Counting Carambolas

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Abstract

We give upper and lower bounds on the maximum and minimum number of certain geometric configurations hidden in a triangulation of n points in the plane. Configurations of interest include *star-shaped polygons* and *monotone paths*. We also consider related problems in *directed* planar straight-line graphs.

1 Introduction

Problems in extremal combinatorics typically ask the minimum or maximum number of certain subconfigurations in a configuration of a given size. We consider extremal problems where both the configuration and the subconfiguration have geometric attributes. Consider a (straight-line) triangulation of n points in the plane. Buchin et al. [1] showed that every triangulation contains $O(2.893^n)$ simple cycles, and there are triangulations that contain $\Omega(2.4262^n)$ simple cycles and $\Omega(2.0845^n)$ Hamilton cycles. Buchin and Schulz [2] proved that every n-vertex triangulation contains $O(5.2852^n)$ spanning trees. These techniques are instrumental for bounding the total number of noncrossing Hamilton cycles and spanning trees that n points in the plane admit [4, 6].

Van Kreveld et al. [7] were the first to consider substructures with geometric attributes. They proved that every triangulation contains $O(1.62^n)$ convex polygons (cycles), and some contain $\Omega(1.5028^n)$ convex polygons. Their upper bound is based on counting *star-shaped* polygons in a "reduced" graph, which is a carefully constructed subgraph of a given triangulation.

In this note, we consider subgraphs of a straight-line triangulation with other common geometric attributes. A *star-shaped* polygon (a.k.a. *carambola*, see Fig. 1) is a simple polygon P such that there is a point o with the property that every ray emanating from o intersects the boundary of P in exactly one point. Star-shaped polygons are closely related to monotone paths. A path P is *monotone* in direction $\underline{u} \in \mathbb{R}^2$, $\underline{u} \neq \underline{0}$, if every line orthogonal to $\underline{u}$ intersects P in at most one point. A special case is an x-monotone path, which is monotone in horizontal direction $\underline{u} = (1, 0)$. Table 1 summarizes known and new results:

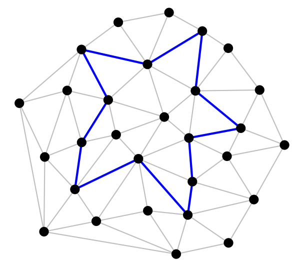

Figure 1: A "carambola" in a triangulation..

configurations	lower bound	upper bound
convex polygons	$\Omega(1.50^n)$ [7]	$O(1.62^n)$ [7]
star-shaped polygons	$\Omega(1.70^n)$	$O(n^3\alpha^n)$
monotone paths	$\Omega(1.70^n)$	$O(n\alpha^n)$
directed simple paths	$\Omega(\alpha^n)$	$O(n^2 3^n)$

Table 1: Bounds for the maximum number of configurations in an n-vertex plane (di)graph. Results in row 1 are and included for comparison; the bounds in rows 2-4 are proved in the paper. Note that $\alpha \approx 1.84$ is the real root of $x^3 = x^2 + x + 1$.

2 Lower Bounds

We construct plane straight-line graphs on n vertices that contain $\Omega(1.70^n)$ x-monotone paths (Fig. 2 and 3). By orienting all edges from left to right, we obtain a directed planar graph that contains $\Omega(1.70^n)$ directed paths. By connecting the leftmost and rightmost vertices by an extra edge, we obtain a plane straight-line graph that contains $\Omega(1.70^n)$ monotone polygons. By arranging three copies of this graph around the origin in a cyclic fashion (Fig. 4), we obtain a plane straight-line graph that contains $\Omega(1.70^n)$ star-shaped polygons.

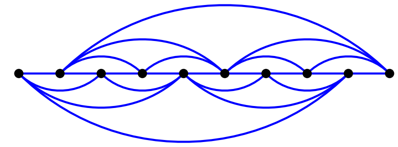

Figure 2: There are 1.70^n monotone paths in this graph.

Let $n = 2^\ell + 2$ for an integer $\ell \in \mathbb{N}$. We define the plane graph G on n vertices $V = \{v_1, \ldots, v_n\}$: it consists

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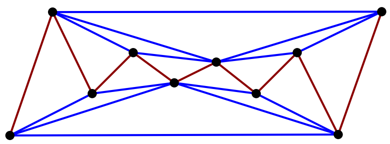

Figure 3: A straight-line embedding of the graph in Fig. 2, where the points lie alternately on two circular arcs, preserving x-monotonicity.

of a path $(v_1, \ldots, v_n)$ and two balanced binary triangulation of the vertices $\{v_1, \ldots, v_{n-1}\}$ and $\{v_2, \ldots, v_n\}$, respectively, one on each side of the path (Fig. 2). A straight-line embedding is shown in Fig. 3, where the odd (resp., even) vertices lie on a concave (convex) polygonal chain.

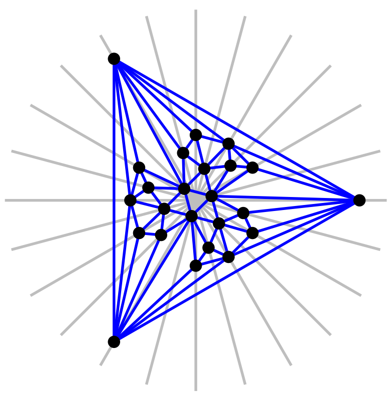

Figure 4: Cyclic embedding of 3 copies of the graph in Fig. 2. The monotone order becomes cyclic order.

Theorem 1 *The graph G described in the above paragraph has $\Omega(1.700^n)$ x-monotone paths.*

Proof. We count the number of x-monotone paths in a sequence of subgraphs of G. Let G_0 be a Hamilton path $(v_1, \ldots, v_n)$; and we recursively define G_k from G_{k-1} by adding the edges (v_i, v_{i+2^k}) for $i = j2^k + 1$ and $i = j2^k + 2$ for $j = 0, 1, \ldots, \ell/k - 1$. Denote by $p_k(v_i)$ the number of x-monotone paths in G_k that end at vertex v_i. Since every monotone path can be extended to the rightmost vertex v_n, the number of maximal[1] monotone paths in G_k is $p_k(v_n)$. We establish recurrence relations for $p_k(v_i)$. The initial values are $p_k(v_1) = p_k(v_2) = 1$ for all $k = 0, \ldots, \ell$. For $k = 1$ and $i \geq 2$, we have $p_1(v_i) = p_1(v_{i-1}) + p_1(v_{i-2})$, therefore $p_1(v_i) = F_i$ is the ith Fibonacci number and $p_1(v_n) = \Theta(1.619^n)$.

The recurrence for $p_k(v_i)$, $k \geq 2$, is more nuanced, due to the asymmetry between the triangulations on the two sides of the Hamilton path. We partition the vertices of G_k into disjoint groups of consecutive vertices of size 2^k. Let $a_i = v_{i2^k+1}$ denote the first vertex of group i, and let $b_i = v_{i2^k+2}$ be the second vertex of group i. We count in how many ways one can route an x-monotone path through a group. A path through group i starts at either a_i or b_i, and ends at either a_{i+1} or b_{i+1}. Thus, it is enough to keep track of four different type of paths. By our choice, the edge (a_i, b_i) belongs to group i but not to group $i+1$. We record the number of paths connecting a_i or b_i to a_{i+1} or b_{i+1} in a 2×2 matrix M_k, such that

$$M_k \cdot (p_k(a_i), p_k(b_i))^T = (p_k(a_{i+1}), p_k(b_{i+1}))^T.$$

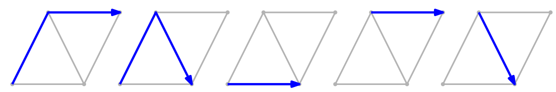

Figure 5: The five possible x-monotone paths in the group of G_2.

Once the matrix M_k is known, we can compute the number of paths by $(p(v_{n-1}), p(v_n))^T = M_k^{n/2^k} \cdot (1,1)^T$. By the Perron-Frobenius theorem, $\lim_{p\to\infty} M_k^p/\lambda^p = A$, for some matrix A, and for λ being the largest eigenvalue of M_k. Hence, we have $\lim_{n\to\infty} T_k(v_n) = \Theta(\lambda^{n/2^k})$ maximal x-monotone paths in G_k.

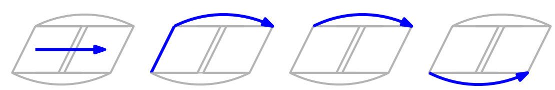

Figure 6: Schematic drawing of the paths counted by M_k.

In the last step we show how to compute the matrices M_k. The matrix M_2 can be easily obtained by hand (see Fig. 5). For a block of size 2^k in G_k, we have all the paths that use only the edges in G_{k-1} (and can therefore be decomposed in two paths of G_{k-1}'s blocks) plus one additional path of type $a_i \to a_{i+1}$, $a_i \to b_{i+1}$, and $b_i \to b_{i+1}$ each (see Fig. 6). Therefore, we can compute the matrices M_k as

$$M_2 := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } M_i := M_{i-1}^2 + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

k	2	3	4	5	6
$\lambda^{2^{-k}}$	1.61803	1.69605	1.70034	1.70037	1.70037

Table 2: The asymptotic growth of x-monotone paths in the graphs G_k. For $k = 6$ it follows that there are $\Omega(1.70037^n)$ monotone paths.

Table 2 shows the values $\lambda^{1/2^k}$ for $k = 2, \ldots, 6$. We observed that when going from $k = 5$ to $k = 6$, there is no change in $\lambda^{1/2^k}$ up to 8 digits after the point. The precise value for $k = 5$ equals $\lambda = 1/2(4885 + 9\sqrt{294153})$. $\square$

[1] *Maximal* for containment.

Directed plane graphs. We also construct n-vertex directed plane straight-line graphs that contain $\Omega(1.83^n)$ directed paths (Fig. 7). The directed paths, however, cannot be extended to a cycle because in a planar embedding the start and end vertex are not in the same face. Similarly, most of the directed paths are not monotone in any direction in a planar embedding.

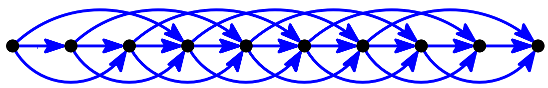

Figure 7: There are $\Theta(\alpha^n)$, $\alpha \approx 1.84$, directed paths in this graph. A plane embedding of the graph is depicted in Fig. 8.

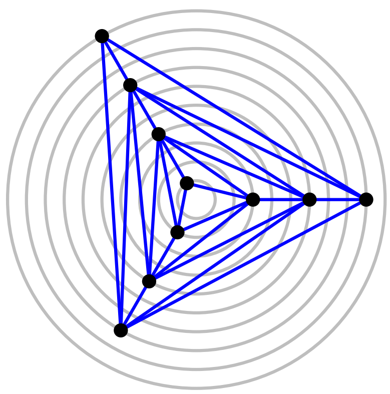

Figure 8: A plane embedding of the graph in Fig. 7. The edges are directed from the outer circles to inner inner circles.

Denoting by $T(i)$ the number of directed paths ending at vertex v_i, we have $T(1) = T(2) = 1$, $T(3) = 2$, and a linear recurrence relation

$$T(i) = T(i-1) + T(i-2) + T(i-3)$$

for $i \geq 4$. The recurrence solves to $T(i) = O(\alpha^i)$, where $\alpha \approx 1.83929$ is the real root of $x^3 = x^2 + x + 1$. Therefore the total number of directed paths, starting from any vertex, is $\Theta(\alpha^n)$.

3 Upper Bounds

Monotone Paths. We start with x-monotone paths in plane straight-line graph. We prove the bound for a broader class of graphs, *plane monotone graphs*, in which every edge is an x-monotone Jordan arc (since some of the operations in our argument may not preserve straight-line edges).

Let $n \in \mathbb{N}$, $n \geq 3$, and let $G = (V, E)$ be a plane monotone graph with $|V| = n$ vertices that maximizes the number of x-monotone paths. We may assume that the vertices have distinct x-coordinates (otherwise we can perturb the vertices without decreasing the number of x-monotone paths). We may also assume that G is fully triangulated (i.e., it is an edge-maximal planar graph), otherwise we add extra edges which only increase the number of x-monotone paths. Label the vertices in V as $v_1, v_2, \ldots, v_n$ sorted by their x-coordinates. Orient each edge $\{v_i, v_j\} \in E$ from v_i to v_j if $i < j$.

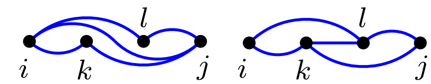

Figure 9: The flip operation.

Consider an edge $\overline{v_i v_j} \in E$ that is not on the boundary. There are two triangular faces incident to $\overline{v_i v_j}$, and two other vertices v_k and v_l that are connected to both v_i and v_j.

Claim 2 *If $i < k < j$ and $i < l < j$, then flipping $\overline{v_i v_j}$ to $\overline{v_k v_l}$ or $\overline{v_l v_k}$ (depending on whether $k < l$ or $l < k$) only increases the number of paths. (See Fig. 9.)*

Since G has the maximum number of x-monotone paths, we may now assume that for all edges in G, there is a vertex either to the left or to the right of the edge that is adjacent to both endpoints. We show next that G contains an x-monotone Hamilton path.

Lemma 3 *All edges $\overline{v_i v_{i+1}}$ are present in G.*

Proof. Suppose, to the contrary, that there are two non-adjacent vertices v_i and v_{i+1}. Since G is a triangulation, there must be an edge $\overline{v_j v_k}$, with $j < k$, that separates v_i and v_{i+1}. Since the edge $\overline{v_j v_k}$ is x-monotone, we have $j < i < i+1 < k$. Assume w.l.o.g. that v_i lies below $\overline{v_j v_k}$ and v_{i+1} lies above $\overline{v_j v_k}$. By Claim 2, the triangle incident to $\overline{v_j v_k}$ from either above or below connects to a vertex v_l either to the left of v_j or to the right of v_k. Assume w.l.o.g. it is the triangle above, and that v_l lies to the right of v_k. Now consider edge $\overline{v_j v_l}$. Since the triangle below it has v_k as third vertex and $j < k < l$, there must be another vertex v_m that connects to $\overline{v_j v_l}$ and lies either to the left of j or to the right of l. This argument repeats, and we never reach v_{i+1}. Contradiction. $\square$

For any pair $i < j$, let V_{ij} denote the set of consecutive vertices $v_i, v_{i+1}, \ldots, v_j$, and let $G_{ij} = (V_{ij}, E_{ij})$ be the subgraph of G induced by V_{ij}.

Since G is planar, we know that $|E| \leq 3|V| - 6$, and furthermore, that $|E_{ij}| \leq 3|V_{ij}| - 6$ for all subgraphs induced by groups of 3 or more consecutive vertices.

In the remainder of the proof we will apply a sequence of operations on G that may create multiple edges and edge crossings. Hence, we consider G as an abstract multigraph. However, the operations will maintain the invariant that $|E_{ij}| \leq 3|V_{ij}| - 6$ for all $i < j$.

Let $i < j < k$ be a triple of indices such that $\overline{v_i v_j}, \overline{v_i v_k} \in E$. The operation $\mathsf{shift}(i, j, k)$ removes the edge $\overline{v_iv_k}$ from E, and inserts the edge $\overline{v_jv_k}$ into E (see Fig. 10). Note that the new edge may already have been present, in this case we insert a new copy of this edge (i.e., we increment its multiplicity by one).

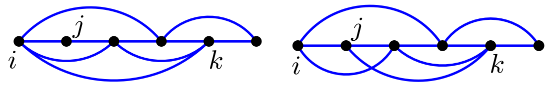

Figure 10: The operation $\mathsf{shift}(i, j, k)$.

Claim 4 *The operation* $\mathsf{shift}(i, j, k)$ *does not decrease the number of* x*-monotone paths in* G*.*

Proof. Clearly, any path that used $\overline{v_iv_k}$ can be replaced by a path that uses $\overline{v_iv_j}$ and (the new copy of) $\overline{v_jv_k}$. $\square$

Now, we apply the following algorithm to the input graph G. We process the vertices from left to right, and whenever we encounter a vertex v_i with outdegree 4 or higher, we identify the smallest index j such that v_i has an edge to v_j and the largest index k such that v_i has an edge to v_k; and then apply $\mathsf{shift}(i, j, k)$. We repeat until there are no more vertices with outdegree larger than 3.

Claim 5 *The algorithm terminates and maintains the invariants that (1) all edges* $\overline{v_iv_{i+1}}$ *are present in* G *with multiplicity one; and (2)* $|E_{ij}| \leq 3|V_{ij}| - 6$ *for all subgraphs induced by* V_{ij}*,* $i < j$*.*

Proof. Initially, invariant (1) holds by Lemma 3, and (2) by planarity. Suppose, to the contrary, that there is an operation that increases the number of edges of an induced subgraph above the threshold. Let $\mathsf{shift}(i, j, k)$ be the first such operation. Since the only new edge is $\overline{v_jv_k}$, any violator subgraph must contain both v_j and v_k; and it cannot contain v_i since the only edge removed is $\overline{v_iv_k}$. Recall that v_j was the leftmost vertex that v_i is adjacent to; and by invariant (1), we know $j = i + 1$. Therefore, the violator subgraph is induced by $V_{jk'}$ for some $k' \geq k$, and we have $|E_{jk'}| > 3|V_{jk'}| - 6$ after the shift. Since v_k was the rightmost vertex adjacent to v_i before the shift, all outgoing edges of v_i went to vertices in $V_{jk'}$. The outdegree of v_i was at least 4 before the shift, hence $G_{ik'}$ had at least $3|V_{ik'}| - 4$ edges. Contradiction. $\square$

Now, after executing the algorithm, we are left with a multigraph where the outdegree of every vertex is at most 3, and no subgraph induced by $|V_{i,j}| \geq 3$ consecutive vertices has more than $3|V_{ij}| - 6$ edges. This, combined with invariant (1), implies that the multiplicity of any edge $\overline{v_iv_{i+2}}$ is at most one. Thus, for every vertex v_i, the three outgoing edges go to vertices at distance at least 1, 2, and 3, respectively, from v_i. Denoting by $T(i)$ the number of x-monotone paths that start at v_{n-i+1}, we arrive at the recurrence

$$T(i) = T(i-1) + T(i-2) + T(i-3)$$

for $i \geq 4$, with initial values $T(1) = T(2) = 1$, $T(3) = 2$. The recurrence solves to $T(n) = O(\alpha^n)$ where $\alpha \approx 1.83929$ is the real root of $x^3 - x^2 - x - 1 = 0$. Therefore, every plane straight-line graph on n vertices admits at most $O(\alpha^n)$ x-monotone paths.

Since the edges of an n-vertex planar straight-line graph have $O(n)$ distinct directions, the number of monotone paths (in any direction) is bounded by $O(n\alpha^n)$.

Star-shaped Polygons. Given a plane straight-line graph G on n vertices, the lines passing through the $O(n)$ edges induce a line arrangement with $O(n^2)$ faces. Choose a face f of the arrangement, and a vertex p of G. We show that G contains $O(\alpha^n)$ star-shaped polygons with vertex v and a star center lying in f. Indeed, pick an arbitrary point $o \in f$. Each edge of G is oriented either clockwise or counterclockwise with respect to o (with the same orientation for any $o \in f$). Order the vertices of G by a rotational sweep around o starting from the ray $\overrightarrow{ov}$. Let G_v be the graph obtained from G by deleting all edges that cross the ray $\overrightarrow{ov}$. We can repeat the argument for x-monotone path for G_f, replacing the x-monotone order by the rotational sweep order about o, and conclude that G admits $O(\alpha^n)$ star-shaped polygons with vertex p and star center o.

Directed Simple Paths. Let $G = (V, E)$ be a directed planar graph. Denote by $\deg^-(v)$ the outdegree of vertex $v \in V$; let $V_0 = \{v_1, \ldots, v_\ell\}$ be the set of vertices with outdegree at least 1, where $1 \leq \ell \leq n$. We show that for every $v_0 \in V_0$, there are $O(3^n)$ maximal[2] directed simple paths starting from v_0. Each maximal directed simple path can be encoded in an ℓ-dimensional vector that contains the outgoing edge of each vertex $v \in V_0$ in the path (and an arbitrary outgoing edge if $v \in V_0$ is not part of the path). The number of such vectors is

$$\prod_{i=1}^{\ell} \deg^-(v_i) \leq \left(\frac{1}{\ell}\sum_{i=1}^{\ell} \deg^-(v_i)\right)^\ell < \left(\frac{3n}{\ell}\right)^\ell \leq 3^n,$$

where we have used the geometric-arithmetic mean inequality, Euler's formula $\sum_{i=1}^{\ell} \deg^-(v_i) \leq 3n - 6 < 3n$, and maximized the function $x \to (3n/x)^x$ over the interval $1 \leq x \leq n$. Since there are $O(n)$ choices for the starting vertex $v_0 \in V_0$, and a maximal simple path contains $O(n)$ nonmaximal paths starting from the same vertex, the total number of simple paths is $O(n^2 3^n)$.

[2] *Maximal* for containment.

4 Minimizing the number of configurations

In this section, we explore the *minimum* number of geometric configurations that a triangulation on n points in the plane can have. Our bounds are summarized in Table 3.

configurations	lower bound	upper bound
convex polygons	$\Omega(n)$	$O(n)$
star-shaped polygons	$\Omega(n)$	$O(n^2)$
monotone paths	$\Omega(n^2)$	$O(n^{3.39})$
directed paths	$\Omega(n)$	$O(n)$

Table 3: Bounds for the minimum number of configurations in a (directed) triangulation with n vertices.

Potatoes. Every n-vertex triangulation has $\Theta(n)$ convex faces, hence $\Omega(n)$ is a natural lower bound for the number of convex polygons. The triangulation in Fig. 11(left) contains $\Theta(n)$ convex polygons, which is the best possible apart from constant factors. The triangulation consists of the join of two paths $P_2 * P_{n-2}$, where path P_{n-2} is realized as a monotone zig-zag path. Every convex polygon is either a triangle or the union of two adjacent triangles that share a flippable edge [5].

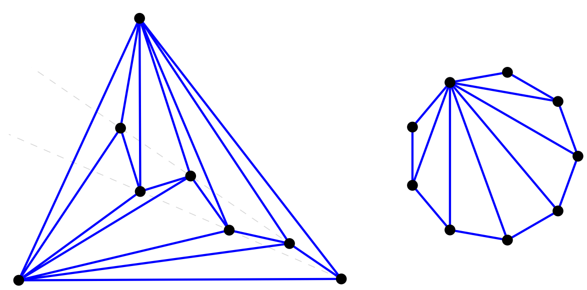

Figure 11: There are $\Theta(n)$ convex polygons and x-monotone paths in the triangulation on the left; it contains $\Theta(n^4)$ star-shaped polygons and monotone paths. There are $\Theta(n^2)$ star-shaped polygons and $\Theta(n^4)$ monotone paths in the triangulation on the right.

Carambolas. The sum of degree squares $\Omega(\sum_{v\in V} \deg^2(v)) = \Omega(n)$ is a natural lower bound for the number of star-shaped polygons, since the union of consecutive triangles incident to a vertex forms a star-shaped polygon. This might suggest that a triangulation that minimizes the number of star-shaped polygons should have bounded degree. Surprisingly, the best construction found so far is a triangulation with a vertex of degree $n-1$ (Fig. 11, right), which admits $\Theta(n^2)$ star-shaped polygons.

Monotone paths. It is not difficult to see that between any two vertices, u and v, in a triangulation there is a monotone path in direction $\overrightarrow{uv}$ [3]. Hence every triangulation contains $\Omega(n^2)$ monotone paths. Two vertices, however, are not always connected by an x-monotone path: a trivial lower bound for x-monotone paths is $\Omega(n)$, since every edge is x-monotone.

The triangulation $P_2 * P_{n-2}$ in Fig. 11(left) is embedded such that P_{n-2} is x-monotone and lies to the right of P_2. With this embedding, it contains $\Theta(n^2)$ x-monotone paths: every x-monotone path consists of a sequence of consecutive vertices of P_{n-2}, and 0, 1, or 2 vertices of P_2. However, both triangulations in Fig. 11 admit $\Theta(n^4)$ monotone paths (in some direction).

Triangulations with a polynomial number of monotone paths are also provided by known constructions in which all monotone paths are "short." Dumitrescu et al. [3] constructed full triangulations with maximum degree $O(\log n/\log\log n)$ such that every monotone path has $O(\log n/\log n\log n)$ edges. Furthermore, there are triangulations with bounded degree in which every monotone path has $O(\log n)$ edges. These constructions contain polynomially many, but $\omega(n^4)$, monotone paths.

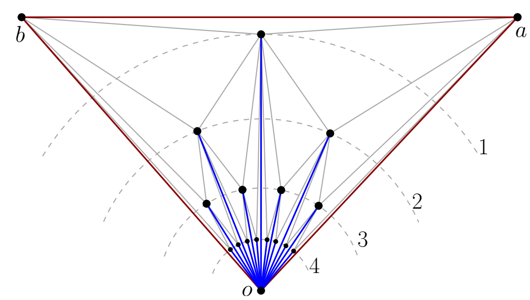

Figure 12: A triangulation from [3] in which every monotone path has $O(\log n)$ edges.

A triangulation that contains only $O(n^\beta \log^2 n)$ monotone paths, where $\beta = 2 + 2\log_2((1+\sqrt{5})/2) \approx 3.3885$ comes from [3]: It has maximum degree $n-1$ and every monotone path has $O(\log n)$ edges. Refer to Fig. 12.

The number of vertices is $n = 2^\ell + 1$ for some $\ell \in \mathbb{N}$. The outer face is a regular triangle abo, where o is the origin. The interior vertices are arranged on $\ell - 1$ circles centered at the origin, with 2^i points on circle i, where the radii of the circles rapidly converge to 0. The vertices on circle i are drawn interspersed (in angular order) with the vertices of the previous layers. The origin is connected to all other vertices, and a vertex on circle i is connected to the two vertices of the previous layers that are closest in angular order. The radii of the circles are chosen recursively such that the edges that connect an interior vertex v to vertices on smaller circles are almost parallel—thus a monotone path can contain two such edges for at most one interior vertex v. It follows that every monotone path contains at most two vertices from each circle, hence the $O(\log n)$ bound on the number of edges [3].

Claim 6 *For every $n \in \mathbb{N}$, there is an nvertex triangulation that admit $O(n^\beta \log^2 n)$ monotone paths, where $\beta = 2 + 2\log_2((1+\sqrt{5})/2) \approx 3.3885$.*

Proof. In the above construction, it is enough to count *maximal* monotone paths, since every monotone path can be extended to the outer triangle abo, and each maximal monotone path contains only $O(\log^2 n)$ subpaths. First, consider paths between a and b. For every path between a and b, we can record the layers of the vertices along the paths, where a and b are at level 0, and o is at level ℓ. This sequence must be unimodal for a monotone path (by construction). An a-b path that avoids the origin is uniquely determined by its modality (the vertex lying on the smallest circle), hence there are at most $O(n)$ such paths (all these paths happen to be monotone).

Consider now the paths incident to o. An a-b path that goes though o is counted as the combination of a path from o to a and one from o to b. By symmetry, it is enough to count monotone paths from o to a. Such a path also corresponds to a unimodal sequence (with o being the only modality). We have $n-1$ choices for the first edge of incident to o, and the remainder of the path is restricted to an outerplanar graph with at most $\ell + 2 = 1 + \log(n-1)$ vertices.

In an outerplanar graph with k vertices, any two vertices are connected by at most $F_k = \Theta((1+\sqrt{5})/2)^k) = O(1.62^k)$ paths, where F_k is the kth Fibonacci number. Therefore, the number of o-a paths is at most $n \cdot F_{\ell+2} = \Theta(n^{1+\log_2((1+\sqrt{5})/2)}) = O(n^{1.70})$. (In fact, all these paths are monotone.) Every path from a to b via o is the combination of two branches: a path from o to a and one from o to b. Hence the number of these paths is bounded by $O(n^\beta \log^2 n)$, where $\beta = 2 + 2\log_2((1+\sqrt{5})/2) \approx 3.3885$. $\square$

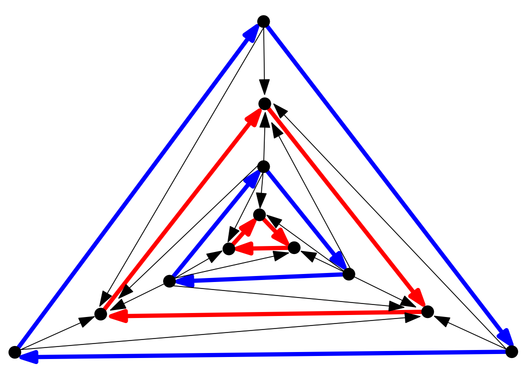

Figure 13: There are $\Theta(n)$ directed paths in this directed planar graph.

Directed paths. Every edge in a planar digraph is a directed path on its own, hence there are $\Omega(n)$ directed paths in every directed triangulation on n vertices. This bound is tight, apart from constant factors. Directed triangulation with $O(n)$ paths consists of a sequence of $n/3$ triangles, and edges between consecutive triangles point either inward or outward alternately (Fig. 13).

5 Conclusion

We considered the maximum and minimum number of star-shaped polygons, monotone paths, and directed paths that a (directed) triangulation of n points in the plane can have. Our results are summarized in Tables 1 and 3. Closing or narrowing the gap between the upper and lower bounds is left for future research.

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References

[1] K. Buchin, C. Knauer, K. Kriegel, A. Schulz, and R. Seidel. On the number of cycles in planar graphs. In *Proc. 13th Annual International Conference on Computing and Combinatorics (COCOON)*, volume 4598 of *Lecture Notes in Computer Science*, pages 97–107. Springer, 2007.

[2] K. Buchin and A. Schulz. On the number of spanning trees a planar graph can have. In M. de Berg and U. Meyer, editors, *Proc. 18th Annual European Symposium on Algorithms (ESA)*, volume 6346 of *Lecture Notes in Computer Science*, pages 110–121. Springer, 2010.

[3] A. Dumitrescu, G. Rote, and C. D. Tóth. Monotone paths in planar convex subdivisions. In *Proc. 18th Annual International Conference on Computing and Combinatorics (COCOON)*, volume 7434 of *Lecture Notes in Computer Science*, pages 240–251. Springer, 2012.

[4] M. Hoffmann, A. Schulz, M. Sharir, A. Sheffer, C. D. Tóth, and E. Welzl. Counting plane graphs: flippability and its applications. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 303–326. Springer, 2013.

[5] F. Hurtado, M. Noy, and J. Urrutia. Flipping edges in triangulations. *Discrete & Computational Geometry*, 22(3):333–346, 1999.

[6] M. Sharir, A. Sheffer, and E. Welzl. Counting plane graphs: Perfect matchings, spanning cycles, and Kasteleyn's technique. *J. Comb. Theory, Ser. A*, 120(4):777–794, 2013.

[7] M. van Kreveld, M. Löffler, and J. Pach. How many potatoes are in a mesh? In *Proc. 23rd International Symposium on Algorithms and Computation (ISAAC)*, volume 7676 of *Lecture Notes in Computer Science*, pages 166–176. Springer, 2012.