A generalization of Apollonian packing of circles

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Abstract

Three circles touching one another at distinct points form two curvilinear triangles. Into one of these we can pack three new circles, touching each other, with each new circle touching two of the original circles. In such a sextuple of circles there are three pairs of circles, with each of the circles in a pair touching all four circles in the other two pairs. Repeating the construction in each curvilinear triangle that is formed results in a generalized Apollonian packing. We can invert the whole packing in every circle in it, getting a "generalized Apollonian super-packing". Many of the properties of the Descartes configuration and the standard Apollonian packing carry over to this case. In particular, there is an equation of degree 2 connecting the bends (curvatures) of a sextuple; all the bends can be integers; and if they are, the packing can be placed in the plane so that for each circle with bend b and center (x, y), the quantities $bx/\sqrt{2}$ and by are integers.

Recently there has been renewed interest in a very old idea, that of Apollonian packing of circles, in which an initial configuration of three mutually tangent circles is augmented by repeatedly drawing new circles in each curvilinear gap. See for example Mumford et al [8]. We can also study "super-Apollonian" packings which are obtained by repeatedly inverting an Apollonian packing in every circle in it. It is a remarkable fact that Apollonian and super-Apollonian packings exist in which all the bends (curvatures) are integers. This property was studied in detail by Graham et al [3], and the group theory associated with these packings has been studied by the same authors [4-6]. Also, if all the bends are integers, the super-Apollonian packing can be placed in the plane so that all the "bend times center" quantities are integers. Several extensions of the Apollonian idea have been studied, for example Mauldon [7] studied configurations in which adjacent circles do not touch but have constant "separation".

Our own interest lies in extending these ideas in new directions, particularly by packing not one but three circles within each triangular gap, thus forming sextuples of circles, and in exploring the degree to which the theory associated with the classical packings can be extended to cover this case. We find that all the bends in such a generalized packing can be integers; and there are results relating to the positions of the centers of the circles that directly generalize those found by Lagarias et al [2] in the classical Descartes-Apollonian case.

Figure 1 shows the four possible configurations of a sextuple. There can be zero, one, or two circles wth bend zero (i.e. straight lines), and at most one bend can be negative, as in case (a).

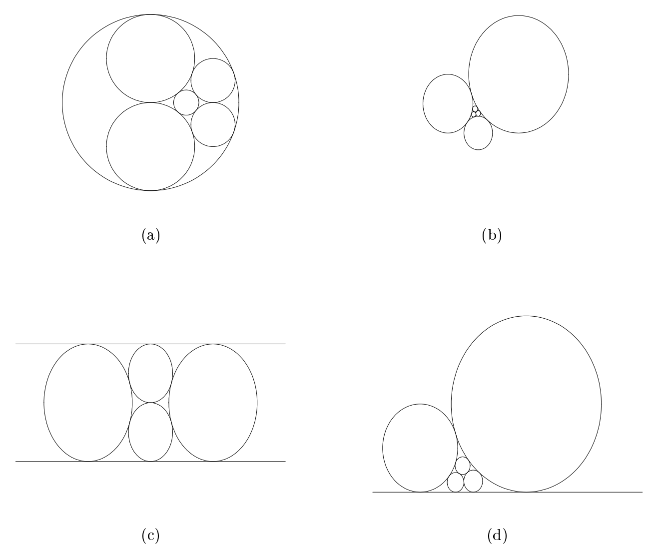

Figure 1: Sextuple configurations

These configurations generalize the classical Descartes configuration, in which just one circle is placed in a curvilinear triangle. Such a sextuple of circles forms an $n = 4$ example of what we call a "ball-bearing" configuration, in which a ring of n circles (each touching two others) have the property that there are "inner" and "outer" circles that each touch all n circles in the ring. The n=3 case reproduces the classical Descartes configuration. With $n = 4$ the circles come in three pairs, with each of the circles in a pair touching all four circles in the other two pairs. The circles of a pair do not touch one another. The sextuple thus has the symmetry of the vertices of an octahedron (or of the faces of a cube), rather than the symmetry of a tetrahedron as in the Descartes case.

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Repeating the construction in each curvilinear triangle that is formed results in a "generalized Apollonian packing". See Figures 2 (based on Figure 1(a)) and 3 (based on Figure 1(c)). Here, and in subsequent figures, we include only circles with bend less than 100.

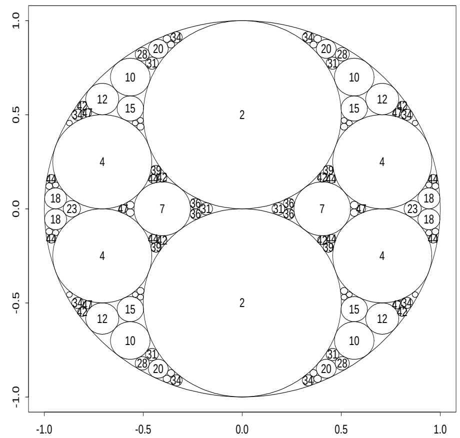

Figure 2: A generalized Apollonian packing

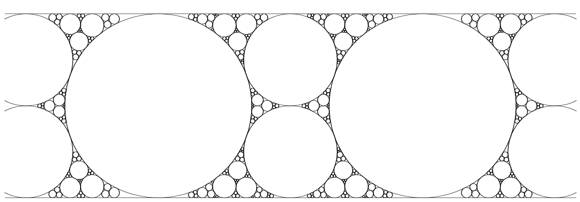

Figure 3: Another generalized Apollonian packing

Many of the properties of the Descartes configuration and the standard Apollonian packing carry over to this case. In particular:

(i) Given a ring of four circles, formed by two pairs of circles in a sextuple, there is a quadratic equation whose coefficients involve the bends of these four circles, the roots of which are the bends of the other pair of circles in the set. Explicitly,

$$2x^2 - x\sigma + \tau - \frac{3}{8}\sigma^2 = 0 \qquad (1)$$

where $\sigma = b_1 + b_1' + b_2 + b_2', \tau = {b_1}^2 + {b_1'}^2 + {b_2}^2 + {b_2'}^2$ This generalizes the classical Descartes equation. Replacing each bend by the corresponding bend*(complex) center gives another result which generalizes the "Complex Descartes Theorem" of [2].

(ii) There is an analog of "Descartes reflection" (see [2]) in which three circles (one from each pair in a sextuple, these three circles occupying a curvilinear triangle formed by the other three circles of the sextuple) are replaced by three circles occupying the other triangle formed by these three circles, thus forming another sextuple. Given a sextuple, this operation can be performed in eight different ways. Iteration of this operation creates a generalized Apollonian packing in which the interiors of all circles are disjoint. A packing is determined by any three touching circles within it.

(iii) all six bends of the circles in a sextuple can be integers. Examples: in Figure 1(c), the bends are (0,2; 0,2; 1,1); in Figure 1(a) they are (-1,7; 2,4; 2,4). This property is inherited by all derived circles.

(iv) if all bends of a sextuple are integers, the sextuple can be placed in the plane so that for each circle with bend b and center (x, y), the "bend times center" quantities (bx, by) have both $bx/\sqrt{2}$ and by integers. This property is inherited by the generalized packing based on this sextuple.

(v) The construction of the generalized packing can be realised by integral linear operations acting on matrices representing the sextuples. These matrices could be 6 x 4 matrices with each row containing the "abbc" or "augmented bend, bend times center" coordinates introduced in [2]. The abbc coordinates of a circle C with bend ($=1/\text{radius}$) b and center (x, y) is the vector $\mathbf{a}(C) = (\bar{b}, b, bx, by)$ where $\bar{b}$ is the bend of the circle that is the inverse of C in the unit circle, namely

$$\bar{b} = b(x^2 + y^2) - 1/b \qquad (2)$$

However it is convenient to represent a sextuple by a 4 x 4 matrix that we call $\mathbf{F}(C)$ in which the first three rows contain the abbc coordinates of three of the circles in the sextuple (one from each pair) and the fourth row is the average of two rows that represent a pair of circles in the sextuple (this average is the same for each of the three pairs). This row does not represent a circle.

(vi) There are dual operations acting on the right, which represent Mobius transformations.

(vii) Among the sextuples with integer bends, there are "root" sextuples (see [3] and [5]) having the property that any application of the reflection operation in (ii) results in circles with larger bends. These root sextuples can be found by applying a reduction algorithm, just as in the Descartes case. Except for the special sextuple with bends (0,2; 0,2; 1,1) (Figure 1(a)), each root sextuple has exactly one circle with negative bend. We have a conjecture as to the number of root sextuples with smallest bend $-n$.

(viii) By inverting a generalized Apollonian packing in each circle in the packing, and then again in every circle, and so on, we obtain a "generalized Apollonian super-packing", directly analogous to the Apollonian super-packing studied in [5]. There is essentially just one super-packing in which all bends are integral. This super-packing can be placed in the plane, in exactly four ways, so that each $bx/\sqrt{2}, by$ is integral. In each version of this super-packing, there is a basic rectangle $(0, \sqrt{2})$ x $(0, 1)$ which repeats by translation and reflection to cover the whole plane. See Figure 4.

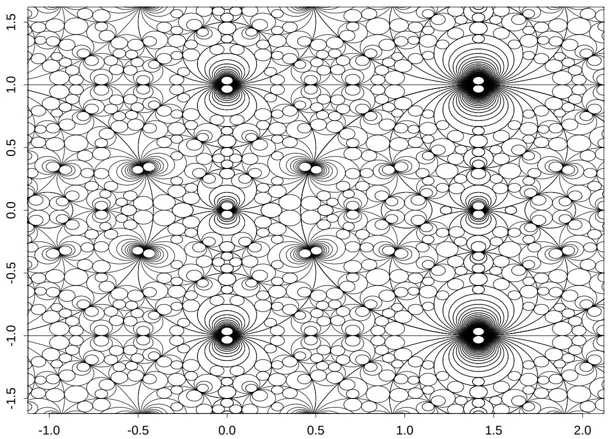

Figure 4: A generalized Apollonian super-packing

(ix) Each primitive integral sextuple appears exactly once in the basic $(0, \sqrt{2})$ x $(0, 1)$ rectangle of the super-packing. Computation suggests that there are some symmetries within the basic rectangle, like those shown in [5].

(x) One can consider "ball-bearing" structures of circles, in which a ring of n balls (each touching two neighbors) have the property that there exist "inner" and "outer" circles that each touch each of the "balls" in the ring. The case $n = 3$ reproduces the Descartes configuration; the case $n = 4$ gives the sextuples studied in this paper. The bends of all $n + 2$ circles can be integral only when $n = 3, 4, 6$. There is a quadratic equation whose roots are the bends of the "inner" and "outer" circles, and whose coefficients involve the bends of the circles in the ring.

(xi) There is an analog of the Farey series and the associated Ford circles, in which at every stage we insert two new fractions (and two new touching circles) instead of just one, between every existing adjacent pair of fractions. See Figure 5.

There are several open questions.

- Is the conjectured formula for the number of root sextuples correct?
- Are the conjectured symmetries within the basic cell of the super-packing valid?
- Do all integers arise as bends of circles in generalized Apollonian packings?
- Is the Hausdorff dimension of our generalized super-packing the same as in the Apollonian case?
- Are there other ways to generalize the classical Apollonian packing?
- Are there other ways to pack integer-bend circles?
- What about higher dimensions?

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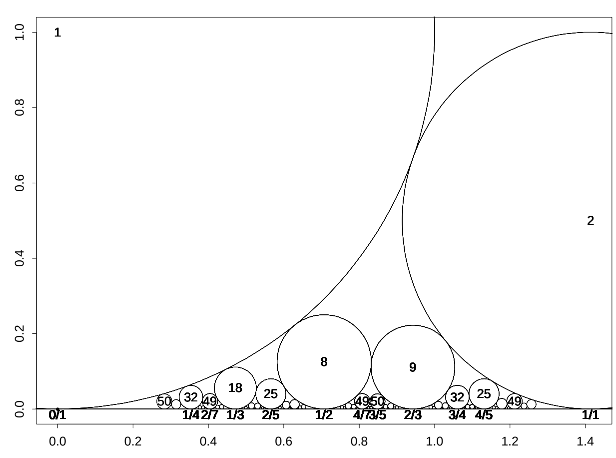

Figure 5: Generalized Ford circles

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