Drawing some 4-regular planar graphs with integer edge lengths

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Abstract

A classic result of Fáry states that every planar graph can be drawn in the plane without crossings using only straight line segments. Harborth *et al.* conjecture that every planar graph has such a drawing where every edge length is integral. Biedl proves that every planar graph of maximum degree 4 that is not 4-regular has such a straight-line embedding, but the techniques are insufficient for 4-regular graphs. We further develop the rigidity-theoretic methods of the author and examine an incomplete construction of Kemnitz and Harborth to exhibit integral drawings of families of 4-regular graphs.

1 Introduction

All graphs in this paper are simple and finite. Let $G = (V, E)$ be a planar graph. A *Fary embedding* $\phi : V \to \mathbb{R}^2$ of a planar graph is an embedding such that the straight-line drawing induced by ϕ has no crossing edges. Fáry [3] proved that all planar graphs have such an embedding. A natural extension of Fáry's theorem is to require that every edge has integral length, but it is not known if every planar graph has such an embedding, which we call an *integral Fary embedding.*

Conjecture 1 (Harborth *et al.* [7]) *All planar graphs have an integral Fary embedding.*

Analogously, we call a Fary embedding with rational edge lengths a *rational Fary embedding.* For the remainder of the paper, we consider only rational Fary embeddings, since an appropriate scaling yields an integral one.

Kemnitz and Harborth [8] show that every planar 3-tree has a rational Fary embedding. However, their solution for the analogous operation for 4-valent vertices does not always work. Geelen *et al.* [4] use a technical theorem of Berry [1] to prove Conjecture 1 for cubic planar graphs. Biedl [2] notes that their proof extends to even more graphs. One family of interest are the *almost 4-regular graphs*, namely the connected graphs of maximum degree 4 that are not 4-regular. These results actually yield rational Fary embeddings with rational coordinates, and we call such embeddings *fully-rational.*

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Biedl strongly conjectures that all 4-regular graphs have rational Fary embeddings. The aforementioned methods all rely on inductively adding vertices of degree at most 3 into a rational Fary embedding, but unfortunately, there are no known general methods for adding vertices of degree 4. It is even unknown whether or not there is a point in the interior of a unit square at rational length from each of the four vertices [6].

A previous paper by the author [11] details a construction of rational Fary embeddings of graphs using elementary results from rigidity theory. We use these rigidity-theoretic techniques for drawing planar graphs with small edge cuts and synthesize the aforementioned results to prove the existence of rational Fary embeddings for two families of 4-regular planar graphs, namely those that are not 4-edge-connected and those with a diamond subgraph.

2 Berry's Theorem and 3-Eliminable Graphs

Perhaps the most general technique known for constructing rational Fary embeddings is the following result of Berry [1].

Theorem 1 (Berry [1]) *Let A, B, and C be points in the plane such that AB, $(BC)^2$, and $(AC)^2$ are rational. Then the set of points P at rational distance with all three points is dense in the plane.*

Geelen *et al.* [4] show that this leads to an inductive method for finding rational Fary embeddings of a certain family of graphs. If G is a graph on n vertices, a sequence of those vertices $v_1, v_2, \ldots, v_n$ is a *3-elimination order* [2] if

1. G is the graph on one vertex, or
2. v_n has degree at most 2 and $v_1, \ldots, v_{n-1}$ is a 3-elimination order for $G - v_n$, or
3. v_n has degree 3 and $v_1, \ldots, v_{n-1}$ is a 3-elimination order for some graph $(G - v_n) \cup uw$, where u and w are two of the neighbors of v_n.

A graph is said to be *3-eliminable* if it is has a 3-elimination order. For any two maps $p, p' : V \to \mathbb{R}^2$, let $d(p, p')$ be the Euclidean distance between p and p', interpreted as points in $\mathbb{R}^{2|V|}$. We say that a Fary embedding ϕ can be *approximated* by a type of Fary embedding (e.g. rational Fary embedding) if for all $\epsilon > 0$, there exists a Fary embedding ϕ' of that type such that $d(\phi, \phi') < \epsilon$. Geelen *et al.* essentially prove the following:

Theorem 2 (Geelen *et al.* [4], Biedl [2]) *Any Fary embedding ϕ of a 3-eliminable graph can be approximated by a fully-rational Fary embedding.*

We do not have a non-inductive characterization of 3-eliminable graphs, but some partial results are known. A graph $G = (V, E)$ is called *(k, l)-sparse* if for every subset of vertices V' of size at least k, the induced subgraph of G on V' has at most $k|V'| - l$ edges.

Theorem 3 (Biedl [2]) *Every $(2, 1)$-sparse graph is 3-eliminable, and hence any Fary embedding of a $(2, 1)$-sparse planar graph can be approximated by a fully-rational Fary embedding.*

The family of $(2, 1)$-sparse graphs contains many other interesting classes of graphs, some of which can be found in [2]. Our interest lies mostly in the following corollary, as it is used in our constructions of rational Fary embeddings for both families of 4-regular planar graphs.

Corollary 4 (Biedl [2]) *Any Fary embedding of an almost 4-regular planar graph can be approximated by a fully-rational Fary embedding.*

3 Rigidity Theory and Graphs With Small Edge Cuts

A *framework* is a pair (G, p) where G is equipped with a *configuration* $p : V(G) \to \mathbb{R}^d$ which sends vertices to points in d-dimensional Euclidean space. A *generic configuration* is one where its $|V|d$ coordinates are independent over the rational numbers, and a *generic framework* is one with a generic configuration. A framework is *flexible* if there is a continuous motion of the vertices preserving edge lengths that does not extend to a Euclidean motion of $\mathbb{R}^d$, and it is said to be *rigid* otherwise.

The rigidity of a framework can be tested by examining its rigidity matrix. Let G be a graph on n vertices and m edges, and fix an ordering of the edges $e_1, \ldots, e_m$. Define $f_G : \mathbb{R}^{nd} \to \mathbb{R}^m$ to be the function that takes a configuration p to a vector $(||p(e_1)||^2, \ldots, ||p(e_m)||^2)$ consisting of the squares of the edge lengths. The *rigidity matrix* of (G, p) is defined to be $\frac{1}{2} df_G(p)$, where d is the Jacobian, and its dimensions are $m \times nd$. Then, the kernel of the rigidity matrix corresponds to so-called "infinitesimal motions" of the framework. A *regular point* is a configuration that maximizes the rank of the rigidity matrix over all possible configurations. It is easy to see that generic configurations are all regular points. We say that a Fary embedding is *regular* if it is a regular point.

An edge is *independent* if the corresponding row in the rigidity matrix is linearly independent from the other rows. Otherwise, it is said to be *redundant*, since deleting it does not change the space of infinitesimal motions. For $d > 2$, it is a long-standing open problem to find a combinatorial characterization of graphs with all independent edges. However, a complete characterization is known in two dimensions.

Theorem 5 (e.g. Graver *et al.* [5]) *A generic framework of a graph in $\mathbb{R}^2$ has all independent edges if and only if it is $(2, 3)$-sparse.*

A framework is *minimally* rigid if it is rigid and deleting any edge makes it flexible. Perhaps the most well-known restatement of this result is known as Laman's theorem.

Corollary 6 (Laman [9]) *A generic framework of a graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if it is $(2, 3)$-sparse and has $2|V| - 3$ edges.*

One consequence of this result is that all planar $(2, 3)$-sparse graphs have rational Fary embeddings, as proved by the author in [11], but we are more concerned with how rigidity theory allows us to draw graphs with small edge cuts. An *edge cut* of a connected graph $G = (V, E)$ is a subset of E whose deletion disconnects the graph. A *minimal* edge cut has no edge cuts as proper subsets. For example, consider a Fary embedding of a planar graph with a bridge uv that splits the graph into subgraphs G_1 and G_2. Deleting uv yields a new flex, namely the one that allows us to translate G_1 or G_2 in a direction parallel to uv, and we move along this flex until the distance between uv is rational and replace the edge. Such a technique can be generalized to cuts of up to three edges, using the main trick from [11].

Lemma 7 (Sun [11]) *Let ϕ be a regular Fary embedding of G, and let uv be an independent edge. Then, ϕ can be approximated by a regular Fary embedding ϕ' such that $||\phi'(u) - \phi'(v)||$ is rational, and all other edge lengths remain the same.*

Theorem 8 *Let $G = (V, E)$ be a graph with a minimal edge cut $\{e_1, e_2, e_3\}$ which separates G into G_1 and G_2. Furthermore, suppose that e_1, e_2, and e_3 are not all incident with the same vertex. Then, each e_i is independent in a generic framework.*

Proof. Assume without loss of generality that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are minimally rigid graphs. If G is also minimally rigid, then each of the edges in the cut must be independent. Furthermore, assume that V_1 and V_2 are just the vertices incident with the e_i's, in which case G_1 and G_2 are one of the complete graphs K_2 or K_3. We can make this assumption because any flex on G induces a rigid motion on G_1 and G_2, so replacing each graph with K_2 or K_3 still results in a flexible graph. There are just three graphs under this assumption, which are depicted in Figure 1. By Corollary 6, all three are rigid. □

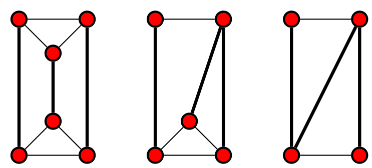

Figure 1: The three minimally rigid graphs in Theorem 8. The edge cuts are thickened.

The previous theorem is the best possible in terms of the number of edges in the cut, since for an edge cut of size 4, there are $|E_1|+|E_2|+4 \geq (2|V_1|-3)+(2|V_2|-3)+4 > 2|V|-3$ edges (we have strict inequality when $|V_1|$ or $|V_2|$ is 1), so one of the edges in the cut is redundant. Furthermore, we require that the e_i's not meet at the same vertex for the same reason.

Using this result and those of the previous section, we obtain our first result for 4-regular graphs.

Theorem 9 *All connected 4-regular planar graphs that are not 4-edge-connected have rational Fary embeddings.*

Proof. By a degree-counting argument, a 4-regular graph cannot have a minimal edge cut of size 3, so the edge cut must consist of two edges. Let G be a 4-regular planar graph that is not 4-edge-connected, and let ϕ be a Fary embedding of G. We can perturb ϕ to a generic (and hence regular) Fary embedding ϕ'. In the framework (G, ϕ'), the edges of the cut are independent by Theorem 8. There exists an open neighborhood around ϕ' consisting of only regular Fary embeddings, so if our perturbations of ϕ' are suitably small, every edge of the cut stays independent.

Deleting the edge cut yields two almost 4-regular graphs G_1 and G_2. By Corollary 4, each G_i can be approximated by a rational Fary embedding. Combining these two approximations yields a Fary embedding of G such that the only edges that are possibly not rational are those in the cut. By applying Lemma 7 on each edge of the cut, we obtain a rational Fary embedding of G. □

4 An Operation of Kemnitz and Harborth

The inductive step in proving Fáry's theorem possibly deletes edges and inserts a new k-valent vertex into the interior of the resulting polygon, as in Figure 2. We call this operation a *k-addition* if we start and end with rational Fary embeddings. Kemnitz and Harborth [8] tried to find k-additions for $k = 3, 4, 5$, following the proof of Fáry's theorem. Geelen *et al.* [4] remarked that Theorem 1 suffices for the case $k = 3$. For adding a vertex of degree 4 into a quadrilateral Q, Kemnitz and Harborth chose to place the new vertex on the diagonal so that one of the constraints is eliminated.

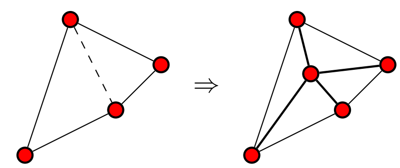

Figure 2: Adding a vertex of degree 4 after deleting an edge.

Consider a quadrilateral Q with a diagonal D of length f, as in Figure 3. Kemnitz and Harborth attempt to find a point P on D such that for rational lengths a, b, c, d, and f, the lengths x, y, and z are rational as well. They do not accomplish this for all quadrilaterals, though they always find a point on the line containing D. We briefly review their solution of the associated Diophantine equations.

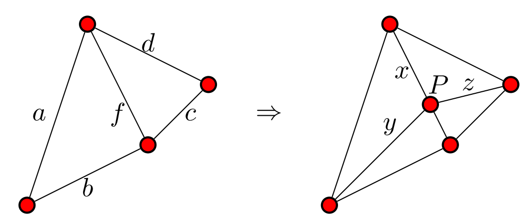

Figure 3: Variables for the Diophantine equations of Kemnitz and Harborth.

Let $s = \frac{y-a}{x}$ and $t = \frac{z-d}{x}$. Note that for nondegenerate Q, s and t cannot be ± 1. We can express x as

$$x = \frac{2afs + a^2 + f^2 - b^2}{f(1-s^2)} = \frac{2dft + d^2 + f^2 - c^2}{f(1-t^2)}.$$

It suffices to find suitable values of s and t such that the second equality holds. Let

$$\begin{aligned} K &= a^2 + f^2 - b^2 \\ L &= 2af \\ M &= d^2 + f^2 - c^2 \\ N &= 2df. \end{aligned}$$

t and s are related by

$$t = \frac{1}{2(K + Ls)}(N(s^2-1) \pm \sqrt{R})$$

where

$$\begin{aligned} R = N^2 s^4 + 4LM^3 + 2(2KM + 2L^2 - N^2)s^2 \\ +4L(2K - M)s + 4K(K-M)^2 + N^2. \end{aligned}$$

We wish for $\sqrt{R}$ to be rational, so if we let

$$\begin{aligned} q &= \sqrt{R}, \\ S &= 4LM, \\ T &= 2(2KM + 2L^2 - N^2), \\ U &= 4L(2K - M), \\ V &= 4K(K - M) + N^2, \end{aligned}$$

we obtain the Diophantine equation

$$N^2 s^4 + Ss^3 + Ts^2 + Us + V = q^2,$$

which has already been solved in Mordell [10]. The solution of this equation gives one for the original Diophantine equation via substitution.

Theorem 10 (Kemnitz and Harborth [8]) *If $4N^2ST - S^3 - 8N^4U \neq 0$, then there exists a solution for P where x, y, and z are all rational that satisfies*

$$\begin{aligned} s &= \frac{64N^6V - 4(N^2T - S^2)^2}{8N^2(4N^2ST - S^3 - 8N^4U)} \\ q &= \frac{4N^2(2N^2s^2 + Ss + T) - S^2}{8N^3}. \end{aligned}$$

Kemnitz and Harborth analyze the case where the denominator of s vanishes, but for our purposes, Theorem 10 is sufficient. Unfortunately the solution to the Diophantine equation does not guarantee that P lies inside the quadrilateral, so it cannot be used as a general operation on rational Fary embeddings. For drawing 4-regular graphs with diamond subgraphs, we make use of *permissible* quadrilaterals, namely those where P does land on the diagonal. All we need is the existence of just one permissible quadrilateral.

Proposition 11 *The quadrilateral with lengths*

$$a = b = 3, \;\; c = d = 4, \;\; f = 5$$

is permissible.

Proof. Tracing through Theorem 10 yields a value of $x = 282240/357599$, which yields a point inside the quadrilateral. □

Directly using Theorem 10 requires a 5-vertex wheel subgraph, but it turns out that we can relax the conditions slightly.

Proposition 12 *Theorem 10 still produces rational solutions even under the weaker condition that only b^2 and c^2 have to be rational.*

When we only require that b^2 and c^2 are rational when adding the new vertex, we call the operation a *generalized* 4-addition. For a fully-rational Fary embedding, the square of the distance between any two vertices is always rational, so Proposition 12 can be used when the corresponding edges are missing. Ultimately, we perform the generalized 4-addition on a slightly perturbed quadrilateral, so we need an additional result for quadrilaterals nearby.

Proposition 13 *Let Q be a permissible quadrilateral. There exists $\epsilon > 0$ such that every Q' satisfying $d(Q, Q') < \epsilon$ is also permissible.*

Proof. The solution for x is a continuous function of the edge lengths of Q, and hence a continuous function of the coordinates of the vertices. □

5 4-Regular Graphs With Diamonds

We use the results of the previous section to find a rational Fary embedding of a 4-regular graph with a diamond subgraph. The *diamond graph* is the simple graph on four vertices and five edges, and the name comes from the common visualization as two triangles sharing an edge. For a 4-regular graph G with a diamond subgraph, label the vertices of that subgraph and the other neighbor of one of the 3-valent vertices as in the leftmost graph in Figure 4. Let G' be the graph formed by deleting P from G and adding the edge v_2v_4, and let G_T be the graph formed by deleting P and adding the edges v_1v_4 and v_3v_4.

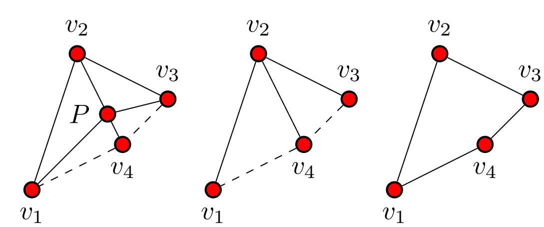

Figure 4: Local drawings of our graphs G, G', G_T. Dashed lines are possibly missing edges.

The main idea of our construction is to perform a generalized 4-addition on a fully-rational Fary embedding of G' to get one of G, but some preliminary results are needed to ensure that the quadrilateral formed by the v_i's is permissible and that adding P does not create any crossing edges. We want to show that the quadrilateral $Q = v_1v_2v_3v_4$ is a face in some planar embedding of G_T, and by using a modification of Tutte's spring theorem, we can devise a Fary embedding where Q is permissible and empty in the interior.

The figure suggests that v_1v_2P and v_2v_3P are faces in the planar embedding. Luckily for the graphs we consider, this is true for any planar embedding.

Proposition 14 *Let G be a planar 4-edge-connected 4-regular graph. Then, every K_3 subgraph bounds a face in any planar embedding of G.*

Proof. Consider any K_3 subgraph with vertices w_1, w_2, and w_3. The edges of the subgraph form a simple cycle C, so consider the remaining six edges incident with the w_i's. We assert that the neighbors of w_1, w_2, and w_3, besides each other, are either all inside C or all outside C. If this is not the case, then there are $g > 0$ and $h > 0$ edges that are incident with vertices inside and outside of C, respectively. Since $g + h = 6$, either g or h is less than 4. However, this implies that one of those sets of edges is a cut of size less than 4, which is a contradiction. Thus, one of g or h must be 0, so C is a face. □

Corollary 15 *Q is a face of some planar embedding of G_T.*

Proof. For any planar embedding of G, Proposition 14 implies that the cyclic rotation of vertices around P is v_1, v_2, v_3, v_4 or its mirror, so we may add the edges v_1v_4 and v_3v_4 into this embedding without violating planarity. Deleting P yields a planar embedding of G_T with Q as a face. □

Now that we know that Q is a face of G_T, we need to draw it in the shape of a permissible quadrilateral. A well-known result in graph theory, sometimes referred to as Tutte's spring theorem, states that for a 3-connected plane graph G with exterior face F, we can make F whatever convex polygon we desire and obtain a Fary embedding with all convex interior faces. If the 3-connectedness condition is dropped, we can add vertices and edges into the graph to make the graph 3-connected, but the faces induced on the original graph might not be convex.

Theorem 16 (Tutte [12]) *Let G be a plane graph with a simple face F and a prescribed convex embedding $\phi_F : V(F) \to \mathbb{R}^2$ of F. Then, there exists a Fary embedding ϕ of G such that ϕ restricted on the vertices of F is equal to ϕ_F and F is the exterior face.*

Corollary 17 *Theorem 16 can be modified so that in ϕ, F is an interior face.*

Proof. Let $P : \mathbb{R}^2 \to S^2$ be the Riemann stereographic projection, where we view S^2 as the unit sphere centered at the origin of $\mathbb{R}^3$ and $\mathbb{R}^2$ as the hyperplane in $\mathbb{R}^3$ that is zero in the last coordinate. Define $r : S^2 \to S^2$ to be the reflection of the sphere across the plane $\mathbb{R}^2$. Let U be the map $P^{-1}rP$, which is defined for all non-origin points in the plane. Intuitively, U "inverts" a Fary embedding, since any face containing the north pole in the sphere will now contain the south pole, and hence, that face goes from being exterior to interior.

Translate ϕ_F so that the origin lies inside the face. The embedding $\phi'_F = U\phi_F$ is an embedding of an inverted face F. Using Theorem 16 on ϕ'_F gives a Fary embedding ϕ' with the inverted F, so $\phi = U\phi'$ restricted to the vertices of F is ϕ_F. Furthermore, F is now an interior face of ϕ. □

We now prove Conjecture 1 for the 4-regular graphs with diamonds.

Theorem 18 *Let G be a connected 4-regular planar graph with a diamond subgraph. Then, G has a rational Fary embedding.*

Proof. If G is not 4-edge-connected, the result follows from Theorem 9. Otherwise, Q is a face of G_T by Corollary 15. By using Corollary 17, we can construct a Fary embedding ϕ_T of G_T such that Q is empty and has the edge lengths prescribed in Proposition 11. ϕ_T is also a valid Fary embedding for G' since Q was drawn as a convex quadrilateral.

The vertex v_1 is 3-valent in G', so G' is almost 4-regular. By Corollary 4, ϕ_T can be approximated by a fully-rational Fary embedding ϕ'. Q is still permissible in ϕ' by Proposition 13, and since ϕ' is fully-rational, Proposition 12 enables us to perform a generalized 4-addition on ϕ', yielding a rational Fary embedding ϕ of G. □

6 Conclusion

In this paper, we construct integral Fary embeddings of some 4-regular planar graphs, making progress on a conjecture of Biedl [2]. Perhaps surprisingly, one of the families we prove Conjecture 1 for has triangles close together, which seemingly make finding integral Fary embeddings difficult. The proof unfortunately does not extend to 4-regular graphs where the triangles are far apart. For every 4-regular planar graph, we can add an edge so that a diamond subgraph is formed, but undoing the generalized 4-addition operation does not always yield a 3-eliminable graph. Nonetheless, we believe that the techniques presented here can be extended to cover all 4-regular planar graphs.

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