by also allowing vertical nodes, we can halve the number of bends in their layout. For $1 \le k \le \log N - 1$, let node $(i,j)_k$ be a vertical node if k is odd and a horizontal node if k is even. Refer to Figure 5(a) of the exact placement and routing of inter-edges.

From Section 4.1, we know that we need $2N-1$ rows and $2N-1$ columns for the intra-edges, and can save one row or column by moving the node of the 1×1-array. For $1 \le k \le \log N - 1$, the nodes of the $N/2^k \times N/2^k$-array are horizontal if k is odd and vertical if k is even, and hence add $N/2^k$ columns if k is odd and $N/2^k$ rows if k is even. These added rows and columns provide enough additional space for the nodes of the $N \times N$-grid and the inter-edges. Hence if $\log N$ is odd, the number of added rows and columns is

$$\sum_{\substack{k=1\\k \text{ even}}}^{\log N-1} \frac{N}{2^k} = \sum_{i=1}^{(\log N-1)/2} \frac{N}{2^{2i}} = \frac{1}{3}N - \frac{2}{3}$$

and $$\sum_{\substack{k=1\\k \text{ odd}}}^{\log N-1} \frac{N}{2^k} = \sum_{i=1}^{(\log N-1)/2} \frac{N}{2^{2i-1}} = \frac{2}{3}N - \frac{4}{3}.$$

We can similarly show that for even $\log N$, the number of added rows and columns is $\frac{1}{3}N - \frac{4}{3}$ and $\frac{2}{3}N - \frac{2}{3}$. By choosing appropriately whether to save a column or a row when moving the node of the 1×1-array, we can achieve a grid size of at most $(\frac{7}{3}N - \frac{7}{3}) \times (\frac{8}{3}N - \frac{7}{3})$ (and one of the dimensions is in fact smaller by a constant of $\frac{1}{3}$).

For $1 \le k \le \log N - 2$ there are $(N/2^{k+1})^2$ inter-edges between the $N/2^k \times N/2^k$ array and the $N/2^{k+1} \times N/2^{k+1}$ array. Each of these edges has one bend, all other edges have no bend, and the total number of bends is

$$\sum_{k=1}^{\log N-2} (N/2^{k+1})^2 = \frac{1}{12}N^2 - \frac{4}{3}.$$

Note in particular that this is half of the number of bends used in [1]. Almost all bends also create an inter-crossing, except that for the $N/2^k \times N/2^k$-array, there are $N/2^{k+1}$ inter-edges (at the top or the left boundary) that have a bend, but no inter-crossing. Hence the number of inter-crossings is

$$\frac{1}{12}N^2 - \frac{4}{3} - \sum_{k=1}^{\log N-2} (N/2^{k+1}) = \frac{1}{12}N^2 - N - \frac{10}{3}$$

4.3 Using Large Segments

We can create a drawing that is entirely without bends by using long nodes, as well as horizontal nodes and dot nodes. For $1 \le k \le \log N - 1$, node $(i,j)_k$ is a long node if both i and j are even, and a horizontal node otherwise. Inter-edges are vertical segments of length 1. See Figure 5(b) for the precise arrangement.

Note that this drawing is in fact a *strong visibility representation*, i.e., all nodes are (possibly degenerate) boxes and there is an edge between two nodes if and

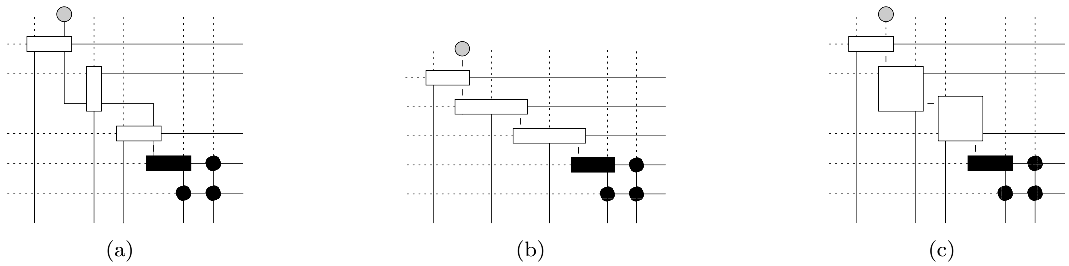


Figure 5: Layout with (a) vertical segments, (b) long segments and (c) box nodes. Dotted lines only exist if we are not at the left/top end.

only if the two nodes can see each other along a horizontal or vertical line. For more information on such representations, see for example [4].

As before, we need $2N-1$ rows and columns each for the intra-edges. To accommodate larger nodes, we need $N/2^k$ additional columns for $1 \leq k \leq \log N$, hence a total of $\sum_{k=1}^{\log N} N/2^k = \sum_{i=0}^{\log N-1} 2^i = N-1$. So the grid size is $(2N-1) \times (3N-3)$. There are no bends and no inter-crossings.

4.4 Using Box Nodes

One could criticize the drawings of the last section as being very unbalanced, in that we use significantly more columns than rows. This can be avoided if we allow box nodes. We use the following types of nodes: for $1 \leq k \leq \log N - 2$, node $(i,j)_k$ is a box node. Node $(i,j)_{\log N-1}$ is a vertical node if $\log N$ is odd and a horizontal node otherwise. See also Fig. 5(c).

Note the similarity of this layout with the layout with vertical nodes (Fig. 5(a)); all that has been changed is to stretch boxes of nodes as to cover incident bends. Exactly as for that layout, one can show that the grid size is $(\frac{7}{3}N - \frac{7}{3}) \times (\frac{8}{3}N - \frac{7}{3})$. There are no bends and no inter-edge crossings.

5 Conclusion

In this paper, we studied how to lay out the multigrid network in an area that is smaller than the layout given by Calamoneri and Massini [1]. We reduced the grid size, eliminated all bends, and all crossings between inter-edges; we also reduced crossings between intra-edges.

The most pressing open question is lower bounds, both for the grid size and the number of crossings. How many crossings between intra-edges are truly needed? Can we decrease their number if we allow more inter-crossings?

We are also interested in 3D layouts. The one in Figure 1 is not optimal. How small a volume can be achieved?

Optimal layouts of other architectures for parallel computing, as initiated for example in [3, 2], also deserve further study.

References

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