Minimum blocking sets of circles for a set of lines in the plane

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Abstract

A circle C is occluded by a set of circles $C_1, \ldots, C_n$ if every line that intersects C also intersects at least one of the $C_i,\ i = 1, \ldots, n$. In this paper, we focus on determining the minimum number of circles that occlude a given circle assuming that all circles have radius 1 and their mutual distance is at least d. As main contribution of this paper, we present upper and lower bounds on this minimal number of circles for $2 \leq d \leq 4$, as well as the algorithms we used to derive them.

1 Introduction

A set of circles $C_1, \ldots, C_n$ occludes a circle C if every line that intersects circle C also intersects at least one of these circles. Such a set of circles is called a blocking set of circles. In this paper, we discuss the case where all the circles have radius 1 and the distance between the circles' centers is at least d. We present the algorithms we used to determine the upper and lower bounds on the minimum cardinality of the blocking sets of circles for different values of d, as well as the results we obtained.

The problem of an occluded convex shape in a two-dimensional plane arises in the process of detecting objects using light beams from arbitrary directions. Here, we limit ourselves to circular objects of equal size in the plane.

In the literature there is related work on blocking sets in a projective plane. [4] defines a similar problem and some solutions are proposed by [2, 1]. Explaining the difference to the problem in a projective plane is beyond the scope of this paper. The related literature indicates that this work is entirely novel - to the best of our knowledge the problem we investigate has not been studied before.

2 Problem Description

Let C be a unit circle in a two-dimensional plane with its center positioned at point p_0 and let $\mathcal{L}_C$ be the set of all lines that intersect C.

Definition 1 (blocking set) *Given a set of lines L, a set B of unit circles is called a blocking set for L if and only if each line $l \in L$ intersects at least one of the circles in B.*

A blocking set for L is denoted as $B(L)$.

Definition 2 (d-apart blocking set) *Given circle C at position p_0, a blocking set $B(L)$ of n unit circles positioned at $p_1, \ldots, p_n$ is called d-apart if and only if for each pair $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$, the Euclidean distance $d(p_i, p_j) \geq d$, for a given distance d.*

Note that not only the mutual distance between the centers of the circles in $B(L)$ needs to be at least d, but also the distance to the center of circle C.

Problem 1 (Minimum blocking set problem) *Given a unit circle C, corresponding set $\mathcal{L}_C$ of lines and distance d, find a d-apart blocking set $B(\mathcal{L}_C)$ of minimum cardinality.*

The cardinality of a minimum blocking set for given d is denoted as N_d.

Let d be a given distance with $2 \leq d \leq 4$. The Minimum blocking set problem has a simple solution [3] of cardinality 4 for $d = 2$; see Figure 8. Each line that has at least one intersection point with the circle C in the middle, also has non-empty intersections with at least one of the 4 circles around it.

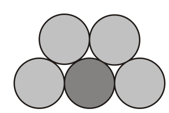

Figure 1: Minimum blocking set for $d = 2$.

Another example of a blocking set is shown in Figure 2. For $d = 4$, the 15 light shaded circles positioned at points of a regular triangular grid, as illustrated below, block all the lines that intersect the circle in the middle.

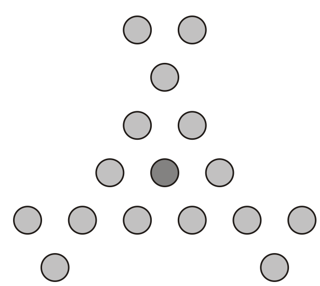

Figure 2: Example of a blocking set: 15 circles block all lines that intersect the dark circle in the middle.

The problem of determining N_d for an arbitrary distance $d > 2$ is difficult. We derive upper and lower bounds on N_d using two different approaches that we explain in detail in Section 3 and Section 4, respectively.

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Next, we elaborate on the properties of a blocking set. Let a d-apart blocking set $B(\mathcal{L}_\mathcal{C})$ be positioned at points $p_1, p_2, \ldots, p_n$. We say that the position p_i of a circle in $B(\mathcal{L}_\mathcal{C})$ is *closest* if and only if $d(p_0, p_i) \leq d(p_0, p)$ for all points p on the line through p_0 and p_i for which it holds that $d(p, p_j) \geq d, j = 0, \ldots, n, j \neq i$.

Definition 3 (maximally shrunk blocking set) *A d-apart blocking set $B(\mathcal{L}_\mathcal{C})$ is maximally shrunk if and only if every circle of $B(\mathcal{L}_\mathcal{C})$ is on one of its closest positions.*

A set B of circles is said to *dominate* another set of circles B' if and only if all lines that are blocked by B' are also blocked by B.

Lemma 1 *Any d-apart blocking set $B(\mathcal{L}_\mathcal{C})$ is dominated by a maximally shrunk d-apart blocking set $B'(\mathcal{L}_\mathcal{C})$.*

Proof. Let p_i be the position of an arbitrary circle of the blocking set $B(\mathcal{L}_\mathcal{C})$ and let p'_i be the corresponding closest position of the circle of the maximally shrunk blocking set $B'(\mathcal{L}_\mathcal{C})$ (see Figure 3).

By elementary calculus it can be shown that every line blocked by the circle at p_i is also blocked by the circle at p'_i. □

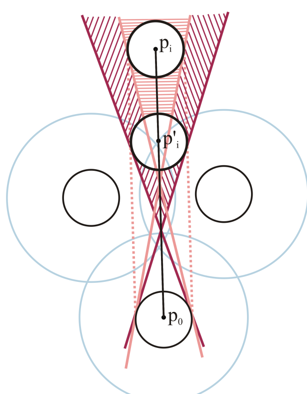

Figure 3: Every line blocked by the circle at p_i is also blocked by the circle at p'_i.

As a consequence of Lemma 1, there are optimal solutions of the minimum blocking set problem within the class of maximally shrunk blocking sets.

3 Deriving upper bounds

In this section, we construct a special class of blocking sets providing upper bounds on N_d. We focus on blocking sets that have an even number k of circles at a distance d from the center of the given circle C, such that they form a regular polygon, with either $k = 4$ or $k = 6$.

Each of these first k circles blocks some lines from the given set $\mathcal{L}_\mathcal{C}$. The remaining set of lines, denoted as $\mathcal{L}'_\mathcal{C}$, can be subdivided into disjunct *bundles of lines*. For $k = 4$ and $k = 6$, we obtain 2 and 3 bundles, respectively (see Figure 4).

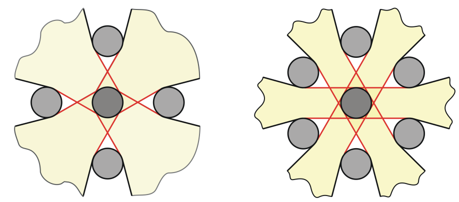
Figure 4: The remaining sets of lines grouped as disjunct bundles of lines.

Definition 4 (bundle of lines) *A bundle of lines $L(p_i, p_j) \subset \mathcal{L}'_\mathcal{C}$ between two adjacent circles positioned at p_i and p_j contains a line $l \in \mathcal{L}'_\mathcal{C}$ if and only if l intersects the line segment $p_i p_j$ and $d(p_i, l) \geq 1$, $d(p_j, l) \geq 1$.*

Note that the pairs of diametrically opposite circles from the first k circles define identical bundles of lines.

Let $L(p_i, p_j)$ be a bundle of lines and let l' be a line such that $p_0 \in l'$ and $l' \notin L(p_i, p_j)$. The two lines t and t' of the bundle $L(p_i, p_j)$ that form the largest and the smallest angle with the line l' are called *extremal lines* and the angle between them is denoted as θ. Note that the lines t and t' are tangent to the circles C_i and C_j positioned at p_i and p_j.

We can now define a subproblem of the minimum blocking set problem as follows.

Problem 2 (Bundle blocking problem) *Given a bundle of lines $L(p_i, p_j) \subset \mathcal{L}'_\mathcal{C}$, find a blocking set $B(L(p_i, p_j))$ of minimum cardinality, such that the blocking set $B(L(p_i, p_j)) \cup \{C_i, C_j\}$ is d-apart.*

Given the restriction on the mutual distance, for each of the circles we define a *boundary circle* that determines the region in which it is not possible to place any additional circles. Therefore, a blocking set for a bundle of lines can be chosen to consist of the circles positioned between the extremal lines and on or outside the boundary circles (see the shaded area in Figure 5 as an example).

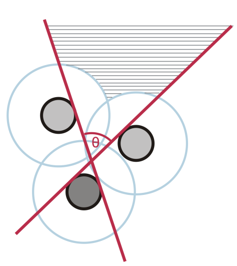

Figure 5: An example of a bundle of lines.

Every additional circle that we place in the shaded area reduces the set of lines of the bundle. However, depending on the position of the added circle, the non-blocked lines can all be in one bundle or can be separated into two disjunct bundles. In both cases, the angle(s) between the extremal lines of the new bundle(s) is/are strictly smaller than the angle between the extremal lines before placing the additional circle.

Next we propose a heuristic algorithm that tries to block a given bundle of lines $L(p_i, p_j)$ by $1, 2, 3, 4$, or 5 circles. We discuss each of the cases separately.

Blocking a bundle by 1 circle. To test whether or not one circle can block all the lines, we use a simple procedure. Let t and t' be the two extremal lines of $L(p_i, p_j)$ and let θ be the angle between them. Let $\bar{p}$ be the intersection point of the bisector of the angle θ and a boundary circle, such that $\bar{p}$ is not in the interior of any other boundary circle. If $d(\bar{p}, t) \leq 1$, it is possible to block the bundle with one circle (see Figure 6 - left).

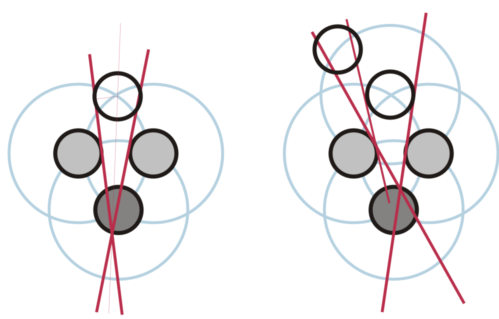

Figure 6: Blocking bundles of lines by one circle (left) and two circles (right).

Blocking a bundle by 2 circles. The essential part of the test whether or not two circles can block a given bundle is the observation that the first added circle can be chosen to be tangent to one of the extremal lines. Otherwise, it would separate the bundle into two disjunct bundles, requiring at least two additional circles for the blocking. Therefore, we add one circle to the closest position such that one of the extremal lines is tangent to the circle and test whether or not the rest of the lines (new bundle) can be blocked by one circle (see Figure 6 - right).

To test whether a bundle can be blocked by 3 or more circles, we need an additional construction method: find the closest position of one circle such that one of the new bundles of lines defined by that circle can be blocked by *exactly* one circle. The specific positions of the two circles can be found using analytic geometry and considering different cases, but we leave out the discussion in the interest of space.

Different positions of the first added circle result in different bundles of non-blocked lines. Therefore, we consider some cases of that positioning for testing whether the given bundle can be blocked by 3 or more circles.

Blocking a bundle by 3 circles. The analysis is by considering two cases. In the first case, we add the first circle such that it is tangent to one of the extremal lines. Then, the non-blocked lines are in one bundle, and we test whether or not these can be blocked by 2 circles.

In the second case, we use the construction method mentioned above to find the position of the first circle such that one of the two new bundles of lines can be blocked by exactly one circle. Then, we check whether the non-blocked bundle of lines can be blocked by 1 circle.

Blocking a bundle by 4 circles. This test consists of checking two cases, as in the test with 3 circles. In the first case, we add one circle such that it is tangent to one of the extremal lines and check whether the new bundle can be blocked by 3 circles.

In the second case, we add two circles using the construction method and check whether the remaining bundle can be blocked by 2 circles.

Blocking a bundle by 5 circles. Besides the two cases similar to those in tests with 3 and 4 circles, we have an additional one: we place the first circle at the intersection point of the angle bisector and a boundary circle and check whether both new bundles of lines can be blocked by 2 circles.

Blocking the lines by 6 or more circles has been considered. However, experimental results show that the bundles of lines defined by the first 4 or 6 circles on regular polygon positions can be blocked by at most 5 circles for $2 \leq d \leq 4$.

4 Deriving lower bounds

In this section, we explain the approach we use to obtain lower bounds on N_d. For this, we consider the set $L \subset \mathcal{L}_\mathcal{C}$ that consists of all the lines from $\mathcal{L}_\mathcal{C}$ that pass through the center p_0 of the given circle C. The cardinality of a minimum blocking set $B(L)$ represents a lower bound on N_d since $L \subset \mathcal{L}_\mathcal{C}$. A minimum blocking set $B(L)$ can be constructed, since one can prove that in the set of minimum blocking sets for L, there are always non-overlapping ones, i.e. blocking sets for which the intersection of lines blocked by any two circles consists of at most one (tangent) line. The number of non-overlapping blocking sets for L of cardinality N can be reduced to a few cases, where for each case, we can determine the largest value of d possible for that case. We will first consider the case where all lines in L can be blocked by four circles.

Let d be a distance for which the cardinality of a minimum blocking set $B(L)$ is $N = 4$. The subset of lines from L blocked by a circle C_i is given by *blocking angle* α_i, defined by the two lines in L that are tangent to C_i. The blocking angle α_i is given by

$$\alpha_i = 2 \arcsin \frac{1}{d_i} \tag{1}$$

where d_i is the distance between p_0 and p_i (See Figure 7).

Let C_1, C_2, C_3 and C_4 be the four circles of a blocking set $B(L)$, and let d_1, d_2, d_3 and d_4 be the distances from p_0 to p_1, p_2, p_3 and p_4, respectively.

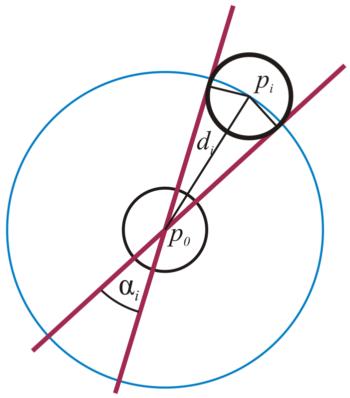

Figure 7: The blocking angle of a circle.

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the blocking angles of the circles C_1, C_2, C_3 and C_4, respectively. We consider non-overlapping blocking sets, thus

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi \tag{2}$$

Without loss of generality we assume

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \tag{3}$$

which implies

$$d_1 \leq d_2 \leq d_3 \leq d_4 \tag{4}$$

From (4) we have that $d = d_1$. Note that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ is not feasible for $d > 2$, since four circles cannot be positioned in a way that each of them has a blocking angle $\pi/4$ and they do not violate a minimum distance condition.

For $\alpha_1 = \alpha_2 = \alpha_3 > \alpha_4$ we have a blocking set $B(L)$ as shown in Figure 8.

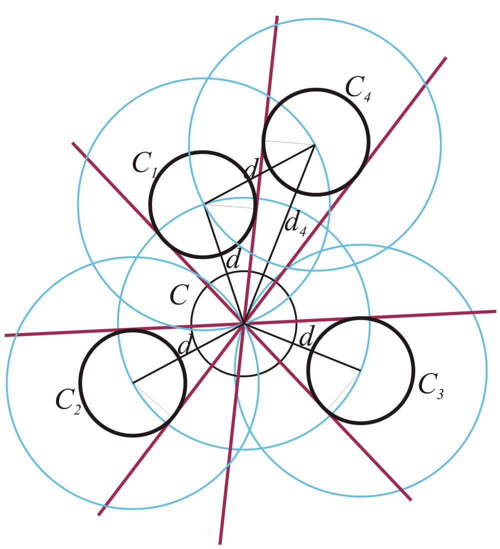

Figure 8: A non-overlapping blocking set of cardinality 4.

From the system of equations

$$\begin{aligned} 3\alpha_1 + \alpha_4 &= \pi \\ \alpha_1 &= 2 \arcsin \frac{1}{d} \\ \alpha_4 &= 2 \arcsin \frac{1}{d_4} \\ d_4 &= 2d \cos \frac{\alpha_1 + \alpha_4}{2} \end{aligned} \tag{5}$$

we have $d = \frac{\sqrt{8}}{\sqrt{5-\sqrt{13}}}$.

We prove now that the derived distance $d = \frac{\sqrt{8}}{\sqrt{5-\sqrt{13}}}$ represents the maximum distance d for which $N = 4$. We consider an arbitrary non-overlapping blocking set $B'(L)$ of cardinality 4 such that for the blocking angles of its circles it holds that

$$\alpha'_1 > \alpha'_2 > \alpha'_3 > \alpha'_4 \tag{6}$$

The minimum mutual distance between the circles of $B'(L)$ is denoted as d'. From (6) we have that the circle C'_1 is on the distance d' from the circle C. We have two cases:

1. If $\alpha_1 \leq \alpha'_1$ then $d \geq d'$, i.e. d is maximum distance.
2. Let $\alpha_1 > \alpha'_1$. From (6) and

$$3\alpha_1 + \alpha_4 = \pi \tag{7}$$

$$\alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 = \pi \tag{8}$$

we have

$$\alpha_4 < \alpha'_4 \tag{9}$$

At least two of four circles are adjacent having one common tangent line (in the same way as circles C_1 and C_4 shown in Figure 8). Then, the furthest circle C'_4 is on the distance d'_4 from the center of the circle C:

$${d'_4}^2 = x^2 + y^2 - 4 + 2\sqrt{y^2-4} \cdot \sqrt{x^2-1} \tag{10}$$

where $x > d$ and $y > d$.

The distance d_4 of the furthest circle in the blocking set $B(L)$ is

$${d_4}^2 = d^2 + d^2 - 4 + 2\sqrt{d^2-4} \cdot \sqrt{d^2-1} \tag{11}$$

From (10) and (11) we conclude that

$$d'_4 > d_4 \tag{12}$$

which implies

$$\alpha'_4 < \alpha_4 \tag{13}$$

which contradicts (9).

Hence, we conclude that $d > d'$. In the same way it can be shown that different relations between the angles $\alpha'_1, \alpha'_2, \alpha'_3$ and α'_4 result in blocking sets with a smaller d. Therefore, the maximum distance d for which the blocking set $B(L)$ is of cardinality 4 is $d = \frac{\sqrt{8}}{\sqrt{5-\sqrt{13}}}$. The maximum distance d for which the blocking set $B(L)$ is of cardinality $N > 4$ can be derived in a similar fashion.

The example in Figure 9 shows the optimal non-overlapping blocking set $B(L)$ for $k = 5$. The maximum distance d for which 5 circles can block the lines from L is simply derived as $d = 1/\sin\frac{\pi}{10}$.

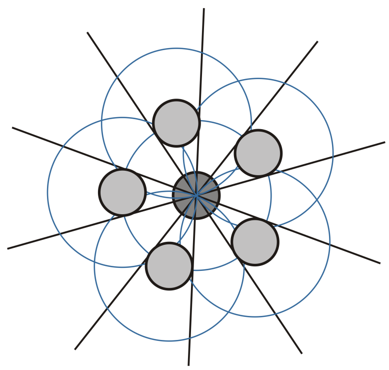

Figure 9: Every line passing through the center of the middle circle is blocked by at least one of the 5 circles around it.

5 Upper and lower bounds - results

In this section we present the upper and lower bounds on minimum blocking sets that we obtained by the methods explained in Sections 3 and 4.

In Figure 10, d is given on the horizontal axis. The number of circles is given on the vertical axis. For example, for $d = 3$, we have $5 \leq N_d \leq 9$.

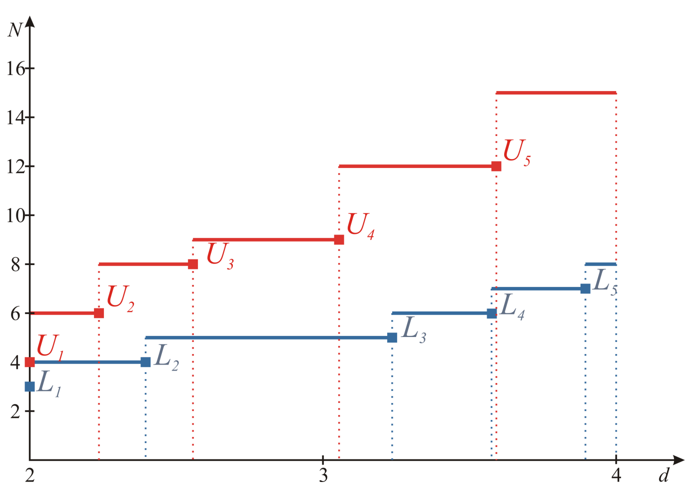

Figure 10: Upper and lower bounds on the cardinality of minimum blocking sets.

Obviously, N_d is a monotonic function of d with non-negative integer values. Table 1 gives the d-values of the points where the bounds on that function change value. The d-value of the lower bound point L_4 is

$$d_4 = 1/\sqrt{\frac{9}{16} - \frac{1}{16}y - \frac{1}{2}\sqrt{\frac{9}{32} + \frac{1}{16\sqrt[3]{18}}x + \frac{1}{8x\sqrt[3]{12}} + \frac{3}{32y}}},$$

where $x = \sqrt[3]{81 - \sqrt{6549}}$ and $y = \sqrt{9 - \frac{4}{x}\sqrt[3]{\frac{2}{3}} - 2x\sqrt[3]{\frac{4}{9}}}$.

i	1	2	3	4	5
U_i	2	2.2361	2.5776	3.0551	3.5914
L_i	$1/\sin\frac{\pi}{6}$	$\frac{\sqrt{8}}{\sqrt{5-\sqrt{13}}}$	$1/\sin\frac{\pi}{10}$	d_4	$4\cos\frac{\pi}{14}$

Table 1: Values of d where upper bounds U_i and lower bounds L_i change value.

By randomly generating maximally shrunk blocking sets, we did not obtain sets with less circles than the corresponding upper bounds. This indicates that probably the lower bounds can be improved.

6 Conclusions

In this paper we investigated the two-dimensional geometrical problem of “blocking” lines that intersect a given unit circle with unit circles. The circles are positioned in such a way that the distance between every two circles is at least a given distance d. We focused on the minimum number N_d of circles that block the lines under given conditions.

The given problem is difficult and we did not find straightforward solutions for it. Our approach has been to consider examples of some simple positionings of circles and downsize the problem to the point where we can easily determine the minimum number of circles needed for the complete blocking. In that way, we provided upper bounds on N_d. We also proposed provable lower bounds on N_d.

The main challenge of this work still remains - finding the minimal number of circles that can cause occlusion for given minimal mutual distance between the circles. The problem can be generalized and investigated in three-dimensional space, as the problem of minimum blocking sets of spheres for a set of lines. Furthermore, one can investigate the asymptotic behavior of N_d (for $d \rightarrow \infty$), generalize to shapes other than circles, or to blocking half-lines instead of lines. Additionally, one can investigate to what extend values of N_d are affected, if circle positions are restricted to points in a grid.

Acknowledgments

We want to thank our colleagues Ramon Clout and Verus Pronk for their suggestions and comments. We are also grateful to Emile Aarts, Johan Lukkien and Zarko Aleksovski for fruitful discussions and inspiring ideas. We especially thank Radivoje Jovanovic for a very useful contribution on finding the lower bounds. Finally, we want to thank the anonymous referees for their in-depth and very useful suggestions that helped us to substantially improve the way we present our work in this paper.

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