Angular rigidity in 3D: combinatorial characterizations and algorithms

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Abstract

Constraint-based CAD software, used by engineers to design sophisticated mechanical systems, relies on a wide range of geometrical constraints. In this paper we focus on one special case: angular constraints in 3D. We give a complete combinatorial characterization for generic minimal rigidity in two new models: *line-plane-and-angle* and *body-and-angle* structures. As an immediate consequence, we obtain efficient algorithms for analyzing angular rigidity.

1 Introduction

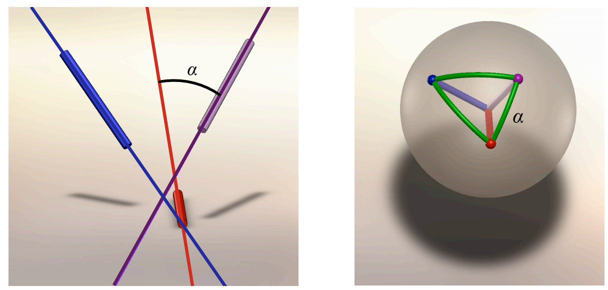
Computer aided design (CAD) software, such as the popular SolidWorks application, provide sophisticated environments for engineers to design complicated systems by using intuitive geometric constraints. They also offer a rich source of interesting open questions in computational geometry. Although an active area of research for over 10 years [11], their study has proven to be very challenging.

Recent work by the authors [2] introduced *body-and-cad* systems, which model 21 **c**oincidence, **a**ngle and **d**istance constraints commonly encountered in CAD systems. These constraints naturally fall into two categories: *angular* and so-called *"blind"* constraints, whose complete theoretical understanding has only just begun. In this paper, we continue their study by fully settling the case for angular constraints. This research lies in the intersection of two areas for which very few results are known: angular constraints and rigidity in dimension 3.

Contributions. We introduce two models of angular constraint structures in dimension 3 and define the natural corresponding concept for angular rigidity. We then proceed to fully characterize generic angular rigidity. To the best of our knowledge, these models and concepts have not been studied before.

Angular constraint structures. A *line-plane-and-angle* structure is composed of lines and planes with pairwise angular constraints between them. A *body-and-angle* structure is composed of rigid bodies with lines and planes rigidly affixed to them; angular constraints are placed between identified lines or planes on a pair of bodies. See Figures 1a and 2a for examples.

We restrict the angular constraints to lie in the range $[0, \pi]$ and remark that this restriction does not limit our model, as an angle α larger than π may be associated to the "small" angle $2\pi - \alpha$. For lack of space, we present here only the $(0, \pi)$ case.



(a) Angular-rigid line-plane-and-angle structure composed of 3 lines with 3 pairwise angle constraints. We highlight one angle constraint with value α.

(b) A natural reduction takes the direction vector of a line to a point on the sphere, and an angle between two lines to a rigid spherical bar. The associated spherical bar-and-joint structure is rigid.

Figure 1: Line-plane-and-angle rigidity.

Angular rigidity. This concept should not be confused with classical rigidity, defined up to trivial rigid body motions. Instead, we take the following point of view. A (classical) *minimally* rigid structure does not allow the addition of any additional independent constraint. In this case, minimality implies rigidity.

In our case, we retain the minimality condition, but do not require rigidity up to trivial 3D rigid body motions. A set of constraints is *independent* if none of them is implied by the others. Our goal is to understand which subsets of angular constraints are independent. An angular structure is *minimally angular-rigid* if the constraints are independent and would imply any additional angular constraints. An angular-rigid structure may still move, but is fully constrained from the angular perspective.

We will give complete combinatorial characterizations for generic angular rigidity by reductions to spherical

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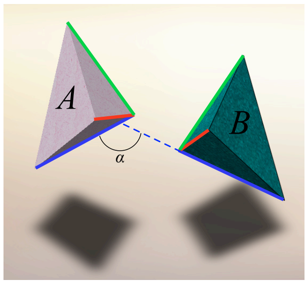
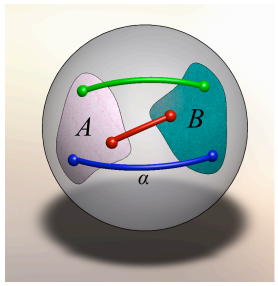
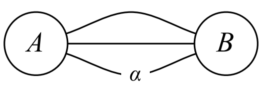
rigidity. This immediately leads to efficient algorithms. See Figures 1b and 2b.



(a) Angular-rigid body-and-angle structure composed of 2 rigid bodies with 3 pairwise angle constraints. Angular constraints are between colored pairs of green, red and blue lines. We highlight the blue angle constraint with value α.



(b) A natural reduction associates each body with a spherical body. Angular constraints are mapped to fixed-length arcs, resulting in a rigid spherical body-and-bar structure.



(c) Associated multigraph is $(3,3)$-tight.

Figure 2: Body-and-angle rigidity.

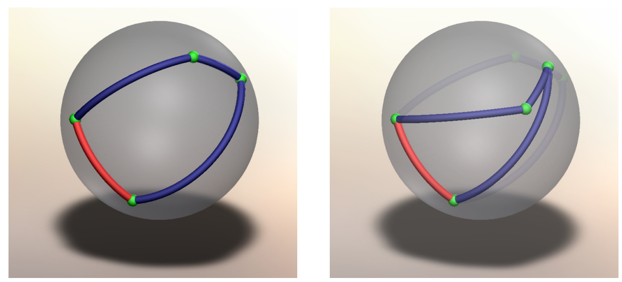
Related work. Full combinatorial characterization results in rigidity theory are rare, with planar bar-and-joint [5] and d-dimensional body-and-bar [13] being essentially the only fully understood models. Bar-and-joint rigidity in dimension 3 remains a conspicuously open problem. While angular constraints have received some attention, the few known results are restricted to the planar case. Zhou and Sitharam [15] characterize a large class of 2D angle constraint systems along with a set of combinatorial construction rules that maintain generic independence. Saliola and Whiteley [8] prove that the complexity of determining the independence of a set of circle intersection angles in the plane is the same as that of generic 3D bar-and-joint rigidity. Related *direction* constraints (where 2 points are required to define a fixed direction, with respect to a global coordinate system) are well-understood and arise from parallel redrawing applications [14]. Servatius and Whiteley present a combinatorial characterization for 2D systems with both length and direction constraints [12]. Our recent work [2] on *body-and-cad* systems in 3D identifies 21 types of constraints; characterizations for most of these remains an open problem.

2 Line-plane-and-angle rigidity theory

We begin with the characterization of line-plane-and-angle rigidity.

By applying a natural reduction, which takes 3D line-plane-and-angle structures to *spherical bar-and-joint structures*, we reduce the new concept of angular rigidity to the classical concept of bar-and-joint rigidity on the sphere.

A *spherical bar-and-joint structure* is composed of fixed-length "bars" or arcs on the unit sphere connected by universal joints. If the bars permit only trivial motions (i.e., rotations of the sphere), then the structure is said to be *rigid*; otherwise, it is *flexible*. This is the classical concept of rigidity, defined up to rigid motions on the sphere. See Figure 3 for an example of a flexible spherical bar-and-joint structure.



(a) (b)

Figure 3: A flexible spherical bar-and-joint structure.

Only direction vectors of the lines and normal vectors to the planes are relevant for line-plane-and-angle rigidity. We normalize the vectors and represent them as points on the unit sphere. Angle constraints are then mapped to fixed-length spherical bars, whose lengths are defined by the dictated angle. In this way, rigidity of line-plane-and-angle structures reduces to spherical bar-and-joint rigidity. See Figure 1a for an example line-plane-and-angle structure; Figure 1b shows the spherical bar-and-joint structure obtained by the reduction.

2.1 Combinatorial characterization and algorithms

The main theorem below characterizes line-plane-and-angle generic rigidity. See [1] for definitions of standard rigidity theory terms, including *Laman graph*.

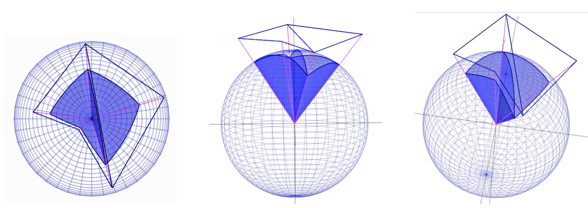
Theorem 1 *A 3D line-plane-and-angle structure with angle constraints in* $(0,\pi)$ *is generically minimally angular-rigid if and only if* $G=(V,E)$ *is a* Laman graph*, where* V *associates a vertex to every line or plane and* E *associates an edge to every angle constraint.*

Algorithms. As an immediate consequence, the original 2D pebble game of Jacobs and Hendrickson [4] decides 3D line-plane-and-angle rigidity and computes angular-rigid components in $O(n^2)$ time.

2.1.1 Proof sketch.

The proof relies on the reduction from 3D line-plane-and-angle structures to *spherical bar-and-joint structures*. In [9], Saliola and Whiteley show the equivalence of generic rigidity for bar-and-joint structures on the sphere with generic rigidity for planar bar-and-joint structures; see also [3]. Figure 4 depicts the correspondence between spherical and planar bar-and-joint structures.

Theorem 2 (Saliola and Whiteley [9]) *Every infinitesimal motion of a spherical bar-and-joint structure maps to an equivalent infinitesimal motion on the corresponding planar bar-and-joint structure. Moreover, every* **trivial** *infinitesimal motion maps to an equivalent* **trivial** *infinitesimal motion on the corresponding planar bar-and-joint structure.*



(a) (b) (c)

Figure 4: Correspondence between spherical and planar bar-and-joint structures.

From Theorem 2 and Laman's classical planar bar-and-joint rigidity result [5], elementary considerations lead to the desired conclusion that generic minimal rigidity of spherical bar-and-joint structures is characterized by the $(2, 3)$-sparsity Laman counts.

3 Body-and-angle rigidity theory

We turn now to the development of the theory for body-and-angle rigidity. As with the previous model, we apply a natural reduction from 3D body-and-angle structures to *spherical body-and-bar structures.*

A *spherical body-and-bar structure* is composed of a set of bodies on the sphere constrained by fixed-length "bars" or arcs. The bars are attached to the bodies by universal joints. The structure is called *rigid* if the only allowable motions are the trivial rotations of the sphere; otherwise, it is *flexible.*

The reduction to spherical body-and-bar rigidity is analogous to the construction from Section 2. Given a 3D body-and-angle structure A, we associate a spherical body-and-bar structure B in the following way. For each 3D body in A, there will be a spherical body in B. A body a with lines and planes rigidly attached to it is associated to a spherical body b with points on the sphere rigidly attached to it; these points correspond to directions of lines or normals to planes. Angle constraints are then mapped to fixed-length arcs or "bars," whose lengths are defined by the dictated angle. Figure 2a depicts a minimally rigid body-and-angle structure; it reduces to the spherical body-and-bar structure shown in Figure 2b. Again, we emphasize that this reduction maps the new concept of angular rigidity to classical rigidity of body-and-bar structures on the sphere.

3.1 Combinatorial characterization and algorithms

The main result characterizing body-and-angle rigidity can now be stated.

Theorem 3 *A 3D body-and-angle structure with angle constraints in* $(0, \pi)$ *is generically minimally angular-rigid if and only if* $G = (V, E)$ *is* $(3, 3)$*-tight, where* V *associates a vertex to every body and* E *associates an edge to every angle constraint.*

Algorithms. As an immediate consequence, the $(3, 3)$-pebble game algorithm developed in [7] decides body-and-angle rigidity and computes angular rigid components in $O(n^2)$ time. See [7] for *sparsity* concepts (including *tight* graphs) and pebble game algorithms.

3.1.1 Proof sketch.

The reduction from 3D body-and-angle structures to spherical body-and-bar structure is the key to the proof.

Spherical body-and-bar rigidity. We are not aware of any publications on spherical body-and-bar rigidity. For completeness, we sketch the full development of the theory, which follows the pattern used in Tay [13] for body-and-bar structures and relies on Lie groups, Grassmann-Cayley algebra, and screw theory. See [10] for standard notation and background of these concepts. Further details can be found in the first author's Ph.D. dissertation [6].

Rigidity theory roadmap. To develop the rigidity theory for a new model, three steps must be accomplished. (1) *Algebraic theory:* Formulate the rigidity concept in algebraic terms, resulting in an algebraic variety. (2) *Infinitesimal theory:* Analyze the local behavior at some point on the algebraic variety. (3) *Combinatorial rigidity:* Whenever possible, find a combinatorial characterization of minimal rigidity in terms of properties of an underlying graph structure.

Algebraic theory. The algebraic theory for spherical body-and-bar structures is expressed in terms of rotational transformation matrices from $SO(3)$ assigned to each body. Angular constraints are captured by corresponding equations, and the **realization problem** asks for points on the resulting variety. Intuitively, a structure is *rigid* if the only motions allowed are the trivial rotations of the sphere. This is the classical concept of rigidity, up to rigid motions.

Infinitesimal theory. By linearizing the system of equations from the algebraic theory, we obtain the infinitesimal theory. An *infinitesimal motion* of a spherical body-and-bar structure is an assignment of elements from the Lie algebra $so(3)$ such that the angular constraints are infinitesimally maintained. If the only allowable motions are the *trivial* rotations on the sphere, the structure is *infinitesimally rigid*.

Equivalence of infinitesimal rigidity of spherical body-and-bar and Euclidean body-and-bar structures. It is straightforward to extend of the results for the equivalence of infinitesimal spherical bar-and-joint and planar bar-and-joint rigidity to the body-and-bar case. The following corollary to Theorem 2 from [9] is the key ingredient.

Corollary 4 *Every infinitesimal motion of a spherical body-and-bar structure has an equivalent infinitesimal motion on the corresponding planar body-and-bar structure. Moreover, every* **trivial** *infinitesimal motion maps to an equivalent* **trivial** *infinitesimal motion on the corresponding planar body-and-bar structure.*

Combining Corollary 4 and the well-known result of Tay [13], we obtain a characterization for generic spherical body-and-bar rigidity.

Theorem 5 *Generic minimal spherical body-and-bar rigidity is characterized by* $(3, 3)$*-sparsity.*

4 Conclusions

Motivated by CAD software, we have introduced two models for angular rigidity in 3D. For both models, we have given combinatorial characterizations that immediately lead to efficient algorithms. Such algorithms have a very practical application to CAD systems by providing a new tool for giving useful feedback to users designing sophisticated systems.

It is straightforward to extend the reductions to take d-dimensional line-hyperplane-and-angle or body-and-angle structures to their spherical analogs. However, since the combinatorial characterization of 3D line-plane-and-angle rigidity relied on Laman's Theorem [5] for planar bar-and-joint structures, analogous characterizations and algorithms remain an open problem. For generic minimal body-and-angle rigidity in dimension d, the results of Tay [13] imply the combinatorial characterization of $(\binom{d}{2}, \binom{d}{2})$-sparsity. Therefore, the associated pebble games of [7] provide efficient algorithms.

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