VC-Dimension of Visibility on Terrains

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Abstract

A guarding problem can naturally be modeled as a set system $(\mathcal{U}, \mathcal{S})$ in which the universe $\mathcal{U}$ of elements is the set of points we need to guard and our collection $\mathcal{S}$ of sets contains, for each potential guard g, the set of points from $\mathcal{U}$ seen by g.

We prove bounds on the maximum VC-dimension of set systems associated with guarding both 1.5D terrains (monotone chains) and 2.5D terrains (polygonal terrains). We prove that for monotone chains, the maximum VC-dimension is 4 and that for polygonal terrains, the maximum VC-dimension is unbounded.

1 Introduction

Terrain Guarding A 1.5D (resp. 2.5D) terrain is a continuous piecewise linear univariate (resp. bivariate) function. In other words, a 1.5D terrain is a simple polygonal chain that intersects any vertical line at at most one point and a 2.5D terrain is a polygonal mesh with no holes that intersects any vertical line at at most one point.

On a terrain T, either 1.5- or 2.5-dimensional, we say that two points see each other if the line segment between them does not pass under T. To guard T optimally we must find a minimum set $G \subset T$ of points on the terrain such that every point on T is seen by a point in G.

Guarding 1.5D terrains is not known to be NP-hard but no polynomial-time exact algorithm has been found. The best polynomial-time algorithm found so far is a 5-approximation algorithm[1] [10]. Guarding 2.5D terrains is NP-complete, as proved by Cole and Sharir [4].

Set Cover and VC-Dimension SET COVER is a well-studied NP-complete optimization problem. Given a universe $\mathcal{U}$ of elements and a collection $\mathcal{S}$ of subsets of $\mathcal{U}$, SET COVER asks for the minimum subset $\mathcal{C}$ of $\mathcal{S}$ such that $\bigcup_{S \in \mathcal{C}} S = \mathcal{U}$. In other words, we want to cover all of the elements with the minimum number of sets in $\mathcal{S}$.

In general, SET COVER is not only difficult to solve exactly (see, *e.g.*, [7]) but is also difficult to approximate – no polynomial time approximation algorithm can have an $o(\log n)$ approximation factor unless $\mathrm{NP} \subseteq \mathrm{DTIME}(\mathrm{n}^{\log \log \mathrm{n}})$ [6].

However, some instances of SET COVER (we refer to instances as *set systems*), are more complex than others. VC-dimension is a measure of the complexity of a set system $(\mathcal{U}, \mathcal{S})$. Consider a set $S \subseteq \mathcal{U}$. There are $2^{|S|}$ possible subsets of S. We say that S is *shattered* by a collection $\mathcal{C}$ of $2^{|S|}$ sets if, for each of the $2^{|S|}$ subsets of S, there is a set in $\mathcal{C}$ that contains those elements of S but no other elements of S. The VC-dimension of a set system $(\mathcal{U}, \mathcal{S})$ is the maximum d for which a set of d elements from $\mathcal{U}$ can be shattered by sets $\mathcal{C} \subseteq \mathcal{S}$.

VC-Dimension and Approximate Set Cover SET COVER is hard to approximate in general, but set systems with low VC-dimension are simpler and, intuitively, should be easier to approximate. Brönnimann and Goodrich [3] provide a polynomial time $O(d \log(d \cdot \mathrm{OPT}))$-approximation algorithm for instances of SET COVER with VC-dimension d, where OPT is the size of the optimum solution. When d is bounded from above by a constant, this gives an $O(\log \mathrm{OPT})$ approximation factor.

Set Systems of Guarding Problems Guarding problems can naturally be expressed as instances of SET COVER. For an instance of a guarding problem, the associated set system $(\mathcal{U}, \mathcal{S})$ is constructed with $\mathcal{U}$ containing the points that need to be guarded and $\mathcal{S}$ containing, for each potential guard g, the set of points that g can see. For the sake of brevity we sometimes refer to the VC-dimension of a guarding problem; by this we mean the maximum possible VC-dimension of a set system associated with an instance of the problem.

The classic *art gallery problem*, the problem of guarding the interior of a polygon, is perhaps the best-known and best-studied guarding problem. If the polygon can have holes, the problem is as hard to approximate as general instances of SET COVER [5]. However, if the polygon to be guarded is simple (*i.e.* contains no holes), the associated set system has constant VC-dimension [9] (it is known to be at least 6 and at most 23 [11]). This means that the technique of Brönnimann and Goodrich leads to an $O(\log \mathrm{OPT})$-approximation algorithm, which is the best known. Guarding simple art galleries is known to be APX-hard [5] but the exact ap-

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[1] An error in the paper was found after publication, and the only fix found so far increases the approximation factor from 4 to 5.

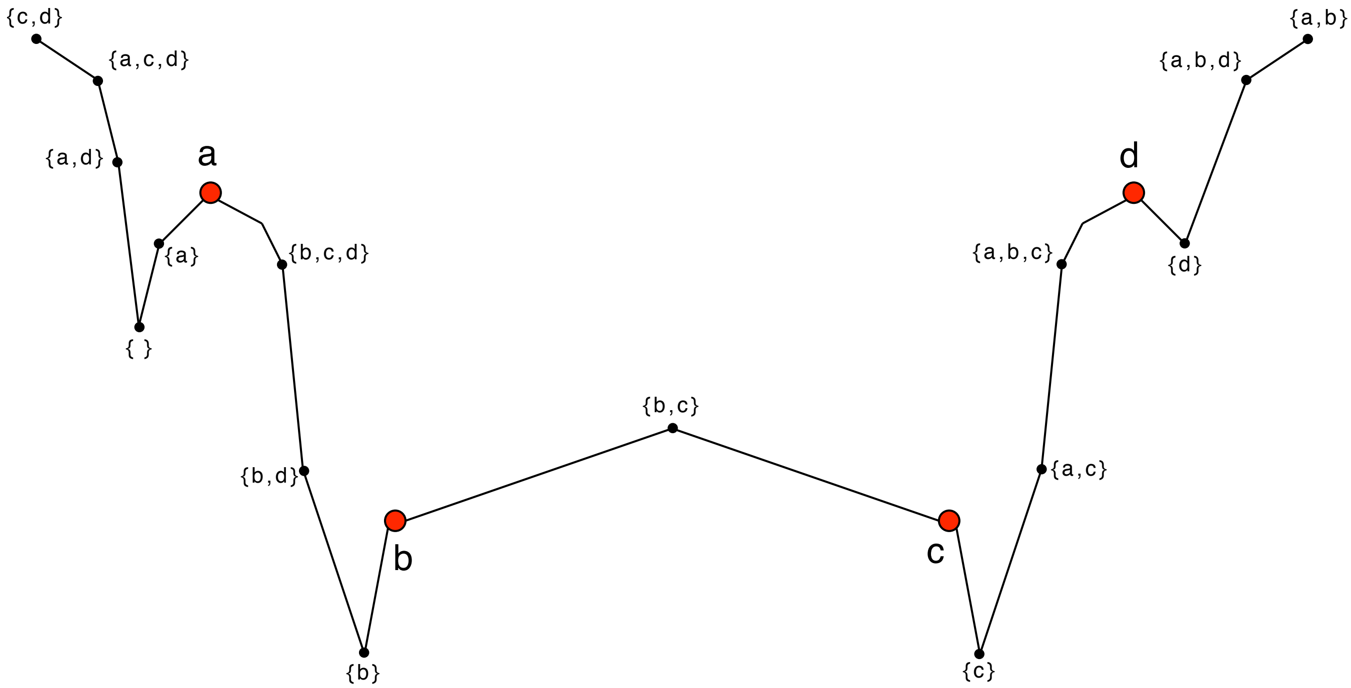


Figure 1: A monotone chain with 4 points, a, b, c, d, that are shattered by 16 guards. The guard seeing $\{a, b, c, d\}$ is not pictured, but a very high vertex on the left end of the terrain would see all other vertices. Each of the other 15 guards is labeled with the subset of $\{a, b, c, d\}$ that it sees.

proximability is unknown.

Isler *et al.* [8] consider guarding the exterior of polygons and polyhedra. For polygons they show that the maximum VC-dimension is 2 when guards must lie on a circle containing the polygon and 5 when guards can lie anywhere outside the convex hull of the polygon. For polyhedral galleries in $\mathbb{R}^3$ they prove that the maximum VC-dimension is unbounded when guards must lie on a sphere containing the gallery.

Our Contribution In section 2 we prove that the maximum VC-dimension of guarding a 1.5D terrain is 4 with matching upper and lower bounds. In section 3 we show that the VC-dimension of guarding a polygonal terrain is unbounded, via a reduction from polygons with holes.

2 VC-Dimension of Guarding 1.5D Terrains

To prove that a monotone chain can have VC-dimension 4, we simply provide an example of a terrain with 4 points that are shattered by 16 guards (see figure 1).

For points a, b on an x-monotone chain, we say that $a < b$ if a is to the left of b. The Order Claim [2] states that, for points a, b, c, d with $a < b < c < d$, if a sees c and b sees d then a sees d.

For any point set P that is shattered by a set of guards G let $g(p_1, \ldots, p_k)$ denote the guard in G that sees $p_1, \ldots, p_k \in P$ but no other points in P. We will now argue, using only the Order Claim, that no set P of size 5 can be shattered. This gives us the upper bound of 4 for the VC-dimension.

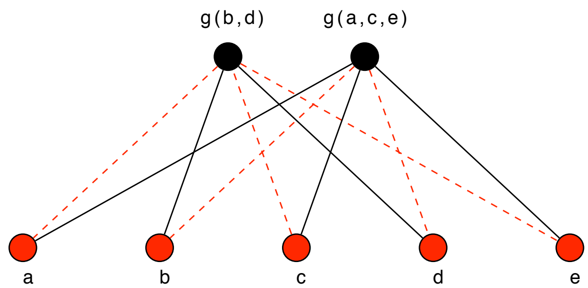
Let $P = \{a, b, c, d, e\}$ and assume without loss of generality that $a < b < c < d < e$. We can see (figures 2(a) and 2(b) may help) that $g(a, c, e)$ and $g(b, d)$ will contradict the order claim unless either

- $g(b, d) < c$ and $d < g(a, c, e)$, or
- $g(a, c, e) < b$ and $c < g(b, d)$.

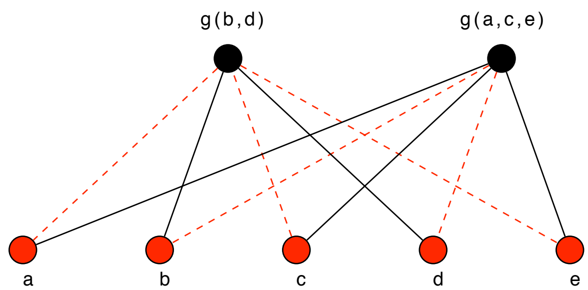
We assume the former without loss of generality. Now consider $g(b, c, e)$. There are four potential ranges that we consider placing $g(b, c, e)$ in:

- left of $g(b, d)$
- between $g(b, d)$ and d
- between d and $g(a, c, e)$
- right of $g(a, c, e)$.

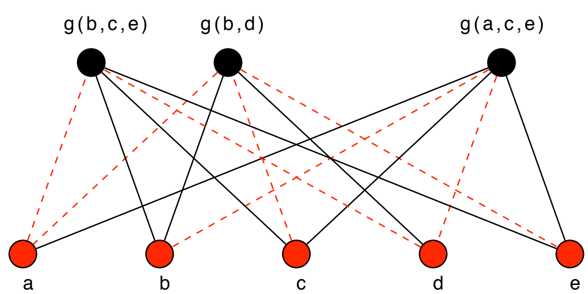
It is not difficult to verify that placing $g(b, c, e)$ in any of these four ranges would contradict the Order Claim (see figure 2(c) for an example). Therefore 5 points on a monotone chain cannot be shattered and no monotone chain can have VC-dimension greater than 4.



(a) In this configuration the Order Claim is contradicted by $g(b, d)$, $g(a, c, e)$, d, and e.



(b) In this configuration the Order Claim is not contradicted.



(c) The Order Claim is now contradicted by the addition of $g(b, c, e)$, regardless of its position. In this configuration the Order Claim is contradicted by $g(b, c, e)$, $g(b, d)$, c, and d.

Figure 2: Examples of configurations of G and P for the proof that no 5 points on a 1.5D terrain can be shattered. Solid lines indicate clear lines of sight. Dashed lines indicate blocked lines of sight.

3 VC-Dimension of Guarding 2.5D Terrains

SET COVER can be reduced to the problem of guarding the perimeter of a polygon with holes using guards on the perimeter (§4 of Eidenbenz *et al.* [5]). As a direct consequence, for any finite set system $(\mathcal{U}_1, \mathcal{S}_1)$, there exists a polygon with holes whose associated set system is $(\mathcal{U}_2, \mathcal{S}_2)$ such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$. This implies that a polygon with holes can have arbitrarily large VC-dimension.

For any polygon A with holes we show how to construct a polygonal terrain of equal or grater VC-dimension. The idea behind building T is simple. Lines of sight between points on A are blocked by the exterior of A. On our terrain T we will build corresponding mountains to block lines of sight.

We start with T as a horizontal rectangle at altitude 0 that will act as a bounding box for A. We then trace the perimeter of A on this rectangle and call it A_T. A_T partitions T into two open sets, T^- which corresponds to the interior of A and T^+ which corresponds to the exterior of A, including the holes.

In terms of vertical projections, A_T, T^- and T^+ will remain fixed as we change T. However, T^- will be lowered and T^+ will be raised. There are many ways to perform this raising and lowering, but perhaps the most elegant is the method of raising roofs from *straight skeletons* (Aichholzer and Aurenhammer [1], in particular §4). We raise T^+ based on its straight skeleton and lower T^- based on its straight skeleton. The result is that every point in T^+ has positive altitude and every point in T^- has negative altitude. Only A_T and the rectangular perimeter of T will be at altitude 0. See figure 3 for an example.

We can now verify that two points p, q on A_T see each other if and only if the corresponding points p', q' on A see each other. Since p and q are both at altitude 0, all of (p, q) is at altitude 0. If p sees q then the open line segment (p, q) contains no point below T so no point on (p, q) can be the vertical projection of a point in T^+. The corresponding open line segment (p', q') therefore cannot intersect the exterior of A, so p' and q' must see each other. Therefore p sees q if and only if p' sees q', and the converse can be proved similarly.

From any polygon with holes, we can construct a 2.5D terrain with equal or greater VC-dimension, so 2.5D terrains have unbounded VC-dimension.

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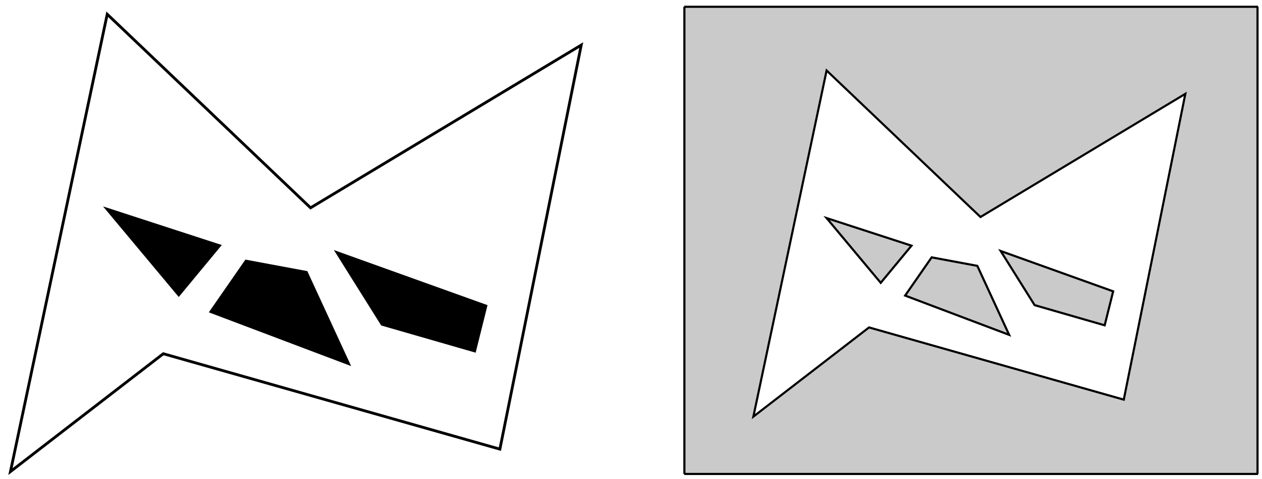

(a) The polygon A with holes indicated in black.

(b) A simplified top view of the associated terrain T. Black lines indicate A_T and the terrain's perimiter, both at altitude 0. T^- (white) has negative altitude while T^+ (shaded) has positive altitude.

Figure 3: A polygon A and a top view of the associated terrain T.

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