Empty Monochromatic Triangles*

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Abstract

We consider a variation of a problem stated by Erdös and Guy in 1973 about the number of convex k-gons determined by any set S of n points in the plane. In our setting the points of S are colored and we say that a spanned polygon is monochromatic if all its points are colored with the same color.

We show that any bi-colored set of n points in $\mathbb{R}^2$ in general position determines a super-linear number of empty monochromatic triangles, namely $\Omega(n^{5/4})$.

We further generalize this result to $\mathbb{R}^d$: Any bi-colored set of n points in $\mathbb{R}^d$ in general position determines $\Omega(n^{d-3/4})$ empty monochromatic simplices.

1 Introduction

Erdös and Guy [6] asked the following question. "What is the least number of convex k-gons determined by any set of n points[1] in the plane?" The trivial solution for the case $k = 3$ is $\binom{n}{3}$. In addition, if we require the triangles to be empty then Katchalski and Meir [8] showed that for all $n \geq 3$ a lower bound is given by $\binom{n-1}{2}$ and that there exists a constant $c > 0$ such that cn^2 is an upper bound. Around the same time Bárány and Füredi [1] showed that any set of n points has at least $n^2 - O(n \log n)$ empty triangles and they also gave an upper bound of $2n^2$ if n is a power of 2.

Valtr [12] described a configuration of n points related to Horton sets [7] with fewer than $1.8n^2$ empty triangles and also provided upper bounds on the number of empty k-gons, e.g. $2.42n^2$ empty quadrilaterals. Later Dumitrescu [5] improved the upper bound for triangles to $\approx 1.68n^2$, which then consequently was further improved by Bárány and Valtr [2] to $\approx 1.62n^2$, the currently best bound. It is still unknown whether the constant could be smaller than 1, that is, whether there exists a family of n-element sets with fewer than n^2 empty triangles.

We consider a related problem, where the points of the given set S are colored. A polygon spanned by points in S is called monochromatic if all its points are colored with the same color. In contrast to the above described race for the best constant for the uncolored case, we are interested in the asymptotic behavior of the number of empty monochromatic triangles for bi-colored point sets.

A result in this direction was obtained by Devillers et al. [4]. They proved that any bi-colored point set in the plane exhibits at least $\lceil \frac{n}{4} \rceil - 2$ interior disjoint empty monochromatic triangles. In a generalization Urrutia [11] showed that in any 4-colored point set in $\mathbb{R}^3$ there is at least a linear number of empty monochromatic tetrahedra.

One might be also interested in the minimum number of colors so that we can color any given set S of n points in a way such that S does not determine an empty monochromatic triangle (or in general an empty monochromatic convex k-gon). In [4] (Theorem 3.3) this question has been settled by showing that already for three colors there are sets not spanning any empty monochromatic triangle.

The remaining question is to determine the asymptotic behavior of the number of empty monochromatic triangles for bi-colored sets. In Section 2 we show that any bi-colored set of n points in $\mathbb{R}^2$ in general position determines $\Omega(n^{5/4})$ empty monochromatic triangles. In Section 3 we generalize this result to $\mathbb{R}^d$ and show that any bi-colored set of n points in $\mathbb{R}^d$ in general position determines $\Omega(n^{d-3/4})$ empty monochromatic simplices. To the best of our knowledge no non-trivial bounds have been known before.

2 Lower Bound Construction

We start with a technical lemma which shows that for point sets with a triangular convex hull there exists a

*Research supported by the Austrian FWF Joint Research Project 'Industrial Geometry' S9205-N12. Part of the work was done while the second, third and fifth author were visiting Graz.

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**Instituto de Matemáticas, Universidad Nacional Autónoma de México. Research supported in part by MTM2006-03909 (Spain) and CONACYT of México, Proyecto SEP-2004-Co1-45876, urrutia@matem.unam.mx

[1]Throughout, all considered point sets are in general position, that is, they do not contain three collinear points.

triangulation such that a sufficient fraction of its triangles are incident to vertices of the convex hull.

Lemma 1 *Let S be a set of n points in general position in the plane with 3 extreme points, that is, with a triangular convex hull and $m = n - 3$ interior points. Then S can be triangulated such that at least $m+\sqrt{m}+1$ triangles have (at least) as one of their vertices an extreme point of S.*

Proof. Let Δ be the convex hull of S, $E(\Delta)$ the edges of Δ, and $M = S \setminus \Delta = \{q_1, ..., q_m\}$ the interior points of S, $|M| = m = n - 3$.

We first define a partial order $\leq_e$ on the elements of M. Two points $p_1, p_2 \in M$ are comparable with respect to an edge $e \in E(\Delta)$ if the open triangle formed by e and p_1 is contained in the closed triangle formed by e and p_2 ($p_1 \leq_e p_2$) or vice versa ($p_2 \leq_e p_1$), see Figure 1 for an example.

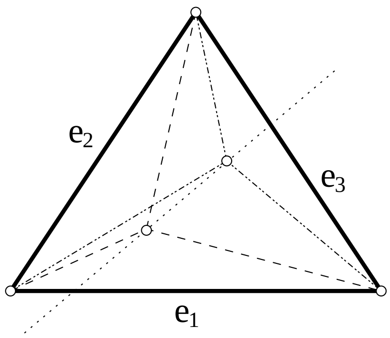

Figure 1: Two points are comparable w.r.t. e_1 and e_3, but incomparable w.r.t. e_2.

Observe that two fixed points $p_1, p_2 \in M$ are comparable w.r.t. exactly 2 out of 3 edges of Δ. This can be seen by considering the supporting line of the edge p_1, p_2, see Figure 1. Two points are comparable w.r.t. an edge e of Δ if and only if this supporting line intersects e. This implies that if two points are not comparable w.r.t. e then they are comparable w.r.t. both other edges.

A chain is an ordered set of (pairwise) comparable points of M and an anti-chain is a set of pairwise incomparable points of M. By Dilworth's Theorem [3] there exists a chain or an anti-chain in M w.r.t. a given edge e of Δ of size $\sqrt{m}$. Because an anti-chain for e is a chain for the other two edges of Δ, we may assume w.l.o.g that there exists a chain $q_{i_1} \leq_e \cdots \leq_e q_{i_{\sqrt{m}}}$ w.r.t. e.

We obtain a triangulation of $\Delta \cup \{q_{i_1}, \ldots, q_{i_{\sqrt{m}}}\}$ by joining each q_{i_j}, $1 \leq j < \sqrt{m}$, to $q_{i_{j+1}}$ and to the endpoints of e, and $q_{i_{\sqrt{m}}}$ to the vertices of Δ, see Figure 2, left. There are $2\sqrt{m}+1$ triangles in this triangulation and all of them have at least one vertex on the convex hull. We now extend the triangulation to cover the remaining points. For each point q_i not in the chain there is precisely one end-point p of e visible to q_i and we add the edge joining q_i and p.

We have, so far, a collection of pairwise non-crossing edges, and we complete this to a triangulation of $\Delta \cup \{q_1, \ldots q_m\}$, see Figure 2, right. There are $2\sqrt{m} + m - \sqrt{m} + 1 = m + \sqrt{m} + 1$ triangles in this triangulation with at least one of its vertices on the convex hull. $\square$

We now generalize the above result to sets with larger convex hulls. Let $CH(S)$ denote the set of vertices of the convex hull of S and $|CH(S)|$ its cardinality, that is, the number of extreme points of S.

Lemma 2 *(Order Lemma) Let S be a set of n points in general position in the plane with $h = |CH(S)|$ extreme points. Then S can be triangulated such that at least $n + \sqrt{n-h} - 2$ triangles have (at least) as one of their vertices an extreme point of S.*

Proof. Consider an arbitrary triangulation of the h convex hull points of S (ignoring inner points). Let $\tau_1, \ldots, \tau_{h-2}$ be the obtained triangles and let s_i be the number of points of S interior to τ_i. By Lemma 1 each triangle τ_i can be triangulated such that at least $s_i + \sqrt{s_i} + 1$ triangles have one of its vertices on the convex hull of τ_i and therefore on the convex hull of S. Taking the sum over all τ_i we have: $\sum_{i=1}^{h-2}(s_i+\sqrt{s_i}+1) = \sum_{i=1}^{h-2} s_i + \sum_{i=1}^{h-2} \sqrt{s_i} + \sum_{i=1}^{h-2} 1 = (n-h) + \sum_{i=1}^{h-2} \sqrt{s_i} + (h-2) \geq n + \sqrt{\sum_{i=1}^{h-2} s_i} - 2 = n + \sqrt{n-h} - 2.$ $\square$

For the next result we consider bi-colored sets. We will show that if the cardinality of the two color classes differs significantly then this implies the existence of a large number of empty monochromatic triangles.

Lemma 3 *(Discrepancy Lemma) Let S be a set of n points in general position in the plane, partitioned in a red set R and a blue set B with $|R| = |B| + \alpha$, $\alpha \geq 2$. Then S determines at least $\frac{(\alpha-2)}{6}(n+\alpha)$ empty monochromatic triangles.*

Proof. Consider a red point $r \in R$ and the star connecting r to all vertices $R \setminus r$. Completing this star to a triangulation of R gives at least $|R| - 2$ triangles having r as a vertex. At least $\alpha - 2$ of these triangles are empty of points from B, as $|B| = |R| - \alpha$. Repeating this process for all points in R we obtain at least $\frac{(\alpha-2)}{3}|R| = \frac{(\alpha-2)}{3}\frac{n+\alpha}{2} = \frac{(\alpha-2)}{6}(n+\alpha)$ empty red triangles, since we over-count a triangle at most 3 times. $\square$

Note that for the monochromatic case the Discrepancy Lemma implies the $\Omega(n^2)$ bound on the number of empty triangles given in [8], although the constants are slightly worse.

We are now ready to prove the main result of this section.

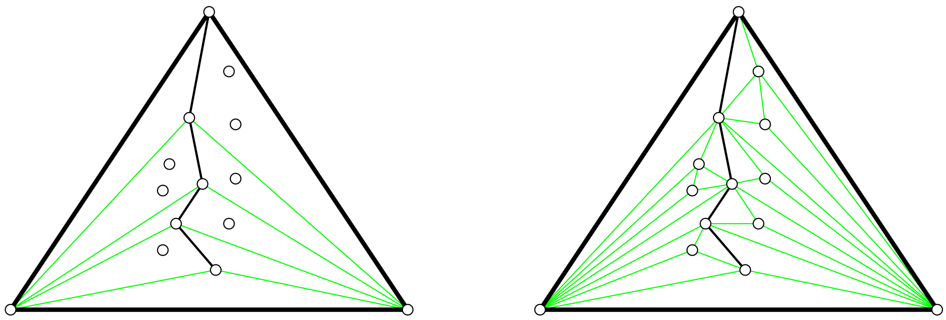

Figure 2: A triangulation for a chain and its extended triangulation

Theorem 4 *Any bi-colored set of n points in the plane in general position determines $\Omega(n^{5/4})$ empty monochromatic triangles.*

Proof. The general idea behind the proof is to iteratively peel a monochromatic convex layer of the point set. For each layer we use the Order Lemma to obtain roughly $\sqrt{n}$ empty monochromatic triangles. If at any moment the difference of the cardinality of the two color classes is too large we utilize the Discrepancy Lemma and terminate the process. Otherwise we stop after at most $\frac{1}{3}n^{3/4}$ steps.

Let S_1 be the given bi-colored set of n points, with R_1 the set of red and B_1 the set of blue points. Let $\tilde{n} = \frac{n}{6}$. For each iteration step we construct smaller sets $S_{i+1} \subset S_i$, $R_{i+1} \subseteq R_i$, and $B_{i+1} \subseteq B_i$, respectively, with $S_{i+1} = R_{i+1} \cup B_{i+1}$. As an invariant we will have that in any step $|S_i| \geq 2\tilde{n}$ holds. The iteration stops either if at some step the discrepancy between the two sets is larger than $\tilde{n}^{1/4}$ or after at most $\frac{1}{3}n^{3/4}$ steps.

Consider the i-th step of the iteration and w.l.o.g. let $|R_i| \geq |B_i|$. There are two possible cases.

(a) If $|R_i| - |B_i| \geq \tilde{n}^{1/4}$ we apply the Discrepancy Lemma and get at least $\frac{(\tilde{n}^{1/4}-2)}{6}(2 * \tilde{n} + \tilde{n}^{1/4}) = \Omega(n^{5/4})$ empty monochromatic triangles.

(b) Otherwise build the convex hull of the red points and let $B'_i \subseteq B_i$ be the blue points outside of this convex hull. We denote by $r_i = |R_i|$ and $b_i = |B_i \setminus B'_i|$. We have $r_i \geq \tilde{n}$ by our invariant assumption, $r_i \geq b_i$, and $r_i \leq b_i + \tilde{n}^{1/4}$, as otherwise we apply the Discrepancy Lemma to $R_i \cup B_i \setminus B'_i$ and terminate the iteration with $\Omega(n^{5/4})$ empty monochromatic triangles as above. Note that the latter inequality implies that $|B'_i| < \tilde{n}^{1/4}$.

We apply the Order Lemma to R_i and get at least $r_i + \sqrt{r_i - |CH(R_i)|} - 2$ monochromatic (red) triangles which are by construction a subset of a triangulation of R_i incident to $CH(R_i)$. At most b_i of these triangles may contain a blue point, so we get at least $r_i - b_i + \sqrt{r_i - |CH(R_i)|} - 2 \geq \sqrt{r_i - |CH(R_i)|} - 2$ empty monochromatic triangles.

Now we show that $|CH(R_i)| < 2\tilde{n}^{1/4}$. Assume to the contrary that $|CH(R_i)| \geq 2\tilde{n}^{1/4}$ and consider the set $(R_i \setminus CH(R_i)) \cup (B_i \setminus B'_i)$. This set has at most $r_i - 2\tilde{n}^{1/4}$ red points and $b_i \geq r_i - \tilde{n}^{1/4}$ blue points, so the difference is at least $\tilde{n}^{1/4}$ and as above we apply the Discrepancy Lemma and terminate.

Thus, if we don't terminate, we get at least $\sqrt{\tilde{n} - 2\tilde{n}^{1/4}} - 2 \geq \frac{\sqrt{\tilde{n}}}{2}$ empty monochromatic triangles in step i. Note that the last inequality holds for sufficiently large $\tilde{n}$.

For the next iteration step let $R_{i+1} = R_i \setminus CH(R_i)$, $B_{i+1} = B_i \setminus B'_i$, and $S_{i+1} = R_{i+1} \cup B_{i+1}$. Note that all the empty monochromatic triangles we have constructed in step i had at least one vertex in $CH(R_i)$, that is, we will not use these vertices for the next iterations, and therefore we do not overcount.

The process ends either by applying the Discrepancy Lemma or after $\frac{1}{3}n^{3/4}$ steps. As in each step we obtain at least $\frac{\sqrt{\tilde{n}}}{2}$ empty monochromatic triangles, we get in both cases a total of $\Omega(n^{5/4})$ empty monochromatic triangles.

It remains to show that the invariant $|S_i| \geq 2\tilde{n}$ holds. In step i we remove $|B'_i| + |CH(R_i)| < \tilde{n}^{1/4} + 2\tilde{n}^{1/4} = 3\tilde{n}^{1/4}$ points. Thus after $\frac{1}{3}n^{3/4}$ steps we have at least $n - \frac{1}{3}n^{3/4} \cdot 3\tilde{n}^{1/4} \geq 2\tilde{n}$ points left. □

Note that the constants in the above proof could be improved, but it is easy to see that the asymptotic behavior is tight within this approach.

3 Empty monochromatic simplices in $\mathbb{R}^d$

In this section we show that our results can be generalized to $\mathbb{R}^d$, where d is a fixed positive integer. We first show an analogous lemma to the Discrepancy Lemma in $\mathbb{R}^d$ and then we outline a proof of the main result of this section. We leave the details of this extension for the full version of this paper.

A set S of points in d-dimensional Euclidean space $\mathbb{R}^d$ is in general position if no $d+1$ of them lie in a $d-1$ dimensional hyperplane. A simplex spanned by points in S is called monochromatic if all its points are colored with the same color. A simplex is called empty if it does not contain points of S in its interior. We denote by $Conv(S)$ the convex hull of a point set S together with its interior.

Lemma 5 *Let S be a set of $n \geq d+2$ points in general position in $\mathbb{R}^d$ and X a subset of S of $d-1$ elements. There exists a set of $n-d$ interior disjoint empty simplices with vertices on S such that every simplex in the set contains all the elements of X as vertices.*

Proof. Let Π be the $d-2$ dimensional hyperplane containing X and let Π' be a 2-dimensional plane orthogonal to Π. Project all of S orthogonally to Π'. Note that X is projected to a single point x in Π'. Let $p_1, p_2, \ldots, p_{n-(d-1)}$ be the images in Π' of the remaining $n-(d-1)$ points in $S-X$. Moreover assume the points are ordered by angle around x. Note that the preimages of $\{p_i, p_{i+1}\} \cup x$, $1 \leq i \leq n-d$, are a family of interior disjoint simplices in $\mathbb{R}^d$, with vertices in S and all containing the elements of X as vertices. □

Lemma 6 *(Generalized Discrepancy Lemma) Let S be a set of n points in general position in $\mathbb{R}^d$, partitioned in a red set R and a blue set B with $|R| = |B| + \alpha$, $\alpha > d$. Then S determines at least $\alpha c_d n^{d-1}$ empty monochromatic simplices, where c_d is a constant depending only on d.*

Proof. Let X be a subset of R of $d-1$ elements. By Lemma 5 there exist $|R|-d$ interior disjoint red simplices, empty of red points such that all simplices contain X in their vertex set. At least $|R|-d-|B| = \alpha - d$ of this red simplices are also empty of blue points. Doing this for every $d-1$ subset of R, we obtain $\binom{|R|}{d-1}(\alpha-d) = \binom{(n+\alpha)/2}{d-1}(\alpha-d)$ empty red simplices, and we over count each of them $\binom{d}{d-1} = \binom{d}{2}$ times. Thus in total there are at least $\binom{(n+\alpha)/2}{d-1}(\alpha-d)/\binom{d}{2}$ empty red simplices and our result follows. □

Theorem 7 *Let S be a set of n points in general position in $\mathbb{R}^d, d \geq 2$, partitioned in a red set R and a blue set B. Then either there are $\Omega(n^{d-3/4})$ empty red simplices or there is a convex set C in $\mathbb{R}^d$, containing a linear number of elements of S and such that $||C \cup R| - |C \cup B|| = \Omega(n^{1/4})$.*

Sketch of Proof. We proceed by induction on d. The result for $d = 2$ is based on a modification of Theorem 4. Let then $d \geq 3$.

Note that if $|R-B| \geq n^{1/4}$, then $Conv(S)$ is a convex set with the desired difference between color classes. Assume then that $|R-B| < n^{1/4}$. Note that this implies that $|R| > n - \frac{n^{1/4}}{2}$.

Let $r \in R$ be a red point. For every other point in $p \in S$ consider the infinite ray $l_{r,p}$ starting at r and containing p. Let Π_r and Π'_r be two parallel $d-1$ dimensional hyperplanes, not parallel to any of the rays $l_{r,p}$; with the additional property that S lies between Π_r and Π'_r. Note that every ray $l_{r,p}$ intersects either Π_r or Π'_r but not both. We project the elements of $S - \{r\}$ to the planes by taking for every point the intersection of its corresponding ray with Π_r of Π'_r.

We apply induction to the images of the elements of $S - \{r\}$ in Π_r and Π'_r. Note that the preimage of an empty red simplex in Π_r (or Π'_r) together with r is an empty simplex in $\mathbb{R}^d$; also note that if P' is the set of projected points inside some convex set in Π_r (or Π'_r) and P is the preimage of P', then $Conv(P) \cap S = P$.

If for one red point r we are not able to find $\Omega(n^{d-1-3/4})$ empty simplices we obtain a convex set C in $\mathbb{R}^d$, containing a linear number of elements of S with $||C \cup R| - |C \cup B|| = \Omega(n^{1/4})$. Otherwise, for every red point we find $\Omega(n^{d-1-3/4})$ empty red simplices in $\mathbb{R}^d$; thus $\Omega(n^{d-3/4})$ in total (since we over count each simplex at most d times).

We are now able to prove the generalization of our result to $\mathbb{R}^d$:

Theorem 8 *Any bi-colored set S of points in $\mathbb{R}^d$ in general position determines $\Omega(n^{d-3/4})$ empty monochromatic simplices.*

Proof. By Theorem 7 either we have $\Omega(n^{d-3/4})$ empty simplices or we obtain a convex set containing a linear number of elements of S and such that the difference between the two color classes is $\Omega(n^{1/4})$. In the latter case we apply the Generalized Discrepancy Lemma and also obtain $\Omega(n^{d-3/4})$ empty monochromatic simplices. □

4 Conclusions and Open Problems

We have not been able to construct a point set with $o(n^2)$ empty monochromatic triangles. Usually Horton sets are a good candidate to provide minimal examples with respect to determining empty convex polygons. But it turns out that every two-coloring of a Horton set has $\Omega(n^2)$ empty monochromatic triangles. A brief sketch of that fact looks as follows. Take any bi-coloring of the Horton set and note that the upper and lower part must have a linear number of red and blue points, as otherwise by the Discrepancy Lemma there would be a quadratic number of empty monochromatic triangles. Now take any triangle of three consecutive points in the upper part which form a cap. Any edge of this triangle, where at least one is monochromatic, say red, and any point from the lower part spans an empty triangle. Thus, together with the $\Theta(n)$ red points from below, it forms a linear number of empty red triangles. Since there is a linear number of such caps we get a quadratic number of empty monochromatic triangles for the Horton set.

Other interesting sets with $O(n^2)$ empty triangles, which are not based on Horton sets, can be found in the constructions of Katchalski and Meir [8].

Considering results in [2] and [10] one can see that in the uncolored case and for sufficiently large n there is always a quadratic number of empty triangles and empty convex quadrilaterals, there is at least a linear number of pentagons but the correct bound seems to be quadratic. The status for convex empty hexagons was a long-standing open problem and it has recently been shown that at least one (and thus a linear number) exists, but the best upper bound is again quadratic. Finally no empty convex 7-gons may exist. So it seems that either none, or a quadratic number of empty k-gons exists, and we believe that somehow this translates to colored point sets. We therefore state the following conjecture.

Conjecture 1 *Any bi-colored set of n points in $\mathbb{R}^2$ in general position determines a quadratic number of empty monochromatic triangles.*

In fact we did not obtain a single family of sets where the asymptotics of the number of empty triangles and empty monochromatic triangles differ. What we have been able to construct are sets which have 5 times fewer empty monochromatic triangles than empty triangles. The idea behind the construction is to start with a set S of n points with $t(S)$ empty triangles. W.l.o.g. S has no two points on a horizontal line. We then add a copy of S which is shifted horizontally to the right by some sufficiently small ε and color the points of S red and their duplicates blue. For each ε-near pair we get $2n-2$ empty bi-chromatic triangles. For each empty triangle in S we get 3 new bi-chromatic triangles (not using an ε-near pair of points), but only one empty monochromatic triangle. Thus the ratio of empty triangles to monochromatic ones is $4+\frac{2n^2-2n}{t(S)}$. Taking the sets constructed by Bárány and Valtr [2] with $t(S) = 1.62n^2$ empty triangles gives a factor of ≈ 5.23.

Another interesting question is to consider empty monochromatic convex k-gons for $k > 3$. Devillers et al. [4] (Theorem 3.4) showed that for $k \geq 5$ and any n there are bi-colored sets where no empty monochromatic convex k-gon exists. So the remaining case are empty monochromatic convex quadrilaterals. For example in [4] they showed that for $n \geq 64$ any bi-colored Horton set contains empty monochromatic convex quadrilaterals. This leads to Conjecture 3.1 in [4] which states that for sufficiently large n any bi-colored set contains at least one monochromatic convex quadrilateral.

We recently learned that based on our approach the lower bound on the number of empty monochromatic triangles can be slightly improved [9].

We finally state an analogous conjecture to Conjecture 1 for $\mathbb{R}^d$.

Conjecture 2 *Any bi-colored set of n points in $\mathbb{R}^d$ in general position determines $\Omega(n^d)$ empty monochromatic simplices.*

Note that, when neglecting colors, Katchalski and Meir showed that each set of n points in $\mathbb{R}^d$ has $\Omega(n^d)$ empty simplices[8] and Bárány and Füredi showed that there are sets of n points in $\mathbb{R}^d$ that do not have more than $\Theta(n^d)$ empty simplices[1].

5 Acknowledgment

We would like to thank Wolfgang Aigner, Sergio Cabello, Bernhard Kornberger and Birgit Vogtenhuber for helpful discussions.

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