Complete enumeration of small realizable oriented matroids

Hiroyuki Miyata *
hmiyata@is.s.u-tokyo.ac.jp

Sonoko Moriyama †
moriso@dais.is.tohoku.ac.jp

Komei Fukuda ‡
fukuda@ifor.math.ethz.ch

1 Introduction

Point configurations and convex polytopes play central roles in computational geometry and discrete geometry. For many problems, their *combinatorial structures*, i.e., the underlying oriented matroids up to isomorphism, are often more important than their metric structures. For example, the convexity, the face lattice of the convex hull and all possible triangulations of a given point configuration are determined by its combinatorial structure. One of the most significant merits to consider combinatorial types of them is that there are a finite number of them for any fixed *sizes* (dimension and number of elements) while there are infinitely many those objects. This enables us to enumerate those objects and study them through computational experiments (for example, see [1, 2, 12]).

Despite its merits, enumerating combinatorial types of point configurations is known to be a quite hard task. In fact, they do not admit good combinatorial characterizations unless $\mathbf{P} = \mathbf{NP}$ [20]. On the other hand, this problem can be overcome in an abstract combinatorial setting of oriented matroids, denoted by OMs shortly. In fact, Finschi and Fukuda [12] performed a large-scale enumeration of OMs including non-uniform ones and of high rank. Aichholzer, Aurenhammer and Krasser [1], and Aichholzer and Krasser [2] enumerated a large class of rank-3 *uniform* OMs, non-degenerate configurations in the abstract setting.

Now, to obtain all possible combinatorial types of point configurations from OM catalogues $\mathrm{OM}(r, n)$, the set of all OMs of rank r on n elements, we only need to extract those OMs that are *acyclic* and *realizable*. Formally, the *realizability problem* is to decide whether a given OM can be realized by a vector configuration or not, which is the most crucial part to detect the combinatorial types of point configurations. It is as difficult as *the Existential Theory of the Reals (ETR)*, the problem to decide whether a given polynomial equalities and inequalities with integer coefficients has a real solution or not [20]. Although it is a very hard task to solve ETR in general, the realizability problem for small size instances appears to be tractable by exploiting sufficient conditions of realizability or those of non-realizability.

	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10	n = 11
r = 3	1 (1)	2 (1)	4 (1)	17 (4)	143 (11)	4,890 (135)	461,053 (4,382)	95,052,532 (312,356)	unknown (41,848,591)
r = 4		1 (1)	3 (1)	12 (4)	206 (11)	181,472 (2,628)	unknown (9,276,601)		
r = 5			1 (1)	4 (1)	25 (1)	6,029 (135)	unknown (9,276,601)		
r = 6				1 (1)	5 (1)	50 (1)	508,321 (4,382)		

Table 1: The numbers of reorientation classes of simple OMs (n: the number of elements, r: rank) (the numbers enclosed by brackets are those of uniform OMs) [12, 1, 2]

1.1 Brief history of related enumeration

The enumeration of realizable OMs has a long history. Realizable OMs of rank 3 on up 11 elements has been enumerated in [17, 15, 24, 14, 1, 2] and those of rank 4 on 8 elements in [7]. For non-uniform OMs, those of rank 3 on up to 8 elements were decided in [16].

The enumeration of combinatorial types of convex polytopes also has a long history. All combinatorial types of d-polytopes with n vertice can be enumerated efficiently if $d \leq 3$ or $n \leq d + 3$ [17, 10]. On the other hand, the enumeration is known to be quite difficult for $d \geq 4$ and $n \geq d + 4$ [25]. For those cases, the enumerations were performed for $n = 4, d = 8$ case [4], and $n = 4, d = 9$ and simplicial case [3].

	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9
d = 2	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
d = 3		1 (1)	2 (1)	7 (2)	34 (5)	257 (14)	2606 (50)
d = 4			1 (1)	4 (2)	31 (5)	1294 (37)	unknown (1142)
d = 5				1 (1)	6 (2)	116 (8)	unknown (unknown)

Table 2: The numbers of convex polytopes (the numbers enclosed by brackets are those of simplicial polytopes) (n: the number of vertices, d: dimension)

However, there is no large database of these objects including degenerate ones or of high dimension currently. Many problems in combinatorial geometry re-

*Graduate School of Information Science and Technology, University of Tokyo, Japan. Research partially supported by Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science, and Grant-in-Aid for JSPS Fellows.

†Graduate School of Information Sciences, Tohoku University, Japan. Research partially supported by Grant-in-Aid for Scientific Research from Ministry of Education, Science and Culture, Japan.

‡ETH Zurich, Switzerland. Research partially supported by the Swiss National Science Foundation Project No. 200021-124752/1.

main open especially for high dimensional cases or degenerate cases, and thus a database of combinatorial types for higher dimensional or degenerate ones will be of great importance. For example, characterizing the f-vectors of d-polytopes is a big open problem for $d \geq 4$ while the same questions for 3-polytopes and for simplicial polytopes have already been solved [17].

Since Finschi and Fukuda developed a database of OMs [11, 12] containing non-uniform ones, the realizability classification of larger oriented matroids including non-uniform case has begun. Various existing certificates [8, 26, 26, 9, 18] and new certificates [13, 22, 19] were applied to OM(4, 8) and OM(3, 9) [13, 21, 22, 23, 19]. However, there are 4803 oriented matroids in OM(4, 8) and 8548 oriented matroids in OM(3, 9) whose realizability has remained unknown as shown in Figure 1.

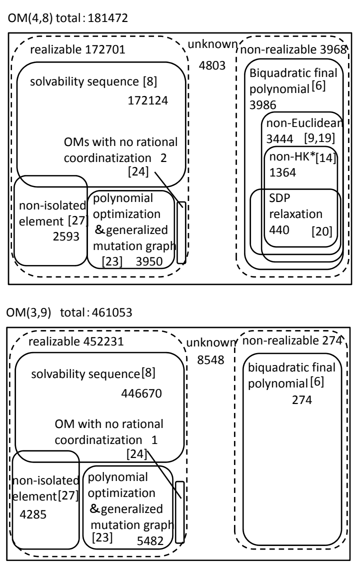

Figure 1: Classifications of OM(4, 8) & OM(3, 9) w.r.t. certificates [13, 21, 22, 23, 19]

1.2 Our contribution

In this paper, we propose a new realizability certificate, motivated by a solvability sequence method [8]. Using it, we manage to realize all realizable OMs in OM(4, 8) and OM(3, 9) except for 3 irrational ones [23] and 8 other OMs. Combining these results and hand computations, we complete the classification of OM(4, 8), OM(3, 9) and OM(6, 9).

Theorem 1.1

(a) *Among* 181, 472 *OMs in OM(*4, 8*) (reorientation class),* 177, 486 *OMs are realizable and* 3, 986 *are non-realizable.*

(b) *Among* 461, 053 *OMs in OM(*3, 9*) (reorientation class),* 460, 779 *OMs are realizable and* 274 *are non-realizable.*

(c) *Among* 508, 321 *OMs in OM(*6, 9*) (reorientation class),* 508, 047 *OMs are realizable and* 274 *are non-realizable.*

As a byproduct, we obtain the following results.

Theorem 1.2

(a) *There are* 15, 287, 993 2*-dimensional point configurations on* 9 *elements,* 105, 128, 749 5*-dimensional point configurations on* 9 *elements and* 10, 559, 305 3*-dimensional point configurations on* 8 *elements.*

(b) *There are* 47, 923 5*-dimensional polytopes with* 9 *vertices.*

2 Oriented matroids and the realizability problem

In this section, we review basic facts about oriented matroids. For further, details about oriented matroids, see [5].

Let $P = (p_1, \ldots, p_n)$ be a point configuration in $\mathbb{R}^{r-1}$. Then we define a map $\chi : \{1, \ldots, n\}^r \to \{+, -, 0\}$ by

$$\chi(i_1, \ldots, i_r) := \operatorname{sign}(\det(v_{i_1}, \ldots, v_{i_r})),$$

where $v_1 := \begin{pmatrix} p_1 \\ 1 \end{pmatrix}, \ldots, v_n := \begin{pmatrix} p_n \\ 1 \end{pmatrix} \in \mathbb{R}^r$ are the associated vectors of $p_1, \ldots, p_n$. We consider the map χ as the *combinatorial type* of P, satisfying the following properties.

Definition 2.1 (Chirotope axioms)
Let E be a finite set and $r \geq 1$ an integer. A *chirotope of rank r on E* is a mapping $\chi : E^r \to \{+1, -1, 0\}$ which satisfies the following properties.

(a) χ is not identically zero.

(b) $\chi(i_{\sigma(1)}, \ldots, i_{\sigma(r)}) = \operatorname{sgn}(\sigma)\chi(i_1, \ldots, i_r)$
for all $i_1, \ldots, i_r$ and every permutation σ, where $\operatorname{sgn}(\sigma) := \prod_{i>j} \frac{\sigma(i)-\sigma(j)}{i-j}$.

(c) For all $i_1, \ldots, i_r, j_1, \ldots, j_r \in E$ such that
$\chi(j_s, i_2, \ldots, i_r) \cdot \chi(j_1, \ldots, j_{s-1}, i_1, j_{s+1}, \ldots, j_r) \geq 0$
for $s = 1, \ldots, r$, we have
$\chi(i_1, \ldots, i_r) \cdot \chi(j_1, \ldots, j_r) \geq 0$.

We define an *oriented matroid*, an *OM* shortly, as a pair of a finite set E and a chirotope $\chi : E^r \to \{+1, -1, 0\}$ satisfying the above axioms. If $|E| = n$, we call the pair $(E, \{\chi, -\chi\})$ a *rank-r OM with n elements.* It is called a *uniform* OM if $\chi(i_1, \ldots, i_r) \neq 0$ for all $1 \leq i_1 < \cdots < i_r \leq n$.

Every vector configuration has the underlying OM, but the converse is not true because "non-realizable" OMs exist.

Definition 2.2 Given a rank-r OM $M = (E, \{\chi, -\chi\})$ with n elements, the *realizability problem for* $(E, \{\chi, -\chi\})$ is to decide whether the following polynomial system has a real solution or not,

$$\mathrm{sign}(det(v_{i_1}, \dots, v_{i_r})) = \chi(i_1, \dots, i_r), \qquad (1)$$

for $v_1, \dots, v_n \in \mathbb{R}^r$ and $1 \leq i_1 < \cdots < i_r \leq n$.

If M arises from a vector configuration, M is said to be *realizable*, otherwise *non-realizable*. *Acyclic* realizable OMs corresponds completely to the combinatorial types of point configurations. We do not explain *acyclicity* here. For details, see [5].

3 Our method to decide realizability

We decide realizability of OMs by solving the polynomial systems (1). This problem is as hard as solving ETR asymptotically, however, the system for realizability contains various kinds of redundancies, and can be solved exploiting special structures of them for small size instances.

First, we apply techniques used in [8, 22] to simplify the polynomial systems. Then we eliminate square-free variables, motivated by a solvability sequence method [8]. We consider the following elimination rule similarly to [8].

Proposition 3.1 *Let $l_1, l_2 \geq 0$ be integers and L_i, R_j rational functions for $i = 1, \dots, l_1$ and $j = 1, \dots, l_2$. Then the feasibility of polynomial system:*

$$\begin{cases} y < R_i(x_1, \dots, x_n) \ (i = 1, \dots, l_1), \\ y > L_j(x_1, \dots, x_n) \ (j = 1, \dots, l_2) \end{cases}$$

is equivaluent to that of the following polynomial system:

$$L_j(x_1, \dots, x_n) < R_i(x_1, \dots, x_n) \ (i = 1, \dots, l_1, j = 1, \dots, l_2).$$

The solvability sequence method applies this elimination rule under *bipartiteness condition* [8] for determinant systems. We consider here the rule for general polynomial inequalities to eliminate more variables.

We note that an elimination rule for systems containing equalities can also be considered easily by substitution operations. A variable y appearing in the proposition can be seen as *redundant*. We try to eliminate as many variables of this type as possible. To apply this elimination rule to as many variables as possible, we consider the following additional rule, which we call *a branching rule*.

Proposition 3.2 *Let $k_1, k_2 \geq 0$. Then polynomial system:*

$$\begin{cases} A_i(x_1, \dots, x_n) y < B_i(x_1, \dots, x_n) \ (i = 1, \dots, k_1), \\ A_j(x_1, \dots, x_n) y = B_j(x_1, \dots, x_n) \ (j = k_1 + 1, \dots, k_1 + k_2) \end{cases}$$

is feasible if and only if one of the following polynomial systems is feasible.

$$\begin{cases} \mathrm{sign}(A_i(x_1, \dots, x_n)) = s(i) \ (i = 1, \dots, k_1 + k_2), \\ y < \frac{B_i(x_1, \dots, x_n)}{A_i(x_1, \dots, x_n)} \ (i \leq k_1, s(i) = +), \\ y > \frac{B_i(x_1, \dots, x_n)}{A_i(x_1, \dots, x_n)} \ (i \leq k_1, s(i) = -), \\ B_i(x_1, \dots, x_n) > 0 \ (i \leq k_1, s(i) = 0), \\ y = \frac{B_j(x_1, \dots, x_n)}{A_j(x_1, \dots, x_n)} \ (j > k_1, s(j) \neq 0), \\ B_j(x_1, \dots, x_n) = 0 \ (j > k_1, s(j) = 0), \end{cases}$$

for $s : \{1, \dots, k_1 + k_2\} \to \{+, -, 0\}$.

We note here that the solvability sequence method consider special cases where the sign of A_i are known in advance for all $i = 1, \dots, k_1 + k_2$.

One can eliminate a square-free variable y by applying the branching rule and then the elimination rule for y and obtain $3^{k_1+k_2}$ polynomial systems with n variables and at most $\frac{k_1^2}{4} + k_1 + k_2$ constraints. Using the branching rule, we can formulate a problem to give realizations as a kind of tree search problems as follows. Starting from the root node, which consists of the original polynomial system, we expand nodes using the elimination rules and the branching rule repeatedly. If we eliminate all variables at some node and obtain a consistent system, we prove the feasibility. It can be viewed as an extension of the solvability sequence method.

In addition, we try to prove the feasibility of polynomial systems such as $x^2 - 2xy + y^2 > 0$ and $a^3 - b^3 > 0, a^3 - 2b^3 < 0$. These systems are *clearly* feasible, but there exist no efficient and unified algorithm to prove the feasibility known to the authors. We propose to use random assignments to variables in order to prove the feasibility. That is, we decide whether we arrive at goal nodes or not using the random assignments instead in the above tree search. In this setting, we define the cost of each node x by $c(x) := (log_2 3)k_1 + k_2$, where k_1, k_2 are the parameters appearing in Proposition 3.2, and apply the iterative lengthening search to it by increasing the limit of the total cost by 1.

4 Experimental results

We apply our method to OM$(4, 8)$ and OM$(3, 9)$. For random assignments, we consider the uniform distribution of $\{n/100 \mid n = 1, 2, ..., 10,000\}^m$, where m is the number of variables and try random assignments 1000 times at every node. Our method manages to find realizations of all realizable OMs in OM$(4, 8)$ and OM$(3, 9)$ except for 3 irrational ones and 8 other OMs. In order to generate realizations for the 8 OMs, we added some hand computations and then applied our method. As a result, we obtained complete classification of OM$(4, 8)$, OM$(3, 9)$. In addition, the classification of OM$(6, 9)$ was obtained by the duality of OMs [5] (Theorem 1.1).

From these results, we obtain the combinatorial types of point configurations (Theorem 1.2) by generating relabeling classes of acyclic realizable OMs. We extract realizable matroid polytopes from them and compute the face lattices to obtain combinatorial types of polytopes (Theorem 1.2).

5 Conclusion

In this paper, we complete the classification of $\mathrm{OM}(4,8)$, $\mathrm{OM}(3,9)$ and $\mathrm{OM}(6,9)$, which almost reaches the limit of today's computational environments. Our Java programs and the classification results are available at
`http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/index.html`

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