Inverting Linkages with Stretch

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Abstract

We consider inversion of linkages on 2-dimensional plane. Inversion of a linkage is a transform by continuous moves of vertices to its mirror image. There exist noninvertible linkages due to fixed-length links. To invert such noninvertible linkages, we have to relax some constraints. We allow variable-length links instead of fixed-length links since any linkage can be invertible by expanding sufficiently all links. We introduce a notion "stretch ratio" as a measure of length change, and analyze upper/lower bounds to invert polygons, outerplanar graphs, and wheels.

1 Introduction

A *linkage* is a collection of line segments joined at their endpoints to form a graph. A segment endpoint is called *joint* or *vertex*. The line segments are called *links*. In ordinary linkage, the length of each link is fixed. While we can ignore the length of edges in general graph embedding, a linkage must be embedded with keeping the lengths of edges. We allow self-intersection, i.e., any pair of links or vertices, they can not only intersect each other, but locate on the same position. In this paper, we consider linkages on Euclidean 2-dimensional plane.

A linkage is considered as a physical model which consists of straight rigid bars and joints permitting arbitrary bend. Linkages have a number of practical applications such as smooth graph deformation, analysis of physical structures, motion planning in robotics and molecular modeling.

A *configuration* of a linkage is a specification of the location of all the vertices. Given an initial configuration and a final configuration, the *reconfiguration* is continuous moves of vertices from the initial configuration to the final one, keeping all links rigid. Due to rigidity of links, there may exist pairs of initial and final configurations without reconfiguration. It is known that it is PSPACE hard to determine whether there exists reconfiguration for a given initial configuration and final configuration of a linkage on 2-dimensional plane [3]. *Inversion* of a linkage is a special reconfiguration from a given configuration to its mirror-image configuration. We say that a linkage is *invertible* if there exists the inversion of a given configuration. For example, any triangle linkage has unique configuration, and thus noninvertible. Lenhart and Whitesides showed it is easy to distinguish invertible polygon on 2-dimensional plane [4].

It is easy to see that every linkage can be invertible if we can change the length of links. Hence, by relaxing the constraints on the length of links, that is, using variable-length links instead of fixed-length links, we can invert even noninvertible linkages. It is practical to consider such linkages. For example, in molecular modeling, the distances between molecules are not strictly fixed [2]. Another example is telescoping arms in robotics. We introduce a notion "stretch ratio" to denote how much each link can be stretched. When we can independently stretch any link with length l from l/α to αl for some constant $\alpha \geq 1$, we say that the *stretch ratio* of the linkage is α. Note that $\alpha = 1$ means ordinary linkages.

Thus the problem now is to obtain the minimum stretch ratio to invert a given linkage, that is, we extend decision problem to optimization problem which are more practical problem of minimizing the stretch ratio.

We analyze upper and lower bounds of stretch ratio to invert polygons, outerplanar graphs, and wheels. For polygons and outerplanar graphs, we show a constant tight upper bound. We can also calculate the optimal stretch ratio for each polygon or outerplanar graph. On the other hand, we show a pessimistic results for general planar graphs. We show there is no constant upper bound of stretch ratio to invert wheels. A good news is that we can invert a wheel with an optimal stretch ratio with few exceptions.

2 Inverting Polygons with Stretch

As mentioned before, Lenhart and Whitesides [4] proved the necessary and sufficient condition of invertible polygon on 2-dimensional plane.

Theorem 1 *[4] A polygon is invertible iff the lengths of the second and third longest links sum to no more than the sum of the lengths of the remaining links.*

Thus we consider noninvertible polygons only. First, we show an upper bound of stretch ratio to invert polygons.

Theorem 2 *Any polygon can be invertible within stretch ratio at most* $\sqrt{2}$.

Proof. First we show that, for any polygon, we can divide into two paths so that the length of the longer path is at most

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twice of the length of the shorter path. For any two vertices v and u, let $L_l(v_1,v_2)$ ($L_s(v_1,v_2)$, resp.) be the length of the longer (shorter, resp.) path. Now suppose v_1 and v_2 be vertices where $L_l(v_1,v_2) - L_s(v_1,v_2)$ is the minimum among all the pairs of two distinct vertices. We can also divide the longer path at a vertex v_3 so that the length $L_s(v_2,v_3)$ of path between v_3 and v_2 (or v_1, but w.l.o.g. we can assume v_2) is at most half of $L_L(v_1,v_2)$ because of the triangle inequality. Thus, if $L_l(v_1,v_2) > 2L_s(v_1,v_2)$, we can easily see

$$\begin{aligned} & L_l(v_1,v_3) - L_s(v_1,v_3) \\ & \quad = (L_s(v_1,v_2) + L_s(v_2,v_3)) - (L_l(v_1,v_2) - L_s(v_2,v_3)) \\ & \quad = L_s(v_1,v_2) - L_l(v_1,v_2) + 2L_s(v_2,v_3) \\ & \quad \le L_s(v_1,v_2) < L_l(v_1,v_2) - L_s(v_1,v_2). \end{aligned}$$

This is a contradiction.

Therefore, by shortening the longer path by factor $1/\sqrt{2}$ and lengthening the shorter path by factor $\sqrt{2}$, we can invert any polygon with in stretch ratio at most $\sqrt{2}$. □

We remark that this upper bound $\sqrt{2}$ is tight. The worst case is an equilateral triangle. While inverting an equilateral triangle, one vertex must cross the opposite link. Thus the stretch ratio $\sqrt{2}$ is necessary and sufficient.

We can also compute the optimal stretch ratio to invert a given polygon.

Theorem 3 *If a polygon with n links is not invertible, the optimal stretch ratio of the polygon is $\sqrt{\frac{l_2+l_3}{\sum_{i=1}^{n} l_i-(l_2+l_3)}}$, where l_i is the length of the i-th longest link.*

Proof. In the inversion procedure, we first lengthen or shorten each link, then invert the polygon and finally restore each link to the original length.

First, we show that we can obtain an optimal stretch ratio by determining the second and third longest links after the first step. Let l'_i be the length of the i-th longest link after first lengthening or shortening. Since the obtained polygon is invertible, we have $l'_2 + l'_3 = \sum_{i=1}^{n} l'_i - (l'_2 + l'_3)$ by Theorem 1. We call the links with length l'_2 and l'_3 by shortened links, and the others are called lengthened links. On the inversion with optimal stretch ratio, we can assume that we all shorten (lengthen, resp.) the shortened (lengthened, resp.) links by an identical ratio. Suppose links with length l_i and l_j are the shorten links ($1 \le i < j \le n$). Then the stretch ratio is given by $\alpha_{i,j} = \sqrt{\frac{l_i+l_j}{\sum_{k=1}^{n} l_k-(l_i+l_j)}}$.

By considering following 3 cases, we show that $\alpha_{2,3}$ is the minimum, which concludes the proof.

Case 1 $i = 1$ and $j \in \{2,3\}$: It is easy calculation to show $\alpha_{1,2} \ge \alpha_{2,3}$ and $\alpha_{1,3} \ge \alpha_{2,3}$.

Case 2 $i = 3$ and $j \ge 4$: In this case, the j-th longest link lengthen to the third longest link. This means that during lengthening and shortening links, the lengths of links with original length l_2 and l_j will match. Let $\beta = \sqrt{l_2/l_j}$ be the stretch ratio at the moment. It is easy to see $\beta \ge \alpha_{2,3}$.

Case 3 otherwise: We can show similar to the case 2 by replacing l_2 with l_3. □

It is easy to see that we can apply these results to outerplanar graphs. We say that a cycle in an outerplanar graph is *minimal* if it does not include other cycle inside in the outerplanar embedding of the graph. Each edge is included in at most two minimal cycles. By constructing a dual graph and removing a vertex corresponding to the outer plane, we have a tree. Therefore we can invert each minimal cycle one by one in depth-first-search order. While inverting one minimal cycle, we stretch the other cycles similar to chords. Therefore, we have the following corollaries.

Corollary 4 *Any outerplanar graph can be invertible within stretch ratio at most $\sqrt{2}$.*

Corollary 5 *If an outerplanar graph with n vertices is not invertible, the optimal stretch ratio is computable in $O(n)$ time.*

3 Hardness of Inverting Wheels with Stretch

It is a natural question to ask whether any planar graph is invertible within some constant stretch ratio. Unfortunately, we can prove that there is no constant upper bound of the stretch ratio to invert general planar graphs.

We show that wheels has no constant upper bound of the stretch ratio. Let W_n be a wheel with an equilateral $(n-1)$-gon $v_0, v_1, v_2, \dots, v_{n-3}, v_{n-2}, v_0$ and a center vertex c on the barycenter. Let $e_i = \{v_i, v_{i+1}\}$ for each i with $0 \le i \le n-3$ and $e_{n-2} = \{v_0, v_{n-2}\}$.

We first show that the center vertex c must traverse one edge of the outer polygon.

Lemma 6 *When W_n is inverted, c traverses a point on e_i for some i with $0 \le i \le n-2$.*

Proof. We employ topological deformations. First, we reduce the cycle $(v_0, v_1, \dots, v_{n-2}, v_0)$ to a closed curve and a center vertex c on the plane (Figure 1(a)(b)). Similarly, the inverse of W_n can be reduced to a closed curve and the same center point c (Figure 1(d)(c)). Then those curves are inverted with respect to the center c. That is, the winding numbers of the curves around c are $+1$ and -1, respectively. Hence, to invert the curve, the winding number has to be changed from $+1$ to -1. We cannot change the number without crossing the center c over the curve. Therefore, c traverses a point on the cycle. □

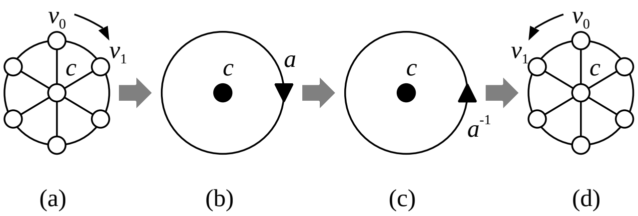

Figure 1: Inverting a wheel

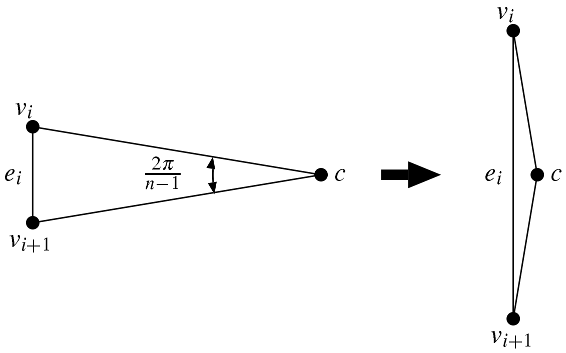

Figure 2: A triangle cv_iv_{i+1}

Theorem 7 *For any constant* $\alpha > 1$*, a wheel* W_n *is not invertible within stretch ratio* α*, where* n *satisfies* $\sqrt{1/\sin\frac{\pi}{n-1}} > \alpha$*, i.e.,* $n > 1 + \pi/\arcsin\frac{1}{\alpha^2}$.

Proof. By Lemma 6, the center vertex c must traverse an edge of outer polygon. Suppose c traverses e_i (Figure 2). Thus we have to shorten links cv_i and cv_{i+1} by factor $\sqrt{\sin\frac{\pi}{n-1}}$ and to lengthen e_i by factor $1/\sqrt{\sin\frac{\pi}{n-1}}$. This stretch ratio is monotone increasing with n. □

On the other hand, we also have the following positive result.

Theorem 8 *For* $n = 7$ *or* $n \geq 9$*, we can invert a wheel* W_n *within stretch ratio* $\sqrt{1/\sin\frac{\pi}{n-1}}$ *with* $O(n)$ *motions.*

Since this upper bound is equal to the lower bound in Theorem 7, this is the optimal stretch ratio.

The proof is constructive. Due to the page limitation, we omit the algorithm description and proof. Hereafter, we show an algorithm sketch only.

Let W_n be a wheel with an equilateral $(n-1)$-gon $v_0, v_1, v_2, \ldots, v_k, v_{-k}, v_{-(k-1)}, \ldots, v_{-1}, v_0$ and a center vertex c on the barycenter, where $k = \lfloor (n-1)/2 \rfloor$ and if n is odd v_k and v_{-k} are identical. Let m be a line passes through c and v_0. We design an algorithm that reconfigures a wheel W_n onto the line m. In each step, we move 2 or 4 vertices onto m. The algorithm consists of three phases.

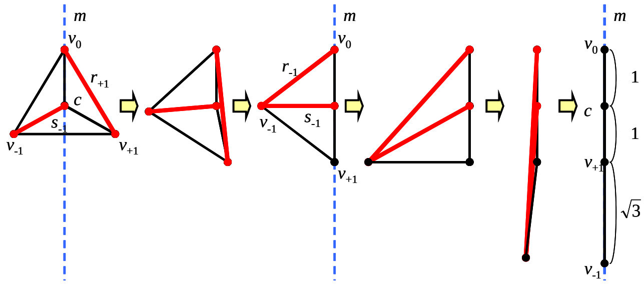

Figure 3: Inversion of W_4

The first phase is used when n is even, i.e., v_k and v_{-k} are not identical. In this phase, we move v_k and v_{k-1} onto m. We remark that, if n is odd, v_k is already on m.

During the second phase, we move 4 vertices on m at once. At i-th iteration, we move v_i, v_{-i}, v_{k-i}, and $v_{-(k-i)}$.

If 2 vertices remains after $\lfloor k/2 \rfloor$ iterations, i.e., if k is even, we proceed to the third phase. In this phase we move $v_{k/2}$ and $v_{-k/2}$ onto m.

Our algorithm, however, cannot invert neither W_4, W_5, W_6, nor W_8 within stretch ratio $\sqrt{1/\sin\frac{\pi}{n-1}}$. For example, a lower bound of the stretch ratio to invert W_4 is $\sqrt{2}$ ($> \sqrt{1/\sin\frac{\pi}{3}}$) since the outer polygon of W_4 forms an equilateral triangle. For W_4, we can also show the following non-trivial upper bound (see Figure 3).

Theorem 9 W_4 *can be invertible within stretch ratio* $\sqrt{1+\sqrt{3}}$.

4 Concluding Remarks

Of course, inversion linkages with stretch in 3 or higher dimension space is also included in future works.

Another future work is the problem to compute the optimal ratio to invert a given general graph. It might be easier to determine whether a given graph can be inverted with a given stretch ratio. We conjecture that the problems are PSPACE-hard in general. For example, fix the stretch ratio to 1, that is, without stretching. Then the problem becomes the decision problem that asks if a given linkage can be inverted in 2D space. This is a special case of the usual linkage reconfiguration problem. The linkage reconfiguration problem is PSPACE-hard even for trees in general [1].

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