A generalization of Apollonian packing of circles

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Abstract

Three circles touching one another at distinct points form two curvilinear triangles. Into one of these we can pack three new circles, touching each other, with each new circle touching two of the original circles. In such a sextuple of circles there are three pairs of circles, with each of the circles in a pair touching all four circles in the other two pairs. Repeating the construction in each curvilinear triangle that is formed results in a generalized Apollonian packing. We can invert the whole packing in every circle in it, getting a "generalized Apollonian super-packing". Many of the properties of the Descartes configuration and the standard Apollonian packing carry over to this case. In particular, there is an equation of degree 2 connecting the bends (curvatures) of a sextuple; all the bends can be integers; and if they are, the packing can be placed in the plane so that for each circle with bend b and center (x, y), the quantities $bx/\sqrt{2}$ and by are integers.

1 Introduction and summary

Recently there has been renewed interest in a very old idea, that of Apollonian packing of circles, in which an initial configuration of three mutually tangent circles is augmented by repeatedly drawing new circles in each curvilinear gap. See for example Mumford et al [9]. We can also study "super-Apollonian" packings which are obtained by repeatedly inverting an Apollonian packing in every circle in it. It is a remarkable fact that Apollonian and super-Apollonian packings exist in which all the bends (curvatures) are integers. This property was studied in detail by Graham et al [3], and the group theory associated with these packings has been studied by the same authors [4-6]. Also, if all the bends are integers, the super-Apollonian packing can be placed in the plane so that all the "bend times center" quantities are integers. Several extensions of the Apollonian idea have been studied, for example Mauldon [8] studied configurations in which adjacent circles do not touch but have constant "separation". Stephenson [10] takes the theory in rather different directions.

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Our own interest lies in extending these ideas in new directions, particularly by packing not one but three circles within each triangular gap, thus forming sextuples of circles, and in exploring the degree to which the theory associated with the classical packings can be extended to cover this case. We find that all the bends in such a generalized packing can be integers; and there are results relating to the positions of the centers of the circles that directly generalize those found by Lagarias et al [7] in the classical Descartes-Apollonian case.

Into either of the curvilinear triangles formed by three mutually tangent circles we can pack three more circles, forming a sextuple. Figure 1 shows the four possible configurations of a sextuple. There can be zero, one, or two circles with bend zero (i.e. straight lines), and at most one bend can be negative, as in case (a).

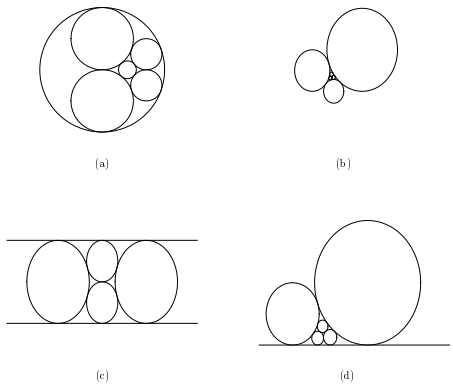

Figure 1: Some sextuple configurations

Figure 1: Sextuple configurations

This generalizes the classical Descartes configuration, in which just one circle is placed in a curvilinear trian-

gle. Such a sextuple of circles forms an $n = 4$ example of what we call a "ball-bearing" configuration, in which a ring of n circles (each touching two others) have the property that there are "inner" and "outer" circles that each touch all n circles in the ring. The n=3 case reproduces the classical Descartes configuration. With $n = 4$ the circles come in three pairs, with each of the circles in a pair touching all four circles in the other two pairs. The circles of a pair do not touch one another. The sextuple thus has the symmetry of the vertices of an octahedron (or of the faces of a cube), rather than the symmetry of a tetrahedron as in the Descartes case. Repeating the construction in each curvilinear triangle that is formed results in a "generalized Apollonian packing". See Figures 2 (based on Figure 1(a)) and 3 (based on Figure 1(c)). In these packings, all the bends are integers. In this and subsequent Figures, we have drawn only the circles with bends less than 100. There are also packings that fill either a half-plane, or the whole plane; but their bends cannot all be integral.

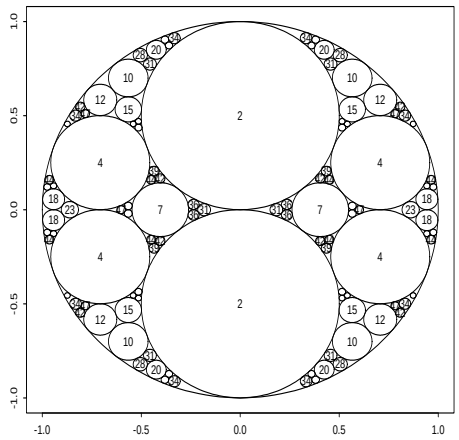

Figure 2: A generalized Apollonian packing

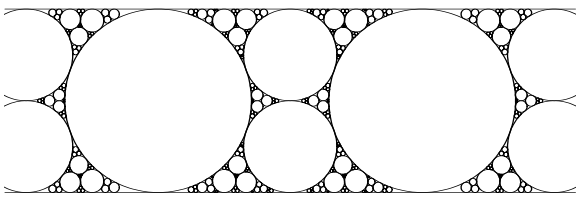

Figure 3: Another generalized Apollonian packing

Many of the properties of the Descartes configuration and the standard Apollonian packing carry over to this case. In later sections we will establish these results:

(i) Given a ring of four circles, formed by two pairs of circles in a sextuple, there is a quadratic equation (equation (7) below) whose coefficients involve the bends of these four circles, the roots of which are the bends of the other pair of circles in the sextuple. This generalizes the classical Descartes equation (equation (2) below). Replacing each bend by the corresponding bend*(complex) center gives another result which generalizes the "Complex Descartes Theorem" of [7].

(ii) There is an analog of "Descartes reflection" (see [7]) in which three circles (one from each pair in a sextuple, these three circles occupying a curvilinear triangle formed by the other three circles of the sextuple) are replaced by three circles occupying the other triangle formed by these three circles, thus forming another sextuple. Given a sextuple, this operation can be performed in eight different ways. See Section 4 below. Iteration of this operation creates a generalized Apollonian packing in which the interiors of all circles are disjoint. A packing is determined by any three touching circles within it.

(iii) all six bends of the circles in a sextuple can be integers. See Section 5. Examples: in Figure 1(c), the bends are (0,2; 0,2; 1,1); in Figure 1(a) they are (-1,7; 2,4; 2,4). The integrality property is inherited by all derived circles.

(iv) if all bends of a sextuple are integers, the sextuple can be placed in the plane so that for each circle with bend b and center (x, y), the "bend times center" quantities (bx, by) are of the form $m\sqrt{2}, n$ with m, n integers. This property is inherited by the generalized packing based on this sextuple. See Section 7.

(v) The construction of the generalized packing can be realised by integral linear operations acting on matrices representing the sextuples. These matrices could be 6 x 4 matrices with each row containing the "abbc" or "augmented bend, bend times center" coordinates introduced in [7], and defined in Section 2 below. However it is convenient to represent a sextuple by a 4 x 4 matrix in which the first three rows contain the abbc coordinates of three of the circles in the sextuple (one from each pair) and the fourth row is the average of the two rows that represent a pair of circles in the sextuple (this average is the same for each of the three pairs, see Lemma 3 below). This row does not represent a circle. See Section 3.

(vi) There are dual operations acting on the right, which represent Moebius transformations. See Section 4.

(vii) Among the sextuples with integer bends, there are "root" sextuples (see [3] and [5]) having the property that any application of the reflection operation in (ii) results in circles with larger bends. These root sextuples can be found by applying a reduction algorithm, just as in the Descartes case. Except for the special sextuple with bends (0,2; 0,2; 1,1) (Figure 1(a)), each root sextuple has exactly one circle with negative bend.

We have a conjecture ((10) below) as to the number of primitive root sextuples with smallest bend $-n$. See Section 6.

(viii) By inverting a generalized Apollonian packing in each circle in the packing, and then again in every circle, and so on, we obtain a "generalized Apollonian super-packing", directly analogous to the Apollonian super-packing studied in [5]. There is essentially just one super-packing in which all bends are integral. This super-packing can be placed in the plane, in exactly four ways, so that each $bx/\sqrt{2}, by$ is integral. In each version of this super-packing, there is a basic rectangle $(0, \sqrt{2})$ x $(0, 1)$ which repeats by translation and reflection to cover the whole plane. See Section 7 and Figure 4.

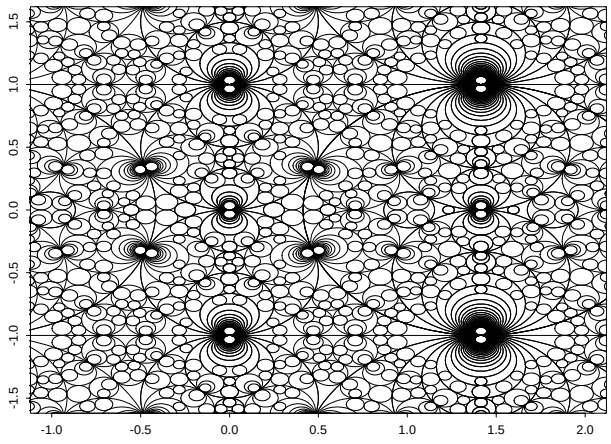

Figure 4: A generalized Apollonian super-packing

(ix) Each primitive integral sextuple appears exactly once in the $(0, \sqrt{2})$ x $(0, 1)$ rectangle of the super-packing. See Section 7. Computation suggests that there are some symmetries within the basic rectangle, like those shown in [5]. See Figure 5 below.

(x) One can consider "ball-bearing" structures of circles, in which a ring of n balls (each touching two neighbors) have the property that there exist "inner" and "outer" circles that each touch each of the "balls" in the ring. The case $n = 3$ reproduces the Descartes configuration; the case $n = 4$ gives the sextuples studied in this paper. The bends of all $n+2$ circles can be integral only when $n = 3, 4, 6$. There is a quadratic equation (equation (15) below) whose roots are the bends of the "inner" and "outer" circles, and whose coefficients involve the bends of the circles in the ring.

(xi) There is an analog of the Farey series and the associated Ford circles, in which at every stage we insert two new fractions (and two new touching circles) instead of just one, between every existing adjacent pair of fractions. See Figure 6 below.

2 Conventions and notation

We adopt the conventions in [3-7]. A circle C with radius r and center (x, y) is described by its "augmented bend, bend*center" (abbc) coordinates $\mathbf{a}(C) = (\bar{b}, b, bx, by)$ where b is the bend (i.e. curvature) $b = 1/r$, and where $\bar{b}$ is the bend of the circle that is the inverse of C in the unit circle, namely

$$\bar{b} = b(x^2 + y^2) - 1/b \tag{1}$$

Sometimes we will write $z = x + iy$ and work in the complex plane.

A straight line is a circle with bend zero, and we need a definition to replace (1). Consider the line with equation $p_1 x + p_2 y = h$ where $p = (p_1, p_2)$ is a unit vector. This is the limit as $\lambda \to \infty$ of a circle with center λp and radius $(\lambda - h)$, so it makes sense to define the abbc coordinates of this line as $(2h, 0, p_1, p_2)$. If the bend of a circle is positive, the "interior" of the circle contains its center. In all cases, the abbc coordinates uniquely determine the circle and its orientation. We say that two circles "touch" or "are tangent" when they have just one point in common and their interiors are disjoint. We are considering sets of six circles such that each touches four others, and the interiors are all disjoint.

We can describe a sextuple S by the 6 x 4 matrix that has the abbc coordinates of the circles in its rows, where the circles are taken in what we call *standard order*, namely rows 1,2 contain the circles in a pair, as do rows 3,4 and rows 5,6. Since we can permute the three pairs, and also can take the circles of a pair in either order, there are 48 different standard orders, all representing the same sextuple.

We introduce the matrix (this is twice the inverse of what is called the "Wilker quadratic form" in [4]), so named in honor of Wilker [11].

$$\mathbf{W} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From [4] and [7] we have

Lemma 1 For any two circles C and C',
$\mathbf{a}(C)\mathbf{W}\mathbf{a}(C')^T = 1$ if $C = C'$
$= -1$ if C and C' are externally tangent

Proof Simple algebra.

We also introduce the matrix of the "Descartes quadratic form" (twice what is defined in [4])

$$\mathbf{D} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Note that $\mathbf{D}^{-1} = \mathbf{D}/4$. We honor Rene Descartes here because in 1643 he derived a result connecting the bends of four mutually tangent circles, equivalent to

$$2(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (b_1 + b_2 + b_3 + b_4)^2 \qquad (2)$$

and this can be written $\mathbf{bDb}^T = 0$ where $\mathbf{b}$ is the vector (b_1, b_2, b_3, b_4). Given a Descartes configuration consisting of four mutually tangent circles C_1, C_2, C_3, C_4 with disjoint interiors, we define the 4 x 4 matrix $\mathbf{A}$ as containing rows $\mathbf{a}(C_j)$, $j = 1, 2, 3, 4$. From Lemma 1 we have

$$\mathbf{AWA}^T = \mathbf{D}$$

Inverting both sides of this relation we immediately obtain the "Augmented Euclidean Descartes Theorem" of [7], namely

$$\mathbf{A}^T\mathbf{DA} = 4\mathbf{W}^{-1} \qquad (3)$$

The Descartes equation (2) is the second diagonal element of this relation. From the (3,4) coordinates we have the "Complex Descartes equation" of [7], namely

$$\mathbf{bzDbz}^T = 0$$

where $\mathbf{bz} = (b_1z_1, b_2z_2, b_3z_3, b_4z_4)$. We will find a generalization of (3) in the sextuple case, but the same argument will not work because we have a 6 x 4 matrix of abbc coordinates, and we get singular matrices.

3 Sextuples

Lemma 2 If C and C' are a pair of (non-tangent) circles in a sextuple configuration, then

$$\mathbf{a}(C)\mathbf{Wa}(C')^T = -3$$

Proof. The relation is easily verified for the special sextuple shown in Figure 1(c), with the origin at the contact-point of the two circles with bend 2, for which the abbc coordinates are the rows of

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \end{pmatrix}$$

Any sextuple can be obtained from this one by a Moebius transformation. The group of Moebius transformations is generated by translations, dilations, conjugation, and inversion in the unit circle. Appendix A of [4] shows that the effect of each of these transformations on the abbc coordinates of a circle is right-multiplication by one of a certain group of matrices, and each matrix $\mathbf{m}$ in this group has the property that $\mathbf{mWm}^T = \mathbf{W}$. The Lemma follows.

This simple result has an important consequence.

Lemma 3 In a sextuple S containing circles C_1, C_1' etc. we have

$$\mathbf{a}(C_1) + \mathbf{a}(C_1') = \mathbf{a}(C_2) + \mathbf{a}(C_2') = \mathbf{a}(C_3) + \mathbf{a}(C_3')$$

Proof. For $j = 1, 2, 3$ define $\mathbf{w}_j = (\mathbf{a}(C_j) + \mathbf{a}(C_j'))/2$, and let $\mathbf{F}_j$ be the 4 x 4 matrix whose rows are $\mathbf{a}(C_1), \mathbf{a}(C_2), \mathbf{a}(C_3), \mathbf{w}_j$. From Lemmas 1 and 2 we have that $\mathbf{w}_1\mathbf{W}\mathbf{w}_1^T = (1 - 3 - 3 + 1)/4 = -1$, and $\mathbf{w}_1\mathbf{W}\mathbf{w}_2^T = (-1 - 1 - 1 - 1)/4 = -1$ also. So

$$\mathbf{F}_1\mathbf{W}\mathbf{F}_1^T = \mathbf{K} = \mathbf{F}_2\mathbf{W}\mathbf{F}_1^T \qquad (4)$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

Since all the matrices in (4) are non-singular, this shows that $\mathbf{F}_1 = \mathbf{F}_2$, so that $\mathbf{w}_1 = \mathbf{w}_2$. Similarly both are equal to $\mathbf{w}_3$.

This result makes possible a convenient representation of a sextuple.

F notation

Given a sextuple S containing six circles $C_1, C_1'; C_2, C_2'; C_3, C_3'$, an F matrix describing the sextuple is a 4 x 4 matrix whose first three rows contain the abbc coordinates of three of the circles (one from each pair) and the fourth row is the average of the abbc coordinates of the two circles in any pair.

Given a sextuple, there are 48 such representations, obtained by choosing the pairs in different orders, and choosing different representatives of each pair. We can move between different representations by premultiplying $\mathbf{F}$ by matrices like

$$\begin{pmatrix} 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which replaces C_1, C_2, C_3 by C_2', C_3', C_1. The set of 48 such integral matrices form a group which we call *Perm*.

We can recover all the 48 6 x 4 standard representations as $\mathbf{G}(S) = \mathbf{EF}(S)$ where

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The next Theorem generalizes the Descartes equation.

Theorem 1 Given the bends b_1, b_1', b_2, b_2' of two pairs of a sextuple, forming a ring, the bends of the other pair,

each of which touches all four of these circles, are the roots of the equation

$$2x^2 - x\sigma + \tau - \frac{3}{8}\sigma^2 = 0 \qquad (5)$$

where $\sigma = b_1 + b_1' + b_2 + b_2'$, $\tau = b_1^2 + {b_1'}^2 + b_2^2 + {b_2'}^2$. The same equation holds when each bend b is replaced by bz, where z is the (complex) center of the circle with bend b.

Proof. Inverting both sides of the equation $\mathbf{FWF}^T = \mathbf{K}$ we find

$$\mathbf{F}^T\mathbf{K}^{-1}\mathbf{F} = \mathbf{W}^{-1}$$

in which

$$\mathbf{K}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

So if $2w = b_1 + b_1' = b_2 + b_2' = b_3 + b_3'$, then

$$w^2 - 2w(b_1 + b_2 + b_3) + b_1^2 + b_2^2 + b_3^2 = 0 \qquad (6)$$

Since $b_1' = 2w - b_1, b_2' = 2w - b_2$, we have

$$\sigma = 4w, \tau = 2(b_1^2 + b_2^2) - 4w(b_1 + b_2) + 8w^2$$

and (5) follows immediately. The bend*center result follows by the same argument. This validates claim (i) of the Introduction. Equation (6) can be regarded as another generalization of the Descartes equation, and gives the following result.

Corollary Given three mutually tangent circles with bends b_1, b_2, b_3, enclosing two triangular gaps, the bends of the three circles that can be inscribed in these gaps are $w - b_1, w - b_2, w - b_3$ and $w' - b_1, w' - b_2, w' - b_3$ where w, w' are the roots of the equation (6). A similar result holds for the bend*centers.

4 Reflection and generalized Apollonian packing

In [7] the operation of "Descartes reflection" was defined, in which one of the circles in a Descartes quadruple is replaced by the other circle that is tangent to the same three circles as the chosen one. Given three mutually tangent circles C_1, C_2, C_3 with bends b_1, b_2, b_3 and (complex) centers z_1, z_2, z_3, the bends b_4, b_4' of the two circles that are each tangent to these three circles are the roots of the Descartes equation (2). So "reflection" of the b_4 circle replaces this circle by one with bend $b_4' = 2(b_1 + b_2 + b_3) - b_4$, and (using the "Generalized Descartes equation") with bend*center $b_4'z_4' = 2(b_1z_1 + b_2z_2 + b_3z_3) - b_4z_4$.

This operation can be performed in four ways, taking each of the circles in the Descartes quadruple in turn.

Using equation (6), we can perform a similar "reflection" operation on the circles in a sextuple. Specifically, let $\mathbf{F}$ be a 4 x 4 matrix describing a sextuple. Then (6) has two roots w, w' say. One of these gives the bends of the other three circles in the sextuple described by $\mathbf{F}$, namely $2w - b_1, 2w - b_2, 2w - b_3$. The other root gives the bends of another set of three circles, namely $2w' - b_1, 2w' - b_2, 2w' - b_3$ which also form a sextuple with the three circles in the first three rows of $\mathbf{F}$. The (complex) centers of these circles can be found by using (6) with the bends replaced by bend*centers. The new sextuple is the result of inverting the original one in the circle that passes through the three points of contact of the three circles that stay invariant. This validates claim (ii).

Since from (6) $w + w' = 2(b_1 + b_2 + b_3)$, we do not need to solve a quadratic explicitly.

Lemma 4 The operation of reflection in circles C_1, C_2, C_3 can be performed by premultiplying an $\mathbf{F}$ matrix that describes the sextuple, with the abbc coordinates of these three circles in its first three rows, by the matrix $\mathbf{R}$, where

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

Proof. R replaces w by w' and $\mathbf{w}$ by $\mathbf{w}'$.

This validates the "reflect" part of claim (v). Note that reflection is an involution: $\mathbf{R}^2 = \mathbf{I}$. $\mathbf{R}$ is the same as the matrix S_4 of [4] that implements reflection of the fourth circle in a Descartes quadruple. To reflect in a different triad of circles in the sextuple, we use the "permute" group *Perm* to bring the required triad of circles into rows 1,2,3 of the representation $\mathbf{F}$. So the operation can be performed in eight ways.

The generalized Apollonian packing based on a given sextuple S_0 is obtained by premultiplying $\mathbf{F}(S_0)$ by "permute" and "reflect" matrices, in all possible ways. Thus a general sextuple in the generalized Apollonian packing based on S_0 is represented by an $\mathbf{F}$ matrix of the form

$$\mathbf{RP}_n\mathbf{RP}_{n-1}\cdots\mathbf{RP}_1\mathbf{F}(S)$$

(Since $\mathbf{R}$ is an involution, we do not need powers of $\mathbf{R}$ higher than the first; since the "permute" matrices form a group, we never need more than one between two $\mathbf{R}$'s.)

Figure 2 shows a generalized Apollonian packing based on the sextuple in Figure 1(a).

As in the Descartes case, there are several ways to regard the generalized Apollonian packing:

(i) as a packing of circles in the plane, each touching four others; these four all being touched by one other circle;

(ii) as a packing of triads of touching circles, each triad surrounding (and being surrounded by) another triad of circles;

(iii) as an arrangement formed by configurations consisting of sextuples of circles, interlocking by overlapping sets of three tangent circles.

The view (iii) gives rise to intriguing group-theory problems, analogous to those studied in Graham et al [4-5]. We remarked above that in [4] it was pointed out that the effect of a Moebius transformation on the abbc coordinates of a circle is right-multiplication by an element of a certain group. It follows that the same is true for the effect on an **F** matrix describing a sextuple.

5 Integral packings

Lemma 5 If the bends of a sextuple are all integers, the same is true for the bends of all the circles in the generalized packing based on this sextuple. If the abbc coordinates of the circles in a sextuple are of the form (integer, integer, integer$\sqrt{2}$, integer), then the same is true for every circle in the packing.

Proof. The matrix **R** and the matrices in the group *Perm* have integer entries.

This lemma is not empty, since we have the examples in figure 1(a) and 1(c).

We will consider only *primitive* sextuples, where the bends have no common factor.

We can generate primitive integral sextuples of bends as follows. Suppose the bends of three mutually tangent circles are integers a, b, c. From equation (6) we have

$$w = a + b + c \pm \sqrt{2(ab + ac + bc)}$$

so w is an integer only if $2(ab+ac+b)$ is an (even) perfect square, say $4m^2$. Only in this case can the bends of the remaining three circles be integers. Write $f = a+b, g = a + c$. Then

$$ab + ac + bc = fg - a^2 = 2m^2$$

To generate integral sextuples we can choose a and m arbitrarily, factorize a^2+2m^2 into fg, and hence obtain b and c. Then the other three bends come from $2w - (a, b, c)$ where

$$w = a + b + c \pm 2m$$

Without loss of generality we can take m non-negative, but a can have either sign.

The sextuple is primitive iff a, f, g have no common factor. Thus w.l.o.g. we can take a odd (of either sign), so that f and g are odd (and positive). Then b and c are even, and w is necessarily odd. This settles claim (iii).

6 Reduction algorithm and root sextuples.

The *size* of a sextuple is conveniently measured by the value of w. Given the bends of a sextuple, by applying a permutation and then reflecting, we may be able to find a sextuple with smaller value of w. If this is not possible, we call this a *root* sextuple. Except for the special sextuple (shown in Figure 1(c)) with bends (0,2;0,2;1,1), a root sextuple has exactly one negative bend. As in the Descartes case,, starting from a given sextuple, there is a unique root sextuple that can be reached by repeated reflections.

Lemma 6 The sextuple described by integer bends $a \le b \le c \le w$ (satisfying (6)) is a root sextuple iff $w \le a + b + c$.

Proof. $w' \ge w$.

Computation finds the numbers of primitive root sextuples with smallest bend $-n$, for $n = 1, \cdots 100$, in the following table.

The number of root sextuples with smallest bend $-n$

i	1	2	3	4	5	6	7	8	9	10
$n = 0 + i$	1	2	1	3	2	4	3	5	2	8
$n = 10 + i$	3	6	4	10	4	9	5	8	5	14
$n = 20 + i$	5	12	7	10	8	16	5	18	8	16
$n = 30 + i$	9	17	6	18	13	14	10	20	8	26
$n = 40 + i$	11	20	11	22	10	26	13	18	15	32
$n = 50 + i$	9	30	14	20	16	34	10	32	15	28
$n = 60 + i$	16	34	13	33	22	24	17	34	13	52
$n = 70 + i$	19	26	19	40	16	38	21	32	21	50
$n = 80 + i$	14	42	21	36	25	44	16	42	23	40
$n = 90 + i$	29	50	17	50	28	34	25	58	16	62

For example, the three primitive root sextuples with smallest bend -4 are:

(-4,26; 7,15; 10,12)
(-4,30; 6,20; 13,13)
(-4,46; 5,37; 20,22).

From the counts in the table and several more with larger values of n, we conjecture a formula for the number of root sextuples.

Conjecture The number of primitive root sextuples with smallest bend $-n$ is

$$k(n) = \frac{1}{4}\left(n \prod_{p|n} \frac{p + \chi(p)}{p} + r(n) \right)$$

where $\chi(2) = 2$, $\chi(p) = -1, -1, 1, 1$ for $p = 1, 3, 5, 7$ (mod 8),and the remainder term depends on the number of different odd prime factors of n, which we name $\rho(n)$:

if n is odd, $\rho(n) = 1$, then $r(n) = \phi(p)$ where $\phi(p) = 4, 2, 2, 4$ when $p = 1, 3, 5, 7 \pmod 8$.
if n is odd and $\rho(n) \geq 2$, $r(n) = 2^{\rho(n)}$
if n is even, $r(n) = 4.2^{\rho(n)}$

This conjecture constitutes claim (vi).

7 Inversion and the generalized super-packing

Given a generalized Apollonian packing, we can invert the whole packing in every circle in the packing, and then again in every circle in the resulting packing, and so on. The result is what we call a "generalized super-packing" (Compare [5]). In this packing, every two circles either touch, or are disjoint. See Figure 4.

It is convenient to regard the super-packing as being composed of sextuples. We can generate the sextuples of the super-packing by adding another generator to the generalized Apollonian group, representing inversion in the first circle of a general sextuple.

Lemma 7 Given a sextuple S described by an F matrix $\mathbf{F}(S)$, the result of inversion in the circle described by the top row of $\mathbf{F}$ is a sextuple S' with

$$\mathbf{F}(S') = \mathbf{V}\mathbf{F}(S)$$

where

$$\mathbf{V} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Proof. Suppose the rows of $F(S)$ are $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{w}$. From (3.26) of [4] (or by simple geometry) we have that since each of the $\mathbf{b}, \mathbf{c}$ circles touches the $\mathbf{a}$ circle, their transformed abbc coordinates are given correctly. Since the $\mathbf{b}'$ circle (whose abbc coordinates are $\mathbf{2w - b}$) also touches the $\mathbf{a}$ circle, its transformed abbc coordinates are $\mathbf{2a + 2w - b}$ so that the new fourth row of F is $(1/2)(\mathbf{2a + b + 2a + 2w - b}) = \mathbf{2a + w}$. Note that $\mathbf{V}$ is an involution. This validates claim (v).

The situation is exactly analogous to that of the standard Apollonian super-packing, see [5]. It is convenient to work with the matrix group B that has eight generators of the form $\mathbf{pRp}^T$, with $\mathbf{p} \epsilon Perm$, which reflect in the eight different triads in a sextuple, and six generators of the form $\mathbf{pVp}^T$ which each invert with respect to one of the circles in a sextuple. Explicitly, given a sextuple with circles $C_1, C_2, C_3, C_1', C_2', C_3'$ described by an F matrix with the first three rows corresponding to the first three of these circles, we have $\mathbf{R}_{000} = \mathbf{R}$ as given above, which reflects leaving the circles C_1, C_2, C_3 invariant,

$$\mathbf{R}_{100} = \begin{pmatrix} -3 & 4 & 4 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 2 & 2 & 3 \end{pmatrix}$$

which reflects leaving C_1', C_2, C_3 invariant, and similarly $\mathbf{R}_{010}$ and $\mathbf{R}_{001}$,

$$\mathbf{R}_{110} = \begin{pmatrix} -3 & -4 & 4 & 12 \\ -4 & -3 & 4 & 12 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 7 \end{pmatrix}$$

which reflects leaving C_1', C_2', C_3 invariant, and similarly $\mathbf{R}_{101}$ and $\mathbf{R}_{011}$, and finally

$$\mathbf{R}_{111} = \begin{pmatrix} -3 & -4 & -4 & 20 \\ -4 & -3 & -4 & 20 \\ -4 & -4 & -3 & 20 \\ -2 & -2 & -2 & 11 \end{pmatrix}$$

which reflects leaving C_1', C_2', C_3' invariant. Also we have $\mathbf{V}_1 = \mathbf{V}$ as above, which inverts with respect to the circle C_1, and similarly $\mathbf{V}_2$ and $\mathbf{V}_3$, and

$$\mathbf{V}_{1'} = \begin{pmatrix} -5 & 0 & 0 & 12 \\ -2 & 1 & 0 & 4 \\ -2 & 0 & 1 & 4 \\ -2 & 0 & 0 & 5 \end{pmatrix}$$

which inverts with respect to the circle C_1', and similarly $\mathbf{V}_{2'}$ and $\mathbf{V}_{3'}$.

We can write a word in the super-group B as $\mathbf{U} = \mathbf{U}_n\mathbf{U}_{n-1}\cdots\mathbf{U}_1$, in which each $\mathbf{U}_i$ is one of these generators $\mathbf{R}_x$ and $\mathbf{V}_y$. Each such word in B has a *normal form*, obtained by cancelling all squares, and moving each appearance of each inversion $\mathbf{V}_y$ as far to the right (i.e. earlier) as possible, so that $\mathbf{V}_y$ follows a reflection that does not leave the circle C_y invariant. Then the sextuples in the super-packing are in 1-1 correspondence with normal-form words in the super-group. We need to show that there are no identities in this group other than those described above.

Theorem 2 The generators $\mathbf{R}_{000}$ etc. and $\mathbf{V}_1$ etc described above are involutions. Also each $\mathbf{V}_j$ commutes with the four $\mathbf{R}$s that leave C_j invariant. There are no other identities among these generators.

Proof. Direct computation verifies the involution and commutativity relations. The hard part is showing that there are no other relations. The following argument mimics the key part of the proof of Theorem 3.1 in [5]. We consider the effect of applying an element of the group (i.e. a word in normal form) to an arbitrary basic sextuple S_0. There are two cases. First, the word contains no inversions. In this case, a reflection $\mathbf{R}$ that leaves circles C_1, C_2, C_3 invariant puts three circles inside one of the curvilinear triangles that they form. We can get out of this triangle only by repeating this reflection, but this is not allowed in a normal-form word until some other reflection has intervened, and then not all of C_1, C_2, C_3 are still members of the sextuple.

Second, suppose an inversion $\mathbf{V}_j$ appears. This puts five circles inside C_j, and we can only get outside of this

circle by another application of $\mathbf{V}_j$. But in a normal form word, $\mathbf{V}_j$ cannot reappear until either (i) another $\mathbf{V}$, or (ii) an $\mathbf{R}$ that does not leave C_j invariant, has intervened. In either case C_j is no longer a member of the sextuple. Thus we can never get back to the original sextuple.

Theorem 3 The integral super-packing can be placed in the plane so that all centers are of the form $(m\sqrt{(}2), n)$ with m and n integral, in exactly four ways.

Proof The argument exactly parallels that in the Descartes case, see Theorems 6.2 and 6.3 of [5]. Given the bends of any primitive integral sextuple S, there is an unique normal-form word U in the generators of the super-Apollonian group that reduces it to the special integral sextuple $s_0 = (0, 2; 0, 2; 1, 1)$. This sextuple can be placed in the plane making $bx/\sqrt{2}$ and by integers for every circle, by making the straight lines parallel to the x-axis, and with the point of contact of the two bend-2 circles at any point $m\sqrt{2}, n$ with m and n integral. Application of inversions in the bend-0 circles and reflections leaving these circles invariant then fills the whole plane, and forms just four different placements of the whole figure, namely with the origin at

(i) the center of a bend-1 circle,
(ii) the point of contact of a bend-1 circle and one of the lines,
(iii) the contact-point of the two bend-2 circles,
(iv) the point of contact of a bend-2 circle and one of the lines.

We get only four placements, not eight as in the Descartes case, because here we cannot reflect in the line $y = x$ while keeping the values of $bx/\sqrt{(}2)$ and by integral. In each placement, the basic rectangle $(0, \sqrt{2})$ x $(0, 1)$ repeats by reflections to cover the whole plane.

Now applying the elements of the word $\mathbf{U}$ in reverse order to the F matrix describing this placed basic sextuple s_0, we find exactly one placing of the sextuple S within each version of the basic rectangle. This validates claims (vii) and (viii) of the Introduction.

Computation suggests that there are some symmetries within the basic rectangle, as in the Descartes case, namely

circles with even bends appear symmetrically about $x = \sqrt{2}/2$
circles with bends $= 0 \bmod 8$ appear symmetrically about $x = \sqrt{2}/4$
circles with bends $= 2$ or $= 6 \bmod 8$ appear symmetrically about $y = 1/2$
a circle with bend $= 4 \bmod 8$ and center x, y has a mate at $\sqrt{2}/2 - x, 1 - y$

See Figures 5(a)-(d). Each circle in these figures is the largest circle of a unique root sextuple. For example, the three root sextuples with smallest bend -4, which we listed in section 6 above, lie within the three largest circles in Figure 5(c). We do not see any symmetries for circles with odd bends.

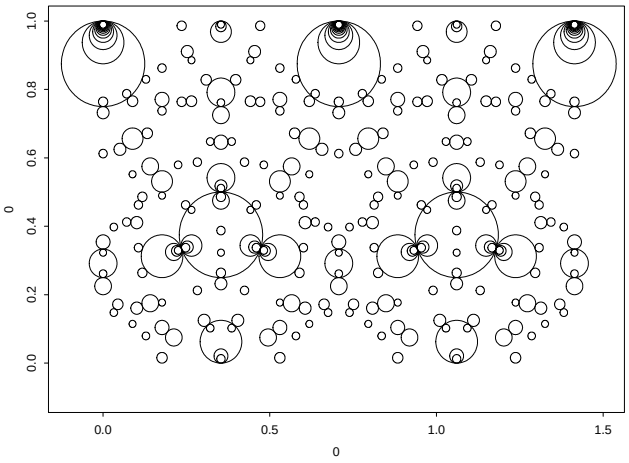

Figure 5: (a) Circles with bend =0 mod 8

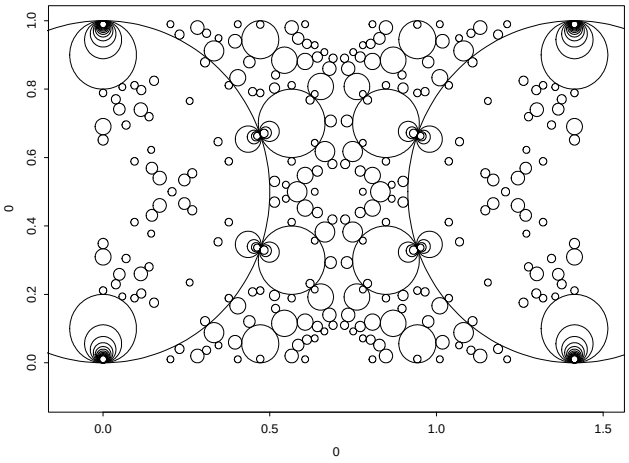

Figure 5: (b) Circles with bend =2 mod 8

8 Ball-bearing configurations

We define a "ball-bearing" structure as consisting of a ring of circles, each touching two neighbors, with the property that there exist "inner" and "outer" circles that each touch each of the circles in the ring.

Theorem 4 In an n-ball ball-bearing structure, where the bends of the "ball" circles are (in order)

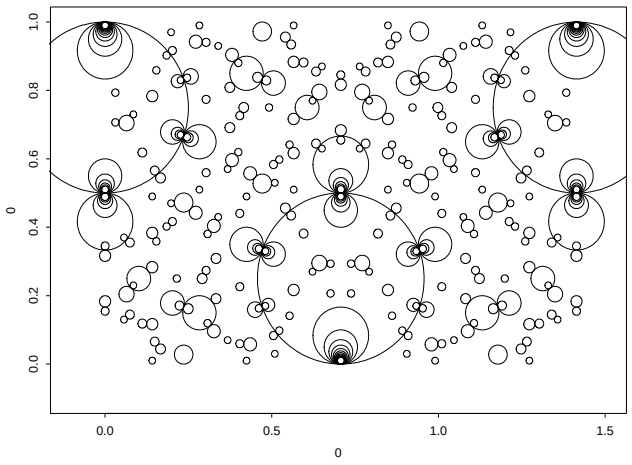

Figure 5: (c) Circles with bend =4 mod 8

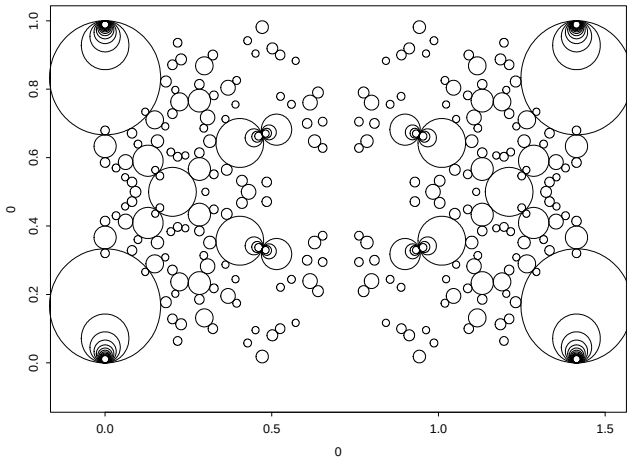

Figure 5: (d) Circles with bend =6 mod 8

$b_1, b_2, ...b_n$, the bends of the "inner" and "outer" circles are the roots of the equation

$$D_n x^2 - x\sigma + \tau - \frac{3}{2n}\sigma^2 = 0 \qquad (7)$$

where $\sigma = \sum(b_j)$, $\tau = \sum(b_j^2)$, and $D_n = (n/2)\cot(\pi/n)^2$. Also, if the centers of the ring circles are the complex numbers $z_1, ..., z_n$, the bend*center coordinates of the inner and outer circles are the roots of the equation

$$D_n X^2 - XS + T - \frac{3}{2n}S^2 = 0 \qquad (8)$$

where $S = \sum b_j z_j$, $T = \sum((b_j z_j)^2)$.

Remark 1 For $n = 3$, $D_3 = 1/2$ and (7) is the Descartes relation that connects the bends of four mutually tangent circles. Also (8) is then the "Complex Descartes Theorem" of [7].

Remark 2 An entry in MathWorld entitled "Soddy Circles" gives a formula that is different from ours. However this must be in error, since it is not homogeneous in the bends.

If the bends $b_1, b_2, ...b_n$ are integers, the roots of (13) can be rational only when $n = 3, 4, 6$ ($D_n = 1/2, 2, 9$). Only in these cases can all the bends be integers.

Proof of the Theorem. We start by proving the relation (7) for a special configuration, in which the "inner" and "outer" circles are centered at the origin, with the j-th ring circle having center ω^{2j}, where $\omega^n = -1$. The radius of each ring circle is $s = \sin\pi/n$; the radii of the inner and outer circles are $1 - s, -(1 + s)$.

Let a_0 contain the abbc coordinates of the "inner" circle, and for $j = 1, \cdots, n$ let a_j contain the abbc coordinates of the j-th circle in the ring, so that

$$\begin{aligned} a_0 &= (s - 1, 1/(1 - s), 0, 0) \\ a_j &= (1/s - s, 1/s, \cos(2\pi j/n)/s, \sin(2\pi j/n)/s) \end{aligned}$$

We assemble these vectors in an $(n + 1)$ x 4 matrix $\mathbf{A}$. Let $\mathbf{Q}$ be the $n + 1$ x $n + 1$ matrix

$$\mathbf{Q} = \begin{pmatrix} D_n & -\frac{1}{2}\mathbf{1}_n^T \\ -\frac{1}{2}\mathbf{1}_n & I_n - \frac{3}{2n}\mathbf{1}_n\mathbf{1}_n^T \end{pmatrix}$$

where $\mathbf{1}_n$ is the n x 1 column vector $(1, 1, \cdots 1)^T$. The equation in the Theorem is $\mathbf{b}^T\mathbf{Q}\mathbf{b} = 0$ where $\mathbf{b}$ is the vector of bends $\mathbf{b}^T = (1/(1 - s), (1/s)\mathbf{1}_n^T)$. Straightforward computation shows that

$$\mathbf{A}^T\mathbf{Q}\mathbf{A} = \frac{n}{s^2}\mathbf{W}^{-1}$$

where $\mathbf{W}$ was defined in Section 2. Picking off the (2,2) element proves the first claim of the theorem (for this special case). The second claim follows from the (3,4) x (3,4) submatrix.

Now let $\mathbf{m}$ be any matrix in the Moebius group M. We remarked in the proof of Lemma 2 that the effect of a Moebius transformation is to replace $\mathbf{A}$ by $\mathbf{Am}$, and each $\mathbf{m}$ in M satisfies $\mathbf{mWm}^T = \mathbf{W}$. It follows that $\mathbf{m}^T\mathbf{W}^{-1}\mathbf{m} = \mathbf{W}^{-1}$, so (8) remains true when $\mathbf{A}$ is replaced by $\mathbf{Am}$. Thus the relation holds for every ball-bearing configuration. This validates claim (ix).

Remark 3 The bends of the ring circles satisfy $n - 3$ linear identities, as do the "bend*center" quantities. For $n = 4$ the identities are $b_1 + b_3 = b_2 + b_4$ and $b_1 z_1 + b_3 z_3 = b_2 z_2 + b_4 z_4$. For $n = 6$ they are $b_1 + b_4 = b_2 + b_5 = b_3 + b_6, b_1 + b_3 + b_5 = b_2 + b_4 + b_6$, etc. For all $n \geq 3$ a ball-bearing configuration is determined by the abbc coordinates of any three mutually tangent circles (one inner or outer, and two "ring" circles) and a "$\mathbf{w}$" vector which is the average of the abbc coordinates of the n "ring" circles. Thus we can work with 4 x 4 matrices in all cases.

Remark 4 Given three mutually tangent circles with integral bends a, b, c, set $q = ab + ac + bc$. Then the

circles can belong to an integral Descartes quadruple iff q is a perfect square; can belong to an integral sextuple (in the sense of this paper) iff $2q$ is a perfect square; and can form part of a $n = 6$ ball-bearing configuration iff $3q$ is a perfect square.

Remark 5 In the $n = 6$ case the basic integral configuration (analogous to the (0,0,1,1) configuration in the $n = 3$ case, and (0,2;0,2;1,1) in the $n = 4$ case) is (1,1; 0,3,6,6,3,0), consisting of two parallel lines with two unit circles touching both, and a pile of four circles between them. This configuration can be placed in the plane so that all values of $bx/\sqrt{3}$ and by are integers. However it is not the case that all derived bends and bends*centers are integral.

Remark 6 One can also consider multiple-ring structures, for example a 1-5-5-1 structure. Here, every circle touches five others, and the circles have the symmetry of the vertices of an icosahedron. This suggests another generalization of Apollonian packing, in which nine circles are placed in every curvilinear triangle. In this case the bends cannot all be integers.

Remark 7 Similar constructions are possible in three dimensions. Between four spheres, each of which touches the other three, we can (uniquely) pack four more spheres, touching each other and each touching three of the original spheres. We leave consideration of these configurations to another occasion.

9 Generalized Farey series and Ford circles

In the generalized Apollonian packing based on the sextuple (0,2;0,2;1,1) depicted in Figure 1(c), the circles that touch the lower bend-zero line are analogous to Ford circles, and their points of contact are analogous to terms in the Farey series. See Figure 6.

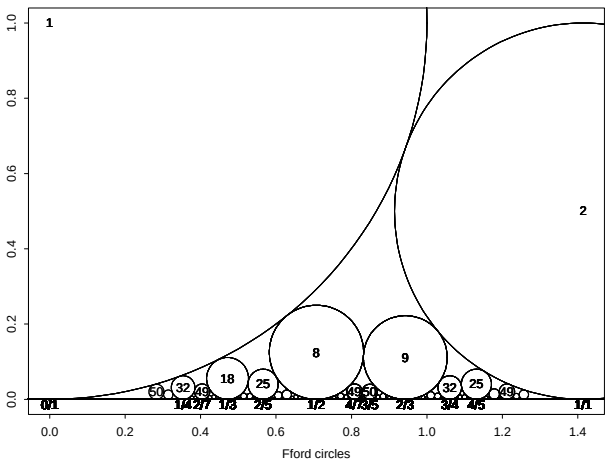

Figure 6: Generalized Ford circles

We start with a circle with bend 1 touching the x-axis at $x = 0$, and another circle with bend 2, touching the first one and touching the x-axis at $x = \sqrt{2}$. Packing three circles between these gives two new circles touching the x-axis, the original circles, and each other, namely a circle with bend 8 touching at $x = (1/2)\sqrt{2}$ and another with bend 9 touching at $x = (2/3)\sqrt{2}$. Note that $8 = 2.2^2$, $9 = 3^2$, and $1/2 = (0+1)/(1+1)$, $2/3 = (0+2.1)/(1+2.1)$. This pattern continues. In general, in a gap between a circle with bend q^2 touching at $x = (p/q)\sqrt{2}$ and another with bend $2s^2$ touching at $x = (r/s)\sqrt{2}$, where p is even and r is odd, we construct a circle with bend $(q+2s)^2$ touching at $x = ((p+2r)/(q+2s))\sqrt{2}$ and another with bend $2(q+s)^2$ touching at $x = ((p+r)/(q+s))\sqrt{2}$. Two fractions $p/q, r/s$ are neighbors iff $|ps - qr| = 1$. At every stage, the two types of circles (with bends q^2 and $2r^2$) alternate. See Figure 6.

10 Some open questions

Is the conjectured formula for the number of root sextuples correct?

Are the conjectured symmetries within the basic cell of the super-packing valid?

Do all integers arise as bends of circles in generalized Apollonian packings?

Is the Hausdorff dimension of our generalized packing the same as in the Apollonian case?

Are there other ways to generalized the classical Apollonian packing?

Are there other ways to pack integer-bend circles?

What about higher dimensions?

References

[1] Aharonov, D. and Stephenson, K. Geometric sequences of discs in the Apollonian packing. *Algebra i Analiz.*, 1997

[2] Descartes, R. *Oeuvres de Descartes, Correspondance IV* ed. C. Adam and P. Tannery, Leopold Cerf, Paris, 1901.

[3] Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., and Yan, C. Apollonian circle packings: number theory. *J. Number Theory* 100 p 1-45, 2003.

[4] Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., and Yan, C. Apollonian circle packings: Geometry and Group Theory I. The Apollonian Group . *Discrete and Computational Geometry* 34, p 547-585, 2005.

[5] Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., and Yan, C. Apollonian circle packings: Geometry and Group Theory II. Super-Apollonian Group and Integral Packings. *Discrete and Computational Geometry* 35, p 1-36, 2006.

[6] Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., and Yan, C. Apollonian circle packings: Geometry and Group Theory III. Higher Dimensions. *Discrete and Computational Geometry* 35, p 37-72, 2006.

[7] Lagarias, J.C., Mallows, C.L., and Wilks, A.R Beyond the Descartes circle theorem. *Amer. Math. Monthly* 109 p.338-361, 2002.

[8] Mauldon, J. G. Sets if equally inclined spheres. *Canadian J. Math* 14:509-516, 1962.

[9] Mumford, D, Series, C, and Wright, D. Indra's Pearls. Cambridge U.P. 2002.

[10] Stephenson, K. Introduction to circle packing. Cambridge U. P. 2005.

[11] Wilker, J. B. Inversive Geometry in *The Geometric Vein* (ed. C.Davis, B.Grunbaum and F.A.Sherk) Springer-Verlag, New York pp. 379-442, 1981.