An Optimal Solution for Dynamic Polar Diagram

Ebrahim Ehsanfar * Bahram Sadeghi Bigham † Najmeh Madadi‡

Abstract

The Polar Diagram [1] of a set of points (i.e. sites) is a partition of the plane. It is a locus approach for problems processing angles. Also, Dynamic Polar Diagram problem is a problem in which some points can be added to or removed from the point set of the Polar Diagram. Sadeghi et al. [4] introduced this problem and solved it using an algorithm which is optimal in the case that some points are deleted from the set. But this algorithm is not optimal when some new points are inserted into the diagram.

In this paper, we present an algorithm to solve the Dynamic Polar Diagram in optimal time in which we insert some new points into the diagram one by one. Our approach applies only on the regions that would be changed and solves the problem for each insertion in $O(k + \log n)$ time, in which $1 \leq k \leq n$ is the number of the sites which their regions would be changed.

1 Introduction

C. I. Grima et al. [1] introduced the concept of the Polar Diagram. This diagram is a plane partition with similar features to those of the Voroni diagram. In fact the Polar Diagram can be seen in the context of generalized Voronoi diagram. The polar angle of a point p with respect to a point s_i, denoted by $ang_{s_i}(p)$, is the angle formed by the positive horizontal line of p and the straight line linking p and s_i. Given a set S of n points in the plane, the locus of points with smaller positive polar angle with respect to $s_i \in S$ is called polar region of s_i and denoted by P_{s_i} [1, 2]. Thus,

$$P_{S_i} = \{(x, y) \in E^2 | ang_{S_i}(x, y) < ang_{S_j}(x, y); \forall j \neq i\}. \quad (1)$$

Each polar region, as observed in the Figure 1, is constructed using two different half-lines or polar edges. There is always a horizontal edge starting at each s_i, $\{(x, y), x < x_i\}$, and another oblique polar edges with the gradient of the straight line crossing s_i and s_k, with $s_i \in P_{s_k}$ that is referred as the *model line.* in this case we call s_k the *parent* of s_i denoted by $s_k = Par(s_i)$ or $s_i \prec s_k$.

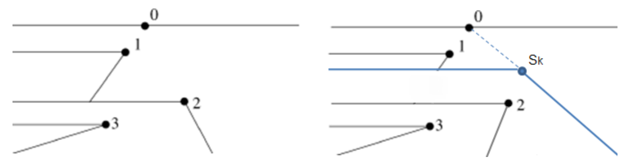

Figure 1: To the left, the Polar Diagram of a set of points in plane. The polar region of each site is constructed by horizontal and model lines. To the Right, the Polar Diagram after insertion of a new site s_k.

The plane is divided into different regions in such a way that if point $(x, y) \in E^2$ lies in P_{s_i}, it is known that s_i is the first site found performing an angular scanning starting from (x, y). The boundary is the horizontal line crossing the top most site of S. Figure 1 depicts the Polar Diagram of an exemplary set of points on the plane. Also for n points on the plane, it is not possible to compute the Polar Diagram in less than $\Theta(nlogn)$ time [1, 3].

There are at least two approaches for the Polar Diagram construction, the Incremental and the Divide and Conquer methods [1], both working in optimal time.

For computing the Polar Diagram using the Incremental method, suppose S is sorted from top to down obtaining the sequence $s_0, s_1, ..., s_{i-1}$. The polar region of s_i is computed after $P_{s_0}, P_{s_1}, ..., P_{s_{i-1}}$ are processed. Suppose s_i is in the region of s_j. The model line of P_{s_i} crosses s_i and s_j, and by drawing the horizontal half-line of s_i, the boundary of P_{s_i} will be introduced [1, 5, 6].

Section 2 consists of general information about Dynamic Polar Diagram. In Section 3, we present some new definitions which are needed through the rest of the papre. In Section 4, we introduce our algorithm. Section 5 analyzes the algorithm and Section 6 present a conclusion for this paper.

2 Dynamic Polar Diagram

Sadeghi et al. [4] introduced a problem called Dynamic Polar Diagram problem. In this problem, some points can be added to, or removed from the point set. So-

*Department of Mathematics and Computer Science, Institute for Advanced Studies in Basic science (IASBS), Zanjan, Iran ehsanfar@iasbs.ac.ir

†Department of Information Technology, Institute for Advanced Studies in Basic science (IASBS), Zanjan, Iran b_sadeghi_b@iasbs.ac.ir

‡Maharan eng. Co., No 21, West Arash blvd., F. Afshar st., Dastjerdi Ave. Tehran, Iran madadi@maharan.com

lution to this problem is to redraw the Polar Diagram in the minimum time. Sadeghi et al. propose to determine an area out of which the Polar Diagram does not change due to the insertion or deletion on the new sites. Although, solving the deletion process was optimum, there are several examples that show this algorithm would not solve the problem in optimum time when some new points are inserted into the point set. Solving the problem in case of insertion has remained an open problem [4].

We introduce an algorithm to find the actual changes in diagram when some new sites are inserted into the point set. We apply the algorithm to each site one by one, the same way as Sadeghi et al. did, so we can say our algorithm is an online solution to the Dynamic Polar Diagram problem.

3 Definitions and Preliminaries

Definition 1 *Complement Area for each polar region P_{s_i} is denoted by $Com(P_{s_i})$ and constructed using two half-lines or edges, one is the horizontal half-line $\{(x, y_i), x > x_i\}$ and the other is a half-line which links s_i to $Par(s_i)$ and extends up to the infinity (Figure 2)*

In fact, the Complement Area for each polar region is built by continuing the region edges. For each polar region P_{s_i}, $Com(P_{s_i})$ is bounded between two lines that we know their equations, in other words:

$$Com(P_{s_i}) = \{(x, y)|x > x_i,$$
$$y_i < y < \frac{y_{Par(s_i)} - y_i}{x_{Par(s_i)} - x_i}(x - x_i) + y_i\} \qquad (2)$$

Definition 2 *Parallel Complement Area for each polar region P_{s_i} is a sub area of $Com(P_{s_i})$ which is above $P_{Par(s_i)}$ and is denoted by $PCom(P_{s_i})$.*

In other words,

$$PCom(P_{s_i}) = \{(x, y)|x > x_{Par(s_i)},$$
$$y_{Par(s_i)} < y < \frac{y_{Par(s_i)} - y_i}{x_{Par(s_i)} - x_i}(x - x_i) + y_i\} \qquad (3)$$

Lemma 1 *The parallel complement area of each polar region P_{s_i} is a subset of complement area of P_{s_j} which $s_j = Par(s_i)$, i.e. $PCom(P_{s_i}) \subset Com(P_{s_j})$.*

Proof. The proof is rather obvious. According to general position condition, s_i would not be located on the model line of P_{s_j}, so for each site s_i, s_j which $s_i \prec s_j$, the model line of region P_{s_i} forms a non-zero angle θ with the model line of region P_{s_j} (Figure 2) and the region $PCom(P_{s_i})$ is always included in $Com(P_{s_j})$. □

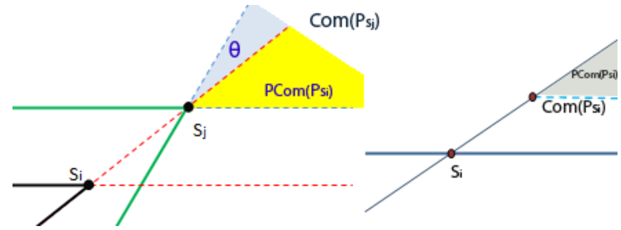

Figure 2: The complement area and the parallel complement area of polar region P_{s_i} are shown on the right. The parallel complement area of each polar region P_{s_i} is a subset of complement area of $P_{Par(i)}$. which is shown on the left.

We are going to find the regions which are be changed by inserting a new site s_k into the diagram. The position of the sites are fixed, so the horizontal edge of regions will remain unchanged. Also, changing of the model line of P_{s_i} leads to changing the shape of P_{s_i}.

The inserted point would effect the polar region P_{s_i} whether by changing its model line (changing the shape of the region), or by falling inside a region or intersecting a region (changing the boundaries of the region). When the regions boundaries are changed, the incremental method would draw the new regions without any change in the algorithm procedure. Suppose s_j is the first lower site from the site s_k which is located on the right site of its horizontal line specifies the boundaries of P_{s_k}. By redrawing $P_{s_k}, ..., P_{s_j}$ according to the incremental method [1], diagram update procedure would be complete. But the changes in the shape of regions should be studied in another way that will be discussed next.

We introduce a lemma that holds a necessary and sufficient condition for changing the shape of a polar region, when a new site is inserted into the diagram.

Lemma 2 *For any new site s_k in Polar Diagram, the shape of polar region P_{s_i} will be changed if and only if $s_k \in Com(P_{s_i})$.*

Proof. Using equation 2, if $s_k \in Com(P_{s_i})$ we have $y_{s_i} < y_{s_k}$. Considering $s_j = Par(s_i)$, there are two conditions, one in which $y_{s_i} < y_{s_k} \leq y_{s_j}$ and the other $y_{s_j} \leq y_{s_k}$, so:

1. In the case where $y_{s_i} < y_{s_k} \leq y_{s_j}$, new site s_k is located between horizontal lines crossing s_i and s_j. Whether the position of s_k is in p_{s_j} or not, s_k is either $s_k \in Com(P_{s_i}) \bigcap P_{s_j}$ or $s_k \notin Com(P_{s_i}) \bigcap P_{s_j}$, so:
   (a) If $s_k \notin Com(P_{s_i}) \bigcap P_{s_j}$, then the horizontal line of P_{s_k} will cut the entire the polar region P_{s_j}, and s_i is not a *child* of s_j any longer, and there is no other sites between s_i and s_k so $s_i \prec s_k$.

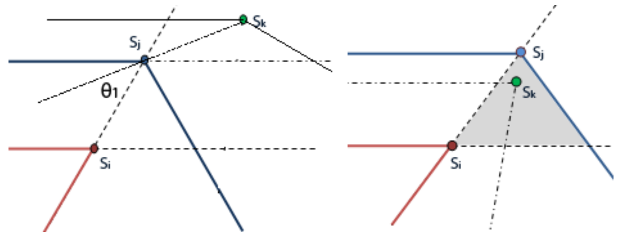

Figure 3: The position of a new site s_k inside the complement area of P_{s_i}.

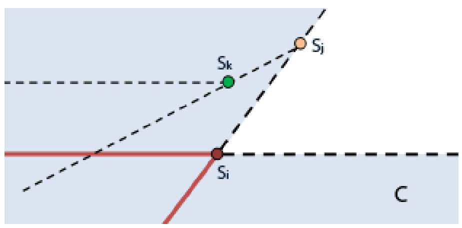

Figure 4: The position of a new site s_k when it is not located in the complement area of P_{s_i}.

(b) If $y_{s_i} < y_{s_k} \leq y_{s_j}$ and $s_k \in P_{s_j}$, then s_k is located in a triangular cell (Figure 3) and the model line of P_{s_k} would be drawn to link s_j and s_k which intersects the base of triangular cell. Then, the polar region of s_k will include s_i and the model line of P_{s_k} would be changed.

2. In the second case, $y_{s_j} \leq y_{s_k}$, suppose sequence $s = s_i, , s_{k-1}, s_k$ is a set of sites that includes s_i and s_k and is between them. Again let $s_j = Par(s_i)$; we show that s_k would be the new parent of s_i. From Lemma 1, $s_k \in PCom(P_{s_i}) \subset Com(P_{s_j})$. Without loss of generality consider s_k to be located between s_i and s_j. From case 1, s_k is the new parent of s_j, then the new model line of P_{s_j} changes in a way that the polar region P_{s_j}, no longer includes s_i (Figure 3). From the general position condition, $\theta_1 > 0$ (s_k and s_j are not on a direct line), this causes P_{s_j} not to include s_i any longer. However $\theta_2 > 0$ as well, where θ_2 is the angle made by the model lines of P_{s_j} and P_{s_k} , and this makes that new polar region P_{s_k} to cover all of the missing points of old P_{s_j}, so s_i will be in P_{s_k} andP_{s_i} would be changed. We claim that our assumption does notinterfere with the generality of the problem, because if s_j bethe indirect parent of s_i, $s_i \prec s_{i+1} \prec ... \prec s_j$, then with the same method and making two angles $\theta_1, \theta_2 > 0$, for each pair of these sites, eventually, we conclude s_k is the new parent of s_i.

On the other hand, we are going to prove if the shape of a polar region P_{s_i} would change by inserting a new site s_k, s_k is located in $Com(P_{s_i})$. Suppose this is not true (proof by contradiction) and s_k is the new parent of s_i which is not located in $Com(P_{s_i})$. In this case, s_k must be located in $C = Plane - Com(P_{s_i})$ which is shown in Figure 4. If s_k is below s_i, there is a contradiction, because it can not be a parent of s_i and below s_i. Else if, s_k is located above s_i, it would be either in a polar region of s_j which $s_j = Par(s_i)$, or in P_{s_j} which s_j is not located in $Com(s_i)$ (if s_j be located in $Com(s_i)$ then from case 1, it should be the parent of s_i). On one hand s_k is located in polar region of parent s_i and it is obvious that P_{s_k} will not include s_i (Figure 4), on the other hand, if s_k is located in the other region, It would not cut the region of s_j and change the model line of P_{s_i}, so there is a contradiction. □

Lemma 2 is a necessity for finding the changing regions when a site is inserted into the diagram. We introduce an algorithm which explore the diagram using some preprocesses. We need a tree that corresponds to the diagram. Each node v_i of the tree corresponds to a site s_i. If s_j be the parent of s_i, then there is an edge $v_i v_j$ in the tree that could model the parent to child relations. This tree could be a Horizon Tree [7] of the sites which could represent the parent-child relation of sites. With the help of searching algorithms, this tree would be useful to seek the changing regions.

Definition 3 *For each site s_i, Symmetry Indexing Region of s_i, denoted by SI_i, is a sub area of $Com(P_{s_i})$ which is not in $PCom(P_{s_i})$,*
i.e. $\forall i,\ SI_i = Com(P_{s_i}) - PCom(P_{s_i})$ (Figure 5).

Definition 4 *We name the area which is not in any SI_i, the Null region of the diagram (Figure 5).*

We use these definitions to produce a new diagram which is used in our algorithm.

Definition 5 *Complemental diagram for Polar Diagram, denoted by CPD, is a partition of the plane which is constructed by two regions, The Null region and the SI region, $SI = \bigcup_i SI_i$ (Figure 5).*

4 Algorithm

Our algorithm functions by finding the starting point for exploring the diagram. When the new site s_k is inserted into the Polar Diagram, it would be located in one of the regions of the CPD, either in Null or in SI regions. It is obvious that, if s_k is located inside the Null region, due to lemma 2, there would not be any changes in the shape of the regions (because Null region has no conjunction with the $Com(s_i)$ for all sites s_i). But if $s_k \in SI_i$, from Lemma 2, P_{s_i} is the first region where the diagram would be changed. The algorithm finds

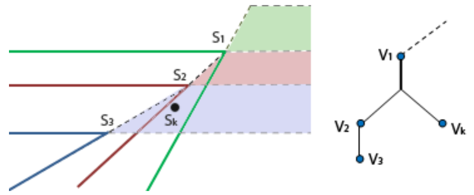

Figure 5: To the left, the Polar Diagram with 3 sites which the CPD is drawn on it, the colored regions is SI and the rest is the Null region. To the Right, Corresponding tree with new site s_k.

other changing regions by exploring the diagram using the already mentioned corresponding tree.

From computational perspective, it is easy to find a SI_i where s_k is located in, using a point location algorithm [8] having the time complexity of $O(\log n)$.

Afterward, if $s_k \in SI_i$ we will continue the algorithm. We will find node v_i in the tree that corresponds to site s_i in the diagram and start exploration from its siblings, left to right. During exploration, every site s_j will be checked by Lemma 2 condition if its region has been changed or not. If a region P_{s_j} is not changed ($s_k \notin Com(P_{s_j})$), for all of its children s_i, the region P_{s_i} would not also change (from lemma 1, $s_k \notin PCom(P_{s_i}) \subset Com(P_{s_j})$). The algorithm ignore them and continue the exploration for its siblings. Similarly, if a site s_j is not changed, all of its right hand's siblings s_p will not also change. Because with the same parent, they have bigger complement areas which cover the complement area of the P_{s_j}, the algorithm continues to find all of the changing regions.

Algorithm 1. inserting a new point to the Polar Diagram

`input:` n sites on the plane, Polar Diagram, corresponding tree, CPD diagram and a new point site s_k.
`output:` Polar Diagram of n+1 points
`step 1:` Find SI_i from the CPD where s_k is located in.
`step 2:` $p \leftarrow s_k$, $S' = \emptyset$
`step 3:` Explore the tree from siblings of p, from left to right
  `3_1:` If region P_{s_i} is changed, $S' \leftarrow s_i$ and $p \leftarrow s_j$ which s_j is one of the children of s_i, goto 3
  `3_2:` If P_{s_i} is not changed and if s_i is in the right side of the last changing region goto 4
`step 4:` Update the diagram redrawing P_{s_i}, for all $s_i \in S'$

5 Analysis

Our algorithm does not need to save a lot of information and reach the required data by exploring the diagram step by step. This algorithm solves the problem for each insertion in $O(k + \log n)$ time in which $1 \leq k \leq n$ is the number of the sites which their regions would be changed and $\log n$ indicates the time complexity of the point location procedure. Clearly this is optimal in all cases. but it can perform better in special cases. The more lower the location of the new site, the less exploration in the diagram is needed, and the algorithm performs more better, since k is much closer to 1. Similarly the more the location of the new site s_k is to the left of the diagram, the more efficient this algorithm would become, because the prospects of s_k located in the Null region of CPD are more and this algorithm would reach the answer sooner.

6 Conclusion

The Polar Diagram has been recently introduced by Grima et al. [1]. The dynamic version of this problem also has been discussed by Sadeghi et al. [4]. They introduced this problem and solved it using an algorithm which is optimal in the case that some points delete from the points set. But this algorithm is not optimal when some new points insert into the diagram.

In this paper, we presented an algorithm, solve the insertion case of Dynamic Polar Diagram in optimal time.

7 Acknowledgment

The authors would like to thank Hesam Yousefipour and Tayyebeh Panahi for their contributions throughout the writing of this paper.

References

[1] Clara I. Grima, Alberto Mrquez, Lidia M. Ortega. *A new 2D tessellation for angle problems: The polar diagram.* Computational Geometry. 34(2), 58-74, 2006.

[2] Antonio J. Rueda Ruiz, Lidia M. Ortega. *Geometric Algorithms on CUDA.* GRAPP, 107-112, 2008.

[3] Lidia M. Ortega, Francisco R. Feito. *Collision detection using polar diagrams.* Computers & Graphics 29(5), 726-737, 2005.

[4] Bahram Sadeghi Bigham, Ali Mohades, Lidia M. Ortega. *Dynamic polar diagram.* Inf. Process. Lett. 109(2), 142-146, 2008.

[5] Bahram Sadeghi Bigham, Ali Mohades. *Polar diagram with respect to a near pole.* 23rd European Workshop on Computational Geometry EWCG07, Austria, 206-209, 2007.

[6] Bahram Sadeghi Bigham, Ali Mohades. *The Dual of Polar diagrams and its extraction.* International Conference of Computational Methods in Sciences and Engineering (ICCMSE) volume 7, Greece, 451-454, 2006.

[7] Herbert Edelsbrunner and Leonidas J. Guibas. *Topologically sweeping an arrangement.* Journal of Computer and System Sciences, Volume 38, 165-194, 1989.

[8] Neil Sarnak, Robert E. Tarjan. *Planar point location using persistent search trees.* Communications of the ACM 29, 669-679, 1986.