On Stretch Minimization Problem on Unit Strip Paper

Ryuhei Uehara*

Abstract

For a given mountain-valley pattern of equidistant creases on a long strip paper, there are many folded states consistent with the pattern. Among these folded states, we like to fold a paper so that the number of the paper layers between each pair of hinged paper segments is minimized. We first formalize this problem as optimization problem. The complexity of the problem is not known. In this paper, we give partial results related to the problem. First, we show that the problem is well-defined even in a simple folding model. The simple folding model is the most primitive model of basic origami models, and hence the folding availability is very restricted. We show a universality theorem of the simple folding model for this problem. That is, every flat folded state consistent with any given pattern can be folded by a sequence of simple foldings. Next, we investigate the number of folded states consistent with a given pattern. For a given random mountain-valley pattern, the expected number of folded states consistent with the pattern is exponential.

1 Introduction

What is the best way to fold an origami model? Origamists around the world struggle with this problem daily. Even if you have a good origami model with its crease pattern, this is not the end. To make the model, we have to search for clever, more accurate, or faster folding sequences and techniques. In this paper, we focus on the problem for accurate folding on a simple kind of one-dimensional creasing, where the piece of paper is a long rectangular strip, which can be abstracted into a line segment, and the creases uniformly subdivide the strip. A mountain-valley pattern is then simply a binary string over the alphabet $\{M, V\}$ (M for mountain, V for valley), which we call a *mountain-valley string*. Of particular interest in origami is the *pleat*, which alternates $MVMVMV\cdots$. The pleat folding is quite unique in the sense that the folded state is unique [Asano et al. 10]. In general, this is not the case. For example, for a string $MMVMMVMVVVV$, surprisingly, there are 100 distinct folded states consistent with this string. Among them, what is the *best* folded state?

*School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa 923-1292, Japan. uehara@jaist.ac.jp

From the practical point of view, it seems better to decrease the number of paper layers between each pair of paper segments hinged at a crease as possible as we can. If we have many paper layers between the hinged papers, it becomes to be difficult to fold with accuracy, and if we have too many, we cannot fold any more.

For a folded state, we define a *stretch* at a crease i is the number of the paper layers between the papers hinged at the crease i. Then, we can consider two optimization problems as follows:

Input: A strip of paper of length $n+1$ with a mountain-valley string s in $\{M, V\}^n$.
Goal: Among the folded states consistent with s, we aim to find a folded state of unit length that (1) minimizes the maximum stretch of all stretches at each crease in the folded state, or (2) minimizes the total stretch of all stretches at each crease in the folded state.

The minimization problem for the average stretch is equivalent to the second optimization problem (by dividing n). These two problems have different solutions in general. For example, among the 100 valid folded states of the string $MMVMMVMVVVV$, the minimum maximum stretch is 3, which is achieved by the folded state [4|3|2|5|6|0|1|7|9|11|10|8] (the details of this notation is described later), the minimum total stretch is 11 by the other state [4|3|2|0|1|5|6|7|9|11|10|8], and moreover, these solutions are unique for this string. Here we state an open problem:

Open Problem: Determine the computational complexity of the minimization problems of the maximum/total stretch for a given string s in $\{M, V\}^n$.

We first show that the problem is well-defined even in a simple folding model introduced by Arkin et al. [Arkin et al. 04]: even in the simple folding model, every folded state consistent with any given mountain-valley string can be folded. This universality theorem of the simple folding model is related to the one-dimensional flat folding problem [Demaine and O'Rourke 07, Sec. 12.1], and the locked chain problem, that has a long history [Demaine and O'Rourke 07, Chap. 6].

The open problem seems to be NP-hard in general. We next prove this intuition by counting.

Theorem 1 *Let s be a mountain-valley string of length n taken uniformly at random, and $f(n)$ the expected*

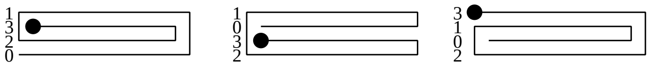

Figure 1: Three foldings for the mountain-valley string VVV.

number of folded states consistent with s. Then experimental results imply that $f(n) = \Theta(1.65^n)$. *We also show the upper and lower bounds;* $f(n) = \Omega(1.53^n)$ *and* $f(n) = O(2^n)$.

The results guarantee that $f(n)$ is an exponential function, and hence the exhaust search approach has no hope in general. Theorem 1 comes from more general counting problem:

Theorem 2 *Let* $F(n)$ *be the number of folded states of a paper of length* $n+1$. *Then experimental results imply that* $F(n) = \Theta(3.3^n)$. *We also have the upper and lower bounds:* $F(n) = \Omega(3.06^n)$ *and* $F(n) = O(4^n)$.

Theorem 1 says that a simple exhaust search runs in an exponential time in general. Unfortunately, we have no idea about the computational complexity of the optimization problems up to now.

A part of this paper was presented as an oral talk at 5th international conference on Origami in science, mathematics, and education (5OSME) [Uehara 10]. All the results in this paper will be published in the future book that collects the works in 5OSME.

2 Preliminaries

The *paper strip* is a one-dimensional line with creases at every integer position. At first, the paper of length $n+1$ with the string of length n is placed at the interval $[0..n+1]$. The paper is *rigid* except the creases on the integer positions; that is, we can only fold the paper at these integer positions. At the end of the folding operations, all creases are folded, the paper becomes unit length, and the direction of each folded crease follows the letter (in $\{M, V\}$). That is, the ith letter of the mountain-valley string of length n indicates the final folded state of the crease at integer point i in $[1..n]$. We call each paper segment between i and $i+1$ the i*th segment*. Each final folded state can be represented by the ordering of the segments; for example, a pleat folding $MVMV$ is described by $[0|1|2|3|4]$ or $[4|3|2|1|0]$, and a crease string VVV produces $[1|3|2|0]$, $[1|0|3|2]$, $[3|1|0|2]$, or their reverses (Figure 1). We distinguish between the left and right endpoints of the paper, but we sometimes ignore the reverse of one folded state since they are essentially the same. In fact, the sides of a folded state sometimes turn upside down when we fold all paper layers at a crease from right to left or from left to right.

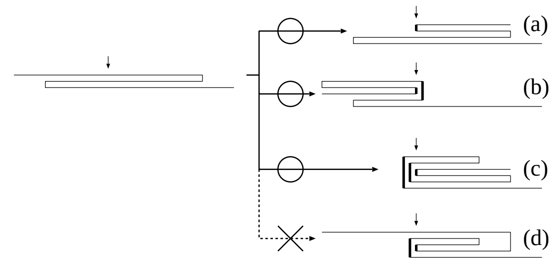

Figure 2: Simple folding model.

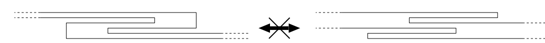

Figure 3: Two legal folded states for a string which cannot be exchanged by local simple (un)foldings.

We employ the *simple folding model* by Arkin et al. [Arkin et al. 04]. Precisely, each simple folding is the folding from a flat folded state to another flat folded state by the following operations: (1) put the flat (folded) paper (on the reverse side, if necessary) in a plane, (2) choose an integer point to fold, and (3) valley fold consecutive most inner paper layers at the crease (see also [Demaine and O'Rourke 07, Sec. 14.1] and [Cardinal et al. 09]). In Figure 2, (a), (b), and (c) are simple foldings, but (d) is not allowed. A simple unfolding is defined by rewinding a simple folding; that is, we can *unfold* a folded state a to a folded state b if and only if a can be obtained from b by a simple folding. We note that a can be unfolded to one of several folded states; that is, a simple unfolding is not just a rewind of the *last* simple folding. (In a sense, conceptually, a simple unfolding can be seen as a simple folding. That is, they are the same operation that flips consecutive most inner paper layers at a crease.)

For a mountain-valley string s, we call a folded state *legal for s* if it is consistent with the string.

3 Universality of the simple folding model

We here show that the simple folding model is strong enough to discuss the strip paper of equidistant creases. More precisely, we show that every legal folded state for any string can be made by a sequence of simple foldings.

Theorem 3 *Let* P *be any legal folded state for a mountain-valley string* s *in* $\{M, V\}^n$. *Then* P *can be folded from the initial state by a sequence of simple foldings.*

Before proving Theorem 3, we comment on the claim of the theorem. One may think that Theorem 3 is "trivial". But it is not so trivial. A typical counterexample is shown in Figure 3; these two folded states are legal for the same mountain-valley string, but they cannot be

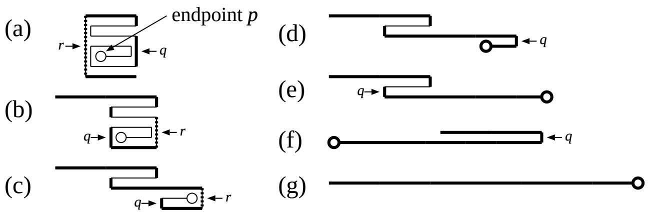

Figure 4: Simple (un)foldings.

exchanged by just local simple (un)foldings. This fact implies that folding of these states from the initial state requires some global strategy.

By definition of unfolding, a folded state P can be folded from the initial state by simple foldings iff P can be unfolded to the initial state. Hence, we prove Theorem 3 by showing how to unfold any folded state P to the initial state. This is strongly related to two well investigated problems in computational origami:
(1) This is a kind of the "(un)locked chain problem in 2D" that has a long history [Demaine and O'Rourke 07, Chap. 6]. There is no locked chain in 2D [Demaine and O'Rourke 07, Sec. 6.6]. However, this fact does not imply Theorem 3 since the operations are restricted to simple unfoldings.
(2) Our problem is also related to "one-dimensional flat foldings" [Demaine and O'Rourke 07, Sec. 12.1]. This problem asks if there exists a flat folded state for a given pattern on a strip paper. The known result says that we can find *one* flat folded state by repeating crimp and end folding if it exists. Hence, the known algorithm cannot construct a given *specified* folded state from the initial state. (In contrast with Theorem 3, this is not always possible for non-unit case; see Concluding Remarks.)

Thus, in a sense, our problem is more difficult than the above problems; the folded state is specified, and we can only use simple foldings to make it. But all links in our "linkage" have unit length. Using this advantage, we can show the universality theorem for the unit strip paper in the simple folding model. Here we prove the following stronger claim than Theorem 3:

Corollary 4 *Let P be a flat folded state of a paper of length $n+1$ s. t. every folded point is placed at an integer point in $[1..n]$ in the initial state. Then P can be folded from the initial state by a sequence of simple foldings that are made at each integer point. Moreover, the total number of simple foldings is bounded by $2n$.*

Proof. We prove the claim by unfolding any folded state P to the initial state. Intuitively, we unbind the last segment and arrange the last consecutive segments in line. But before unbinding, we have to peel off the papers covering the last segment. To describe in detail, let p be the last endpoint of the paper, that will be placed at integer point $n+1$ in the initial state. We abuse the symbol P to denote the current flat folded state. We here define *visibility* of a point on P; a point is *visible* on P if and only if it appears on a surface of P. All visible points are drawn in thick lines in Figure 4. According to the visibility of the last endpoint p, we have two cases.
Case 1: The point p is not visible in the folded state P (Figure 4(a)-(c)). Let q be the largest visible crease. That is, all points $r > q$ (including p) are invisible. Let q' be the smallest folded crease with $q' > q$. If there is no folded crease greater than q, set $q' = p$.

First suppose the crease q is flat. Then, by the visibility of q, the papers on the visible side of q can be flipped by a simple (un)folding at the crease point q'. Then, the largest visible crease is updated from q to $q' > q$.

Next suppose q is a folded crease. Without loss of generality, the crease $q+1$ is placed at left of q as in Figure 4(a). Then, the papers on the opposite side of $q-1$ with respect to the segment $q(=[q,q+1])$ covers the point $q+1$ but do not cover $q-1$ since q is visible. This fact implies that these papers can be flipped by a simple (un)folding at the crease point $q' \geq q+1$.

In any case, the largest visible crease is updated from q to $q' > q$ by one (un)folding. We repeat this process until the point p becomes visible. The number of repeating is at most n, and hence the total number of (un)foldings in case 1 is at most n.
Case 2: The point p is visible in the folded state P (Figure 4(d)-(f)). Let q be the largest folded crease. If q is not visible, since there is no folded crease between visible p and q, we can make q visible by just one simple (un)folding at the point q by using the same technique in case 1. Now we can assume that all points in $[q,p]$ are visible. Moreover, these points can be seen from one side (otherwise, the paper is disconnected). Hence we can unfold at the folded crease q and make it flat. This does not change the visibility of p. Thus we can repeat this process until the whole creases become flat. We can observe that these two (un)foldings (to make q visible if necessary, and to make q flat) can be done at once. Hence the total number of (un)foldings in this case is at most n.

Hence we have Theorem 3 and Corollary 4. $\square$

By Theorem 3, the optimization problems are well-defined for any mountain-valley string, and it is worth considering on the simple folding model. Furthermore, if we have an optimal solution, it can be folded in linear time by Corollary 4.

4 The number of folded states

In this section, we will prove Theorems 1 and 2. Using Theorem 2, Theorem 1 follows easily. Hence we first focus on Theorem 2.

Recall that $F(n)$ is the number of folded states of a paper of length $n+1$. A simple algorithm can compute $F(n)$ in a straightforward way for small n, but we can find the correct values for larger n at "The On-Line Encyclopedia of Integer Sequences" with id:A000136[1]. Plotting the sequence, we have an experimental result $F(n) = \Theta(3.3^n)$. Now we turn to the upper and lower bounds of $F(n)$.

Lemma 5 $F(n) = O(4^n)$.

Proof. We first assume that n is even, say $n = 2k$, and each folded state of unit length is placed on the interval [0..1]. We see the relationship among the papers at the point 0. The papers should not be penetrated through each other. That is, at the point 0, k creases with one end (of the left end of the segment 0) make a nest structure. The number of nest structure with k pairs is given by the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{(2k)!}{(k+1)!k!}$ (see, e.g., [Stanley 97]). Once the left end is connected to the right nest structure at the point 1, the paper order is automatically determined. The number of the possible connections of the left end to the right nest structure is k. Hence the number of folding ways can be bounded above by kC_kC_k. The other case (n is odd) is similar, and hence omitted. $\square$

Lemma 6 $F(n) = \Omega(3.065^n)$.

Proof. We imagine folding the last k creases for some $k \ll n$. After folding the last k creases into unit length, we glue it, and obtain a paper of length $n-k+1$ with $n-k$ creases. Let $G(k)$ be the number of the folding ways of this last k creases under the constraint that the $(n-k)$th crease is not *covered*, which means the segments $(n-k-1)$ and $(n-k)$ are not separated by the other papers between $[n-k..n+1]$. Repeating this process, we have a lower bound: $F(n) > (G(k))^{\frac{n}{k}} = (G(k)^{\frac{1}{k}})^n$. This function $G(k)$ is also listed at "The On-Line Encyclopedia of Integer Sequences" with id:A000682[2]. Since the function $G(k)$ is a monotone increasing function for k, we use the largest value $G(43) = 830776205506531894760$, and obtain the lower bound $F(n) > (830776205506531894760^{\frac{1}{43}})^n = 3.06549^n$ for sufficiently large n. $\square$

By the experimental results listed on "The On-Line Encyclopedia of Integer Sequences" with Lemmas 5 and 6, we have Theorem 2 immediately. Here, the number of mountain-valley strings of length n is 2^n. Hence, dividing the values in Theorem 2 by 2^n, we have Theorem 1.

[1] http://www.research.att.com/~njas/sequences/A000136

[2] http://www.research.att.com/~njas/sequences/A000682

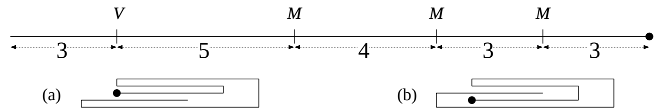

Figure 5: Two legal folded states; (a) is foldable by simple foldings, and (b) is not foldable by simple foldings.

5 Concluding Remarks

In this paper, we state an open problem that asks the computational complexity of the minimization problems of the maximum/total stretch of a strip paper with a given mountain-valley string. We first show that the problem is well-defined even in a simple folding model. That is, we show that any given folded state of a strip paper can be folded by a sequence of simple (un)foldings. The proof of the universality gives us a linear time algorithm that requires at most $2n$ (un)foldings. The improvement of this bound $2n$ to n remains to be open.

Extending the proof of Theorem 3, for the one-dimensional flat foldings, one might wonder if any specified legal folded state can be folded from the initial state even if we allow nonuniform intervals. However, this is not the case. In Figure 5, both of (a) and (b) are legal folded states for the above mountain-valley string $VMMM$. Although (a) is foldable by a sequence of simple foldings, (b) is not. In fact, (b) cannot be unfolded at all from this position by a simple unfolding. From the viewpoint of industry, the characterization of folded states that can be folded by a sequence of simple foldings seems to be a nice future work.

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