Cell-Paths in Mono- and Bichromatic Line Arrangements in the Plane*

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Abstract

We show that in every arrangement of n red and blue lines — in general position and not all of the same color — there is a path through a linear number of cells where red and blue lines are crossed alternatingly (and no cell is revisited). When all lines have the same color, and hence the preceding alternating constraint is dropped, we prove that the dual graph of the arrangement always contains a path of length $\Theta(n^2)$.

1 Introduction

Given an arrangement of n red and blue lines in the Euclidean plane, we consider sequences of cells of the arrangement such that consecutive cells share an edge and no cell appears more than once in the sequence. We refer to such sequences as *cell-paths*, or simply *paths*. A path is called *alternating* if the common edges of consecutive cells alternate in color. The *length* of a path is defined to be one less than the number of cells involved. Cell-paths can also be seen as paths in the dual graph of the arrangement, in which there is a node for every cell in the arrangement, and an edge connects two nodes when the corresponding cells are adjacent.

We consider the following question: Is there a function $p(n)$ tending to infinity so that every arrangement of n blue and red lines in general position (i.e., no three lines share a point and no two lines parallel) and not all of the same color has an alternating path of length at least $p(n)$? In Section 2 we answer this question in the affirmative proving that $p(n) \geq n$ (and give an upper-bound example with $p(n) = 2n - O(1)$).

If the n lines in the arrangement have the same color — a monochromatic arrangement — we ask a similar question: Is there a growing function $f(n)$ so that every arrangement of n lines in general position has a cell-path of length at least $f(n)$? One would expect that dropping the requirement for the path to be alternating from the previous problem would lead to a stronger result. Indeed, in Section 3 we prove that $f(n) = \Theta(n^2)$.

Previous work. Arrangements of (uncolored) lines have been thoroughly studied for decades [7, 10, 13, 14, 15, 22]. For example, properties of monotone paths in the arrangement have been studied (see, e.g., [9]). Substantial emphasis has been put into studying degenerate arrangements in which, e.g., the number of vertices decreases dramatically. Further, the kind of cells one may obtain as well as their extremal number were investigated (for example how many triangles appear in any simple arrangement). In another direction one can study the graph having as nodes the intersection points of lines, which are adjacent if they are consecutive in one of the lines [6], and study its basic properties as a graph, such as edge-colorings or whether it can be decomposed into Hamiltonian cycles (in projective space) [11]. For later use recall that the cells in any arrangement can be 2-colored chessboard-like, i.e., no two cells with the same color are adjacent [23] (see also [18]).

Not many problems on colored arrangements of lines were considered in the early times, in contrast to the rich (and still growing) literature on combinatorial problems on red and blue points [7, 20]. The first publications considered bichromatic sets of lines and studied the number and distribution of the intersection points of lines with the same color [16, 17, 26]. There is a recent line of research on problems in which lines have to be colored to achieve some property, or are already colored and one looks at the kind of cells that appear, regarding the color of their sides [3, 4, 5]. Our problem

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on alternating paths adds to this trend.

The presented work on bichromatic line arrangements was inspired by a well known (still open) problem on points: consider a set R of n red points and a set B of n blue points in convex position, what is the longest alternating path spanned by these points (a crossing-free alternating Hamiltonian path on $R \cup B$ does not always exist)? Erdős (see [21]) proposed to study the value $\ell(n)$ such that a plane alternating path of length at least $\ell(n)$ always exists for any such pair R and B. Independently, Akiyama and Urrutia [2] considered the same problem and gave a necessary and sufficient condition for the existence of an alternating Hamiltonian path and an $O(n^2)$ algorithm to find one, if it exists. Abellanas et al. [1], and independently Kynčl et al. [21], proved that $\ell(n) \leq \frac{4}{3}n + O(\sqrt{n})$. Cibulka et al. [8] proved that $\ell(n) \geq n + \Omega(\sqrt{n})$. The gap is still to be closed. Other variations and related problems appear in Mészáros' PhD Thesis [24].

Finally, our results are also related to a long-standing open problem about paths in planar graphs. In 1963, Moon and Moser [25] showed that there exist three-connected planar graphs with n vertices in which the longest simple path has length at most $cn^{\log 2/\log 3}$, where c is some constant. It is conjectured (see [19]) that this is a lower bound as well, hence that every three-connected planar graph contains a path of this length. Our result on $f(n)$ shows that considering dual graphs of arrangements instead, we always get a path of length $\Omega(n^2)$.

2 Long Alternating Paths in Bichromatic Arrangements

In this section we prove the existence of long alternating paths in bichromatic arrangements. First observe that general position is important to allow a positive answer: Assume $n \geq 2$ and take all the n lines in the arrangement to go through a common point, so that all of the red lines have slope between 0 and 1, and all of the blue lines have positive slope larger than 1. Now every alternating path has length at most two. This holds since each cell on an alternating path, except for the first and the last one, has to be *bichromatic*, i.e., has to have a red and a blue edge on its boundary, and the constructed arrangement has only four bichromatic cells, which do not share an edge among each other.

Further consider an arrangement of n lines, all blue except for exactly one of them red, in general position. Then the length of the longest alternating path is $2n - O(1)$: In the arrangement (as edge set) there are n red edges which can be used at most once in a path, so in an alternating path at most $2n+1$ edges can be used; hence the upper bound. For the lower bound we go through the $2n$ cells along the red line, crossing the red line in every other step, and a blue edge for entering a red-line-incident cell not yet visited in the steps in between; hence a path of length at least $2n - 1$.

The following lemma directly implies our main result for the stated problem on bichromatic arrangements. For the sake of convenience we delay the proof of Lemma 1 to Section 2.1.

Lemma 1 *Any pair of bichromatic cells in an arrangement of red and blue lines in general position is connected by an alternating path.*

Consider two antipodal bichromatic infinite cells in an arrangement ("*antipodal*" means that the cells are separated by all n lines). As long as there is at least one line of each color, such a pair of cells has to exist and, by Lemma 1, is connected by an alternating path. Clearly, such a path has to cross every line at least once.

Theorem 2 *In a set of n blue and red lines — in general position and not all of the same color — there is an alternating path of length n.*

By the example with exactly one red line, the bound in the theorem is asymptotically tight. However, if we require the same number of red and blue lines, we do not know whether longer alternating paths always exist.

2.1 Proof of Lemma 1

The graph underlying our problem is the dual graph of the arrangement: the $\binom{n+1}{2} + 1$ cells are the nodes of the graph, two nodes are adjacent if their corresponding cells share an edge in the arrangement. In order to capture the 'alternating' property, we can orient the edges as follows. First, observe that the graph has a proper 2-coloring (choose the color of a cell according to the parity of the number of lines above the cell), for which we choose the colors "r-out" and "b-out". Now direct the edges of the dual graph by directing red edges (i.e., edges dual to red edges in the arrangement) from color r-out to color b-out and blue edges from color b-out to color r-out. This is illustrated in Figure 1. It is an easy exercise to verify that every (undirected) alternating path in the arrangement can be directed in one way so that it appears as a directed path in this oriented version of the dual of the arrangement — and vice versa, every directed path is clearly alternating.

Let us fix an arbitrary bichromatic cell z in the arrangement and consider the set of all cells that can be reached by a directed path in the just defined directed graph. Note that the construction of this directed graph is not unique, since we can obtain a second one by changing directions of all edges. We arbitrarily choose one and proceed. Now consider the closure of the union, $\mathsf{reach}(z)$, of all the cells that are reachable by a directed path starting from z in the oriented graph.

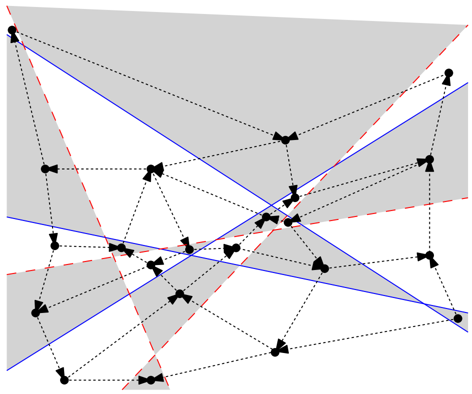

Figure 1: Orientation of the dual graph of a bichromatic line arrangement. Every bichromatic vertex of the arrangement yields a four-cycle in the oriented graph.

Here comes the crucial observation: Let us call a vertex in the arrangement *bichromatic*, if it is the intersection of a red and a blue line. Then the four cells incident to a bichromatic vertex form a directed cycle in the oriented graph and thus either all four of them are contained in $\mathsf{reach}(z)$ or none of them is. (Here we implicitly use the fact that the existence of a directed walk — i.e., with repetitions of vertices allowed — from node x to node y implies the existence of a directed path from x to y.) Therefore, every bichromatic vertex is interior either to $\mathsf{reach}(z)$ or to its complement.

We have now established the following fact.

Lemma 3 *Let E be the set of edges of the arrangement that separate cells in $\mathsf{reach}(z)$ from its complement. No red edge in E shares a vertex with a blue edge in E.*

We will show that bichromatic vertices cannot be interior to the complement of $\mathsf{reach}(z)$. Thus all of them have to be interior to $\mathsf{reach}(z)$ and therefore, all bichromatic cells are in $\mathsf{reach}(z)$ from which Lemma 1 follows. To this end consider a set X of separating edges that form a cycle or maximal path in the graph of separating edges. A maximal path has to start and end with an edge that extends to infinity. Consequently the union of edges in X separates the plane into two parts, one part of which contains the seed cell z. Suppose the edges in X are all blue (recall that they all have the same color). Then it is not possible that both sides contain a point on a red line, as otherwise, since we can travel on red lines between these two points, a red line had to cross the union of edges in X, which we know is not possible, since all of them are blue. Thus, the side that does not contain z must be completely monochromatic, i.e., all cells there are bounded by edges of the same color.

Since every point p in the complement of $\mathsf{reach}(z)$ must have such a cycle or path X separating p from $\mathsf{reach}(z)$, it follows that bichromatic vertices cannot be interior to the complement of $\mathsf{reach}(z)$, as claimed. As argued before, this implies Lemma 1.

2.2 Discussion

(1) As mentioned before, the linear bound on the length of the alternating path is probably not tight if an equal number of red and blue lines is required. However, abandoning the general position assumption, we have an example with the same number of red and blue lines, half of the lines vertical and half of them horizontal, so that the longest alternating path has only length $O(n)$.
(2) A closer inspection of the proof given shows that we have actually established the following.

Theorem 4 *Let C be a set of red and blue simple closed or biinfinite curves, each of which separates the plane into two parts. If the union of red curves is connected, the union of blue curves is connected, and no point is contained in more than two of the curves, then any pair of bichromatic cells in the arrangement is connected by an alternating path.*

(3) Similarly, Lemma 1 can be generalized to higher dimensions: Consider two antipodal bichromatic cells in a $(d+1)$-dimensional arrangement. Intersect these two cells (and the arrangement) with a hyperplane H. The intersection of the arrangement with H gives a d-dimensional bichromatic arrangement, in which the antipodal cells are connected by induction.

3 Long Paths in (Monochromatic) Arrangements

We are given a set S of $n \geq 2$ lines in general position. Our aim is to find bounds on the length of the longest simple path in the dual graph of the arrangement. Let $A(S)$ be the arrangement associated with S, and let G be the dual graph of $A(S)$. Recall that the number of vertices of G is $N := \binom{n+1}{2} + 1$. We define $f(S)$ as the length of the longest simple path in G, and let $f(n) = \min_{|S|=n} f(S)$. In this section we show:

Theorem 5 $f(n) = \Theta(n^2)$.

This theorem is a direct consequence of Theorem 14 and Proposition 15 below. The lower bound actually holds for simple arrangements of *pseudolines* in the Euclidean plane. The main idea for its proof is to perform local transformations to G to make it four-connected, and then apply Tutte's Theorem, which states that every four-connected planar graph is Hamiltonian.

3.1 Lower Bound

G is a planar bipartite quasi-quadrangulation (i.e., every face of G has size four except for the unbounded one). It is easy to check that G is two-connected. We consider the natural embedding of G given by S, in which every vertex of G is located in the corresponding face of $A(S)$, every edge of G intersects exactly one edge of $A(S)$, and the face corresponding to the unbounded cells of $A(S)$ is the outer face. Recall that a *vertex cutset* of a graph is a set of vertices whose removal disconnects the graph.

Let C be a simple cycle of G. By Jordan's Theorem, the removal of C from G decomposes the remaining vertices into two subsets, which we call *outer* and *inner*, where the outer one is the component that contains the unbounded cells of the arrangement. Given a vertex $v \in C$, we define its *inner degree* as the number of neighbors of v that belong to the inner component.

Lemma 6 *In any cycle C of G of length $2k$, the inner degree of any vertex of C is at most $k-2$ and the number of vertices in the inner part is at most $(k-1)(k-2)/2$.*

Proof. Consider the set $S(C) \subseteq S$ of lines associated with edges of C. Since C is a cycle, every line in $S(C)$ is intersected by C an even number of times, and at least twice. Hence, there are $|S(C)| \leq k$ such lines. As any vertex v of C is incident to two edges on C, there are exactly $|S(C)| - 2$ lines left that could correspond to edges incident to v and in the inner part of C. Further, the number of vertices in the inner part of C is at most the number of bounded cells of the arrangement formed by $S(C)$ and thus at most $(k-1)(k-2)/2$. □

Let P be a simple path of G whose first and last vertex are incident to the outer face, while all other vertices are interior vertices of G. Then the removal of P splits the remaining vertices of G into two subsets as well. We refer to P as a *separating path* and to the induced subsets as *separated vertex sets*. The proof of the following lemma is similar to the one of Lemma 6.

Lemma 7 *For any separating path P of G with k vertices, one of the separated vertex sets of G has cardinality at most $(k-1)(k-2)/2$.*

Lemma 8 *Let C be a simple cycle of G, and let $I(C)$ be the set of vertices in its interior. If $I(C) \neq \emptyset$, there exists a simple cycle C' with the same set $I(C)$ in its interior such that no two consecutive vertices of C' have inner degree zero. For any separating path P which splits the remaining vertices of G into V_1 and V_2, there exists a separating path P' such that (1) for each side of P', at least one out of any two consecutive vertices of P' has an emanating edge to this side, (2) P' splits the remaining vertices of G into $V_1' \supseteq V_1$ and $V_2' \supseteq V_2$, and (3) the first and the last vertex of P' have emanating edges to both sides of P'.*

Proof. Suppose $I(C) \neq \emptyset$. As all faces of G in the interior of C have size four, at most two consecutive vertices of C can have inner degree zero. Moreover, the only possibility of having two consecutive such vertices is that C uses three consecutive edges of a face in its interior. Iteratively replacing these edges by the fourth edge of this face wherever such a situation occurs, we obtain a simple cycle C'. C' has exactly $I(C)$ in its interior as well, and no two consecutive vertices of C' have inner degree zero. Similarly, in a separating path P, two consecutive vertices without emanating edges on one side can occur only if P uses three consecutive edges of a face on that side. Replacing all such occurrences on both sides gives the claimed properties. □

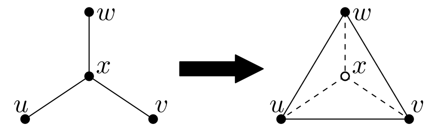

Figure 2: The Y-Δ transformation.

Lemma 9 *All vertex cutsets of size two of G consist of the two neighbors of a degree-two vertex.*

Proof. Consider a cutset $C = \{c_1, c_2\}$ and the at least two resulting sets V_1, V_2 of the remaining vertices of G. First consider the case in which a component, say V_1, contains only inner vertices of G. Then c_1 and c_2 are part of a cycle which has V_1 as its inner part. By Lemma 8 such a cycle with length at most four exists. Hence Lemma 6 implies that $V_1 = \emptyset$ and thus C is not a cutset. If both V_1 and V_2 contain vertices of the outer face, then by Lemma 8, c_1 and c_2 are the end points of a separating path with at most three vertices. Thus, by Lemma 7, $\min\{|V_1|, |V_2|\} \leq 1$, implying that c_1 and c_2 are the two neighbors of a degree-two vertex or C is not a cutset. □

Lemma 10 *If C is a vertex cutset of size three of G where one of the separated sets does not contain any vertex of the outer face, then C consists of the three neighbors of a degree-three vertex.*

Proof. Consider a minimal cutset $C = \{c_1, c_2, c_3\}$ and let V_1 be a connected component that contains only interior vertices of G. By Lemma 8, C must be contained in a cycle of length at most six that has V_1 in its interior. Thus, Lemma 6 implies $|V_1| \leq 1$. □

In order to construct a long path in G, we will use the following well-known result.

Theorem 11 (Tutte [27]) *Every four-connected planar graph is Hamiltonian.*

Given a degree-three vertex x in a graph, adjacent to u, v, w, the corresponding Y-Δ transformation consists of removing vertex x, adding edges uv, uw, and vw, and removing any parallel edges. This is illustrated in

Figure 2. We define a new graph G' by applying the following two transformations to G, in the given order (see Figure 3):

1. Add an extra vertex v_∞, and make it adjacent to all vertices of G dual to the unbounded faces of $A(S)$.
2. For all vertices of degree three, perform a Y-Δ transformation. Note that after adding v_∞, no two vertices of degree three are adjacent. Hence, the transformation is well-defined.

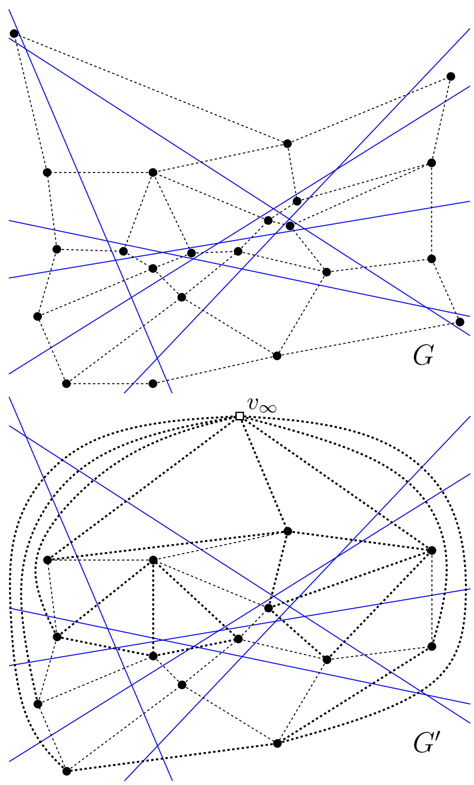

Figure 3: Transforming the dual graph G into G'.

Lemma 12 *G' is a four-connected planar graph, hence it is Hamiltonian.*

Proof. We need to show that G' does not have any cutset of size three or less. We first have to make the following observation: if C is a cutset of G', then $C \setminus \{v_\infty\}$ is a cutset of G. To check this, we need to show that the vertices of degree three that are eliminated by the Y-Δ transformations do not reconnect the separated vertex sets in G'. Each such Y-Δ transformation involves three vertices u, v, w that are pairwise adjacent in G'. Hence the degree-three vertex of G that was eliminated can be assigned to the same side of the partition in G as the vertices $\{u, v, w\} \setminus C$.

Now, we rule out the existence of a cutset C of size two in G', thereby showing that G' is at least three-connected. From our observation, $C \setminus \{v_\infty\}$ would be a cutset of G. If v_∞ belongs to C then $C \setminus \{v_\infty\}$ has size one, which is impossible since G is known to be two-connected. Otherwise, we have a cutset of size two in G, which from Lemma 9 must be neighbors of a degree-two vertex in G. This vertex, however, after adding v_∞ became of degree three, thus it must have been eliminated by a Y-Δ transformation, hence again C cannot be a cutset in G'.

Now suppose that C is a cutset of size exactly three in G', and let us first suppose that C does not contain v_∞. From our observation, C is also a cutset of G, and from Lemma 10 it either (i) consists of the neighbors of a degree-three vertex of G, or (ii) is such that both separated sets contain vertices of the outer face. In case (i), since degree-three vertices of G are eliminated by the Y-Δ transformations, C cannot be a cutset of G' and we have a contradiction. In case (ii), since the vertex subsets separated in G are connected by v_∞ in G', C cannot be a cutset in G' and we have a contradiction again. Hence we can assume that $v_\infty \in C$, and that $C \setminus \{v_\infty\}$ is a cutset of size two of G. But this case was already ruled out above, hence G' cannot have a cutset of size at most three, and therefore is four-connected. □

Lemma 13 *G' has at least $n^2/6 - 5n/6 + 2$ vertices.*

Proof. Recall that G has exactly $N = \binom{n+1}{2} + 1$ vertices. Further, it is known that the maximum number of degree-three interior vertices in G, that is, of bounded triangular faces in the arrangement $A(S)$, is at most $n(n-2)/3$ [12]. Also, the number of degree-two vertices is at most $2n$. These are exactly the vertices that are eliminated by the Y-Δ transformation to obtain G'. Hence the number of vertices of G' is at least $n(n+1)/2+2-(n(n-2)/3)-2n = n^2/6-5n/6+2$. □

Theorem 14 $f(n) \geq n^2/6 - 5n/6$.

Proof. Consider a Hamiltonian cycle in G'. This cycle can be transformed into a simple path of length at least $n^2/6-5n/6$ in G, by eliminating v_∞ and replacing every portion of the cycle using one or two edges of a triangle obtained from a Y-Δ transformation by two edges incident to the degree-three vertex (see Figure 4). □

3.2 Upper Bound

We show that the previous lower bound on the length of the longest simple path in G is within a factor two of the optimum.

Proposition 15 $f(n) \leq n^2/3 + O(n)$.

Proof. It is well-known that the cells of any line arrangement can be properly two-colored, hence that G is a bipartite graph. We will refer to the colors as black and white. Füredi and Palásti [13] give an example of

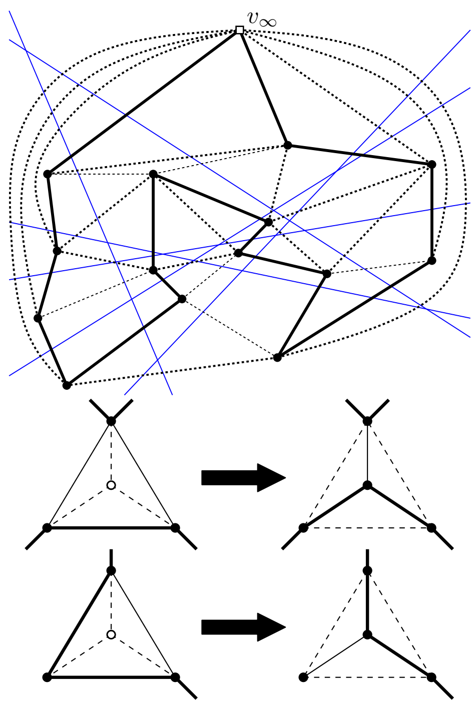

Figure 4: Obtaining a long path in G (operations shown at the bottom) from a Hamiltonian cycle in G' (top).

an arrangement of n lines in which there are $n^2/3+O(1)$ black cells and $n^2/6+O(n)$ white cells. Hence, there are roughly twice as many black cells as white cells, which is known to be asymptotically tight [18]. Now observe that any simple path or cycle in G will alternatingly traverse white and black cells, hence cannot have length greater than $n^2/3 + O(n)$. □

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