Compatible Pointed Pseudo-Triangulations

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Abstract

For a given point set S (in general position), two pointed pseudo-triangulations are compatible if their union is plane. We show that for any set S there exist two maximally disjoint compatible pointed pseudo-triangulations, that is, their union is a triangulation of S. In contrast, we show that there are point sets S and pointed pseudo-triangulations T such that there exists no pointed pseudo-triangulation that is compatible to and different from T.

1 Introduction and Preliminaries

Let S be a set of n (labeled) points in the Euclidean plane in general position, that is, no three points of S lie on a common line. We denote the convex hull of S with CH(S), with h the number of extreme points of S, that is, the points of S that are on the boundary of CH(S), and with $i = n-h$ the number of non-extreme (interior) points of S.

Let $G = (S,E)$, $E \subseteq (S \times S)$, be a geometric (or straight-line) graph whose vertex set is S, and whose edges are straight-line segments spanned by points of S. In the following we will solely consider straight-line graphs and thus simply refer to them as graphs. A graph is called *plane*, if no two of its edges cross (i.e., share a point $p \notin S$).

Two plane straight-line graphs $G = (S,E)$ and $G' = (S,E')$ on top of (the same) point set S are *compatible*, if their union $G \cup G' = (S, E \cup E')$ is plane. Accordingly, we call an edge $e \in (S \times S)$ *compatible (to G)* if $(S, E \cup \{e\})$ is a plane graph. For recent work on compatible graphs see e.g. [1] and references therein. An overview of results with different types of graph compatibility can be found in [2].

A *pseudo-triangle* is a simple polygon which has exactly three *corners* (vertices with interior angle smaller than π). A path along the boundary of a pseudo-triangle that has two of the corners as end points and does not contain the third one is called a *side-chain* of this pseudo-triangle, and the non-incident corner is called *opposite* (to this side-chain). A *pseudo-triangulation* $T = (S,E)$ is a plane straight-line graph on top of S whose edges partition the convex hull CH(S) into pseudo-triangles.

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A vertex $p \in S$ in a pseudo-triangulation T is called *pointed* (towards a face f of T), if p is a reflex vertex in f and thus has an incident angle larger than π without emanating edges. The whole pseudo-triangulation is called pointed, if all its vertices are pointed. Note that for any pseudo-triangulation, all extreme vertices of S are pointed towards the outer face.

Pseudo-triangulations have been first introduced by Pocciola and Vegter [3] in a more general framework and by Streinu [5] in the context of geometric graphs. They are a rather young structure, with interesting properties and applications. See the recent survey [4] and references therein.

Streinu [6] showed that pointed pseudo-triangulations are minimal pseudo-triangulations, and at the same time they are maximal pointed plane straight-line graphs (where minimal and maximal is with resprect to the number of edges). Further, any pointed pseudo-triangulation on top of a point set S with n points has $2n-3$ edges, independent of the number of interior points of S.

Proposition 1 *If two pointed pseudo-triangulations $T = (S,E)$ and $T' = (S,E')$ of a point set S are compatible, then they differ by at most i edges: $|E \backslash E'| = |E' \backslash E| \leq i$.*

Proof. The number of edges in any pointed pseudo-triangulation of a point set S with n points is $2n-3$. As S has i interior and $h = n-i$ extreme points, the number of edges in any maximal plane graph (triangulation) of S is $3n-h-3 = 2n+i-3$. Thus, considering an arbitrary pointed pseudo-triangulation T of S, the number of edges that can be added to obtain a maximal plane graph is $2n+i-3-2n+3 = i$. This implies that *any* plane straight-line graph compatible with T, and thus also any such pointed pseudo-triangulation, can have at most i edges that are not in T. □

We call two pointed pseudo-triangulations $T = (S,E)$ and $T' = (S,E')$ *maximally disjoint compatible*, if their union $T \cup T' = (S, E \cup E')$ is a triangulation of S. In other words, T and T' differ by exactly i edges.

2 Two Compatible Pointed Pseudo-Triangulations

We start with the following question. Given a point set S with n points, can we find two compatible pointed pseudo-triangulations which are maximally disjoint? Recall that maximally disjoint means that the two pointed pseudo-triangulations have exactly $2n-3-i$ edges in common, where i is the number of interior points of S.

Theorem 2 *For every point set S with n points, i of them interior, there exist two pointed pseudo-triangulations $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ such that T_1 and T_2 are maximally disjoint compatible, that is, $|E_1 \backslash E_2| = |E_2 \backslash E_1| = i$.*

Proof. We prove the statement by construction of the two pointed pseudo-triangulations T_1 and T_2 by iteratively adding edges. We color edges of T_1 blue (dashed), those of T_2 red (dotted), and edges that are in both, T_1 and T_2, black (solid), see Figures 2-4. For the sake of brevity we sometimes refer to edges belonging to T_1, T_2, or both, by their color.

Assume that S has a triangular convex hull pqr, and $i > 0$ interior points[1]. We start by adding the boundary of the convex hull to both, T_1 and T_2. Now choose an extreme point $p \in S$, and consider the boundary of the convex hull of $S \backslash \{p\}$, consisting of the edge qr and a concave chain C connecting q and r in the interior of CH(S). The chain C together with p forms a pseudo-triangle with corners pqr that does not contain any point of S in its interior, see Figure 1.

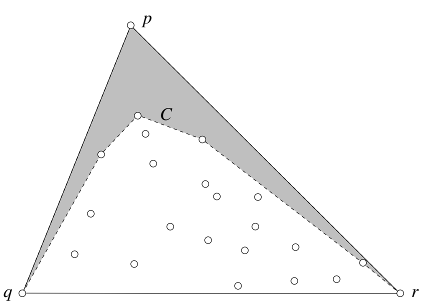

Figure 1: Empty pseudo-triangle (shaded) formed by p and the chain C (dashed).

For our construction, we add black edges from q to all points of C except r (to both T_1 and T_2), red edges from p to all points of C except q and r (to T_2), and all edges on the chain C (except the one incident to q) with color blue (to T_1). See Figure 2 for the set of edges added in this step.

The union of T_1 and T_2 splits CH(S) into a set of triangles. Each triangle that contains further points of S in its interior has q as one corner, two of the triangle

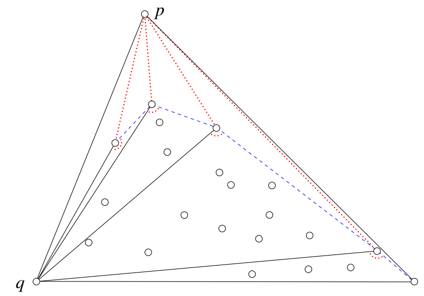

Figure 2: First step of adding edges to T_1 and T_2.

edges are black, and the third is blue. In each such triangle, one of the corners adjacent to the blue edge becomes pointed in the whole graph when removing this blue edge. We mark this corner red (indicated by a small arc in the figures).

We consider these triangles iteratively, in a similar way as the starting triangle. Throughout the process we keep the following invariants for each interior triangle Δ: q is a corner of Δ and both triangle edges incident to q are black. One of the other two corners is marked with color $c'_\Delta \in \{\text{red, blue}\}$ (red after the first step). Let this colored corner be p_Δ and the remaining corner r_Δ. The triangle edge $p_\Delta r_\Delta$ has the color $c_\Delta \in \{\text{red, blue}\} \backslash \{c'_\Delta\}$ (blue after the first step).

Let $S_\Delta \subset S$ be the set of points inside Δ plus the corners of Δ. Like in the first step, we consider the chain C_Δ on the convex hull of $S_\Delta \backslash \{p_\Delta\}$. We add black edges from q to all points of C_Δ except r_Δ, edges with color c_Δ from p_Δ to all points of C_Δ except q and r_Δ, and all edges on the chain C_Δ (except the one incident to q) with color c'_Δ. Inside every nonempty triangle, we mark the corner that becomes pointed in $T_1 \cup T_2$, when removing the c'_Δ-colored triangle edge, with color c_Δ. See Figure 3 for one iterative step (inside the shaded triangle).

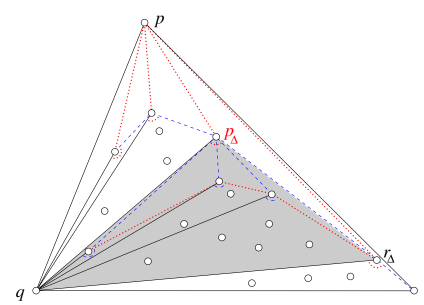

Figure 3: Iterative construction step.

The construction results in a red-blue-black colored triangulation of S, and thus T_1 and T_2 are compatible.

Concerning the pointedness of the interior points of S,

[1]If S has more than three extreme points, we first triangulate these extreme points arbitrarily (adding the edges to both, T_1 and T_2), and then process each resulting non-empty triangle independently.

note that in each construction step, all points on the chain C_Δ (except q and r_Δ) become pointed towards p_Δ with respect to color c'_Δ. This pointedness cannot be destroyed in further recursive steps because there is nothing left to be processed between C_Δ and p_Δ (see again Figure 3). With respect to color c_Δ, each point on C_Δ (except q and r_Δ) is pointed towards one of its adjacent c'_Δ-colored edges on C_Δ. If the according triangle is not empty then the point is marked with c_Δ for the next iteration and thus cannot get any additional incident c_Δ-colored edges in the relevant area. Thus all points of S are pointed in both T_1 and T_2.

Finally, every interior point of S appears on exactly one chain C_Δ as non-endpoint. For every non-endpoint of a chain C_Δ, exactly one red, one blue, and one black edge is added, which, including the initial three black edges on the convex hull of S, adds up to $|E_1| = |E_2| = 2n - 3$. Together with pointedness and planarity, this proves that T_1 and T_2 are pointed pseudo-triangulations. Figure 4 shows the two resulting maximally disjoint compatible pointed pseudo-triangulations. □

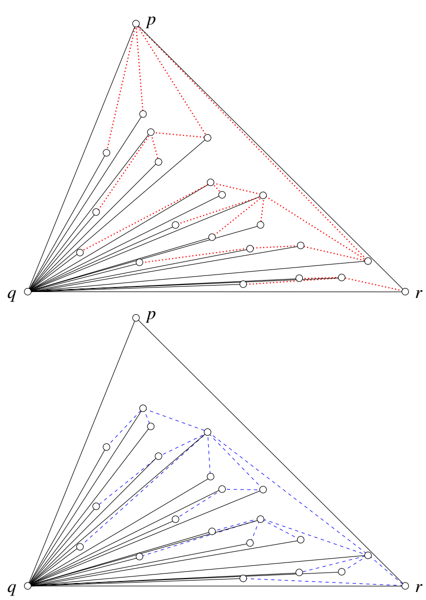

Figure 4: Two maximally disjoint compatible pointed pseudo-triangulations.

3 Compatible Pointed Pseudo-Triangulations for a given Pointed Pseudo-Triangulation

We now consider a more restrictive setting. Given a point set S and a pointed pseudo-triangulation $T = (S, E)$, can we find a pointed pseudo-triangulation $T' = (S, E')$, such that T and T' are compatible and differ by at least k edges for some $k \geq 1$?

Before giving a general answer to this question, let us introduce the concept of flips. A *flip* in a pointed pseudo-triangulation $T = (S, E)$ is the exchange of an edge $e \in E$ by an edge $e' \in (S \times S) \backslash E$ such that the resulting graph $T' = (S, E \backslash \{e\} \cup \{e'\})$ is again a pointed pseudo-triangulation. The flip is called *compatible flip* if e' is compatible with T.

In a pointed pseudo-triangulation, every edge e that is not a convex hull edge is flippable, and there is a unique edge e' to which e can be flipped [4].

Two questions naturally arise. Given a point set S and a pointed pseudo-triangulation on top of S, can we always perform a compatible flip? And if this is true, is the flip graph of pointed pseudo-triangulations (w.r.t. compatible flips) connected, that is, can we flip any pointed pseudo-triangulation to any other by only using compatible flips? Note that this is true in the unrestricted case [4]. Unfortunately, for compatible flips the answer to both questions is negative. There exist point sets (also with interior points) and pointed pseudo-triangulations that do not admit any compatible flip. Thus the according flip graph is not connected (as it has isolated vertices). Figure 5(left) shows a small point set and a pointed pseudo-triangulation (black/thin lines) which does not admit any compatible flip. The only compatible edge in this example is drawn in red/bold. As the interior point has two incident edges on each side of the dotted line, adding the red edge and removing one black edge always leaves this point non-pointed and thus the resulting graph is not a pointed pseudo-triangulation.

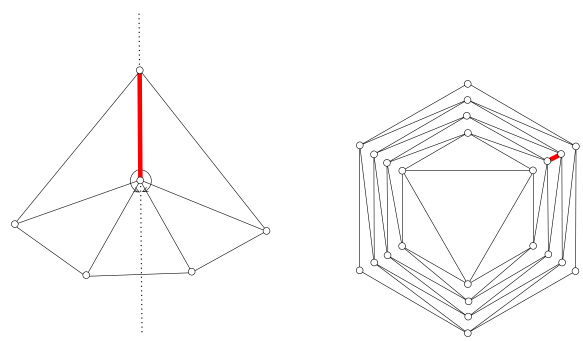

Figure 5: Point sets and pointed pseudo-triangulations that do not admit any compatible flip.

Figure 5(right) contains a more complex example, showing that the basic concept of the small example can be used to build point sets with many interior points and according pointed pseudo-triangulations such that not a single edge can be compatibly flipped. Again, the black/thin edges form a pointed pseudo-triangulation. The red/bold edge, as well as all other compatible edges, cannot be part of any compatible flip.

Actually, the example in Figure 5(right) can be modified to contain an arbitrary number of (at least six) points. In the interior, it contains the graph from Figure 5(left) as a subgraph (the central triangle plus two of the adjacent triangles plus the pseudo-triangle adjacent to these triangles). Starting from this subgraph, we can iteratively make the example larger by adding first the last missing triangle, and then the desired number of pseudo-triangles (with the only restriction that one layer of pseudo-triangles has to be completed before the next one is started).

Corollary 3 *For every $n \geq 6$ and $i \leq \max\{1, n-6\}$, there exists a point set S with n points, i of them interior, such that there exists a pointed pseudo-triangulation for S that does not admit any compatible flip.*

In the graphs from Figure 5, every interior vertex p has degree four. Further, for each p and each compatible edge $e = pq$, the supporting line of e partitions the edges incident to p into two groups such that each group contains two edges. This provides the argument for why none of the compatible edges can be involved in a compatible flip.

Using the contrary argumentation, we can derive sufficient conditions for compatibly flippable edges.

Proposition 4 *Given a pointed pseudo-triangulation $T = (S, E)$ and a compatible edge $e = pq \in (S \times S) \backslash E$ that destroys the pointedness of p but not of q, consider the two non-empty subsets into which the set of incident edges of p is split by the supporting line of e. If one of these subsets contains only one edge e', then e' can be flipped to e in a compatible way.*

Proof. Consider the two pseudo-triangles Δ_1 and Δ_2 that are incident to e' and the corners c_1 of Δ_1 and c_2 of Δ_2 that are opposite to the side-chains on which e' lies. As the edge e does not destroy the pointedness of q, and as e' is the only edge incident to p on one side of the supporting line of e, e lies on the geodesic from c_1 to c_2 in $\Delta_1 \cup \Delta_2 \backslash e'$. Thus e' is flippable to e in a compatible way. □

Corollary 5 *A necessary condition for a pointed pseudo-triangulation $T = (S, E)$ to not admit a compatible flip is that every interior point p of S has vertex degree $d(p) \geq 4$ in T. Every point with vertex degree $d(p) \leq 3$ has at least one incident edge that can be compatibly flipped.*

Proof. We have seen in the arguments for Corollary 3 that a vertex with degree four might not admit any compatible flip. In the other direction, if T contains vertices with degree less than four, consider such a vertex p and the pseudo-triangle Δ in which p is pointed. The geodesic from p to its opposite corner in Δ spans a compatible edge $e = pq$ that destroys the pointedness of p and does not destroy the pointedness of q. The line through this edge e has at least one side with only one edge e' incident to p, and thus e' can be compatibly flipped to e. □

Let us come back to the question of compatible pointed pseudo-triangulations. Consider again the example in Figure 5(left). Any pointed pseudo-triangulation on top of this point set that contains the red/bold edge must not contain two of the black/thin edges in order to keep the interior vertex pointed. Thus it has to also contain another non-black edge. But any non-black edge on top of S, except for the red one, is incompatible to the black pointed pseudo-triangulation. Accordingly, in the example in Figure 5(right), none of the compatible edges can be part of a pointed pseudo-triangulation compatible to the black one.

Corollary 6 *For every $n \geq 6$ and $i \leq \max\{1, n-6\}$, there exists a point set S with n points, i of them interior, such that there exists a pointed pseudo-triangulation $T = (S, E)$ that is incompatible to any pointed pseudo-triangulation $T' = (S, E')$ with $E' \neq E$.*

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