Symmetry Restoration by Stretching

Misha Kazhdan[*] Nina Amenta[†] Shengyin Gu[†] David F. Wiley[†] Bernd Hamann[†]

Abstract

We consider restoring the bilateral symmetry of an object which has been deformed by compression. This problem arises in paleontology, where symmetric bones are compressed in the process of fossilization. Our input is a user-selected set P of point-pairs on the deformed object, which are assumed to be mirror-images in some undeformed set AP, with some added noise. We carefully formulate the problem, and give a closed-form solution.

1 Introduction

Much of what we know about evolution comes from the study of fossils. From the shapes of the bones of extinct animals we form hypotheses about how they moved, what they ate, how they are related to each other, and so on. Yet these shapes are usually deformed by the geological processes which occur during fossilization, for example the skull in Figure 1. For some fossils, for example skulls and vertebrae, we can assume that the original shape was roughly bilaterally symmetric. We can use this assumption to reverse the deformation, or at least limit the family of possible reconstructions. This process is sometimes called retrodeformation.

Usually the input for retrodeformation is a set of point-pairs, chosen by the paleontologist on the deformed specimen. We assume the point-pairs are stored in a $3 \times 2n$ matrix P with the assumption that point p_{2i} was the mirror image of p_{2i+1}, on the original object before deformation. The point-pairs are chosen using the expert's understanding of the biological shape. Developing automatic methods for finding point-pairs or other useful descriptions of the input data is a different, also well-studied, research question (see below).

Under the assumption that the object was *compressed*, we assume that the inverse deformation should be what we call a *single axis stretch*. A single axis stretch is produced by choosing a direction vector and scaling only in that direction; it is represented by a symmetric matrix A for which two of its eigenvalues are one and the third is greater than one. Single-axis stretches are important, since the simplest hypothesis for how a fossil is deformed is that it is compressed in a single direction. We want to find a single-axis stretch A such that AP is as symmetric as possible.

[*]Department of Computer Science, Johns Hopkins University

[†]Department of Computer Science, University of California at Davis

Figure 1: A deformed dinosaur skull in the Carnegie Museum of Science, Pittsburgh.

Problem 1 *Let P be a set of point-pairs. Find the single-axis stretch A, a translation vector t, and a plane of reflection, such that the mean-squared error*

$$E(A, w, t) = \sum_{i=1}^{n} ||A(p_{2i}+t) - \mathit{Refl}_w(A(p_{2i+1}+t)||^2 \quad (1)$$

is minimized. Here Refl_w is the affine transformation reflecting space across the plane with normal w passing through the origin.

The choice of mean-squared error is natural, and consistent with usual practice in paleontology.

But there is a problem with this formulation: there is not a unique solution in the absence of noise. Instead, as we shall see, there is a one-dimensional set of single-axis stretches that produce different, but equally optimally symmetric, shapes. As an analogy, think of fitting a plane to set of points that lie on a line; there is no unique solution. If noise is added, there is a unique solution, but it provides information only about the noise, not about the unknown plane that contains the points. Similarly, when P is noisy the unique minimum error solution selects one of the possible symmetrizing single-axis stretches, but based on the noise rather than on any information about the original shape.

In the absence of any other information, the best of these possible solutions would be the one requiring minimum deformation from the input shape: the smallest stretch (alternatives such as maintaining the volume or minimizing the squared distance from the input data are not reasonable choices assuming compression). When other information is available - comparison with other fossils, or perhaps similarity to extant species - it can be used to select a solution [6, 11].

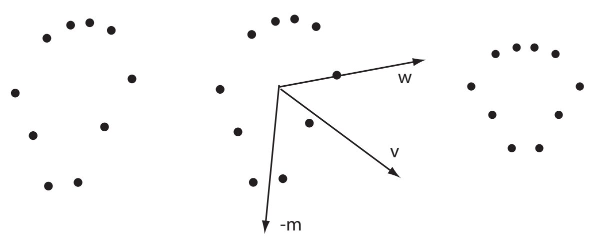

Figure 2: Two ideas for retrodeformation. On the left, a perfectly symmetric set of point-pairs, deformed by compression along a single axis. Center, it seems intuitively clear that stretching in direction v is the most efficient way to make w and $-m$ perpendicular. Right, making the entire point set isotropic also makes it symmetric.

Our approach: In this paper we combine two ideas for restoring symmetry, illustrated in Figure 2. The first is a "well known" idea in the area of symmetry detection: if there is any linear transformation which makes P perfectly symmetric, then any linear transformation which takes P to an isotropic set $\tilde{P}$ (that is, the principal components of $\tilde{P}$ are all vectors of length one), also makes P perfectly symmetric. This means that the set of all perfectly symmetric solutions are exactly the transformations of $\tilde{P}$ which preserve symmetry. We apply this idea to a any approximately symmetric set P by first transforming P into an isotropic set $\tilde{P}$ and then finding the best of plane of symmetry of $\tilde{P}$.

We then consider the single-axis stretches as a subset of this family, and describe how to find the minimal single axis stretch. This method is essentially the same as a procedure for retrodeformation suggested by the physical anthropolgists Zollikofer and Ponce de León [13] (Appendix E): given a vector w estimating the average direction of the vectors $p_{2i} - p_{2i+1}$, and an estimate m of the projection of that vector on the sagittal plane of reflection, stretch in the direction v bisecting $\angle w, -m$ until w and $-m$ become perpendicular. This method was presented without a proof of optimality. We use the first idea to select v and m, and prove that the solution is indeed optimal in 3D.

Other related work: In paleontology, this problem has been approached in different ways. An article by Motatni [6] gave a closed-form solution in two dimensions, using a somewhat different set-up. Other 2D methods which have been used to study, for example, trilobites and turtles, are compared experimentally by Angielczyk and Sheets [1]. More free-form non-linear deformations have also been considered [7]. In morphometrics, the problem of measuring symmetry has been studied [4, 3].

Research in computer science has focused on detecting symmetry; see prior work by the first author [2] and references therein, [9], [10], and [12]. A notable exception is [5], where detected approximate symmetries were grouped and aligned to restore the symmetry of bent objects (ie, straightening out a snake).

2 Isotropy and symmetry preserving transformations

We assume throughout that P is not co-planar and, without loss of generality, that P is translated so that its center of mass is at the origin.

We say a set of points $\tilde{P}$ is *isotropic* if its 3×3 covariance matrix $\tilde{P}\tilde{P}^t = I$ (all of its principle components are one). Given any set P of points, there is a transformation $M^{-1/2}$ such that $\tilde{P} = M^{-1/2}P$ is isotropic. Details of the definition of $M^{-1/2}$, which is standard, can be found in the long version of this paper. It is important to note, however, that the center of mass remains fixed at the origin.

We say a set P of point-pairs is *symmetric* if there exists a plane T through the origin such that P has reflective symmetry across T. We say P is *perfectly symmetrizable* if there exists any matrix A such that AP is symmetric. We will use a key idea which follows from the work of [8]:

Fact 2 *If P is perfectly symmetrizable, then the isotropic set $\tilde{P} = M^{-1/2}P$ is symmetric.*

Let us first consider the set of transformations that preserve symmetry across a plane T. Let R be any rotation matrix which takes T into the plane $x = 0$. The symmetry of a set of point-pairs $\tilde{P}$ is preserved by the multiplication $SFR\tilde{P}$ where S is any rotation and F is any matrix of the form

$$F = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix} \quad (2)$$

This gives us a set of transformations $V = SFR$, such that any $V\tilde{P}$ is symmetric: the *symmetry preserving transformations* of $\tilde{P}$. This is a seven dimensional set of transformations; although there are five degrees of freedom in choosing F and three degrees of freedom in choosing S, the fact that rotating one choice of F about the x-axis produces some other choice of F reduces the dimensionality to seven.

3 Cross-covariance

We define the 3×3 *cross-covariance* matrix C_Q of a set of point pairs Q as:

$$C_Q = \sum_i q_{2i} q_{2i+1}^t + q_{2i+1} q_{2i}^t$$

The cross-covariance matrix of a *symmetric* set of point pairs has the following property.

Lemma 3 *Let Q be a symmetric set of point-pairs. The cross-covariance matrix C_Q has exactly one negative eigenvalue.*

Intuitively, this eigenvalue corresponds to the reflection; the proof is omitted here but will be in the long version. An arbitrary set Q of point-pairs might not have this property, in which case Q would not much resemble a set of symmetric point-pairs. We say that a point set P is *approximately symmetrizable* if $C_{\tilde{P}}$ has exactly one negative eigenvalue, where $\tilde{P} = M^{-1/2}P$ is isotropic.

When the input P is approximately symmetrizable, we can use the cross-covariance matrix $C_{\tilde{P}}$ of $\tilde{P}$ to find an approximate plane of symmetry T. We define T to the be the plane through the origin with normal u, where u is the unique negative eigenvalue of $C_{\tilde{P}}$.

4 Noise and optimality

We now consider the symmetry error across T, as defined by Equation 1. Since T passes through the origin, which remains the center of mass of the deformed input P, the translation parameter t will be zero. Recall that the symmetry-preserving transformations have the form $V = SFR$, where R is a rotation taking T into the plane $x = 0$, F preserves symmetry across that plane, and S is an arbitrary rotation.

Theorem 4 *Let P be an approximately symmetrizable set of point-pairs. Then T is the plane minimizing the symmetry error of Equation 1 for $\tilde{P}$, and VT is the plane minimizing the symmetry error for set $V\tilde{P}$, where $V = SFR$ is a symmetry preserving transformation.*

Proof. We expand Equation 1 giving the reflection error as a function of the linear transformation A and T's unit normal w:

$$\begin{aligned} E(A, w) &= \sum_{i=1}^{n} ||A(p_{2i}) - \mathrm{Refl}_w(A(p_{2i+1}))||^2 \\ &= \sum_{i=1}^{n} ||V(\tilde{p}_{2i}) - \mathrm{Refl}_w(V(\tilde{p}_{2i+1}))||^2 \\ &= \sum_{i=1}^{n} ||V(\tilde{p}_{2i} - \tilde{p}_{2i+1}) + 2\langle V(\tilde{p}_{2i+1}), w\rangle w||^2 \\ &= \sum_{i=1}^{n} \|V(\tilde{p}_{2i} - \tilde{p}_{2i+1})\|^2 + 4\langle V(\tilde{p}_{2i+1}), w\rangle^2 \\ &\qquad + 4\langle V(\tilde{p}_{2i} - \tilde{p}_{2i+1}), w\rangle\langle V(\tilde{p}_{2i+1}), w\rangle \\ &= \sum_{i=1}^{n} \|V(\tilde{p}_{2i} - \tilde{p}_{2i+1})\|^2 + 2w^t V C_{\tilde{P}} V^t w. \end{aligned}$$

Thus, the plane minimizing the symmetry error is the plane whose normal is the eigenvector of $VC_{\tilde{P}}V^t$ with smallest eigenvalue. When $V = I$, this is the unique negative eigenvector u_1 of $C_{\tilde{P}}$, the normal of T. The transformation $VC_{\tilde{P}}V^t$ cannot introduce additional negative eigenvalues, so that $VC_{\tilde{P}}V^t$ will also have a unique negative eigenvalue. This eigenvector with be the unit vector parallel to Vu_1: $VC_{\tilde{P}}(V^tVu_1)$ has to be parallel to Vu_1 since V^tVu_1 is parallel to u_1, and u_1 in turn is an eigenvector of $C_{\tilde{P}}$. Thus VT is the plane of reflection minimizing the error for $V\tilde{P}$. □

To summarize, we find the isotropic $\tilde{P} = M^{-1/2}P$, check that it is approximately symmetrizable, and take the plane through the origin with normal u as the symmetry plane T. The linear transformations of P that are approximately symmetric form the set $V\tilde{P}$.

5 Single-axis stretches

Since we know the plane of symmetry T, we can employ it to define the set of single-axis stretches rather than all linear transformations, and to find the single-axis stretch that minimizes the deformation of P.

Single-axis stretches have the special form:

$$A = (\alpha - 1)vv^t + I$$

where v is the unit vector in the direction of stretching, and the stretching factor is α, which we define to be greater than one.

The entire set of single-axis stretches is three-dimensional, but not all single-axis stretches are symmetrizing transformations. When A is also a symmetrizing transformation, of the form $VM^{-1/2}$, the negative eigenvector u_1 of $\tilde{P} = M^{-1/2}P$ is perpendicular to the plane of symmetry T spanned by the other two eigenvectors. This introduces two additional constraints, so that the dimension of the set of symmetrizing single-axis stretches is only one.

To specify these two constraints, let $(w1, w2, w3) = (M^{-1/2})^{-1}(u_1, u_2, u_3)$, where u_1, u_2, u_3 are all the eigenvectors of $C_{\tilde{P}}$. Single-axis stretches are symmetric matrices, so the condition that Aw_1 should be perpendicular to Aw_2 is $w_1^t A^t A w_2 = w_1 A^2 w_2 = 0$, and similarly for w_3. Since we can write $A^2 = (\alpha^2 - 1)vv^t + I$, the constraints on v and α are:

$$\begin{aligned} w_1((\alpha^2-1)vv^t + I)w_2 &= 0 \\ w_1((\alpha^2-1)vv^t + I)w_2 &= 0 \\ v^t v = 1 & \end{aligned}$$

The set of solutions for α, v determine the symmetrizing single-axis stretches.

Lemma 5 *Let $n = w_2 \times w_3$, the normal of the plane spanned by w_2 and w_3. The vector v in the symmetrizing single-axis stretch with minimal stretching factor α lies in the plane spanned by n and w_1.*

This proof can be found in the long version.

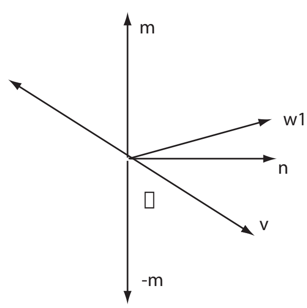

Figure 3: The vector v along which stretching occurs lies in the plane spanned by n, the normal to the plane spanned by w_2, w_3, and w_1. The optimal choice for v is the vector half-way between w_1 and $-m$.

The single-axis stretch which minimizes the deformation is the one which minimizes the stretching factor α. Since, by Lemma 5, the solution will be found in the plane spanned by $n = w_2 \times w_3$, we can apply the two-dimensional procedure of [13], pictured in Figure 3. Indeed, we show in the long version of this paper that solving for the minimal α produces exactly the solution they describe. Let m be the unit vector in the direction of w_1 perpendicular to n, and let $\beta = \angle w_1, -m$. Then we take:

$$\begin{aligned} v &= (w_1 - m)/2 \\ \alpha^2 &= \tan\beta \end{aligned}$$

With this, we have a closed-form expression for both the direction and magnitude of the minimally-distorting single-axis stretch returning the object to its (approximately) symmetric form.

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