Ref.TH.3689-CERN

CHIRAL PERTURBATION THEORY TO ONE LOOP *)

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ABSTRACT

We expand the Green's functions of QCD in powers of the external momenta and of the quark masses. The Ward identities of chiral symmetry determine the expansion up to and including terms of order p^4 (at fixed ratio m_{quark}/p^2) in terms of a few constants, which may be identified with the coupling constants of a unique effective low energy Lagrangian. We then calculate the low energy representation of several Green's functions and form factors and of the $\pi\pi$ scattering amplitude. The values of the low energy coupling constants are extracted from available experimental data. The corrections of order M_π^2 to the $\pi\pi$ scattering lengths and effective ranges turn out to be substantial and the improved low energy theorems agree very well with the measured phase shifts. The observed differences between the data and the uncorrected soft pion theorems may even be used to measure the scalar radius of the pion, which plays a central role in the low energy expansion.

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*) Work supported in part by Schweizerischer Nationalfonds.

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Ref.TH.3689-CERN
August 1983

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1. Introduction

If the quark masses are set equal to zero, the QCD Hamiltonian is symmetric under the chiral group $SU(N_f)xSU(N_f)$. One assumes that the ground state of the theory spontaneously breaks this symmetry (Nambu 1960; Nambu and Jona-Lasinio 1961; Koyama 1967; Glashow and Weinberg 1968; Gell-Mann, Oakes and Renner 1968) down to $SU(N_f)$. The hidden symmetry then manifests itself in the occurrence of $N_f^2 - 1$ pseudoscalar Goldstone bosons. Furthermore, the symmetry fixes the low energy couplings of these particles (Weinberg 1966) in terms of the matrix element $\langle 0|A_\mu|\pi\rangle$. For massless quarks these low energy theorems are exact.

In reality, the QCD Hamiltonian contains a quark mass term which breaks the symmetry. The vector and axial currents are not exactly conserved:

$$\begin{aligned} \partial_\mu(\bar{u}\gamma^\mu s) &= i(m_u - m_s)\bar{u}s \\ \partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) &= (m_u + m_d)\bar{u}\, i\gamma_5 d \end{aligned} \tag{1.1}$$

Since the masses of u, d and s are however small, the divergence of the currents which generate SU(3)xSU(3) approximately vanishes. Accordingly the low energy theorems of SU(3)xSU(3) should be approximately valid in the real world. The subgroup SU(2)xSU(2) is an exact symmetry if $m_u = m_d = 0$. Since the masses of u and d are tiny, the low energy theorems associated with this subgroup should show even smaller deviations. The deviations from chiral symmetry may be studied by treating the quark mass term in the Hamiltonian as a perturbation, with massless QCD as the unperturbed system (chiral perturbation theory) (Dashen 1969; Dashen and Weinstein 1969; Pagels 1975). To have a rough estimate of the order of magnitude of the perturbations to be expected one may compare the size of the quark mass (for a review see Gasser and Leutwyler 1982)

$$\hat{m} = \frac{1}{2}(m_u + m_d) \tag{1.2}$$

($\hat{m} \sim 7$ MeV) with the typical energy of a light quark participating in the low energy process in question. This energy is of the order of the characteristic scale M of QCD, say M = 500 MeV or 1 GeV. The corrections to the soft pion theorems are expected to be of the form $1 + \hat{m}/M$: the low energy theorems of SU(2)xSU(2) should be valid to within one or two percent. This rule of thumb is confirmed if one compares SU(2)xSU(2) with SU(3). In order of magnitude the matrix elements of the densities $\bar{u}s$ and $\bar{u}i\gamma_5 d$ are expected to be the same. According to (1.1) this implies that the breaking of SU(2)xSU(2) should be smaller than the typical breaking of SU(3) by the factor $(m_u + m_d) : (m_s - m_u) = 1 : 12$ which measures the relative size of the symmetry breaking terms in the Hamiltonian. Since the SU(3) relations generally hold at the 20% level, the predictions of SU(2)xSU(2) should have an accuracy of 20% : 12 $\underset{-}{\sim}$ 2%.

Note that these crude order of magnitude estimates only concern the size of the matrix elements of the perturbation $m_u\bar{u}u + m_d\bar{d}d$. The effect of the perturbation may be amplified by small energy denominators. As pointed out by Li and Pagels (1971) the fact that the unperturbed system contains massless particles (Goldstone bosons) implies that the energy denominators may vanish - chiral perturbation theory contains infrared singularities which may enhance the size of the perturbation and lead to deviations from the soft pion theorems that are substantially larger than what is indicated by the above rule of thumb. Furthermore, we will show that some of the corrections generated by the quark mass term are accompanied by large numerical factors. In the case of the scattering length a_1^1 e.g., we find corrections of order 7 $M_\pi^2/M_\rho^2 \sim 20\%$. Although this ratio is algebraically of the type $\hat{m}/M$ it happens to be larger numerically by about an order of magnitude than what is suggested by the rule of thumb.

The low energy theorems provide us with very sensitive tests (Gasser and Leutwyler 1983) of QCD, allowing us, in particular, to test whether the observed low energy structures are consistent with the standard picture of a spontaneously broken chiral symmetry, as proposed by Nambu, Jona-Lasinio, Glashow, Weinberg, Gell-Mann, Oakes and Renner. To quantitatively compare the theoretical predictions with experimental data it is however necessary to first calculate the corrections.

In (Gasser and Leutwyler 1983) we have proposed a method which allows one to systematically determine the low energy structure of the Green's functions in QCD. The method extends Weinberg's analysis of S-matrix elements (Weinberg 1979) to an expansion of the Green's functions in powers of the momenta and of the quark masses. Since QCD does not contain any free parameters apart from the renormalization group invariant scale Λ and from the quark masses the expansion coefficients are fixed by these basic parameters of the theory. Unfortunately, we are not able to exploit the full content of this information in a quantitative manner. What our method allows us to do is to exploit the symmetry properties of the theory. Chiral symmetry implies a set of Ward identities which link the various Green's functions and therefore interrelate the expansion coefficients. As is well-known the leading terms in the low energy expansion are determined by two constants, viz. the pion decay constant and the vacuum expectation value of the scalar quark density (Weinberg 1966; Gell-Mann, Oakes and Renner 1968) (both of these constants are determined by Λ; crude numerical evaluations on a lattice (Hamber and Parisi 1981, 1982; Marinari, Parisi and Rebbi 1981; Hamber, Marinari, Parisi and Rebbi 1982; Weingarten 1982; Bowler, Marinari, Pawley, Rapuano and Wallace 1983) indicate that the observed values of F_π and of $\langle 0|\bar{q}q|0\rangle$ are indeed consistent with the theory).

As will be shown in the first part of this paper (sections 2 - 9) the relations among the expansion coefficients which follow from the Ward identities allow one to work out the low energy expansion of all Green's functions to next-to-leading order in terms of a few effective coupling constants which chiral symmetry leaves undetermined. In the second part (sections 10 - 18) we determine the explicit low energy representation of some Green's functions, form factors and of the $\pi\pi$ scattering amplitude; in particular, we calculate the corrections of order M_π^2 to the current algebra low energy theorems for the $\pi\pi$ scattering lengths. Finally, we show that the values of the low energy coupling constants may be extracted from available experimental information and confront the improved low energy theorems with the experimental $\pi\pi$ phase shifts.

2. Symmetries of the Green's functions - anomalies

We consider the Green's functions of the vector, axial vector, scalar and pseudoscalar currents. These Green's functions are generated by the vacuum-to-vacuum amplitude

$$e^{iZ[v,a,s,p]} = \langle 0_{out}|0_{in}\rangle_{v,a,s,p} \tag{2.1}$$

associated with the Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{QCD}_{m_{quark}=0} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - i\gamma_5 p)q \\ \mathcal{L}^{QCD}_{m_{quark}=0} &= -\frac{1}{2g^2}\,\mathrm{tr}_c\, G_{\mu\nu}G^{\mu\nu} + \bar{q}\,i\gamma^\mu(\partial_\mu - iG_\mu)q\end{aligned} \tag{2.2}$$

where the external fields $v_\mu(x)$, $a_\mu(x)$, $s(x)$, $p(x)$ are hermitean, colour neutral matrices in flavour space. Note that the quark mass matrix M is included in $s(x)$:

$$s(x) = m + \dots$$

Formally, the Lagrangian (2.2) is invariant under independent unitary transformations of the right- and left-handed components of q:

$$q(x) \rightarrow \left\{ \tfrac{1}{2}(1+\gamma_5) V_R(x) + \tfrac{1}{2}(1-\gamma_5) V_L(x) \right\} q(x) \qquad (2.3)$$

provided the external fields are subject to the gauge transformation

$$\begin{aligned} v_\mu' + a_\mu' &= V_R (v_\mu + a_\mu) V_R^+ + i V_R \partial_\mu V_R^+ \\ v_\mu' - a_\mu' &= V_L (v_\mu - a_\mu) V_L^+ + i V_L \partial_\mu V_L^+ \\ s' + ip' &= V_R (s + ip) V_L^+ \end{aligned} \qquad (2.4)$$

As is well-known the $U(N_f)xU(N_f)$ symmetry of the Lagrangian is however afflicted with anomalies (Adler 1969; Bell and Jackiw 1969; Jackiw and Johnson 1969; Adler and Bardeen 1969) - the generating functional $Z[v, a, s, p]$ is not invariant under the full group of chiral transformations. The general structure of these anomalies was given by Bardeen (1969) and by Wess and Zumino (1971). For completeness we sketch a simple derivation of their results based on an analysis of the fermion determinant in Appendix A.

The anomalies are due to the fact that the determinant of the Dirac operator, det D, which embodies the quark contributions to the vacuum-to-vacuum amplitude, requires renormalization. There is no regularization of this object which preserves the full chiral symmetry. Consider an infinitesimal $U(N_f)xU(N_f)$ transformation:

$$\begin{aligned} V_R(x) &= 1 + i\alpha(x) + i\beta(x) + \dots \\ V_L(x) &= 1 + i\alpha(x) - i\beta(x) + \dots \end{aligned} \qquad (2.5)$$

The matrices α, β are hermitean; α is the infinitesimal transformation generated by the vector currents, β is a chiral transformation. The corresponding gauge transformation of the external fields is given by

$$\begin{aligned} \delta v_\mu &= \partial_\mu \alpha + i\,[\alpha, v_\mu] + i\,[\beta, a_\mu] \\ \delta a_\mu &= \partial_\mu \beta + i\,[\alpha, a_\mu] + i\,[\beta, v_\mu] \\ \delta s &= i\,[\alpha, s] - \{\beta, p\} \\ \delta p &= i\,[\alpha, p] + \{\beta, s\} \end{aligned} \qquad (2.6)$$

Since there are regularization schemes which preserve the subgroup $U(N_f)$ generated by the vector currents, the fermion determinant may be renormalized in such a manner that it is invariant under the transformations generated by α. The same is true of the generating functional $Z[v, a, s, p]$. The chiral transformations β on the other hand necessarily affect the fermion determinant. In particular, the flavour singlet piece of β generates a change in $\det D$ which involves the gluon field through the winding number density $G_{\mu\nu}\tilde{G}^{\mu\nu}$. Since the Green's functions of this object are not known explicitly, the Ward identities for the singlet axial current $\bar{q}\gamma_\mu\gamma_5 q$ merely allow us to relate the Green's functions involving $\bar{q}\gamma_\mu\gamma_5 q$ to the Green's functions involving $G_{\mu\nu}\tilde{G}^{\mu\nu}$. To explicitly exhibit this relation in terms of the generating functional we supplement the Lagrangian with an external field contribution proportional to the winding number density:

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{(4\pi)^2}\,\Theta(x)\,\mathrm{tr}\,G_{\mu\nu}\tilde{G}^{\mu\nu} \qquad (2.7)$$

such that the generating functional now also depends on the function $\theta(x)$. In this extended framework the change of $Z[v, a, s, p, \theta]$ under an infinitesimal $U(N_f)\mathrm{x}U(N_f)$ flavour transformation is known explicitly (see Appendix A): if the fields v_μ, a_μ, s, p are subject to the infinitesimal transformation (2.6) and θ is shifted by

$$\delta\Theta = -2\,\mathrm{tr}\,\beta(x) \qquad (2.8)$$

then the resulting change in $Z[v, a, s, p, \theta]$ is given by

$$\delta Z = -\frac{N_c}{(4\pi)^2}\int dx\, \epsilon^{\alpha\beta\mu\nu} \operatorname{tr}_f\left[\beta\left(v_{\alpha\beta}v_{\mu\nu} + \frac{4}{3}\nabla_\alpha a_\beta \nabla_\mu a_\nu + \frac{2i}{3}\{v_{\alpha\beta}, a_\mu a_\nu\} + \frac{8i}{3}a_\mu v_{\alpha\beta}a_\nu + \frac{4}{3}a_\alpha a_\beta a_\mu a_\nu\right)\right] \tag{2.9}$$

(The sign of δZ is convention dependent; we use the metric + --- and take $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$, $\epsilon_{0123} = +1$). N_c is the number of colours, $v_{\alpha\beta}$ denotes the field strength associated with v_α:

$$v_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha - i[v_\alpha, v_\beta] \tag{2.10}$$

and $\nabla_\alpha a_\beta$ stands for the $U(N_f)$-covariant derivative of a_α:

$$\nabla_\alpha a_\beta = \partial_\alpha a_\beta - i[v_\alpha, a_\beta] \tag{2.11}$$

The relation (2.9) guarantees that the Green's functions which one obtains by expanding the functional Z in powers of the external fields, obey the relevant Ward identities. Conversely, this relation allows us to characterize the general solution of the Ward identities: Suppose that Z_1, Z_2 are two generating functionals whose Green's functions obey the Ward identities. The relation (2.9) then implies that the difference $Z_1 - Z_2$ is invariant with respect to local chiral transformations. The Ward identities thus determine the generating functional of QCD up to a contribution that is gauge invariant with respect to $U(N_f)xU(N_f)$.

3. Low energy expansion

The low energy structure of the theory strongly depends on the size of the quark masses. In particular, heavy quarks only play a minor role, because their degrees of freedom are frozen at low energies. In the following we restrict our attention to currents built out of the first two flavours u, d. (We plan to discuss the extension to currents involving the strange quarks in a subsequent paper.) Accordingly we restrict the external fields v, a, s, p to the 2x2 subspace associated with the first two flavours. (In the notation used in the last section we reduce the $N_f x N_f$ matrix s(x) to a space-dependent 2x2 block supplemented by the constants m_s, m_c, ... along the diagonal.) Likewise we restrict the flavour transformations to U(2)xU(2). The transformation law of the generating functional Z under this subgroup is again given by (2.9) with the only difference that the trace over the flavour indices is reduced to a trace over 2x2 matrices. In this restricted framework we will be able to study the dependence of the Green's functions on m_u and m_d, but we will not be able to discuss the dependence on the remaining quark masses which are fixed at their physical values from now on. To simplify life further we disregard the isoscalar vector and axial currents as well as the winding number density (i. e. take $\mathrm{tr}\, v_\mu = \mathrm{tr}\, a_\mu = \theta = 0$) and correspondingly restrict the symmetry group to SU(2)xSU(2).

One easily checks that there is no anomaly in the subset of Green's functions we are left with: the right-hand side of (2.9) vanishes if $\mathrm{tr}\, \beta = 0$ and if the external vector and axial currents are of the form

$$v_\mu = v_\mu^i \frac{\tau^i}{2} \quad ; \quad a_\mu = a_\mu^i \frac{\tau^i}{2} \tag{3.1}$$

The information contained in the Ward identities thus amounts to the statement that the generating functional $Z[v, a, s, p]$ is gauge invariant under SU(2)xSU(2):

$$\delta Z = 0 \tag{3.2}$$

To analyze the structure of Z at low energies (external fields of long wavelength) we momentarily put $s(x) = p(x) = 0$, i.e. look at the Green's functions of the vector and axial currents in the chiral limit. If the theory did not contain massless particles the low energy expansion of Z would start with the terms of lowest dimension that can be built out of the fields $v_\mu(x)$, $a_\mu(x)$ and their derivatives in accordance with gauge invariance (3.2) and with parity:

$$Z = h \int dx \{ \mathrm{tr}\, F_{\mu\nu}^R F^{\mu\nu R} + \mathrm{tr}\, F_{\mu\nu}^L F^{\mu\nu L} \} \tag{3.3}$$

where $F_{\mu\nu}^R$, $F_{\mu\nu}^L$ are the field strengths associated with the external gauge fields $v_\mu \pm a_\mu$. In particular, the axial vector two point function which is given by the second derivative of Z with respect to a_μ would be of order p^2 at small momenta p. In QCD the spontaneous breakdown of the symmetry however generates

Goldstone bosons which, if the matrix element $\langle 0|A_\mu|\pi\rangle$ does not vanish in the chiral limit, produce a pole in this two point function: $\langle 0|TA_\mu A_\nu|0\rangle$ is of order one rather than of order p^2. The corresponding contribution to Z reads

$$Z = F^2 \int dx\,dy\ \mathrm{tr}\{(\partial_\mu a_\nu - \partial_\nu a_\mu)_x\, \Box^{-1}_{xy}\, (\partial^\mu a^\nu - \partial^\nu a^\mu)_y\} + \cdots \tag{3.4}$$

where F is the value of the pion decay constant in the chiral limit. The term (3.4) by itself is not gauge invariant - it requires the presence of corresponding low energy contributions in Green's functions with more than two currents. A technique that allows one to displace these contributions in concise form is well known (Weinberg 1967, 1968; Coleman, Wess and Zumino 1969; Callan, Coleman, Wess and Zumino 1969; Dashen and Weinstein 1969; Weinberg 1979; Boulware and Brown 1982): one considers the action of a suitable classical effective Lagrangian.

4. Effective Lagrangian

To solve the Ward identities for the vector and axial currents in the chiral limit to leading order in the low energy expansion we consider the non-linear σ-model coupled to external vector and axial fields. We denote by $U^A(x)$ a four-component real O(4) vector field of unit length, $U^TU = 1$, define its covariant derivative as

$$\begin{aligned} \nabla_\mu U^0 &= \partial_\mu U^0 + a^i_\mu(x)\, U^i \\ \nabla_\mu U^i &= \partial_\mu U^i + \varepsilon^{ikl}\, v^k_\mu(x)\, U^l - a^i_\mu(x)\, U^0 \end{aligned} \tag{4.1}$$

and consider the effective action

$$Z_1 = F^2 \int dx\ \tfrac{1}{2}\, \nabla_\mu U^T \nabla^\mu U \tag{4.2}$$

The field $U^A(x)$ is determined by the external fields $v^\mu(x)$, $a^\mu(x)$ through the classical equations of motion which follow from the requirement that Z_1 be an extremum:

$$\nabla^\mu \nabla_\mu U^A - U^A(U^T \nabla^\mu \nabla_\mu U) = 0 \tag{4.3}$$

(Note that the solution of the equations of motion is not unique, even if positive (negative) frequency boundary conditions are imposed at $t \to +\infty$ $(-\infty)$. One in addition needs to specify from which direction in flavour space the chiral limit is to be taken in QCD. The situation is analogous to a system that develops spontaneous magnetization: the direction of the magnetization is arbitrary. If we specify the chiral limit as the limit of the massive theory with $s(x) = \hat{m}\mathbb{1}$, $\hat{m} \to +0$ then the corresponding prescription that specifies the classical solution uniquely is $U^A(x) \to \delta^A_0$ as $t \to \pm\infty$.) This determines Z_1 as a functional of the external fields - one easily checks that this functional is gauge invariant and reproduces the low energy behaviour of $<0|TA_\mu A_\nu|0>$ as given in (3.4): in the chiral limit the leading low energy behaviour of the vector and axial vector Green's functions is determined by a single constant F (the symbol F_π is reserved for the value of F in the real world, $F_\pi \simeq$ 93 MeV). One may e.g. determine the leading low energy behaviour of the four point function $<0|TA_\mu A_\nu A_\rho A_\sigma|0>$ by working out the value of the classical action to fourth order in the external field a_μ. The residue of the pion poles in this four point function describes the leading low energy behaviour of the $\pi\pi$ scattering amplitude (Weinberg 1966) in the chiral limit [$A(s,t,u) = s/F^2 + \ldots$].

In the above the external scalar and pseudoscalar fields s(x), p(x) were switched off. The low energy structure of the scalar and pseudoscalar densities is not determined by the pion decay constant alone, but involves a second low energy constant B which measures the vacuum expectation value (Gell-Mann, Oakes and Renner 1968)

$$\langle 0|\bar{u}u|0\rangle_0 = \langle 0|\bar{d}d|0\rangle_0 = -F^2 B \tag{4.4}$$

of the scalar densities in the chiral limit. The expansion of Z in powers of the external field s(x) contains a linear term:

$$Z = -\int dx \, \mathrm{tr}\, s(x) \langle 0|\bar{u}u|0\rangle_0 + \ldots \tag{4.5}$$

which, by itself, is again not gauge invariant, because tr s(x) transforms like a component of a chiral four vector. Writing

$$s(x) = s^o(x)\cdot\mathbb{1} + s^i(x)\tau^i$$
$$p(x) = p^o(x)\cdot\mathbb{1} + p^i(x)\tau^i \tag{4.6}$$

the transformation law (2.4) shows that (s^0, p^i) and (p^0, $- s^i$) transform as independent O(4) four vectors. The linear term in Z is proportional to s^0. To reproduce this term in a gauge invariant manner it suffices to add the contribution $2F^2 B(s^0U^0 + p^iU^i)$ to the effective Lagrangian (4.2) which then becomes

$$Z_1 = F^2 \int dx \left\{ \frac{1}{2} \nabla_\mu U^T \nabla^\mu U + \chi^T U \right\} \tag{4.7}$$

where we have absorbed the constant B in the external fields,

$$\chi^A(x) = 2B\left(s^o(x), p^i(x)\right). \tag{4.8}$$

If the effective action Z_1 is evaluated at its extremum by solving the corresponding classical equations of motion

$$\nabla_\mu \nabla^\mu U^A - U^A(U^T \nabla^\mu \nabla_\mu U) = \chi^A - U^A(U^T\chi) \tag{4.9}$$

one obtains a functional $Z_1[v, a, s, p]$ of the external fields. Expanding this functional in powers of the external fields around $v = a = s = p = 0$ we get a set of Green's functions which satisfy the relevant Ward identities and reproduce the value of the pion decay constant F and the vacuum expectation value $\langle 0|\bar{u}u|0\rangle_0 = - F^2B$. The difference between the full generating functional Z of QCD and Z_1 must be a gauge invariant object which leaves the leading low energy behaviour of $\langle 0|TA_\mu A_\nu|0\rangle_0$ untouched and does not contribute to $\langle 0|\bar{u}u|0\rangle_0$. We will show in the next section that the general gauge invariant functional which admits a local series expansion in terms of the fields U, v, a, s, p and their derivatives reduces to Z_1 in leading order : (4.7) represents the most general effective Lagrangian of order p^2 consistent with Lorentz invariance, parity and chiral symmetry (since the fields $v_\mu(x)$ and $a_\mu(x)$ occur on the same level as the derivative ∂_μ it is convenient to count them as objects of order p, whereas the external scalar and pseudoscalar fields s(x) and p(x) are booked as quantities of order p^2; see below). Alternatively, one may derive the leading low energy behaviour of these Green's functions directly by solving the Ward identities step by step - the result is the same (Weinberg 1968; Dashen and Weinstein 1969):

in the chiral limit the leading low energy behaviour of the Green's functions is determined by the two constants F and B. Whether one uses the effective Lagrangian technique or directly solves the Ward identities one makes the same basic assumption: one postulates that the Green's functions may be expanded in powers of the momenta once the leading low energy singularities (which follow from clustering in a theory that contains Goldstone bosons) are extracted. It has not been shown that the Lagrangian of QCD spontaneously breaks chiral symmetry, much less that the ensuing Goldstone bosons dominate the low energy scene in the above technical sense. We refer the reader to (Coleman, Wess and Zumino 1969; Callan, Coleman, Wess and Zumino 1969; Dashen and Weinstein 1969; Weinberg 1979) for a discussion of the generality of the effective Lagrangian technique.

The effective Lagrangian (4.7) thus allows us to calculate the leading low energy behaviour of the Green's functions associated with V_μ^i, A_μ^i, $\bar{q}q$ and $\bar{q}\tau^i i\gamma_5 q$ in the chiral limit. The information contained in this Lagrangian is however not restricted to the chiral limit. To take the quark mass term in the Lagrangian of QCD into account we simply have to expand the external field s(x) around $s = \mathcal{M}$ rather than around s = 0. Consider e.g. the two point function of the pseudoscalar density $\bar{q}\tau^i i\gamma_5 q$. To calculate this object we need the value of the functional Z_1 to second order in p(x) for $v_\mu = a_\mu = 0$, $s^0 = \hat{m}$ (we disregard the mass difference $m_u - m_d$ for the moment and put $m_u = m_d = \hat{m}$). It suffices to determine the field U^i to first order in $\chi^i = 2Bp^i$. In this approximation the field equation becomes

$$\Box U^i + \chi^0 U^i = \chi^i \tag{4.10}$$

with $\chi^0 = 2\hat{m}B$. The quark mass thus shifts the pion pole according to the well-known Gell-Mann-Oakes-Renner relation:

$$M_\pi^2 = 2\hat{m}B \tag{4.11}$$

and we get

$$\begin{aligned} U^i &= 2B\int dx\, \Delta_c(x-y; M_\pi^2)\, p^i(y) \\ U^0 &= 1 - \tfrac{1}{2}U^iU^i + \ldots \end{aligned} \tag{4.12}$$

The leading low energy behaviour of the two point function may now be read off from the expansion (see (2.1) and (2.2))

$$Z_A = \frac{1}{2} i \int dx\,dy\, p^i(x) p^k(y) \langle 0|T\, \bar{q}_x \tau^i i\gamma_5 q_x\, \bar{q}_y \tau^k i\gamma_5 q_y |0\rangle + \dots \tag{4.13}$$

with the result

$$i \int dx\, e^{ip(x-y)} \langle 0|T\, \bar{q}_x \tau^i i\gamma_5 q_x\, \bar{q}_y \tau^k i\gamma_5 q_y |0\rangle = \delta^{ik} \frac{4F^2B^2}{M_\pi^2 - p^2} + \dots$$

In the language of chiral perturbation theory which expands the amplitudes in powers of the quark mass, the factor $(M_\pi^2 - p^2)^{-1}$ represents the sum of a geometric series which arises from repeated quark mass insertions connected by a single pion line. Even if the quark mass is very small it thus produces a substantial change in the Green's functions at low energy: it shifts the poles from $p^2 = 0$ to $p^2 = M_\pi^2$. To maintain a coherent low energy expansion for $\hat{m} \neq 0$ we have to treat M_π^2 as a term of the same order as p^2 and expand in powers of the momenta at fixed ratio M_π^2/p^2 (Weinberg 1979). This is the reason why we count the external field $s(x) \sim \hat{m} \sim M_\pi^2$ as a quantity of order p^2.

5. General form of effective Lagrangian to order p^4

Consider a general effective Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(U, v, a, s, p;\, \partial_\mu U, \dots, \partial_\mu p;\, \partial_{\mu\nu} U, \dots) \tag{5.1}$$

The low energy properties of this Lagrangian are governed by the terms with the least number of derivatives. We count the field $U(x)$ as a quantity of order 1. $v_\mu(x)$, $a_\mu(x)$ and $\partial_\mu U(x)$ as order p, $s(x)$, $p(x)$, $\partial_\nu v_\mu(x)$, $\partial_{\mu\nu} U(x)$ as order p^2 etc. To discuss the restrictions imposed by gauge invariance it is convenient to replace the ordinary derivatives by covariant ones. The covariant derivative of $U(x)$ is defined in (4.1). Since the vectors (s^0, p^i) and $(p^0, -s^i)$ transform like U we may define their covariant derivative in an analogous manner. (Equivalently, one may use the relations (A.21).) Gauge invariance then implies that the Lagrangian involves the fields v_μ, a_μ only through the covariant derivatives of U, s, p and through the field strengths $F^R_{\mu\nu}$, $F^L_{\mu\nu}$ built out of the right and left-handed gauge fields $v_\mu \pm a_\mu$:

$$\mathcal{L} = \mathcal{L}\,(U;\ \nabla_\mu U;\ \nabla_\mu \nabla_\nu U,\ s,\ p,\ F^R_{\mu\nu},\ F^L_{\mu\nu};\ \nabla_\lambda F^R_{\mu\nu},\dots)$$

To lowest order (p^0) the Lagrangian can only depend on the field U(x). Since U(x) transforms like an O(4) vector, the only invariant that can be built out of it is the length of this vector which equals one. To order p^0 the effective Lagrangian thus is an uninteresting constant. At order p^2 Lorentz invariance permits four types of contributions:

$$f_1^{AB}(U)\,\nabla_\mu U^A \nabla^\mu U^B + f_2^A(U)\,\nabla^\mu \nabla_\mu U^A + f_3^A(U)\,s^A + f_4^A(U)\,p^A$$

For this expression to be invariant under SU(2)xSU(2) the functions f_1 and f_2 must be of the form

$$\begin{aligned} f_1^{AB}(U) &= c_1 \delta^{AB} + c_2 U^A U^B \\ f_2^A(U) &= c_3 U^A \end{aligned} \tag{5.2}$$

Since $(U^T \nabla_\mu U)$ vanishes, the constant c_2 may be dropped. Furthermore, on account of

$$(U^T \nabla^\mu \nabla_\mu U) = -(\nabla_\mu U^T \nabla^\mu U)$$

the constant c_3 may be absorbed in c_1. Finally, the terms linear in s and p must be proportional to the scalar products $s^0U^0 + p^iU^i$ and $p^0U^0 - s^iU^i$. The latter term has the wrong parity. To order p^2 the general effective Lagrangian consistent with Lorentz invariance, parity and chiral symmetry therefore involves only two low energy constants:

$$\mathcal{L}_1 = \frac{F^2}{2} \nabla_\mu U^T \nabla^\mu U + 2BF^2\,(s^0 U^0 + p^i U^i) \tag{5.3}$$

As claimed in the last section, the lowest order effective Lagrangian coincides with the nonlinear σ-model coupled to external fields.

It is straightforward to extend this analysis to the terms of order p^4. Lorentz invariance and chiral symmetry again restrict the Lagrangian to a linear combination of terms involving O(4) - covariant tensors $f^{A_1 \dots A_n}(U)$ of the field U^A. These tensors may be expressed in terms of a few constants in a manner analogous to (5.2). (Note that the O(4)-invariant tensor ε^{ABCD} breaks G-parity and does therefore not occur.) Invariance with respect to space-reflections reduces the number of allowed coupling constants by about a factor of two. Finally, the field equations (4.9) imply the following identities (the use of the classical field equations associated with $\mathcal{L}_1$ will be justified below):

$$(\chi^T \nabla_\mu \nabla^\mu U) + (\chi^T U)(\nabla^\mu U^T \nabla_\mu U) - (\chi^T \chi) + (\chi^T U)^2 = 0$$
$$(\nabla^\mu \nabla_\mu U^T \nabla^\nu \nabla_\nu U) - (\nabla^\mu U^T \nabla_\mu U)^2 - (\chi^T \chi) + (\chi^T U)^2 = 0 \tag{5.4}$$

Using these relations to eliminate two of the invariants the general effective Lagrangian of order p^4 consistent with Lorentz invariance, parity, chiral symmetry and G-parity may be written in the form

$$\begin{aligned}\mathcal{L}_2 = {} & \ell_1 (\nabla^\mu U^T \nabla_\mu U)^2 + \ell_2 (\nabla^\mu U^T \nabla^\nu U)(\nabla_\mu U^T \nabla_\nu U) \\ & + \ell_3 (\chi^T U)^2 + \ell_4 (\nabla^\mu \chi^T \nabla_\mu U) + \ell_5 (U^T F^{\mu\nu} F_{\mu\nu} U) \\ & + \ell_6 (\nabla^\mu U^T F_{\mu\nu} \nabla^\nu U) + \ell_7 (\tilde{\chi}^T U)^2 + h_1 \chi^T \chi + h_2 \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \\ & + h_3 \tilde{\chi}^T \tilde{\chi}\end{aligned} \tag{5.5}$$

where the tensor $F^{AB}_{\mu\nu}$ defined by

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) U = F_{\mu\nu} U \tag{5.6}$$

contains the external fields v_μ, a_μ and their derivatives. The vectors χ^A, $\tilde{\chi}^A$ are proportional to the external scalar and pseudoscalar fields:

$$\chi^A = 2B(s^0, p^i)$$
$$\tilde{\chi}^A = 2B(p^0, -s^i) \tag{5.7}$$

Note that the external fields $p^o(x)$ and $s^i(x)$ do not occur in L_1 and only enter quadratically in L_2. There is therefore no contribution linear in the quark mass difference $m_u - m_d$ to the Green's functions of the operators V_μ^i, A_ν^i, $\bar{q}q$, $\bar{q}\tau^i i\gamma_5 q$ at leading and first nonleading order in the low energy expansion. The isospin breaking piece of the QCD Lagrangian has very little effect on these Green's functions - the symmetry properties of the vacuum protect isospin symmetry almost perfectly (Weinberg 1979).

6. Loops

In the preceding sections we have used classical fields to construct the leading terms in the low energy expansion of the Green's functions of QCD. The classical field theory associated with a given Lagrangian is equivalent to the set of tree graphs of the corresponding quantum field theory: if we use the classical field equations to determine the value of the action Z_1 as a functional of the external fields we are in effect calculating the vacuum-to-vacuum amplitude of the nonlinear σ-model in the tree graph approximation (Feynman 1963; DeWitt 1967). Z_1

What about graphs with loops ? On the one hand, if we simply disregard these graphs the theory violates unitarity. On the other hand, since the nonlinear σ-model is not renormalizable, the contributions from graphs involving loops are not well-defined. This apparent dilemma has a remarkably simple solution (Weinberg 1979): Consider for definiteness graphs containing one loop. The infinities which arise if one calculates these graphs require counter terms. Using dimensional regularization (which preserves chiral symmetry) one finds that the counter terms necessary to renormalize the one loop graphs are of order p^4. Since the regularization is consistent with Lorentz invariance, parity and chiral symmetry, the counter terms have the structure of the general order p^4 Lagrangian L_2 determined in the last section. With a suitable renormalization of the constants $\ell_1,\ldots,h_3$ which occur in L_2 one thus gets finite results for all Green's functions to one loop order. The fact that the nonlinear σ-model requires counter terms which do not have the same structure as the Lagrangian L_1 one starts with is characteristic of the low energy expansion: one needs only two constants F, B to specify the low energy behaviour of the Green's functions to leading order, one needs 10 additional constants $\ell_1,\ldots, h_3$ to characterize the behaviour at next-to-leading order. One needs counter terms of increasing complexity as one evaluates the σ-model to higher orders, one finds an increasing number of independent low energy constants if one carries the low energy expansion to higher orders. (In principle, all of these low energy constants are determined by the parameters Λ, m_s, m_c, ... which specify the underlying renormalizable theory; we are however not attempting to make use of this implicit information, but only exploit the symmetry properties of QCD.) Nonrenormalizability is a problem only if one elevates the nonlinear σ-model to a theory of its own. In the context of the low energy expansion chiral symmetry only guarantees that the σ-model describes the leading order correctly - there is no reason for this model to reproduce the Green's functions of QCD to all orders in the momenta, even if it could be elevated to a mathematically self-consistent framework.

The crucial point here is that the loop contributions are of higher order in the momenta than the tree graphs. This is due to the fact that chiral symmetry requires the pion couplings to vanish at zero momentum. The T-matrix elements for any scattering process involving pions tends to zero at zero momentum (exact chiral symmetry, massless pions). At low energy the T-matrix is therefore small. (For processes that exclusively involve pions which furthermore all have small momenta of order p the T-matrix is of order p^2.) It is this property which allows us to solve the constraints of unitarity, clustering and chiral symmetry in a perturbative manner by expanding the Green's functions in powers of the momenta.

General power counting arguments (Weinberg 1979) show that graphs containing n loops are suppressed by $(p^2)^n$ in comparison to the tree graphs. The loop graphs do therefore not modify the leading low energy behaviour which is given by the classical theory, the <u>one</u> loop graphs do however contribute at first nonleading order. To solve the constraints imposed on the Green's functions by chiral symmetry and by unitarity to first nonleading order we may therefore proceed as follows. The generating functional is given by the vacuum-to-vacuum amplitude in the presence of external fields. The first two terms in the low energy expansion of the Green's functions are obtained by evaluating i) tree and one loop graphs in the nonlinear σ-model coupled to external fields (L_1) ii) tree graphs which contain one vertex of L_2 together with any number of σ-model vertices. The sum of these contributions is finite, provided the constants ℓ_1, ℓ_2,..., h_3 are suitably renormalized. Since the vertices of the Lagrangian L_2 only occur in tree graphs, the contribution from L_2 to the vacuum-to-vacuum amplitude may be calculated by evaluating the action $\int dx\, L_2$ at the classical solution of the equations of motion. This justifies the use of the field equations in simplifying the structure of L_2.

In terms of the Feynman path integral the above prescription may be written in the form

$$e^{iZ} = e^{i\int dx\,\mathcal{L}_2} \int d\mu[U]\, e^{i\int dx\,\mathcal{L}_1} \tag{6.1}$$

where the integral over the field U(x) is to be calculated in one loop approximation. The explicit form of the Lagrangians L_1 and L_2 was given in the last section; L_1 contains the two constants F and B which determine the behaviour of the Green's functions involving the currents $\bar{q}\gamma_\mu\tau^i q$, $\bar{q}\gamma_\mu\gamma_5\tau^i q$ and the densities $\bar{q}q$, $\bar{q}i\gamma_5\tau^i q$ to leading order in the low energy expansion. To determine the behaviour of these Green's functions to first nonleading order we need 8 additional constants $\ell_1,\dots,\ell_6$, h_1, h_2. The two remaining parameters ℓ_7 and h_3 specify the leading low energy behaviour of Green's functions which involve the second chiral multiplet $\bar{q}i\gamma_5 q$ and $\bar{q}\tau^i q$.

The three constants h_1, h_2 and h_3 multiply terms which do not contain the pion field. These constants are inessential in the following sense. The Lagrangian of QCD must be supplemented with counter terms proportional to tr $F_{\mu\nu}F^{\mu\nu}$, $s^{o2} + p^{i2} \sim \chi^T\chi$ and $p^{o2} + s^{i2} \sim \tilde{\chi}^T\tilde{\chi}$. The values of the finite pieces of these counter terms which remain after renormalization of the QCD loops depend on the renormalization prescription. For this reason the constants h_1, h_2, h_3 are not directly measurable and accordingly do not occur in the low energy expansion of physical quantities.

We add a remark about the scale dependence of the scalar and pseudoscalar densities in QCD. The normalization of these fields also depends on the conventions used (MS, $\overline{\text{MS}}$, MOM, choice of scale μ etc.) - the renormalization group transformation law is contragredient to the transformation law of the quark masses. The products $m\bar{q}q$, $m\bar{q}\gamma_5 q$ etc. are convention independent. The value of the constant B e.g. depends on the renormalization prescription, the quantity $(m_u + m_d)B$ is convention independent. We circumvent this problem by absorbing the constant B in the external scalar and pseudoscalar fields. In contrast to s and p the quantities χ and $\tilde{\chi}$ do not change under a change of the renormalization prescription used in QCD - the same is true of the constants $\ell_1,\dots,\ell_7$. The quantities F, $M^2 = (m_u + m_d)$ B and $\ell_1,\dots,\ell_7$ (more precisely the sum of ℓ_i and of the corresponding pion loop contribution) are the proper physical low energy parameters at order p^4.

7. Nonlinear σ-model to one loop

The evaluation of the Feynman path integral in the one loop approximation is a standard problem ('t Hooft 1973; Ramond 1981). One expands the action

$$Z_1 = \int dx\,\mathcal{L}_1 = F^2 \int dx \left\{ \frac{1}{2} \nabla_\mu U^T \nabla^\mu U + \chi^T U \right\} \tag{7.1}$$

in the vicinity of the classical solution $\bar{U}(x)$, which is determined by the external fields through the classical equations of motion:

$$\nabla^\mu \nabla_\mu \bar{U}^A - \bar{U}^A (\bar{U}^T \nabla_\mu \nabla^\mu \bar{U}) = \chi^A - \bar{U}^A (\bar{U}^T \chi) \tag{7.2}$$

We write the expansion of U in the form

$$U^A = \bar{U}^A + \xi^\alpha \bar{\varepsilon}^A_\alpha - \frac{1}{2} \xi^\alpha \xi^\alpha \bar{U}^A + \dots \tag{7.3}$$

where the three vectors $\bar{\varepsilon}^A_1$, $\bar{\varepsilon}^A_2$, $\bar{\varepsilon}^A_3$ are orthogonal to $\bar{U}^A$ and are normalized by

$$(\bar{\varepsilon}^T_\alpha \bar{\varepsilon}_\beta) = \delta_{\alpha\beta} \tag{7.4}$$

The quadratic term in (7.3) insures that the vector U satisfies the constraint $U^T U = 1$ to order ξ^2. The field equations guarantee that there is no term linear in ξ in the expansion of the action:

$$Z_1 = \bar{Z}_1 - \frac{F^2}{2} \int dx\, \xi^\alpha D^{\alpha\beta} \xi^\beta + \dots \tag{7.5}$$

The differential operator $D^{\alpha\beta}$ may be written in the form

$$D^{\alpha\beta} \xi^\beta = D^\mu D_\mu \xi^\alpha + \sigma^{\alpha\beta} \xi^\beta \tag{7.6}$$

where D_μ is a covariant derivative defined by

$$\begin{aligned} D_\mu \xi^\alpha &= \partial_\mu \xi^\alpha + \Gamma^{\alpha\beta}_\mu \xi^\beta \\ \Gamma^{\alpha\beta}_\mu &= (\bar{\varepsilon}^T_\alpha \nabla_\mu \bar{\varepsilon}_\beta) \end{aligned} \tag{7.7}$$

and the mass matrix $\sigma^{\alpha\beta}(x)$ is given by

$$\sigma^{\alpha\beta} = \delta^{\alpha\beta}(\nabla^{\mu}\bar{U}^{T}\nabla_{\mu}\bar{U} + \chi^{T}\bar{U}) - (\bar{\varepsilon}_{\alpha}^{T}\nabla^{\mu}\bar{U})(\bar{\varepsilon}_{\beta}^{T}\nabla_{\mu}\bar{U}) \tag{7.8}$$

Next, we express the measure $d\mu[U]$ in terms of the variable ξ. The explicit expression for the SU(2)xSU(2)-invariant measure reads

$$d\mu[U] = N \prod_{x} \frac{du^{1}du^{2}du^{3}}{u^{0}} \tag{7.9}$$

Inserting the expansion (7.3) we get

$$du^{1}du^{2}du^{3} = u^{0}d\xi^{1}d\xi^{2}d\xi^{3}(1 + O(\xi^{2}))$$

To the required accuracy the measure is therefore translation invariant in the space of ξ-variables:

$$d\mu[U] = N \prod_{x} d^{3}\xi \tag{7.10}$$

and the functional integral reduces to a Gaussian integral

$$N\int \prod_{x} d^{3}\xi \, \exp -i\frac{F^{2}}{2}\int dx\, (\xi^{\alpha}D^{\alpha\beta}\xi^{\beta}) = \bar{N}(\det D)^{-1/2} \tag{7.11}$$

The generating functional therefore becomes

$$Z = \int dx \mathcal{L}_{1} + \int dx \mathcal{L}_{2} + \frac{i}{2}\ln \det D + O(p^{6}) \tag{7.12}$$

where all quantities are to be evaluated at the classical solution $\bar{U}(x)$.

What remains to be done to have an explicit representation of the Green's functions to first nonleading order in the low energy expansion is to evaluate the determinant of the differential operator D. This determinant is a formal object which requires renormalization. In the following section we determine the relevant counter terms which may be given in closed form. (The finite part of det D cannot be given in closed form; we will work out the first few terms in an expansion in powers of the external fields later on.)

8. Dimensional regularization

The short distance properties of the operator $D^{\alpha\beta}$ are governed by the d'Alambertian. In d-dimensional Minkowski space the exponential kernel associated with $\Box$ is given by

$$(x|e^{-\lambda\Box}|y) = i\,(4\pi\lambda)^{-\frac{d}{2}}\, e^{\frac{1}{4}\frac{z^2}{\lambda}} \tag{8.1}$$

where λ is taken on the positive imaginary axis and $z = x - y$. Extracting the leading short distance behaviour we write the kernel $\exp(-\lambda D)$ in the form (Schwinger 1951)

$$(x|e^{-\lambda D}|y) = i\,(4\pi\lambda)^{-\frac{d}{2}}\, e^{\frac{1}{4}\frac{z^2}{\lambda}}\; H(x|\lambda|y) \tag{8.2}$$

where H is a 3x3 matrix which satisfies the differential equation

$$\begin{gathered}\frac{\partial}{\partial\lambda} H + \frac{1}{\lambda} z^{\mu} D_{\mu} H + D^{\mu} D_{\mu} H + \sigma H = 0 \\ H(x|0|x) = 1\end{gathered} \tag{8.3}$$

The d-dimensional determinant of D may then be defined as

$$\begin{aligned}\ln \det D &= -\int_0^{i\infty} \frac{d\lambda}{\lambda}\, \mathrm{Tr}\, e^{-\lambda D} \\ &= -i(4\pi)^{-\frac{d}{2}} \int_0^{i\infty} d\lambda\, \lambda^{-1-\frac{d}{2}} \int dx\, \mathrm{tr}\, H(x|\lambda|x)\end{aligned} \tag{8.4}$$

In this representation the ultraviolet divergences produced by the loops show up at the lower end of the integration over λ: $\det D$ has poles at $d = 0, 2, 4, \ldots$. To identify the residues of these poles we split the integration over λ into an integral from 0 to λ_0 and a remainder. Using the Taylor expansion

$$H(x|\lambda|y) = H_0(x|y) + \lambda H_1(x|y) + \lambda^2 H_2(x|y) + \ldots \tag{8.5}$$

in the integral from 0 to λ_0 we find

$$\frac{i}{2} \ln \det D = -\int dx \left\{ \frac{1}{d} \operatorname{tr} H_0(x|x) + \frac{1}{4\pi(d-2)} \operatorname{tr} H_1(x|x) \right.$$
$$\left. + \frac{1}{(4\pi)^2(d-4)} \operatorname{tr} H_2(x|x) + \ldots \right\} \tag{8.6}$$

Note that we are interested in an expansion of the determinant in powers of the external fields around

$$v_\mu = a_\mu = \chi^i = 0 \ , \ \chi^0 = 2\hat{m}B = M^2$$

If the external fields are switched off, the differential operator D reduces to $\Box + M^2$ and the function H therefore tends to exp $- \lambda M^2$ - there is no infrared divergence at the upper end of the integration as long as the quark mass is not set equal to zero. The contributions omitted in (8.6) are finite for $d < 6$.

The differential equation (8.3) implies the following recursion relations:

$$(z^\mu D_\mu + n+1) H_{n+1} + D^\mu D_\mu H_n + \sigma H_n = 0$$
$$z^\mu D_\mu H_0 = 0 \tag{8.7}$$

Solving these relations (cf. Appendix A) we get

$$H_0(x|x) = 1$$
$$H_1(x|x) = -\sigma \tag{8.8}$$
$$H_2(x|x) = \frac{1}{12} \Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2 - \frac{1}{6}[D^\mu, [D_\mu, \sigma]]$$

where

$$\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] \tag{8.9}$$

9. Renormalization

The counter terms necessary to renormalize the one loop determinant may now be worked out by calculating the traces over the matrices σ and $\Gamma_{\mu\nu}$ which occur in (8.6), (8.8). This may be done without invoking an explicit representation of the vectors ε^A_α. These vectors perform the change of basis from the four-dimensional linear representation to the three-dimensional nonlinear representation of SU(2)xSU(2). The derivative ∇_μ defined in (4.1) is covariant with respect to the linear representation, the derivative D_μ defined in (7.7) is the corresponding covariant derivative of the nonlinear representation. In contrast to ∇_μ the derivative D_μ involves the pion field. The connection between the two may be obtained by expanding the vector $\nabla_\mu(\varepsilon_\alpha\xi^\alpha)$ in the basis ε_1, ε_2, ε_3, U:

$$\nabla_\mu(\varepsilon^A_\alpha \xi^\alpha) = \varepsilon^A_\alpha D_\mu \xi^\alpha - U^A(\varepsilon^T_\alpha \nabla_\mu U)\xi^\alpha \qquad (9.1)$$

(We omit the bars here - all fields are to be evaluated at the solution to the classical field equations, denoted by $\bar{U}$ in section 7.) The field strength $\Gamma_{\mu\nu}$ of the gauge field Γ_μ which specifies the nonlinear covariant derivative contains two contributions:

$$\begin{aligned} \Gamma^{\alpha\beta}_{\mu\nu} &= \varepsilon^T_\alpha (F_{\mu\nu} + U_{\mu\nu})\varepsilon_\beta \\ U^{AB}_{\mu\nu} &= \nabla_\mu U^A \nabla_\nu U^B - \nabla_\nu U^A \nabla_\mu U^B \end{aligned} \qquad (9.2)$$

The first term is proportional to the field strength $F_{\mu\nu}$ associated with the external fields v_μ, a_μ; the second term involves the derivatives of the pion field.

Using these properties and the completeness relation

$$\sum_\alpha \varepsilon^A_\alpha \varepsilon^B_\alpha = \delta^{AB} - U^A U^B \qquad (9.3)$$

one finds

$$\mathrm{tr}\,\sigma = 2\nabla^\mu U^T \nabla_\mu U + 3\chi^T U$$

$$\mathrm{tr}\,\sigma^2 = (\nabla^\mu U^T \nabla_\mu U)^2 + (\nabla_\mu U^T \nabla_\nu U)(\nabla^\mu U^T \nabla^\nu U)$$
$$+ 4(\chi^T U)(\nabla_\mu U^T \nabla^\mu U) + 3(\chi^T U)^2 \qquad (9.4)$$

$$\mathrm{tr}\,\Gamma_{\mu\nu}\Gamma^{\mu\nu} = \mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} - 2U^T F_{\mu\nu}F^{\mu\nu}U - 4\nabla_\mu U^T F^{\mu\nu}\nabla_\nu U$$
$$-2(\nabla^\mu U^T \nabla_\mu U)^2 + 2(\nabla_\mu U^T \nabla_\nu U)(\nabla^\mu U^T \nabla^\nu U)$$

The trace over the last term in (8.8) does not contribute, because it is a total derivative. The poles of the determinant at d = 0, 2, 4 are therefore given by

$$\frac{i}{2}\ln\det D = \int dx\Big[-\frac{3}{d} + \frac{1}{4\pi}\frac{1}{d-2}\{2\nabla^\mu U^T \nabla_\mu U + 3\chi^T U\}$$
$$+\frac{1}{(4\pi)^2}\frac{1}{(d-4)}\Big\{-\frac{1}{12}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{6}U^T F_{\mu\nu}F^{\mu\nu}U$$
$$+\frac{1}{3}(\nabla_\mu U^T F^{\mu\nu}\nabla_\nu U) - \frac{1}{3}(\nabla^\mu U^T \nabla_\mu U)^2$$
$$-\frac{2}{3}(\nabla_\mu U^T \nabla_\nu U)(\nabla^\mu U^T \nabla^\nu U) - 2(\chi^T U)(\nabla^\mu U^T \nabla_\mu U)$$
$$-\frac{3}{2}(\chi^T U)^2\Big\} + \dots\Big] \qquad (9.5)$$

The residue of the pole at d = 2 only contains terms which are already present in the Lagrangian of the nonlinear σ-model, in agreement with the well-known fact that this model is renormalizable in two dimensions. The divergences produced by the two-dimensional loop integrals merely renormalize the pion decay constant F and the field χ. In four dimensions F is not renormalized: the counter terms are of order p^4 rather than of order p^2. Using the first one of the two identities (5.4) one finds that the pole at d = 4 is removed by the following renormalization of the constants which occur in the Lagrangian L_2:

$$\ell_i = \ell_i^r + \gamma_i \lambda \qquad i = 1, \dots, 7$$
$$h_i = h_i^r + \delta_i \lambda \qquad i = 1, 2, 3$$
$$\lambda = (4\pi)^{-2} \mu^{d-4} \left\{ \frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1 \right) \right\}$$
$$\gamma_1 = \frac{1}{3},\ \gamma_2 = \frac{2}{3},\ \gamma_3 = -\frac{1}{2},\ \gamma_4 = 2,\ \gamma_5 = -\frac{1}{6},\ \gamma_6 = -\frac{1}{3},\ \gamma_7 = 0$$
$$\delta_1 = 2,\ \delta_2 = \frac{1}{12},\ \delta_3 = 0 \tag{9.6}$$

(We have included a finite piece in λ for later convenience.) The occurrence of counter terms which are not linear in the external field χ (or contain derivatives thereof) is related to the problems which one has to solve (Honerkamp 1972; Tataru 1975; Kazakov, Pervushin and Pushkin 1977, 1978; de Wit and Grisaru 1979; Bardeen, Lee and Shrock 1976; Appelquist and Bernard 1981) if one calculates the Green's functions of the <u>pion field</u> in the standard manner. In contrast to that procedure the external field technique retains the full symmetry of the theory at every stage of the calculation and, furthermore, specifies the Green's functions associated with the currents. Note also, that an effective Lagrangian which only allows one to deal with <u>on shell</u> matrix elements does not determine the manner in which the low energy parameters depend on the quark mass. In our framework all low energy constants refer to the massless theory; the quark mass appears as an explicit symmetry breaking parameter contained in the external fields:

$$\begin{aligned} \chi^0 &= 2Bs^0 = (m_u + m_d)B + \dots \\ \tilde{\chi}^3 &= -2Bs^i = (m_d - m_u)B + \dots \end{aligned} \tag{9.7}$$

10. One loop integrals for 1-, 2-, 3- and 4-point functions

We calculate the finite part of the determinant by treating the external fields as perturbations. If the external fields are switched off ($v_\mu = a_\mu = p = 0$, $s = M$) the differential operator D reduces to

$$D_M = \Box + M^2 \tag{10.1}$$

We expand the determinant of D in powers of the difference $D - D_M$:

$$\begin{aligned} D &= D_M + \delta \\ \delta &= \{\Gamma_\mu, \partial^\mu\} + \Gamma_\mu \Gamma^\mu + \hat{\sigma} \\ \hat{\sigma} &= \sigma - M^2 \end{aligned} \tag{10.2}$$

To second order in δ the expansion reads

$$\ln \det D = \ln \det D_M + \mathrm{Tr}\, D_M^{-1}\delta - \frac{1}{2}\mathrm{Tr}(D_M^{-1}\delta D_M^{-1}\delta) + \ldots \tag{10.3}$$

The term $\mathrm{Tr}\, D_M^{-1}\delta$ is the set of all tadpole graphs (pion loop interrupted only at one point). The next term collects all graphs with two vertices in the loop etc. To see how far we have to go in this expansion in order to calculate the loop contribution to a given Green's function we first look at the classical equation of motion which determines the pion field U^i in terms of the external fields. Retaining only terms linear in the external fields v_μ, a_μ, χ^i and $\hat{\chi}^0 = \chi^0 - M^2$ this equation reduces to

$$(\Box + M^2)U^i = \partial^\mu a_\mu^i + \chi^i \tag{10.4}$$

The quantities v_μ, $\hat{\chi}^0$ affect the pion field only through terms of order va, $v\chi^i$, $\hat{\chi}^0 a$, $\hat{\chi}^0\chi^i$. It is therefore convenient to associate a weight factor with the different external fields. We count a_μ and χ^i as quantities of order ϕ whereas v_μ and $\hat{\chi}^0$ are counted as $O(\phi^2)$. The field equation shows that U^i is of order ϕ, hence U^0 is of the form $1 + O(\phi^2)$. One easily checks that in this manner of counting weights the quantities Γ_μ and $\hat{\sigma}$ are both of order ϕ^2 and the same therefore applies to δ. The terms given in (10.3) thus allow us to calculate the determinant to order ϕ^4. This information suffices to extract the one loop contributions to all two-point functions as well as to the four-point function of the axial current or of the pseudoscalar density. The one-point function $\langle 0|\bar{q}q|0\rangle$ or the three-point function $\langle 0|TV_\lambda A_\mu A_\nu|0\rangle$ are also included. The 3- or 4-point functions associated with the vector current or with the scalar density $\bar{q}q$ are not accounted for at this level of the expansion: these objects receive contributions from loops containing more than two vertices. Note that the Green's functions of odd weight such as $\langle 0|\, T V_\lambda V_\mu A_\nu |0\rangle$ vanish; this is due to the fact that the G-parity of these objects is negative.

In the following we restrict ourselves to order ϕ^4. To this order the determinant is given by

$$\frac{i}{2} \ln \det D = \text{const} + \frac{i}{2} \Delta(0) \int dx \, \mathrm{tr}\, \hat{\sigma}(x) + \int dx\, dy \left\{ M^{\mu\nu}(x-y)\, \mathrm{tr}\, \Gamma_\mu(x) \Gamma_\nu(y) - \frac{i}{4} \Delta^2(x-y)\, \mathrm{tr}\, \hat{\sigma}(x) \hat{\sigma}(y) \right\} \tag{10.5}$$

where $\Delta(z) = \Delta_c(z; M^2)$ is the Feynman propagator of a scalar field with mass M in d-dimensions and $M_{\mu\nu}(z)$ stands for

$$M_{\mu\nu}(z) = \frac{i}{2} \left\{ \partial_\mu \Delta\, \partial_\nu \Delta - \Delta\, \partial_{\mu\nu} \Delta + g_{\mu\nu} \Delta(0)\, \delta(z) \right\} \tag{10.6}$$

The functions $\Delta^2(z)$ and $M_{\mu\nu}(z)$ develop a pole at $d \to 4$. The Fourier transform of $\Delta^2(z)$ is the standard loop integral

$$\begin{aligned} J(q^2) &= \frac{1}{i} \int d^d z\, e^{iqz} \Delta^2(z) \\ &= \frac{1}{i} (2\pi)^{-d} \int d^d k\, (M^2 - k^2)^{-1} (M^2 - (q-k)^2)^{-1} \end{aligned} \tag{10.7}$$

The quantity $\bar{J}(q^2)$ defined by

$$\bar{J}(q^2) = J(q^2) - J(0) \tag{10.8}$$

remains finite as $d \to 4$:

$$\begin{aligned} \bar{J}(q^2) &= \frac{1}{16\pi^2} \left\{ \sigma \ln \frac{\sigma - 1}{\sigma + 1} + 2 \right\} \\ \sigma &= \left(1 - \frac{4M^2}{q^2} \right)^{1/2} \end{aligned} \tag{10.9}$$

The pole is contained in J(0):

$$J(0) = -2\lambda - \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu^2} + 1\right)$$
$$\lambda = \frac{1}{16\pi^2}\,\mu^{d-4}\left\{\frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1\right)\right\} \tag{10.10}$$

The result may be rewritten in the form

$$\frac{1}{i}\,\Delta^2(z) = \bar{J}(z) - \left\{2\lambda + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu^2} + 1\right)\right\}\delta(z) \tag{10.11}$$

where $\bar{J}(z)$ is the Fourier transform $\bar{J}(q^2)$. The function $M_{\mu\nu}(z)$ may be evaluated in an analogous manner. With

$$\frac{1}{i}\,\Delta(0) = M^2\left\{2\lambda + \frac{1}{16\pi^2}\ln\frac{M^2}{\mu^2}\right\} \tag{10.12}$$

one finds

$$M_{\mu\nu}(z) = (\partial_{\mu\nu} - g_{\mu\nu}\Box)\left[\bar{M}(z) - \frac{1}{12}\left\{2\lambda + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu^2} + \frac{1}{3}\right)\right\}\delta(z)\right] \tag{10.13}$$

where $\bar{M}(z)$ is given by

$$\int dz\, e^{iqz}\,\bar{M}(z) = \frac{1}{12}\left(1 - \frac{4M^2}{q^2}\right)\bar{J}(q^2) \tag{10.14}$$

The generating functional Z, obtained by adding $\int dx(\mathcal{L}_1 + \mathcal{L}_2)$ to $\frac{i}{2}\ln\det D$ then takes the form

$$Z = Z_t + Z_u + O(\phi^6) \tag{10.15}$$

where Z_t is the sum of the tree graph and tadpole contributions

$$Z_t = \int dx \Big[\frac{F^2}{2}\Big(1+\frac{M^2}{8\pi^2F^2}\Big)\nabla_\mu U^T \nabla^\mu U + F^2\Big(1+\frac{3M^2}{32\pi^2F^2}\Big)\chi^T U$$

$$+\frac{1}{32\pi^2}\Big\{\gamma_1(\bar{\ell}_1-\tfrac{4}{3})(\nabla_\mu U^T\nabla^\mu U)^2 + \gamma_2(\bar{\ell}_2-\tfrac{5}{6})(\nabla_\mu U^T\nabla_\nu U)(\nabla^\mu U^T\nabla^\nu U)$$

$$+\gamma_3(\bar{\ell}_3-1)(\chi^T U)^2 + \gamma_4(\bar{\ell}_4-1)\nabla_\mu\chi^T\nabla^\mu U$$

$$+\gamma_5(\bar{\ell}_5-\tfrac{1}{3})U^T F_{\mu\nu}F^{\mu\nu}U + \gamma_6(\bar{\ell}_6-\tfrac{1}{3})\nabla_\mu U^T F^{\mu\nu}\nabla_\nu U$$

$$+\delta_1(\bar{h}_1-1)\chi^T\chi + \delta_2(\bar{h}_2-\tfrac{1}{3})\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}\Big\}$$

$$+\ell_7(\tilde{\chi}^T U)^2 + h_3\tilde{\chi}^T\tilde{\chi}\Big] \tag{10.16}$$

and Z_u is the unitarity correction which contains the finite part of the loop integrals:

$$Z_u = \int dx\,dy \Big\{\frac{1}{2}\bar{M}(x-y)\,\mathrm{tr}\,\Gamma_{\mu\nu}(x)\Gamma^{\mu\nu}(y) + \frac{1}{4}\bar{J}(x-y)\,\mathrm{tr}\,\hat{\sigma}(x)\hat{\sigma}(y)\Big\} \tag{10.17}$$

(We have made use of (9.4) and (5.4) to evaluate the traces over $\hat{\sigma}(x)$, $\hat{\sigma}(x)^2$ and $\Gamma_{\mu\nu}(x)\Gamma^{\mu\nu}(x)$.) In the above representation of the generating functional the renormalization scale μ which occurs in the dimensional regularization scheme, has disappeared: in contrast to the constants ℓ_i^r, h_i^r, which logarithmically depend on μ, the parameters $\bar{\ell}_i$, $\bar{h}_i$, defined by

$$\ell_i^r = \frac{\gamma_i}{32\pi^2}\Big(\bar{\ell}_i + \ln\frac{M^2}{\mu^2}\Big) \qquad i=1,\dots,6$$

$$h_i^r = \frac{\delta_i}{32\pi^2}\Big(\bar{h}_i + \ln\frac{M^2}{\mu^2}\Big) \qquad i=1,2 \tag{10.18}$$

are scale independent. (The coefficients $\gamma_1 = \frac{1}{3}$ etc. are given in (9.6)). Up to a numerical factor the quantity $\bar{\ell}_i$ is the value of the renormalized coupling constant ℓ_i^r at scale $\mu = M \simeq M_\pi$. The price to pay is that $\bar{\ell}_i$ does not exist in the chiral limit, but contains a chiral logarithm with unit coefficient (in the limit $M_\pi \to 0$ the constants $\bar{\ell}_i$, $\bar{h}_i$ tend to infinity like $-\log M_\pi^2$).

The representation (10.17) for the unitarity correction Z_u is not fully explicit, because it contains the quantities $\Gamma_{\mu\nu}$ and $\hat{\sigma}$ which involve the polarization vectors ε_α^A. An explicit representation for these vectors is

$$\varepsilon_\alpha^0 = -U^\alpha \;; \quad \varepsilon_\alpha^i = \delta_\alpha^i - (1+U^0)^{-1} U^i U^\alpha \tag{10.19}$$

To order ϕ^2 the matrices $\Gamma_{\mu\nu}$ and $\hat{\sigma}$ become

$$\begin{aligned}\Gamma_{\mu\nu}^{\alpha\beta} &= \partial_\mu U^\alpha \partial_\nu U^\beta - \partial_\nu U^\alpha \partial_\mu U^\beta - \partial_\mu (U^\alpha a_\nu^\beta - U^\beta a_\nu^\alpha) \\ &\quad + \partial_\nu (U^\alpha a_\mu^\beta - U^\beta a_\mu^\alpha) - \varepsilon^{\alpha\beta i}(\partial_\mu v_\nu^i - \partial_\nu v_\mu^i) + O(\phi^4) \\[1em] \hat{\sigma}^{\alpha\beta} &= \delta^{\alpha\beta}\left[(\partial_\mu U^\ell - a_\mu^\ell)^2 - \tfrac{1}{2} M^2 U^\ell U^\ell + \hat{\chi}^0 + \chi^\ell U^\ell\right] \\ &\quad - (\partial_\mu U^\alpha - a_\mu^\alpha)(\partial^\mu U^\beta - a^{\mu\beta}) + O(\phi^4)\end{aligned} \tag{10.20}$$

Finally, the pion field U is determined by the equations of motion (4.9). To order ϕ^3 the solution reads

$$\begin{aligned} U^i &= U_1^i + U_3^i + O(\phi^5) \\ U^0 &= 1 - \tfrac{1}{2} U_1^i U_1^i + O(\phi^4) \\ U_1^i &= (\Box + M^2)^{-1} \{ \partial^\mu a_\mu^i + \chi^i \} \end{aligned}$$

$$U_3^i = (\Box + M^2)^{-1}\Big[-\tfrac{1}{2}\{\partial^\mu, a_\mu^i\} U_1^\ell U_1^\ell + a_\mu^i a^{\mu\ell} U_1^\ell$$
$$+ \varepsilon^{ik\ell}\left(-\{\partial^\mu, v_\mu^k\} U_1^\ell + v_\mu^k a^{\mu\ell}\right)$$
$$- U_1^i \left\{ (\partial_\mu U_1^\ell - a_\mu^\ell)^2 - \frac{M^2}{2} U_1^\ell U_1^\ell + \hat{\chi}^0 + \chi^\ell U_1^\ell \right\}\Big] \tag{10.21}$$

What remains to be done to obtain the Green's functions is to read off the Taylor coefficients in the expansion of the generating functional in powers of the external fields.

11. The expectation values $<0|\bar{u}u|0>$, $<0|\bar{d}d|0>$

To determine the one-point functions $<0|\bar{u}u|0>$, $<0|\bar{d}d|0>$ we need to calculate the change in the generating functional produced by a change in the scalar external field $s(x)$:

$$\delta Z = -\int dx \left\{ \delta s^0 <0|\bar{q}q|0> + \delta s^i <0|\bar{q}\tau^i q|0> \right\} \tag{11.1}$$

around the value $s^0 = \frac{1}{2}(m_u + m_d)$, $s^3 = \frac{1}{2}(m_u - m_d)$, all other external fields being switched off. In the representation of the generating functional given in the preceding section we have replaced $s(x)$ by the fields $\chi^0 = 2\,Bs^0$ and $\tilde{\chi}^i = -\,2\,Bs^i$. We therefore need to consider the changes $\delta\chi^0 = 2\,B\delta s^0$, $\delta\tilde{\chi}^i = -\,2B\delta s^i$ around the values $\chi^0 = (m_u + m_d)\,B$, $\chi^3 = (m_d - m_u)\,B$. Since $\delta\chi^0$ counts as order ϕ^2 we only need the generating functional to order ϕ^2. The unitarity correction Z_u is of order ϕ^4: the vacuum expectation values therefore only receive contributions from tree graphs and tadpoles. As shown in the preceding section the pion field U^i vanishes for $a_\mu = \chi^i = 0$: the same is true of $\nabla_\mu U$. This implies that only the terms proportional to $\chi^T U$, $(\chi^T U)^2$, $\chi^T\chi$ and $\tilde{\chi}^T\tilde{\chi}^T$ in (10.16) contribute to $<0|\bar{u}u|0>$,$<0|\bar{d}d|0>$. The change in the generating functional is given by

$$\delta Z = \int dx \Big[\delta\chi^0 F^2 \Big[1 + \frac{M^2}{32\pi^2 F^2}\{3 + 2\gamma_3(\bar{\ell}_3 - 1) + 2\delta_1(\bar{h}_1 - 1)\}\Big] \tag{11.2}$$
$$+ \delta\chi^3\, 2h_3 (m_d - m_u) B\Big]$$

and we therefore obtain

$$\langle 0|\bar{u}u+\bar{d}d|0\rangle = -2F^2B\left\{1+\frac{M_\pi^2}{32\pi^2F_\pi^2}(4\bar{h}_1-\bar{\ell}_3)+O(M_\pi^4)\right\}$$
$$\langle 0|\bar{u}u-\bar{d}d|0\rangle = +4(m_d-m_u)B^2h_3+O(M_\pi^4) \tag{11.3}$$

The contributions of order m_{quark} involve the "high energy constants" h_1, h_3: in contrast to the vacuum expectation value in the chiral limit, $\langle 0|\bar{u}u|0\rangle_0 = \langle 0|\bar{d}d|0\rangle_0 = -F^2B$, the vacuum expectation values in the real world involve an additive, convention dependent renormalization (the same remark applies to the difference $\langle 0|\bar{u}u - \bar{s}s|0\rangle$). The origin of this ambiguity may be seen in the first order perturbation theory formula

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{u}u|0\rangle_0 - i\int dx\,\langle 0|T\,\bar{q}\mathcal{M}q\,\bar{u}u|0\rangle + O(\mathcal{M}^2) \tag{11.4}$$

The renormalization dependence of $\langle 0|\bar{u}u|0\rangle$ reflects the fact that the scalar two-point function explodes like $x^{-6}(\log x)^{-2\gamma}$ as $x \to 0$ - the representation (11.4) does not make sense as it stands. (The situation is somewhat similar to the problems one encounters if one evaluates the quantity $\langle 0|\mathrm{tr}G_{\mu\nu}G^{\mu\nu}|0\rangle$ on the basis of QCD sum rules: one has to make sure that the perturbative contributions one neglects are smaller than the nonperturbative contributions one retains.)

In the following we do not make use of the vacuum expectation values $\langle 0|\bar{u}u|0\rangle$, $\langle 0|\bar{d}d|0\rangle$ except in the chiral limit where they are unambiguous. As pointed out by Novikov, Shifman, Vainshtein and Zakharov (1981) the nonanalytic contribution proportional to $M_\pi^2 \log M_\pi^2$ is also unambiguous:

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{u}u|0\rangle_0\left\{1-\frac{3M_\pi^2}{32\pi^2F_\pi^2}\ln M_\pi^2+\ldots\right\} \tag{11.5}$$

Since this term dominates the corrections for small quark mass we may conclude that the modulus of the expectation value increases as the quark mass is turned on. The quantity $|\langle 0|\bar{u}u|0\rangle|$ reaches a maximum at a quark mass value of the order of the scale of QCD and then decreases like $(m_u)^{-1}$ as $m_u \to \infty$ (Shifman, Vainshtein and Zakharov 1979).

12. Axial vector and pseudoscalar two-point functions, M_π, F_π

In the terminology introduced in section 10 the two-point functions containing the axial current and the pseudoscalar density are determined by the generating functional at order ϕ^2. To first nonleading order in the low energy expansion these Green's functions are therefore determined by the tree and tadpole graphs and one may extract them from (10.16) in the manner described in the preceding section. We first disregard isospin breaking effects by putting $m_u = m_d$; we will discuss the modifications produced by $m_u - m_d$ below. Expanding Z to second order in the field a_μ (which also fixes the pion field U^i through (10.21)) one obtains the following explicit expression for the two-point function of the axial current:

$$A^i_\mu = \bar{q}\gamma_\mu\gamma_5\frac{\tau^i}{2}q$$

$$i\int dx\, e^{ip(x-y)} \langle 0|T\, A^i_\mu(x)A^k_\nu(y)|0\rangle \tag{12.1}$$

$$= \delta^{ik}\left\{\frac{p_\mu p_\nu F_\pi^2}{M_\pi^2-p^2} + g_{\mu\nu}F_\pi^2 + (p_\mu p_\nu - g_{\mu\nu}p^2)\frac{1}{48\pi^2}(\bar{h}_2-\bar{\ell}_5)\right\} + O(p^4)$$

with

$$M_\pi^2 = M^2\left\{1-\frac{M^2}{32\pi^2F^2}\bar{\ell}_3 + O(M^4)\right\}$$
$$F_\pi = F\left\{1+\frac{M^2}{16\pi^2F^2}\bar{\ell}_4 + O(M^4)\right\} \tag{12.2}$$

(The symbol $O(p^4)$ in (12.1) includes terms of order p^4, p^2M^2 and M^4.) The tree graph contribution from ℓ_3 (and the tadpole graphs which renormalize this constant to $\bar{\ell}_3$) thus shift the position of the pion pole from $M^2 = 2\,\hat{m}B$ to M_π^2. The constant ℓ_4 determines the physical value of the pion decay constant which measures the matrix element

$$\langle 0|A^i_\mu|\pi^k\rangle = i\,p_\mu\delta^{ik}F_\pi \tag{12.3}$$

(We use the normalization $F_\pi = 93.3$ MeV.) Both M_π and F_π contain a chiral logarithm (Langacker and Pagels 1973)

$$M_\pi^2 = M^2\left\{1 + \frac{M^2}{32\pi^2 F^2}\ln M^2 + \dots\right\}$$
$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2 F^2}\ln M^2 + \dots\right\} \tag{12.4}$$

For small values of the quark mass these terms dominate the corrections and we get $M_\pi < M$, $F_\pi > F$. Note that the terms of order p^2 in $\langle 0|TA_\mu A_\nu|0\rangle$ involve the high energy constant h_2. The fact that this constant is convention dependent reflects the high energy behaviour of the spectral functions - the free quark loop produces a quadratic divergence in $\langle 0|TA_\mu A_\nu|0\rangle$.

The terms proportional to $a_\mu^i \chi^k$ and to $\chi^i\chi^k$ in the generating functional lead to

$$P^i = \bar{q}\, i\gamma_5 \tau^i q$$

$$i\int dx\, e^{ip(x-y)}\langle 0|T A_\mu^i(x) P^k(y)|0\rangle = \delta^{ik} F_\pi G_\pi \frac{ip_\mu}{M_\pi^2 - p^2} + O(p^3)$$

$$i\int dx\, e^{ip(x-y)}\langle 0|T P^i(x) P^k(y)|0\rangle = \delta^{ik}\left\{\frac{G_\pi^2}{M_\pi^2 - p^2} + \frac{B^2}{2\pi^2}(\bar{h}_1 - \bar{l}_4)\right\}$$
$$+ O(p^2) \tag{12.5}$$

where

$$G_\pi = 2BF\left\{1 - \frac{M^2}{32\pi^2 F^2}(\bar{l}_3 - 2\bar{l}_4) + O(M^4)\right\} \tag{12.6}$$

is the coupling constant of the pseudoscalar density to the pion:

$$\langle 0|\bar{q}\, i\gamma_5 \tau^i q|\pi^k\rangle = \delta^{ik} G_\pi \tag{12.7}$$

The relation between the divergence of the axial current and the pseudoscalar density implies

$$F_\pi M_\pi^2 = \hat{m} G_\pi \tag{12.8}$$

One easily checks that the Ward identities linking the two-point functions (12.1) and (12.5) are indeed satisfied. (The Ward identity for $<0|TA_\mu P|0>$ contains the expectation value $<0|\bar{q}q|0>$ determined in the preceding section.)

We now consider the effect of the quark mass difference $m_u - m_d$ on these quantities. The quark mass difference enters the generating functional through the terms involving the field $\tilde{\chi}^3 = (m_d - m_u)B + \ldots$ We are interested here in the expansion of Z with respect to a_μ and χ. Only the term $\ell_7\,(m_u - m_d)^2 B^2 (U^3)^2$ contributes to the two-point functions of the axial and pseudoscalar densities (the field U^3 is determined by $\partial^\mu a_\mu^3 + \chi^3$ through the equation of motion). The only effect of the quark mass difference is therefore a change in the mass of the π^0:

$$M_{\pi^0}^2 = M^2 - \frac{M^4}{32\pi^2 F^2}\bar{\ell}_3 - (m_u - m_d)^2 \frac{2B^2}{F^2}\ell_7 + O(m_{quark}^3) \tag{12.9}$$

We merely have to replace M_π^2 by $M_{\pi^0}^2$ in the Green's functions of A_μ^3 and P^3. There is no isospin symmetry breaking effect in the pion decay constant F_π or in G_π at this order of the low energy expansion.

Finally, the two-point functions involving the isoscalar

$$P^0 = \bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d$$

are obtained by expanding Z in powers of the field $\tilde{\chi}^0$. We find

$$\begin{aligned}
i\int dx\, e^{ip(x-y)} \langle 0|T A_\mu^3(x) P^0(y)|0\rangle &= \frac{ip_\mu}{M_{\pi^0}^2 - p^2} F_\pi \tilde{G}_\pi + O(p^3) \\
i\int dx\, e^{ip(x-y)} \langle 0|T P^3(x) P^0(y)|0\rangle &= \frac{G_\pi \tilde{G}_\pi}{M_{\pi^0}^2 - p^2} + O(p^2) \qquad (12.10) \\
i\int dx\, e^{ip(x-y)} \langle 0|T P^0(x) P^0(y)|0\rangle &= 8B^2(\ell_7 + h_3) + O(p^2)
\end{aligned}$$

where the coupling constant $\tilde{G}_\pi$ of the isoscalar to the pion

$$\langle 0|\bar{q}\, i\gamma_5 q|\pi^0\rangle = \tilde{G}_\pi \tag{12.11}$$

is of order $m_u - m_d$:

$$\tilde{G}_\pi = -(m_u - m_d)\,\frac{4B^2}{F}\,\ell_7 \tag{12.12}$$

To check the Ward identities one has to take the term of order $m_u - m_d$ in the divergence of the axial current into account:

$$\partial^\mu\left(\bar{q}\gamma_\mu\gamma_5\frac{\tau^i}{2}q\right) = \frac{m_u+m_d}{2}\,\bar{q}\, i\gamma_5\tau^i q + \delta^{i3}\,\frac{m_u-m_d}{2}\,\bar{q}\, i\gamma_5 q \tag{12.13}$$

This relation implies, in particular

$$F_\pi M^2_{\pi^0} = \frac{m_u+m_d}{2}\, G_\pi + \frac{m_u-m_d}{2}\,\tilde{G}_\pi \tag{12.14}$$

13. Vector and scalar two-point functions

Since the fields v^i_μ and $\hat{\chi}^0 = \chi^0 - M^2$ count as quantities of order ϕ^2 we need Z to order ϕ^4 to get the two-point functions containing the vector and scalar densities. In particular, the unitarity correction Z_u does contribute to these objects. The pion field U^i and the covariant derivative $\nabla_\mu U$ vanish if the fields a^i_μ and χ^i are switched off and we obtain

$$\begin{aligned} \mathrm{tr}\,\Gamma_{\mu\nu}(x)\Gamma^{\mu\nu}(y) &= -2(\partial_\mu v^i_\nu - \partial_\nu v^i_\mu)_x(\partial^\mu v^{\nu i} - \partial^\nu v^{\mu i})_y \\ \mathrm{tr}\,\hat{\sigma}(x)\hat{\sigma}(y) &= 3\hat{\chi}^0(x)\hat{\chi}^0(y) \end{aligned} \tag{13.1}$$

Inserting these expressions in the representation (10.17) for Z_u we get

$$V_\mu^i = \bar{q}\gamma_\mu \frac{\tau^i}{2} q \;;\; S^0 = \bar{q}q \;;\; S^i = \bar{q}\tau^i q$$

$$i\int dx\, e^{ip(x-y)} \langle 0|T\, V_\mu^i(x) V_\nu^k(y)|0\rangle =$$

$$= \delta^{ik}(p_\mu p_\nu - g_{\mu\nu}p^2)\left\{\frac{1}{3}\left(1-\frac{4M^2}{p^2}\right)\bar{J}(p^2) + \frac{1}{48\pi^2}\left(\bar{h}_2 - \frac{1}{3}\right)\right\} + O(p^4)$$

$$i\int dx\, e^{ip(x-y)} \langle 0|T\, S^0(x) S^0(y)|0\rangle = \tag{13.2}$$

$$= 6B^2\bar{J}(p^2) + \frac{B^2}{8\pi^2}(4\bar{h}_1 - \bar{l}_3 - 3) + O(p^2)$$

$$i\int dx\, e^{ip(x-y)} \langle 0|T S^i(x) S^k(y|0\rangle = \delta^{ik}\, 8B^2 h_3 + O(p^2)$$

where the two body phase space integral $\bar{J}(p^2)$ is given in (10.9). The contributions proportional to $\bar{J}(p^2)$ arise from two pion intermediate states which produce a cut starting at $p^2 = 4M_\pi^2$.

There are no isospin breaking effects in the vector and scalar two-point functions at the level of the low energy expansion we are considering here. The low energy expansions of $\langle 0|TV_\mu^i S^0|0\rangle$ and $\langle 0|TV_\mu^i S^k|0\rangle$ e.g. start with terms of order $(m_u - m_d) \cdot p_\mu = O(p^3)$ which are beyond the reach of our accuracy, because these terms correspond to contributions of order p^6 in the generating functional (the external fields v_μ^i and χ, $\tilde{\chi}$ count as quantities of order p and p^2 respectively).

The contact terms in all of the above Green's functions are sensitive to the high energy constants h_1, h_2, h_3. These constants however drop out in the differences ($M_1^2 = M_2^2 = M_{\pi^+}^2$; $M_3^2 = M_{\pi^0}^2$):

$$i\int dx\, e^{ip(x-y)} \langle 0|T\, V_\mu^i(x) V_\nu^k(y) - T A_\mu^i(x) A_\nu^k(y)|0\rangle =$$

$$= \delta^{ik}\left\{-\frac{p_\mu p_\nu}{M_i^2 - p^2} F_\pi^2 - g_{\mu\nu} F_\pi^2\right\} + \delta^{ik}(p_\mu p_\nu - g_{\mu\nu}p^2)\left\{\frac{1}{3}\left(1 - \frac{4M^2}{p^2}\right)\bar{J}(p^2)\right.$$

$$\left. + \frac{1}{48\pi^2}\left(\bar{l}_5 - \frac{1}{3}\right)\right\} + O(p^4)$$

$$i\int dx\, e^{ip(x-y)}\langle 0|\delta^{ik}\, T S^0(x) S^0(y) - T P^i(x) P^k(y)|0\rangle$$

$$= \delta^{ik}\left\{-\frac{G_\pi^2}{M_i^2-p^2} + 6B^2\bar{J}(p^2) - \frac{B^2}{8\pi^2}(\bar{l}_3 - 4\bar{l}_4 + 3)\right\} + O(p^2)$$

$$i\int dx\, e^{ip(x-y)}\langle 0|T S^i(x) S^k(y) - \delta^{ik}\, T P^0(x) P^0(y)|0\rangle$$

$$= -\delta^{ik}\, 8B^2 l_7 + O(p^2) \tag{13.3}$$

because the leading high energy behaviour of the relevant spectral functions is the same.

14. Spectral representations

Expressed in terms of the corresponding Källen-Lehmann spectral functions the above low energy representations for the two-point functions contain two different pieces of information: (i) they specify the leading low energy contributions to the spectral functions and (ii) they imply a set of sum rules which the spectral functions must satisfy (Weinberg 1967; Das, Mathur and Okubo 1967; Glashow, Schnitzer and Weinberg 1967; Oakes and Sakurai 1967).

For simplicity we disregard isospin breaking. There are then six independent spectral functions which we normalize as follows:

$$\frac{1}{2\pi}\int dx\, e^{ip(x-y)}\langle 0|V^i_\mu(x) V^k_\nu(y)|0\rangle = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\delta^{ik}\,\rho^1_V(p^2)$$

$$\frac{1}{2\pi}\int dx\, e^{ip(x-y)}\langle 0|A^i_\mu(x) A^k_\nu(y)|0\rangle = p_\mu p_\nu\,\rho^0_A(p^2) + (p_\mu p_\nu - g_{\mu\nu}p^2)\,\rho^1_A(p^2)$$

$$\frac{1}{2\pi}\int dx\, e^{ip(x-y)}\langle 0|S^0(x) S^0(y)|0\rangle = \rho_S(p^2)$$

$$\frac{1}{2\pi}\int dx\, e^{ip(x-y)}\langle 0|S^i(x)S^k(y)|0\rangle = \delta^{ik}\tilde{\rho}_S(p^2)$$

$$\frac{1}{2\pi}\int dx\, e^{ip(x-y)}\langle 0|P^0(x)P^0(y)|0\rangle = \tilde{\rho}_P(p^2) \qquad (14.1)$$

The spectral function $\rho_P(s)$ involving the pseudoscalar $P^i(x)$ may be expressed through the spin zero piece of the axial spectral function

$$\rho_P(s) = s^2\rho_A^0(s)/\hat{m}^2$$

The low energy structure of ρ_V^1, ρ_A^0 and ρ_S is obtained by evaluating the imaginary part of the corresponding two-point functions:

$$\rho_V^1(s) = \frac{1}{48\pi^2}\Theta(s-4M_\pi^2)\left(1-\frac{4M_\pi^2}{s}\right)^{3/2} + O(p^2)$$

$$\rho_A^0(s) = F_\pi^2\,\delta(s-M_\pi^2) + O(p^2) \qquad (14.2)$$

$$\rho_S(s) = \frac{3B^2}{8\pi^2}\Theta(s-4M_\pi^2)\left(1-\frac{4M_\pi^2}{s}\right)^{1/2} + O(p^2)$$

The remaining three quantities ρ_A^1, $\tilde{\rho}_S$ and $\tilde{\rho}_P$ are of order p^2 at low energies.

At high energies perturbation theory shows that ρ_V^1 and ρ_A^1 tend to a constant, whereas ρ_A^0 tends to zero:

$$\rho_V^1(s) \longrightarrow (8\pi^2)^{-1}$$

$$\rho_A^1(s) \longrightarrow (8\pi^2)^{-1} \qquad (14.3)$$

$$\rho_A^0(s) \longrightarrow 3\hat{m}^2(s)(4\pi^2 s)^{-1}$$

where $\hat{m}(s)$ is the running quark mass at scale $\sqrt{s}$. The other spectral functions, ρ_S, $\tilde{\rho}_S$ and $\tilde{\rho}_P$ all grow in proportion to $s(\log s)^{-2\kappa}$ with $\kappa = 4\,(11 - \frac{2}{3}N_f)^{-1}$. The formal dispersion relation for the scalar two-point function e.g.

$$i\int dx\, e^{ip(x-y)}\langle 0|T S^0(x)S^0(y)|0\rangle = \int_0^\infty \frac{ds}{s-p^2}\rho_S(s) \qquad (14.4)$$

requires two subtractions to converge. This is why the low energy representation (13.2) for the two-point functions involves the convention-dependent high energy constants h_1, h_2, h_3.

Whereas the leading short distance behaviour of the individual two-point functions is of the type z^{-6} (up to logarithms) the leading contributions to the differences which occur on the left-hand side of (13.3) are of order $(\hat{m})^2 z^{-4} + \hat{m} \langle 0|\bar{q}q|0\rangle z^{-2} + \ldots$ (Fritzsch and Leutwyler 1974; Hagiwara and Mohapatra 1975; Bernard et al. 1975; Weisberger 1976; Ong 1977; Sazdjian 1977; Floratos, Narison and de Rafael 1979; Broadhurst 1981). The renormalization group improved leading singularity is given by the same expression, provided $\hat{m}$ is replaced by the running quark mass $\hat{m}(s)$. Thanks to the presence of anomalous dimensions, the dispersion relations involving the differences $\rho_S - \rho_P$ and $\tilde{\rho}_S - \tilde{\rho}_P$ do therefore not require subtractions. On the other hand, the high energy behaviour of the spectral function $s(\rho_V^1 - \rho_A^1)$ is not improved by an additional logarithm from anomalous dimensions. The contribution proportional to z^{-4} is however of order $(\hat{m}(s))^2$ and only shows up if one carries the low energy expansion beyond first nonleading order. At the order we are considering here the leading short distance behaviour is proportional to z^{-2} - the dispersion relations involving the difference $s(\rho_V^1 - \rho_A^1)$ does not require subtractions (one may e.g. cut the integrals off at some large value s_0 - the result depends on s_0 only through terms of order $(\hat{m})^2$. An analogous remark applies to the sum rules eqs. (14.6), (14.7).) We may therefore rewrite the low energy theorems (13.3) in the form of sum rules, obtained by evaluating the corresponding unsubtracted dispersion relations at $p^2 = 0$:

$$\int_0^\infty ds \left\{ \rho_V^1(s) - \rho_A^1(s) \right\} = F_\pi^2 + O(M_\pi^4)$$

$$\int_0^\infty \frac{ds}{s} \left\{ \rho_V^1(s) - \rho_A^1(s) \right\} = \frac{1}{48\pi^2}(\bar{l}_5 - 1) + O(M_\pi^2)$$

$$\int_0^\infty \frac{ds}{s} \rho_A^0(s) = F_\pi^2 M_\pi^{-2} + O(M_\pi^2)$$

$$\int_0^\infty \frac{ds}{s} \hat{m}^2 \left\{ \rho_S(s) - \rho_P(s) \right\} = -F_\pi^2 M_\pi^2 \left\{ 1 + \frac{M_\pi^2}{32\pi^2 F_\pi^2}(\bar{l}_3 - 4\bar{l}_4 + 3) \right\} + O(M_\pi^6)$$

$$\int_0^\infty \frac{ds}{s} \hat{m}^2 \left\{ \tilde{\rho}_S(s) - \tilde{\rho}_P(s) \right\} = -2 M_\pi^4 l_7 + O(M_\pi^6) \qquad (14.5)$$

(To obtain the sum rules associated with the low energy theorems for the difference VV - AA one separately considers the coefficients of $g_{\mu\nu}$ and of $p_\mu p_\nu$.)

The first one of these relations is equivalent to the first Weinberg sum rule (Weinberg 1967). The second Weinberg sum rule holds up to terms of order $\hat{m}$ (see the references quoted above):

$$\int_0^\infty ds\, s\,\{\rho_V^1(s) - \rho_A^1(s)\} = O(M_\pi^2) \tag{14.6}$$

and the same is true of the sum rules for the differences of scalar and pseudoscalar densities:

$$\begin{aligned}\int_0^\infty ds\,\{\rho_S(s) - \rho_P(s)\} &= O(M_\pi^2)\\ \int_0^\infty ds\,\{\tilde{\rho}_S(s) - \tilde{\rho}_P(s)\} &= O(M_\pi^2)\end{aligned} \tag{14.7}$$

(The relations (14.6) and (14.7) follow from the short distance behaviour of the relevant two-point functions.)

15. Vertex functions and form factors

The explicit expression for Z given in section 10 allows one to read off the low energy representations for the vertex functions VAA, VAP, VPP, SAA, SAP and SPP. In particular, one finds the following expression for the Green's function <0|TVAP|0>:

$$\begin{aligned}&i^2\int dx\,dy\, e^{iqx + ip'y - ipz}\langle 0|T V_\mu^i(x) A_\nu^k(y) P^\ell(z)|0\rangle\\
&= \frac{\varepsilon^{ik\ell} F_\pi G_\pi}{M_\ell^2 - p^2}\Big[g_{\mu\nu} + \frac{(p'_\mu + p_\mu)p'_\nu}{M_k^2 - p'^2}\\
&\quad - \frac{1}{3F_\pi^2}\left(1 - \frac{4M^2}{q^2}\right)\bar{J}(q^2)\left\{q_\mu q_\nu - g_{\mu\nu}q^2 + \frac{(q_\mu p\cdot q - p_\mu q^2)p'_\nu}{M_k^2 - p'^2}\right\}\\
&\quad + \frac{1}{48\pi^2 F_\pi^2}\left(\bar{\ell}_5 - \frac{1}{3}\right)\{p'_\mu q_\nu - g_{\mu\nu}p'\cdot q\}\\
&\quad - \frac{1}{48\pi^2 F_\pi^2}\left(\bar{\ell}_6 - \frac{1}{3}\right)\left\{p_\mu q_\nu - g_{\mu\nu}p\cdot q + \frac{p_\mu p'\cdot q - p'_\mu p\cdot q}{M_k^2 - p'^2}p'_\nu + O(p^4)\right\}\Big]\end{aligned} \tag{15.1}$$

with $p = p' + q$. (Recall that the symbol $O(p^2)$ includes terms of order $(\text{momentum})^2$ as well as terms of order m_u or m_d.) The only effect of isospin breaking at this order of the low energy expansion is that the masses occurring in the pole factors $(M_\ell^2 - p^2)^{-1}$, $(M_k^2 - p'^2)^{-1}$ depend on the electric charge carried by the axial current and by the pseudoscalar field: $M_1 = M_2 = M_{\pi^+}$, $M_3 = M_{\pi^0}$.

The residue of the pion poles at $p^2 = M_\ell^2$, $p'^2 = M_k^2$ is proportional to the vector form factor of the pion, defined by

$$\langle \pi^i p' | V_\mu^k | \pi^\ell p \rangle = i\,\varepsilon^{ik\ell}(p'_\mu + p_\mu) F_V(q^2) \tag{15.2}$$

Explicitly, we obtain from (15.1)

$$F_V(t) = 1 + \frac{1}{6F_\pi^2}(t - 4M_\pi^2)\bar{J}(t) + \frac{t}{96\pi^2 F_\pi^2}\left(\bar{\ell}_6 - \frac{1}{3}\right) + O(p^4) \tag{15.3}$$

There are no isospin breaking contributions to F_V at this order of the low energy expansion (Ademollo-Gatto theorem) (Behrends and Sirlin 1960; Ademollo and Gatto 1964). The low energy representation of the form factor in particular determines the electromagnetic charge radius of the pion:

$$\begin{aligned} F_V(t) &= 1 + \frac{1}{6} t \langle r^2 \rangle_V^\pi + O(t^2) \\ \langle r^2 \rangle_V^\pi &= \frac{1}{16\pi^2 F_\pi^2}(\bar{\ell}_6 - 1) + O(M_\pi^2) \end{aligned} \tag{15.4}$$

The constant $\bar{\ell}_6$ thus measures the electromagnetic charge radius. Note that this radius explodes as $M_\pi \to 0$ (Bēg and Zepeda 1972; Volkov and Pervushin 1974,1975)

$$\langle r^2 \rangle_V^\pi = -\frac{1}{16\pi^2 F_\pi^2} \ln M_\pi^2 + \ldots \tag{15.5}$$

The reason why there is no charge radius in the chiral limit is that the pion cloud surrounding any particle, in particular, the cloud surrounding the pion becomes long range as $M_\pi \to 0$: there is no Yukawa factor $\sim \exp(-M_\pi r)$ to cut it off at large distances. In the chiral limit the charge distribution only falls off like a power of the distance, the mean square radius of the distribution diverges.

The Green's function $<0|TVAP|0>$ also determines the amplitude $<0|TVA|\pi>$ which contributes to the decay $\pi \to e\nu\gamma$. Extracting the pion pole and setting the photon momentum q on the mass shell, $q^2 = 0$, we get

$$i\int dx\, e^{iqx+ip'y} \langle 0|T\, V_\mu^i(x) A_\nu^k(y)|\pi^\ell p\rangle \cdot \varepsilon^\mu$$

$$= \varepsilon^{ik\ell} F_\pi \left\{ \varepsilon_\nu + \frac{\varepsilon \cdot p\,(p-q)_\nu}{p \cdot q} + \frac{1}{48\pi^2 F_\pi^2}(\bar{\ell}_6 - \bar{\ell}_5)(\varepsilon_\nu\, p\cdot q - q_\nu\, \varepsilon \cdot p) \right\} + O(p^4) \tag{15.6}$$

where ε_μ is the polarization vector of the photon, $\varepsilon q = 0$. This result is equivalent to a low energy theorem established by Das, Mathur and Okubo (1967): the constants $\bar{\ell}_6$ and $\bar{\ell}_5$ may be expressed in terms of the electromagnetic charge radius and in terms of the spectral function difference $\rho_V^1 - \rho_A^1$ respectively (cf. (15.4) and (14.5)). The structure term in (15.6) then becomes

$$\frac{1}{48\pi^2 F_\pi^2}(\bar{\ell}_6 - \bar{\ell}_5) = \frac{1}{3}\langle r^2\rangle_V^\pi - \frac{1}{F_\pi^2}\int_0^\infty \frac{ds}{s}(\rho_V^1 - \rho_A^1) + O(M_\pi^2) \tag{15.7}$$

The three point function $<0|TSPP|0>$, which occurs in the Ward identities for the off-shell pion scattering amplitude, may be obtained by expanding the generating functional to order $\hat{\chi}^o \chi^i \chi^k$. The result reads

$$i^2\int dx\,dy\, e^{iqx+ip'y-ipz} \langle 0|T\, S^0(x) P^i(y) P^k(z)|0\rangle$$

$$= \delta^{ik} \frac{1}{M_i^2 - p'^2}\, \frac{1}{M_k^2 - p^2}\, \frac{G_\pi^3}{F_\pi}\, G(q^2, p'^2, p^2)$$

$$G(q^2,p'^2,p^2) = 1 + \frac{1}{2F^2}(2q^2 + p'^2 + p^2 - 3M^2)\bar{J}(q^2) \tag{15.8}$$

$$+ \frac{1}{16\pi^2 F^2}(\bar{\ell}_4 - 1)(q^2 - p'^2 - p^2)$$

$$+ \frac{1}{32\pi^2 F^2}(4\bar{\ell}_4 - \bar{\ell}_3 - 3)(p'^2 + p^2 - M^2)$$

Again, the quark mass difference $m_u - m_d$ only affects the positions of the poles - the residue, which is proportional to the scalar form factor of the pion:

$$\langle \pi^i p' | \bar{q}q | \pi^k p \rangle = \delta^{ik} F_S(t)$$

$$F_S(t) = F_S(0) \left\{ 1 + \frac{1}{2F_\pi^2}(2t - M_\pi^2)\bar{J}(t) + \frac{t}{16\pi^2 F_\pi^2}(\bar{\ell}_4 - 1) + O(t^2) \right\} \tag{15.9}$$

$$F_S(0) = 2B \left\{ 1 - \frac{M_\pi^2}{16\pi^2 F_\pi^2}(\bar{\ell}_3 - \tfrac{1}{2}) + O(M_\pi^4) \right\}$$

does not show any isospin breaking at this order of the low energy expansion. The scalar charge radius which measures the slope of $F_S(t)$ is given by

$$\langle r^2 \rangle_S^\pi = \frac{3}{8\pi^2 F_\pi^2}\left(\bar{\ell}_4 - \frac{13}{12}\right) + O(M_\pi^2) \tag{15.10}$$

Note that the coefficient of the chiral logarithm contained in $\langle r^2 \rangle_S^\pi$

$$\langle r^2 \rangle_S^\pi = -\frac{3}{8\pi^2 F_\pi^2} \ln M_\pi^2 + \dots$$

is six times larger than the corresponding coefficient in $\langle r^2 \rangle_V$. Since the constant $\bar{\ell}_4$ also measures the difference between F and F_π we have the following relation:

$$\frac{F_\pi}{F} = 1 + \frac{1}{6} M_\pi^2 \langle r^2 \rangle_S^\pi + \frac{13}{192\pi^2} \frac{M_\pi^2}{F_\pi^2} + O(M_\pi^4) \tag{15.11}$$

It is clear from this relation that the value of the scalar radius plays a central role in the analysis of the low energy theorems.

The value of the form factor $F_S(t)$ at t = 0 is proportional to the expectation value of the quark mass term in the Hamiltonian of QCD. According to the Feynman-Hellman theorem (Hellman 1937; Feynman 1939; Epstein 1954) the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation. In the present case the perturbation is caused by $\hat{m}\bar{q}q$ and the theorem says that the response of the pion $(\text{mass})^2$ to a change in the quark mass is given by the expectation value of $\bar{q}q$ in the physical eigenstate describing a massive pion:

$$\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi | \bar{q} q | \pi \rangle \tag{15.12}$$

Indeed, $F_S(0)$ is the derivative of M_π^2 with respect to $\hat{m}$ (note that $\bar{\ell}_3$ logarithmically depends on $\hat{m}$).

The same argument, applied to the perturbation produced by the quark mass difference $m_u - m_d$ leads to

$$\frac{\partial M_\pi^2}{\partial (m_u - m_d)} = \frac{1}{2} \langle \pi | \bar{q} \tau^3 q | \pi \rangle \tag{15.13}$$

The change in $M_{\pi^0}^2$ is given in (12.9); M_{π^+} is not affected at order $(m_u - m_d)^2$. This leads to

$$\langle \pi^i p' | \bar{q} \tau^k q | \pi^\ell p \rangle = (m_u - m_d) \left\{ -\frac{4B}{F^2} \ell_7 (\delta^{i3} \delta^{k\ell} + \delta^{\ell 3} \delta^{ik}) + O(p^2) \right\} \tag{15.14}$$

(Note that the form factor vanishes for $m_u = m_d$ on account of isospin invariance and Bose symmetry; the low energy representation of the generating functional given in the preceding sections does therefore not contain any information about the slope of this form factor which remains hidden in the higher order terms.)

16. Four-point function

As a final application of the low energy representation given in section 10 we consider the four-point function $\langle 0 | T P^i P^k P^\ell P^m | 0 \rangle$ and first look at the contribution to this quantity generated by the quark mass difference $m_u - m_d$. If the external fields $\tilde{\chi}^0$, $\tilde{\chi}^i$ (which are the sources of P^0 and S^i) are switched off, the mass difference enters the generating functional only through the term

$$\int dx \, (m_u - m_d)^2 B^2 \ell_7 (U^3)^2 \tag{16.1}$$

We need this term only in tree graph approximation - since it is quadratic in the pion field U^3 it may be absorbed in the pion mass term contained in the zero order generating functional. The mass of the π^0 is shifted from M^2 to $M^2 - 2\,(m_u - m_d)^2 B^2 \ell_7/F^2$; otherwise isospin breaking has no effects at first nonleading order in the low energy expansion. We therefore merely have to equip the pion propagators associated with the four legs of the four-point function with the proper mass; the amputated four-point function is isospin symmetric. For this reason we in the following disregard the mass difference $m_u - m_d$ altogether.

The four-point function is of the form

$$
\begin{aligned}
& i^3 \int dx_1 dx_2 dx_3\, e^{ip_1x_1+ip_2x_2+ip_3x_3+ip_4x_4} \langle 0|T\, P^i(x_1) P^k(x_2) P^\ell(x_3) P^m(x_4)|0\rangle \\
&= \frac{G_\pi^4}{\prod_i (M_\pi^2 - p_i^2)} \Big\{ \delta^{ik}\delta^{\ell m} A(s,t,u;p_1^2,p_2^2,p_3^2,p_4^2) \\
&\qquad + \delta^{i\ell}\delta^{km} A(t,s,u;p_1^2,p_3^2,p_2^2,p_4^2) \\
&\qquad + \delta^{im}\delta^{k\ell} A(u,t,s;p_1^2,p_4^2,p_3^2,p_2^2) \Big\}
\end{aligned}
\tag{16.2}
$$

with

$$
\begin{aligned}
& p_1 + p_2 + p_3 + p_4 = 0 \\
& s = (p_1+p_2)^2 \;;\; t = (p_1+p_3)^2 ;\; u = (p_1+p_4)^2 \\
& s+t+u = \sum_i p_i^2
\end{aligned}
\tag{16.3}
$$

Explicitly, the amplitude A is given by

$$
\begin{aligned}
A(s,t,u;p_1^2,p_2^2,p_3^2,p_4^2) &= \frac{1}{F^2}(s-M^2) + B(s,t,u;p_1^2,\dots,p_4^2) \\
&\quad + C(s,t,u;p_1^2,\dots,p_4^2) + O(p^6)
\end{aligned}
$$

$$B(s,t,u;p_1^2,\ldots,p_4^2) = (6F^4)^{-1}\Big[3(s-M^2)(2s+t+u-3M^2)\bar{J}(s)$$
$$+\Big\{2M^2(u-s)-2tu-st+3(p_1^2p_2^2+p_3^2p_4^2)-\Delta_{13}\Delta_{24}\Big(1+\frac{2M^2}{t}\Big)\Big\}\bar{J}(t)$$
$$+\Big\{2M^2(t-s)-2ut-su+3(p_1^2p_3^2+p_2^2p_4^2)-\Delta_{14}\Delta_{23}\Big(1+\frac{2M^2}{u}\Big)\Big\}\bar{J}(u)$$
$$\Delta_{ik} = p_i^2 - p_k^2$$
$$C(s,t,u;p_1^2,\ldots,p_4^2) = (96\pi^2F^4)^{-1}\Big[2\Big(\bar{\ell}_1-\frac{4}{3}\Big)\Big\{-s(t+u)+(p_1^2+p_2^2)(p_3^2+p_4^2)\Big\}$$
$$+2\Big(\bar{\ell}_2-\frac{5}{6}\Big)\Big\{-s(t+u)-2tu+(p_1^2+p_2^2)(p_3^2+p_4^2)+2(p_1^2p_2^2+p_3^2p_4^2)\Big\}$$
$$+3(\bar{\ell}_3-1)\Big\{M^2(s+t+u)-(p_1^2+p_2^2)(p_3^2+p_4^2)\Big\}$$
$$+6(\bar{\ell}_4-1)(s-M^2)(s+t+u-4M^2)$$
$$-12M^2s+15M^4\Big] \tag{16.4}$$

$B(s,\ t,\ u;\ p_1^2,\ \ldots,\ p_4^2)$ is the unitarity correction, proportional to the phase space integral $\bar{J}(p^2)$ given in (10.9); $C(s,t,u;p_1^2,\ldots,p_4^2)$ contains tree graphs and tadpoles. Note that we have expressed the leading contribution in terms of F and M^2, not in terms of F_π and M_π.

As a check one may verify that $A(s,\ t,\ u;\ p_1^2,\ \ldots,\ p_4^2)$ satisfies the Ward identity

$$A(s,t,u;0,s,t,u) = \frac{s-M_\pi^2}{F_\pi^2}\,G(s,t,u) \tag{16.5}$$

where $G(q^2,\ p'^2,\ p^2)$ is the vertex function <0|TSPP|0> given explicitly in (15.8). G(s,t,u) itself satisfies a Ward identity which, in particular requires

$$G(M_\pi^2, 0, M_\pi^2) = 1 \tag{16.6}$$

17. $\pi\pi$-scattering amplitude

The scattering amplitude is easily obtained from (16.4) by putting all momenta on the mass shell:

$$A(s,t,u) = \frac{s-M^2}{F^2} + B(s,t,u) + C(s,t,u) + O(p^6)$$

$$\begin{aligned} B(s,t,u) = (6F^4)^{-1} \Big[& 3(s^2 - M^4)\bar{J}(s) \\ & + \{t(t-u) - 2M^2 t + 4M^2 u - 2M^4\}\bar{J}(t) \\ & + \{u(u-t) - 2M^2 u + 4M^2 t - 2M^4\}\bar{J}(u) \end{aligned} \tag{17.1}$$

$$\begin{aligned} C(s,t,u) = (96\pi^2 F^4)^{-1} \Big[& 2(\bar{\ell}_1 - \tfrac{4}{3})(s - 2M^2)^2 \\ & + (\bar{\ell}_2 - \tfrac{5}{6})\{s^2 + (t-u)^2\} - 12M^2 s + 15M^4 \Big] \end{aligned}$$

This representation involves only four of the low energy constants: F, M^2, $\bar{\ell}_1$ and $\bar{\ell}_2$ (recall that F is the value of F_π in the chiral limit and M^2 stands for $B(m_u + m_d)$ rather than for the square of the physical pion mass which we denote by M_π).

In the chiral limit (17.1) takes the form first given by Lehmann (1972) (see also (Ecker and Honerkamp 1973; Lehman and Trute 1973)):

$$\begin{aligned} A(s,t,u) = \frac{s}{F^2} + (96\pi^2 F^4)^{-1} \Big\{ & 3s^2 \ln\left(\frac{\mu_1^2}{-s}\right) \\ & + t(t-u)\ln\left(\frac{\mu_2^2}{-t}\right) + u(u-t)\ln\left(\frac{\mu_2^2}{-u}\right)\Big\} + O(p^6) \end{aligned} \tag{17.2}$$

where the scales μ_1 and μ_2 are given by

$$\ln \frac{\mu_1^2}{M_\pi^2} = \frac{2}{3}\bar{\ell}_1 + \frac{1}{3}\bar{\ell}_2 + \frac{5}{6}$$
$$\ln \frac{\mu_2^2}{M_\pi^2} = \bar{\ell}_2 + \frac{7}{6} \qquad (17.3)$$

(Note that μ_1 and μ_2 remain finite as $M_\pi \to 0$.)

The low energy representation (17.1) provides us with a generalization of this result to nonvanishing quark masses. The general structure of the amplitude A(s, t, u) as given in (17.1) is a consequence of unitarity (Iliopoulos 1967, 1968; Morgan and Shaw 1972; Volkov and Pervushin 1974, 1975; Bel'kov, Bunyatov and Pervushin 1979, 1980; Truong 1981). Indeed, it is straightforward to check that the contribution B(s, t, u) does generate the proper imaginary parts required by unitarity. What unitarity does not determine is the structure of the polynomial C(s, t, u); the general crossing symmetric polynomial of order p^4 contains four unknowns:

$$C(s,t,u) = c_1 s^2 + c_2 (t-u)^2 + c_3 s M^2 + c_4 M^4$$

The constants c_1, c_2, c_3, c_4 correspond to the coupling constants at order p^4 occurring in a general effective Lagrangian for on-shell pions (Weinberg 1979) - the effective on-shell Lagrangian leaves these constants also undetermined. The analysis of the off-shell amplitude carried out above furnishes additional information: it fixes the constants $c_1, \ldots, c_4$ in terms of only two unknowns $\bar{\ell}_1$ and $\bar{\ell}_2$. In particular, this analysis shows that the constants c_1, ..., c_4 all contain a chiral logarithm, i.e. explode if the quark mass is sent to zero. In the case of c_1 and c_2 it is easy to see where the chiral logarithm comes from: we have chosen to normalize the "unitarity correction" B(s, t, u) by subtracting the dispersion relation for $\bar{J}(s)$ at s = 0. This normalization cannot be maintained if the pion mass vanishes - the contributions proportional to log M_π^2 in c_1 and c_2 precisely insure that the sum B + C remains finite in the chiral limit. The chiral logarithms in c_3 and c_4 are more subtle - their presence can only be understood on the basis of the Ward identity (16.5) satisfied by the <u>off-shell</u> amplitude.

18. Partial wave expansion and threshold parameters

To compare the calculated amplitude with low energy data on $\pi\pi$-scattering (Petersen 1971; Martin, Morgan and Shaw 1976) one expands the combinations with definite isospin in the s-channel

$$
\begin{aligned}
T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\
T^1(s,t) &= A(t,u,s) - A(u,s,t) \qquad\qquad (18.1) \\
T^2(s,t) &= A(t,u,s) + A(u,s,t)
\end{aligned}
$$

into partial waves:

$$
\begin{aligned}
T^I(s,t) &= 32\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\Theta)\, t_\ell^I(s) \\
s &= 4(M_\pi^2 + q^2) \qquad\qquad (18.2) \\
t &= -2q^2(1-\cos\Theta)
\end{aligned}
$$

Unitarity implies that in the elastic region $4\,M_\pi^2 < s < 16\,M_\pi^2$ the partial wave amplitudes t_ℓ^I are described by real phase shifts δ_ℓ^I:

$$t_\ell^I(s) = \left(\frac{s}{s-4M_\pi^2}\right)^{\frac{1}{2}} \frac{1}{2i}\left\{e^{2i\delta_\ell^I(s)} - 1\right\} \qquad (18.3)$$

The behaviour of the partial wave amplitudes near threshold is of the form

$$\mathrm{Re}\, t_\ell^I(s) = q^{2\ell}\left\{a_\ell^I + q^2 b_\ell^I + O(q^4)\right\} \qquad (18.4)$$

The quantities a_ℓ^I are referred to as the $\pi\pi$-scattering lengths. (Note that the standard definition of the S-wave scattering length A, based on the effective range formula

$$q \,\mathrm{ctg}\, \delta = -\frac{1}{A} + \frac{1}{2} r_0 q^2 + \cdots$$

is of opposite sign: $A = -a_0/M_\pi$.)

It is straightforward to extract the threshold parameters a_ℓ^I, b_ℓ^I from the representation (17.1) of the amplitude. To leading order in the low energy expansion (B = C = 0) the amplitude is a linear function of s and therefore only the S- and P-wave amplitudes are different from zero. At the next order in the low energy expansion all partial waves contribute. For those threshold parameters which are nonzero in leading order we get corrections of relative order M_π^2:

$$a_0^0 = \frac{7M^2}{32\pi F^2}\left[1 + \frac{5}{84\pi^2}\frac{M^2}{F^2}\left\{\bar{\ell}_1 + 2\bar{\ell}_2 - \frac{9}{10}\bar{\ell}_3 + \frac{21}{8}\right\} + O(M^4)\right]$$

$$b_0^0 = \frac{1}{4\pi F^2}\left[1 + \frac{1}{12\pi^2}\frac{M^2}{F^2}\left\{2\bar{\ell}_1 + 3\bar{\ell}_2 - \frac{13}{16}\right\} + O(M^4)\right]$$

$$a_0^2 = -\frac{M^2}{16\pi F^2}\left[1 - \frac{1}{12\pi^2}\frac{M^2}{F^2}\left\{\bar{\ell}_1 + 2\bar{\ell}_2 + \frac{3}{8}\right\} + O(M^4)\right] \tag{18.5}$$

$$b_0^2 = -\frac{1}{8\pi F^2}\left[1 - \frac{1}{12\pi^2}\frac{M^2}{F^2}\left\{\bar{\ell}_1 + 3\bar{\ell}_2 - \frac{5}{16}\right\} + O(M^4)\right]$$

$$a_1^1 = \frac{1}{24\pi F^2}\left[1 - \frac{1}{12\pi^2}\frac{M^2}{F^2}\left\{\bar{\ell}_1 - \bar{\ell}_2 + \frac{65}{48}\right\} + O(M^4)\right]$$

Note that we have expressed the result in terms of F and M rather than in terms of the physical values F_π and M_π. (The definition of a_0^0 however involves the physical value of the pion mass - this is why the quark mass expansion of this quantity contains the constant $\bar{\ell}_3$ which relates M_π to M.)

For the remaining threshold parameters which vanish in lowest order we obtain new low energy theorems which specify the leading term in an expansion of these quantities in powers of the quark mass (or, equivalently, the pion mass). The P-wave slope and the D-wave scattering lengths are given by

$$b_1^1 = (288\pi^3 F^4)^{-1}\left\{-\bar{\ell}_1 + \bar{\ell}_2 + 97/120\right\} + O(M^2)$$

$$a_2^0 = (1440\pi^3 F^4)^{-1}\left\{\bar{\ell}_1 + 4\bar{\ell}_2 - 53/8\right\} + O(M^2) \tag{18.6}$$

$$a_2^2 = (1440\pi^3 F^4)^{-1}\left\{\bar{\ell}_1 + \bar{\ell}_2 - 103/40\right\} + O(M^2)$$

Similar expressions may be given for the scattering lengths of all higher partial waves (Gasser and Leutwyler 1983), e.g.

$$a_\ell^1 = \frac{M^{4-2\ell}}{512\pi^3 F^4} \frac{\ell!\,(\ell-3)!}{[(2\ell+1)!!]^2}(13\ell^2 + 5\ell - 22)\{1 + O(M^2)\}$$

$$\ell = 3, 5, 7, \ldots \tag{18.7}$$

These low energy theorems of course satisfy the Martin inequalities (Martin 1967, 1968)

$$a^I_{\ell+2} \leq a^I_\ell \frac{(\ell+1)(\ell+2)}{4(2\ell+3)(2\ell+5)}$$

The improved soft pion theorems (18.5) imply that the quark mass expansions of the quantities a_0^0, b_0^0, a_0^2, b_0^2 and a_1^1 contain chiral logarithms (Gasser and Leutwyler 1983). Consider, e.g., the S-wave scattering length a_0^0. Rewriting the low energy theorem in terms of the physical pion decay constant and the physical pion mass we get

$$a^0_0 = \frac{7}{32\pi}\frac{M_\pi^2}{F_\pi^2}\left[1 + \frac{5}{84\pi^2}\frac{M_\pi^2}{F_\pi^2}\left\{\bar{\ell}_1 + 2\bar{\ell}_2 - \frac{3}{8}\bar{\ell}_3 + \frac{21}{10}\bar{\ell}_4 + \frac{21}{8}\right\} + O(M^4)\right] \tag{18.8}$$

As $M_\pi \to 0$ the low energy constants $\bar{\ell}_1, \ldots, \bar{\ell}_4$ all tend to infinity like $-\log M_\pi^2$. The quark mass expansion of a_0^0 is therefore of the form

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1 - \frac{9}{32\pi^2}\frac{M_\pi^2}{F_\pi^2}\ln\frac{M_\pi^2}{\mu^2} + \ldots\right\} \tag{18.9}$$

The correction to the soft pion theorem is not of order M_π^2, it is of order $M_\pi^2 \log M_\pi^2$. Taking the scale of the logarithm at μ = 1 GeV we get a correction of order 25% rather than 1 or 2% as suggested by the rule of thumb given in the introduction. This example shows that general order of magnitude estimates are quite misleading if they are applied to quantities like a_0^0 for which the quark mass expansion contains a nonanalytic term with sizeable coefficient. Note that the nonanalytic correction to a_0^0 goes in the right direction to decrease the discrepancy between the soft pion prediction

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

and the observed value

$$a_0^0\Big|_{Exp.} = 0.26 \pm 0.05$$

The numerical value of the correction to the soft pion result depends on the choice of the scale μ in (18.9), i.e. on the size of the analytic terms of order M_π^2. In contrast to the representation (18.9) which only exhibits the nonanalytic term the full expression (18.8) is scale independent. To evaluate the correction of order M_π^2, including both nonanalytic and analytic terms we need an estimate of the coupling constants $\bar{\ell}_1, \ldots, \bar{\ell}_4$ to be inserted in (18.8).

19. Phenomenology of the low energy coupling constants

In principle, the coupling constants $\ell_1, \ell_2, \ldots, \ell_7$ which occur in the effective low energy Lagrangian are fixed by the scale Λ of QCD and by the quark masses. In the absence of a computational scheme that allows one to extract the values of these constants from the Lagrangian of QCD we retreat to a phenomenological evaluation based on experimental information at low energies. We analyze the constants $\ell_1, \ell_2, \ldots, \ell_7$ in turn.

As can be seen from (18.6) the constants ℓ_1 and ℓ_2 are measurable through the D-wave scattering lengths:

$$\begin{aligned} \bar{\ell}_1 &= 480\pi^3 F_\pi^4 (-a_2^0 + 4a_2^2) + 49/40 \\ \bar{\ell}_2 &= 480\pi^3 F_\pi^4 (a_2^0 - a_2^2) + 27/20 \end{aligned} \tag{19.1}$$

With the experimental values given by Petersen (1977):

$$\begin{aligned} a_2^0 &= (17 \pm 3) \cdot 10^{-4} \\ a_2^2 &= (1.3 \pm 3) \cdot 10^{-4} \end{aligned} \tag{19.2}$$

(in units of M_{π^+}) we thus get

$$\begin{aligned} \bar{\ell}_1 &= -2.3 \pm 3.7 \\ \bar{\ell}_2 &= 6.0 \pm 1.3 \end{aligned} \tag{19.3}$$

The constant $\bar{\ell}_3$ cannot be measured directly. In (Gasser and Leutwyler 1983) we have given a crude estimate based on the additivity rule: $\bar{\ell}_3 = 1.8 \pm 2.2$. A more sophisticated analysis based on chiral SU(3)xSU(3) runs as follows (Gasser and Leutwyler, to be published). Denote by $\hat{M}_\pi$ the value of the pion mass in a world in which the strange quark is decreased from its physical value to $\hat{m} = \frac{1}{2}(m_u + m_d)$. The OZI rule (for a review see Okubo 1978) implies that the difference between M_π and $\hat{M}_\pi$

$$M_\pi^2 = \hat{M}_\pi^2 (1+\delta) \tag{19.4}$$

is small. Next, consider the ratio $(M_{K^0}^2 - M_{K^+}^2) : (M_K^2 - M_\pi^2)$ in pure QCD. In lowest order of the quark mass expansion this ratio is given by $(m_d - m_u) : (m_s - \hat{m})$. Denote the difference to the lowest order formula by ε:

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} (1+\varepsilon) \tag{19.5}$$

The constant $\bar{\ell}_3$ is related to ε and δ by

$$\bar{\ell}_3 = \ln \frac{M_\eta^2}{M_\pi^2} + \frac{1}{9} - \frac{32\pi^2 F_\pi^2}{M_K^2 - M_\pi^2} (\varepsilon + \delta) \tag{19.6}$$

The observed value of the $K^0 - K^+$ mass difference (corrected for electromagnetic effects with Dashen's theorem) leads to $(M_{K^0}^2 - M_{K^+}^2) : (M_K^2 - M_\pi^2) = (43.7 \pm 2.7)^{-1}$. (An independent determination is provided by the rate of the decay $\eta \to 3\pi$ which, within the errors, leads to the same value (Roiesnel and Truong 1981; Minkowski 1982; Gasser and Leutwyler, to be published).) Since this value agrees very well with the quark mass ratio $(m_d - m_u) : (m_s - \hat{m})$ obtained (Gasser and Leutwyler 1982) from the baryon mass splittings or from ρ-ω-mixing the quantity ε must be small. (Note, incidentally, that ε also measures the deviation of the ratio $M_K^2 : M_\pi^2$ from the lowest order mass formula

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}} (1+\varepsilon) \tag{19.7}$$

A small value of ε means that the ratio $m_s : \hat{m}$ must be close to 26.) In order for the sum $\varepsilon + \delta$ not to exceed 20% the value of $\bar{\ell}_3$ must be in the range

$$\bar{\ell}_3 = 2.9 \pm 2.4 \tag{19.8}$$

confirming the estimate quoted above. Although the uncertainty in $\bar{\ell}_3$ is rather large, we will see that the effect of this uncertainty on the scattering lengths is extremely small - all that counts is that we are not underestimating $\bar{\ell}_3$ by an order of magnitude. Note that the value (19.8) requires the difference between $M^2 = B(m_u + m_d)$ and M_π^2 to be very small:

$$M = (1.01 \pm 0.01) M_\pi \tag{19.9}$$

The constant ℓ_4 is related to the scalar radius of the pion (cf. (15.10)). In (Gasser and Leutwyler 1983) we have given the estimate $\bar{\ell}_4 = 4.6 \pm 1.2$ based on data for the analogous scalar form factor $\langle\pi|\bar{u}s|K\rangle$ which is measured in K_{ℓ_3} decay (Particle data group 1982).

$$\langle r^2\rangle_s^{\pi K} = 6\lambda_0 = (0.23 \pm 0.05)\,\mathrm{fm}^2 \tag{19.10}$$

The essential point here is that it is misleading to apply plain SU(3) to these quantities (i.e. to take $\langle r^2\rangle^{\pi K} = \langle r^2\rangle^{\pi}$), because they contain chiral logarithms with large coefficients. Since M_π^2 is much smaller than M_K^2 the chiral logarithms contributing to $\langle r^2\rangle^{\pi}$ are numerically quite different from the analogous contributions to $\langle r^2\rangle^{\pi K}$. In (Gasser and Leutwyler 1983) we have argued that these effects lead to a value for $\langle r^2\rangle^{\pi}$ which is more than twice as large as $\langle r^2\rangle^{\pi K}$. A more systematic analysis of the problem is given in (Gasser and Leutwyler, to be published) where we show that $\langle r^2\rangle^{\pi}$ may be determined in two independent ways in the framework of chiral perturbation theory of SU(3)xSU(3) with the result (using either experimental information on K_{μ_3} or on K_{μ_2} decays and applying the OZI rule)

$$\langle r^2\rangle_s^{\pi} = 0.6 \pm 0.15\,\mathrm{fm}^2 \tag{19.11}$$

This value implies

$$\bar{\ell}_4 = 4.6 \pm 0.9 \tag{19.12}$$

In view of (15.11) the pion decay constant therefore decreases by about 6% if the masses of the u and d quarks are sent to zero:

$$F = 0.94\, F_\pi = 88\,\mathrm{MeV} \tag{19.13}$$

The constant ℓ_5 may be estimated by saturating the two sum rules (14.5) for the difference $\rho_V^1 - \rho_A^1$ with π, ρ and A_1. Using the measured values $F_\rho = 144$ MeV, $M_\rho = 770$ MeV, $M_{A_1} = 1275$ MeV one finds

$$\bar{\ell}_5 = 14 \tag{19.14}$$

(In this estimate the two pion continuum which is responsible for the chiral logarithm in $\bar{\ell}_5$ is neglected. With the low energy representation (14.2) one finds that the integral over this continuum up to $s = M_\rho^2$ increases the value of $\bar{\ell}_5$ by less than one unit.) A direct experimental determination of $\bar{\ell}_5$ will be given below.

The constant ℓ_6 determines the electromagnetic charge radius of the pion (cf. (15.4)) which is measured. The most recent value (Dally 1982)

$$\langle r^2 \rangle_V^\pi = 0.439 \pm 0.03 \, \mathrm{fm}^2 \tag{19.15}$$

is consistent with the analysis of the older data performed by Heyn and Lang (1981). (See also Gourdin 1974; Adylov et al. 1974, 1977; Zovko 1975; Dally et al. 1977, 1982; Perez-y-Jorba and Renard 1977; Quenzer et al. 1978; Dubnicka, Meshcheryakov and Milko 1981.) The value (19.15) implies

$$\bar{\ell}_6 = 16.5 \pm 1.1 \tag{19.16}$$

As a check we note that the structure term in the amplitude associated with the decay $\pi \to e\nu\gamma$ is determined by the difference $\bar{\ell}_6 - \bar{\ell}_5$ (cf. (15.6)). There is an ambiguity in the experimental value of

$$\gamma = \frac{1}{6}(\bar{\ell}_6 - \bar{\ell}_5) \tag{19.17}$$

(see Stetz et al. 1978; Bryman, Depommier and Leroy 1982):

$$\gamma = 0.44 \pm 0.12 \quad \text{or} \quad \gamma = -2.36 \pm 0.12 \tag{19.18}$$

The positive solution implies

$$\bar{\ell}_6 - \bar{\ell}_5 = 2.64 \pm 0.72 \tag{19.19}$$

in good agreement with (19.14) and (19.16). The negative solution leads to $\bar{\ell}_6 - \bar{\ell}_5 = -14.2 \pm 0.7$; this solution is inconsistent with our framework. (See also Nasrallah, Papadopoulos and Schilcher 1982. A refined experiment which should resolve the ambiguity is currently performed by Bay et al. at SIN.) We replace the crude estimate (19.14) by the experimental value which follows from (19.16) and (19.19):

$$\bar{\ell}_5 = 13.9 \pm 1.3 \tag{19.20}$$

Finally, the constant ℓ_7 may be estimated on the basis of the sum rule (14.5) for the difference $s^{-1}(\tilde{\rho}_S - \tilde{\rho}_P)$. In this sum rule the contribution of the η-meson is enhanced by a small energy denominator: in the limit $m_s \to 0$ the η becomes massless. The contribution from this state represents the leading term in the quark mass expansion of ℓ_7 with respect to m_s. Using SU(3) to relate the η coupling constant to F_π we find

$$\ell_7 \simeq \frac{F_\pi^2}{6 M_\eta^2} \simeq 5 \cdot 10^{-3} \tag{19.21}$$

This estimate agrees with the familiar lowest order formula for π^0-η mixing (Gross, Treiman and Wilczek 1979)

$$M_{\pi^+}^2 - M_{\pi^0}^2 = \frac{1}{4} \frac{(m_u - m_d)^2}{m_s - \hat{m}} \cdot B \tag{19.22}$$

(compare (12.2) and (12.9)). As pointed out in (Gasser and Leutwyler 1982) the contribution of the $K\bar{K}$ and $\eta\eta$ continua to $M_{\pi^+}^2 - M_{\pi^0}^2$, although of relative order $m_s \log m_s$, is not negligible. Since we do not need an accurate value for ℓ_7 here we do not analyze the problem further, but emphasize that (19.21) should only be taken as an estimate for the order of magnitude of this constant.

Are the values of the low energy constants given above consistent with the general picture underlying our analysis or are they unduly large ? If the pion poles and cuts were the only low energy singularities of importance we should expect the main contribution to the low energy constants to be given by the chiral logarithms ($\bar{\ell}_1 = \ldots = \bar{\ell}_6 = \ln(\mu^2/M_\pi^2)$), which with a scale μ somewhere in the range from 500 MeV to 1 GeV gives $\bar{\ell}_i = 2{,}6 \div 4$. The observed values of $\bar{\ell}_5$ and $\bar{\ell}_6$ are clearly outside this range. The constant $\bar{\ell}_6$ measures the electromagnetic charge radius. The observed value is consistent with ρ-dominance:

$\langle r^2\rangle_V^\pi \simeq 6\ M_\rho^{-2} = 0.4\ \mathrm{fm}^2$, but is not consistent with the estimate $\langle r^2\rangle_V^\pi \lesssim 0.1\ \mathrm{fm}^2$, based on the chiral logarithm (Bég and Zepeda 1972; Volkov and Pervushin 1975). To understand the size of $\bar{\ell}_6$ one therefore needs to understand how the presence of an exited $q\bar{q}$ state at M_ρ=770MeV affects the low energy structure of the Green's functions. In the literature there is some confusion about the importance of the ρ-meson contribution in the context of the low energy theorems. Chiral symmetry of course does not exclude the presence of such a state, as long as the mass of the ρ-meson remains finite as m_u, $m_d \to 0$. To study the contribution of this state to the low energy structure of the Green's functions we construct an effective low energy Lagrangian which includes the ρ-meson and is consistent with chiral symmetry (Appendix C). We show that in the region $p^2 \ll M_\rho^2$ the ρ manifests itself only indirectly through a contribution to the low energy constants $\ell_1, \ldots, \ell_6$. Indeed the observed values of these constants are quite well accounted for by supplementing the chiral logarithms with the contribution from ρ-exchange. In this sense the ρ is the only non-Goldstone singularity seen in the low energy expansion at one loop order.

20. Measuring the scalar radius of the pion

With the experimental information about the coupling constants of the effective low energy Lagrangian given in the last section we could now work out the predictions for the $\pi\pi$ scattering lengths. Instead of simply quoting the resulting numerical values (which perfectly agree with the data, see (Gasser and Leutwyler 1983)) we analyze the available data in the following manner. We first observe that the improved low energy theorems for a_1^1, b_0^0, b_0^2 and for the combination $2\,a_0^0 - 5\,a_0^2$ of S-wave scattering lengths do not involve the low energy constant ℓ_3. Expressing the quantities F, M, $\bar{\ell}_1$, $\bar{\ell}_2$ and $\bar{\ell}_4$ in terms of F_π, M_π, a_2^0, a_2^2 and $\langle r^2\rangle_S^\pi$ the low energy theorems may be written in the form

$$1 + \frac{1}{3} M_\pi^2 \langle r^2\rangle_S^\pi = 24\pi F_\pi^2 \left\{ a_1^1 - \frac{10}{3} M_\pi^2 \left(a_2^0 - \frac{5}{2} a_2^2\right)\right\} - \frac{19}{576}\frac{M_\pi^2}{\pi^2 F_\pi^2} + O(M_\pi^4) \tag{20.1}$$

$$1 + \frac{1}{3} M_\pi^2 \langle r^2\rangle_S^\pi = 4\pi F_\pi^2 \left\{ b_0^0 - 10 M_\pi^2 \left(a_2^0 + 5 a_2^2\right)\right\} - \frac{39}{64}\frac{M_\pi^2}{\pi^2 F_\pi^2} + O(M_\pi^4) \tag{20.2}$$

$$1+\frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = -8\pi F_\pi^2\left\{b_0^2 - 10M_\pi^2\left(a_2^0+\frac{1}{2}a_2^2\right)\right\} + \frac{89}{320}\frac{M_\pi^2}{\pi^2F_\pi^2} + O(M_\pi^4) \tag{20.3}$$

$$1+\frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi = \frac{4\pi}{3}\frac{F_\pi^2}{M_\pi^2}\left\{2a_0^0 - 5a_0^2\right\} - \frac{41}{192}\frac{M_\pi^2}{\pi^2F_\pi^2} + O(M_\pi^4) \tag{20.4}$$

Since the threshold parameters appearing on the right-hand side of these relations are measured, we obtain four independent determinations of the quantity $1+\frac{1}{3}M_\pi^2\langle r^2\rangle$. The results are shown in Table I. [We use the value (Sirlin 1972) $F_\pi = 93.3$ MeV and identify M_π with the mass of the charged pion. The experimental values of the threshold parameters are taken from (Petersen 1977): $a_1^1 = 0.038 \pm 0.002$, $b_0^0 = 0.25 \pm 0.03$, $b_0^2 = -0.082 \pm 0.008$, $a_2^0 = (17 \pm 3)\cdot 10^{-4}$, $a_2^2 = (1.3 \pm 3)\cdot 10^{-4}$. In the case of the combination $2a_0^0 - 5a_0^2$ we have evaluated the "universal curve" (Morgan and Shaw 1968) at $a_0^0 = 0.20$ with the result $2a_0^0 - 5a_0^2 = 0.614 \pm 0.028$. The error bars are obtained by treating the experimental data on the various threshold parameters as uncorrelated. This is probably not correct: the errors are presumably correlated with a_0^0, but we are not aware of an analysis that exhibits these correlations.]

For comparison we list the values of the right-hand sides in (20.1...4) which one obtains if the corrections of order M_π^2 are dropped (last column of the table). The fact that the numbers in this column differ from 1 is equivalent to the statement that the data show deviations from the soft pion theorems.

Note that the quantity $1+\frac{1}{2}M_\pi^2\langle r^2\rangle$ is larger than 1 in all four cases. The mean value 1.12 ± 0.04 may be taken as a measurement of $\langle r^2\rangle$:

$$\langle r^2\rangle_S^\pi = 0.7 \pm 0.2\ \mathrm{fm}^2 \tag{20.5}$$

consistent with the SU(3)xSU(3) estimate $\langle r^2\rangle_S^\pi = 0.6 \pm 0.15\ \mathrm{fm}^2$ obtained from either K_{μ_3} decay or from F_K/F_π, neglecting Zweig rule violating contributions. (This estimate leads to the prediction quoted in the last row of Table I. The agreement of the two values confirms the approximate validity of the OZI rule in this context.)

Apart from the D-wave corrections which essentially eliminate the contributions quadratic in s, t, u, the main effect responsible for the observed deviations from the soft pion theorems is the fact that the pion decay constant in the chiral limit is smaller than the physical decay constant. The soft pion predictions are proportional to F_π^{-2} and hence systematically underestimate the threshold parameters by about 13% (the quantity $1 + \frac{1}{3} M_\pi^2 \langle r^2 \rangle$ roughly represents the ratio F_π^2/F^2, see (15.11)).

The improved low energy theorem for the S-wave scattering length a_0^0 involves the low energy constant ℓ_3:

$$a_0^0 = \frac{7}{32\pi} \frac{M_\pi^2}{F_\pi^2} \left\{ 1 + \frac{1}{3} M_\pi^2 \langle r^2 \rangle_s^\pi - \frac{M_\pi^2}{672\pi^2 F_\pi^2} (15\bar{\ell}_3 - 353) \right\}$$
$$+ \frac{25}{4} M_\pi^4 (a_2^0 + 2a_2^2) + O(M_\pi^6) \qquad (20.6)$$

With the estimate $\bar{\ell}_3 = 2.9 \pm 2.4$ given in section 19 we obtain

$$a_0^0 = 0.20 \pm 0.01 \qquad (20.7)$$

to be compared with the experimental value $a_0^0 = 0.26 \pm 0.05$ and with the soft pion prediction $a_0^0 = 0.16$. The main point here is that the contribution from ℓ_3 is minute; the prediction for a_0^0 is quite accurate, despite the fact that ℓ_3 is poorly known. (Note that $\bar{\ell}_3$ would have to be of the order of - 70 for a_0^0 to have the value 0.26 ! In the notation used in (Gasser and Leutwyler 1983) this value of $\bar{\ell}_3$ corresponds to $B_1 = 50$ instead of $B_1 \sim 1$ as estimated there.)

21. Summary and concluding remarks

In QCD the Green's functions of the quark currents are functions of the renormalization group invariant scale Λ, of the quark masses and of the external momenta: $G(\Lambda, m_{quark}, p)$ (if fields of anomalous dimension are involved, the Green's functions in addition depend on the renormalization point). We have analyzed the behaviour of these Green's functions in the region of small momenta and small quark masses. More precisely, we have studied the dependence of the Green's functions on the momenta and on m_u and m_d at fixed Λ, m_s, m_c, ..., treating p, m_u and m_d as small in comparison with the scale Λ.

This analysis exploits only part of the information contained in the Lagrangian of QCD, viz. its symmetry properties. Lattice calculations are of a wider scope: they allow one to calculate the full set of Green's functions at any value of the momenta in terms of the scale of the theory and of the quark masses. In principle, these calculations include all the low energy structures we are painfully sorting out here; furthermore, in these calculations the quark masses are free parameters that one may vary at will. One may pin their physical values down by simply calculating those physical quantities which are measured most accurately. Unfortunately, we think that one will have to wait a long time before this program achieves the accuracy we are aiming at. The reason for this pessimistic view is precisely in the structure of the low energy singularities we are analyzing here. To calculate F_π, e.g., one may attempt to evaluate the two point function of the axial current, $\langle 0|TA^i_\mu(x)\; A^K_\nu(y)|0\rangle$ on a lattice. The fact that M_π is small however implies that this quantity receives important contributions from field configurations which extend far away from x and y. In Fig. 1 we indicate a typical graph which contributes to F_π: graphs of this type are responsible for the fact that F_π is not an analytic function of $\hat{m}$, but contains a chiral logarithm of the type $\hat{m}\log\hat{m}$. To make sure that the lattice calculation reproduces contributions of this sort correctly, the lattice must extend to distances large in comparison to M_π^{-1}. What is worse is that one cannot see these contributions in the quenched approximation - the calculation has to include the fermion determinant to reproduce the chiral logarithms. It seems fair to say that a lattice calculation that accurately accounts for these long range effects is not in sight. Note, incidentally, that these problems are not specific to a calculation of F_π. All hadrons are surrounded by a cloud of pions, which, by virtue of their small mass, are allowed to walk away rather far from the object which generates them. In particular, the mass of the proton receives a contribution of order 140 MeV from the long range part of the meson cloud (Gasser and Leutwyler 1982, Appendix C). There is no reason to be worried if lattice calculations of the proton mass in the quenched approximation tend to produce too high a value.

The dominating feature in the low energy region is the occurrence of Goldstone bosons: in the chiral limit ($m_u = m_d = 0$) the Green's functions develop poles at $p^2 = 0$ and cuts starting at $p^2 = 0$. If the masses of the u and d quarks are turned on, the poles move to $p^2 = M_\pi^2 = O(m_u + m_d)$, the cuts start at $4\,M_\pi^2$, $9\,M_\pi^2$ etc. To describe the structure of these low energy singularities we treat the momenta as quantities of the same algebraic order as M_π, i.e. consider an expansion in powers of the momenta and of the quark masses at fixed ratio $M_\pi^2/p^2 \sim (m_u + m_d)/p^2$.

The low energy properties of the Goldstone bosons are to a large extent fixed by the underlying, spontaneously broken symmetry group SU(2)xSU(2). In fact, the Ward identities associated with this symmetry determine the leading low energy behaviour of the Green's functions associated with the vector (isovector), axial vector (isovector), scalar (isoscalar) and pseudoscalar (isovector) currents in terms of only two low energy constants: the pion decay constant F_π and the pion mass M_π (or, equivalently, the vacuum expectation value $\langle 0|\bar{u}u|0\rangle$). Instead of solving the Ward identities directly, we use an effective Lagrangian, a technique which has been shown to be very useful in the context of soft pion theorems long ago (Weinberg 1967, 1968; Coleman, Wess and Zumino 1969; Callan, Coleman, Wess and Zumino 1969; Dashen and Weinstein 1969; Weinberg 1979). The main difference between the effective Lagrangian technique we are using and the standard pion field Lagrangians is that in our context the pion field does not play a crucial role - what we are using the Lagrangian for is to calculate the Green's functions associated with quark currents, not Green's functions associated with a pion field.

Note that the effective Lagrangian only involves the O(4)-vector U^A, it does not depend on the choice of the pion field coordinates used to parametrize this vector. Our technique avoids the problems which one has to solve (Honerkamp 1972; Tataru 1975; Bardeen, Lee and Shrock 1976; Kazakov, Pervushin and Pushkin 1977, 1978; de Wit and Grisaru 1979; Appelquist and Bernard 1981) if one calculates the Green's functions of the pion field in the standard manner. In contrast to that procedure the external field technique retains the full symmetry of the theory at every stage of the calculation.

The leading low energy behaviour of the Green's functions is given by the tree graphs of the effective Lagrangian. One loop graphs are suppressed by two powers of the momenta and thus contribute at first nonleading order in the low energy expansion. Graphs with two or more loops are suppressed by four or more powers of the momenta and can therefore be ignored at the accuracy at which we are working.

The choice of the effective Lagrangian is not unique. At leading order either the nonrenormalizable, nonlinear σ-model or the renormalizable, linear σ-model may be used - the results are the same, because all that counts is that the tree graphs of the effective Lagrangian in question lead to the proper value of the two low energy constants F_π and M_π which completely determine the leading low energy behaviour of the Green's functions in QCD. At the one loop level the two models however differ. In the case of the nonlinear σ-model we need a set of counter terms which are not present in the lowest order Lagrangian, whereas there is no need for such additional terms in the case of the renormalizable σ-model. This difference should however not be misinterpreted: even though the renormalizable σ-model does specify the Green's functions also at one loop order, there is no reason for these Green's functions to be correct, i.e. to coincide with the Green's functions of QCD. Chiral symmetry only guarantees that the leading low energy behaviour of the two theories is the same (as long as F_π and M_π are the same), it does not imply that this is true to all orders of the momenta. At first nonleading order the general solution to the Ward identities of SU(2)xSU(2) contains 10 additional constants $\ell_1,\ldots\ell_7$, h_1, h_2, h_3. The low energy expansion involves a set of new unknown constants at every level of the expansion - this is not what happens in a renormalizable theory, it is precisely what happens in a nonrenormalizable theory. (In QCD all of these constants, including F_π and M_π are fixed by Λ and by the quark masses; we are however only exploiting the chiral symmetry properties of QCD which leave these constants unspecified.) To explicitly demonstrate that the renormalizable σ-model by itself is not a reliable effective low energy theory we show in Appendix B that this model leads to relations among the low energy constants $\ell_1,\ldots, h_3$ which are not borne out by experiment. The model does however provide us with a useful check on our low energy expansions: it does contain the proper logarithms characteristic of chiral perturbation theory.

The effective Lagrangian is unique if it does not contain any dynamical degrees of freedom other than the pion field. We give the general expression for this unique effective low energy Lagrangian of QCD up to and including terms of order p^4. The Lagrangian is characterized by F_π, M_π, $\ell_1,\ldots, \ell_7$, $h_1,\ldots, h_3$. We calculate the low energy representation to first nonleading order of several Green's functions, form factors and of the $\pi\pi$ scattering amplitude (equations (10.16) and (10.17) provide an explicit representation of the generating functional to order p^4 in the momenta and to order ϕ^4 in the fields). Only the constants $\ell_1,\ldots, \ell_7$ appear in quantities of physical interest - the constants h_1, h_2, h_3 are contact terms which depend on the conventions used to specify the time ordered products, they do not occur in observable quantities. We show how to extract the values of the constants $\ell_1, \ldots, \ell_7$ from experimental data (D-waves in $\pi\pi$ scattering, SU(3) mass formulae, the decays $K \to \pi e\nu$ and $\pi^+ \to e\nu\gamma$ and elastic πe scattering, see Table II). The phenomenological values obtained in this manner show that the most important low energy singularity which is not explicitly included in the low energy expansion is the ρ-resonance. In Appendix C we exhibit an effective Lagrangian which contains the ρ degrees of freedom and is consistent with chiral symmetry. We show that in a systematic low energy expansion in powers of the momenta the ρ does not play any special role - its presence only manifests itself indirectly in the values of the low energy constants. In fact, the observed values of these constants are quite well reproduced if one assumes that the renormalized low energy coupling constants at a scale of order $\mu = 500$ MeV or $\mu = 1$ GeV are exclusively due to ρ-exchange (see Table III).

We then confront the low energy expansion for the $\pi\pi$ scattering amplitude with experiment. The soft pion predictions for the threshold parameters receive corrections of order M_π^2 and $M_\pi^2 \log M_\pi^2$, which turn out to be sizeable: in contrast to the soft pion theorems the five improved low energy predictions agree with the data to within 1 1/2 standard deviations (Gasser and Leutwyler 1983). In fact, the experimental information on the threshold parameters is so accurate that it allows one to measure the deviations from the soft pion predictions and thereby obtain an independent value for the low energy constant ℓ_4. This constant determines the scalar radius of the pion: Whereas the radius of the electric charge distribution is measured in elastic πe scattering, the scalar radius of the pion is measured in elastic $\pi\pi$ scattering. The value one obtains in this manner is consistent with theoretical estimates based on SU(3)xSU(3).

We conclude that the low energy theorems do provide us with very sensitive tests of the chiral structure of QCD - to compare these low energy theorems with experiment at the available experimental accuracy it is however necessary to work out the corrections of order M_π^2; in some cases the corrections modify the lowest order (soft pion) predictions by more than 20%.

It might be worthwhile to reanalyze the experimental data using the low energy representation given here as a constraint. In fact, the information contained in this representation goes considerably beyond the threshold parameters of the S- and P-waves. We expect such a reanalysis not only to provide a rather accurate value of the scalar radius of the pion, but also to lead to a more precise determination of the D-wave scattering lengths.

Appendix A: Fermion determinant in the presence of external fields

To work out the anomalies of the Green's functions in QCD it is convenient to integrate the fermions out (Fujikawa 1980). In Euclidean space the vacuum-to-vacuum amplitude then takes the form

$$e^{-Z} = \int d\mu[G]\, e^{-\frac{1}{2g^2}\int dx\, \mathrm{tr}_c\, G_{\mu\nu} G_{\mu\nu}} \det D \tag{A.1}$$

where $\det D$ is the determinant of the Dirac operator:

$$D = \gamma_\alpha(\partial_\alpha - i\hat{v}_\alpha - i a_\alpha \gamma_5) + s - i\gamma_5 p \tag{A.2}$$

The quantity $\hat{v}_\alpha(x)$ includes the gluon field $G_\mu(x)$

$$\hat{v}_\alpha = v_\alpha + G_\alpha \tag{A.3}$$

(v_α, a_α, s and p are external fields, represented by colour neutral matrices in flavour space). The γ-matrices are normalized by

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \;;\; \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \tag{A.4}$$

They may be chosen hermitean, $\gamma_5^2 = 1$.

The determinant of D is defined only up to counter terms of dimension less than or equal to four. In the present context, dimensional regularisation is not a convenient method to handle the singularities, because the anomalies are connected with γ_5 and with $\varepsilon_{\alpha\beta\mu\nu}$ which do not admit a straightforward extension to $d \neq 4$. Instead we use the ζ-function technique to specify the finite part of the determinant. The definition

$$\ln \det_\mu A = -\frac{d}{ds} F(s)\Big|_{s=0}$$
$$F(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty d\lambda\, \lambda^{s-1}\, \mathrm{Tr}\, e^{-\lambda A} \tag{A.5}$$

makes sense for a large class of operators (a necessary condition is that the real parts of the eigenvalues of A are nonnegative). For finite dimensional matrices $\det_1 A$ is the ordinary determinant. The formula (A.5) cannot immediately be applied to D or to - D^2; the operator iD is not hermitean and - D^2 is not positive. In the present context we are however only interested in the Taylor series expansion of det D with respect to the external fields. It therefore suffices that the relevant differential operator becomes positive if the external fields are set equal to zero. The operator

$$A = \bar{D}^2 \quad ; \quad \bar{D} = \gamma_5 D \tag{A.6}$$

does have this property ($\bar{D}$ is hermitean for $a_\alpha = p = 0$; neither the gluon fields nor the quark mass term contained in s spoil the hermiticity). We may therefore define the finite part of det D by

$$\ln \det_\mu D = -\frac{d}{ds}\Big|_{s=0} \frac{1}{2} \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty d\lambda\, \lambda^{s-1}\, \mathrm{Tr}\left(e^{-\lambda \bar{D}^2}\right) \tag{A.7}$$

The operator $\bar{D}^2$ is of the form

$$\bar{D}^2 = -D_\mu D_\mu + \sigma \tag{A.8}$$

where D_μ is a covariant derivative:

$$\begin{aligned} D_\mu &= \partial_\mu + \Gamma_\mu \\ \Gamma_\mu &= -i\left(\hat{v}_\mu + \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]\gamma_5 a_\nu + \gamma_\mu p\right) \end{aligned} \tag{A.9}$$

and the mass matrix σ is given by

$$\begin{aligned} \sigma = {} & 2a_\alpha a_\alpha + s^2 + 3p^2 - \gamma_\alpha\left(\partial_\alpha s - i[\hat{v}_\alpha, s] + 2\{a_\alpha, p\}\right) \\ & + \tfrac{1}{2}[\gamma_\alpha, \gamma_\beta]\left(i\partial_\alpha \hat{v}_\beta + \hat{v}_\alpha \hat{v}_\beta + a_\alpha a_\beta\right) + i\gamma_\alpha\gamma_5[a_\alpha, s] \\ & + i\gamma_5\left(\partial_\alpha a_\alpha - i[\hat{v}_\alpha, a_\alpha] - \{s, p\}\right) \end{aligned} \tag{A.10}$$

In the representation (A.7) the ultraviolet divergences of the determinant are related to the behaviour of the integrand near $\lambda = 0$. To study this behaviour we note that the short distance properties of $\bar{D}^2$ are governed by the four-dimensional Laplacian. Accordingly the leading short distance behaviour of exp $(-\lambda\bar{D}^2)$ is given by the free heat kernel exp $\lambda\Box$. We therefore put

$$(x|e^{-\lambda\bar{D}^2}|y) = (4\pi\lambda)^{-2} \exp\left(-\frac{1}{4\lambda}z^2\right) H(x|\lambda|y) \qquad \text{(A.11)}$$

with z = x - y. The kernel H satisfies the differential equation

$$\frac{\partial}{\partial\lambda}H + \frac{1}{\lambda}z_\mu D_\mu H - D_\mu D_\mu H + \sigma H = 0 \qquad \text{(A.12)}$$

and the boundary condition

$$H(x|0|x) = 1$$

The differential equation for H may be solved recursively with the expansion

$$H(x|\lambda|y) = H_0(x|y) + \lambda H_1(x|y) + \lambda^2 H_2(x|y) + \dots$$

$$(n+1 + z_\mu D_\mu) H_{n+1} + (-D_\mu D_\mu + \sigma) H_n = 0 \qquad \text{(A.13)}$$

$$z_\mu D_\mu H_0 = 0$$

The first three terms in the series expansion of H produce poles in the integral (A.7) at s = 2, s = 1 and s = 0 respectively. The pole at s = 0 is responsible for the scale dependence of the finite part of det D:

$$\mu\frac{\partial}{\partial\mu} \ln \det_\mu D = -(4\pi)^{-2}\int dx \, \mathrm{tr}\, H_2(x|x) \qquad \text{(A.14)}$$

The quantity H_2 also determines the anomalies. Consider an infinitesimal local flavour transformation of the type (2.6):

$$\begin{aligned} \delta v_\mu &= \partial_\mu\alpha + i[\alpha, v_\mu] + i[\beta, a_\mu] \\ \delta a_\mu &= \partial_\mu\beta + i[\alpha, a_\mu] + i[\beta, v_\mu] \\ \delta s &= i[\alpha, s] - \{\beta, p\} \\ \delta p &= i[\alpha, p] + \{\beta, s\} \end{aligned} \qquad \text{(A.15)}$$

where α and β are hermitean matrices in flavour space; the transformation α is generated by the vector currents, β is a chiral transformation. The corresponding change in the Dirac operator is given by

$$\delta \bar{D} = -i[\bar{D},\alpha] - i\{\bar{D},\beta\gamma_5\} \tag{A.16}$$

and the change in Tr exp $(-\lambda\bar{D}^2)$ is

$$\begin{aligned}\delta \mathrm{Tr}\exp(-\lambda\bar{D}^2) &= -2\lambda \mathrm{Tr}\{\delta\bar{D}\bar{D}\exp(-\lambda\bar{D}^2)\}\\ &= 4i\lambda\frac{d}{d\lambda}\mathrm{Tr}\{\beta\gamma_5\exp(-\lambda\bar{D}^2)\}\\ &= 4i\lambda\frac{d}{d\lambda}(4\pi\lambda)^{-2}\int dx\, \mathrm{tr}\{\beta(x)\gamma_5 H(x|\lambda|x)\}\end{aligned}$$

The transformations generated by the vector currents thus leave the determinant unaffected, whereas chiral transformations produce the change

$$\delta \ln \det{}_\mu D = -2i(4\pi)^{-2}\int dx\, \mathrm{tr}\{\beta(x)\gamma_5 H_2(x|x)\} \tag{A.17}$$

What remains to be done is to determine $H_2(x|x)$ from the recursion relations (A.13). Putting $n = 1$, $x = y$ we get

$$H_2(x|x) = \tfrac{1}{2}D_\mu D_\mu H_1(x|x) - \tfrac{1}{2}\sigma(x)H_1(x|x)$$

The value of $D_\mu D_\mu H_1$ may be obtained by first applying $D_\mu D_\mu$ to the recursion relation with $n = 0$ and then taking $x = y$:

$$\begin{aligned}3D_\mu D_\mu H_1(x|x) &= (D_\mu D_\mu)^2 H_0 - [D_\mu,[D_\mu,\sigma]]H_0\\ &\quad -2[D_\mu,\sigma]D_\mu H_0 - \sigma D_\mu D_\mu H_0\end{aligned}$$

$$H_1(x|x) = -\sigma(x)H_0(x|x)$$

Finally, the derivatives of H_0 may be obtained by repeatedly applying the differential operator D_μ to the differential equation satisfied by H_0 and using the property

$$D_\mu D_\nu - D_\nu D_\mu = \Gamma_{\mu\nu}$$

In this manner one easily finds

$$H_0(x|x) = 1$$
$$D_\mu H_0(x|x) = 0$$
$$D_\mu D_\nu H_0(x|x) = \frac{1}{2}\Gamma_{\mu\nu}$$
$$(D_\mu D_\mu)^2 H_0(x|x) = \frac{1}{2}\Gamma_{\mu\nu}\Gamma_{\mu\nu}$$

Putting things together we obtain

$$H_2(x|x) = \frac{1}{12}\Gamma_{\mu\nu}\Gamma_{\mu\nu} + \frac{1}{2}\sigma^2 - \frac{1}{6}\left[D_\mu,\left[D_\mu,\sigma\right]\right] \tag{A.18}$$

It is now a straightforward, although somewhat tedious matter to calculate the relevant traces of H_2. The last term in (A.18) does not contribute to $\int dx\, \mathrm{tr}\, H_2$, because it gives rise to a total derivative. Evaluating the traces of the first two terms one obtains the following result:

$$\mu\frac{\partial}{\partial\mu}\ln \det{}_\mu D = (4\pi)^{-2}\int dx\Big\{\frac{2}{3}N_f\, \underset{c}{\mathrm{tr}}\, G_{\mu\nu}G_{\mu\nu} + \frac{N_c}{3}\underset{f}{\mathrm{tr}}\left(F^R_{\mu\nu}F^R_{\mu\nu} + F^L_{\mu\nu}F^L_{\mu\nu}\right) \tag{A.19}$$
$$-2N_c\,\underset{f}{\mathrm{tr}}\left(\nabla_\mu s\nabla_\mu s + \nabla_\mu p\nabla_\mu p\right) - 2N_c\,\underset{f}{\mathrm{tr}}\,(s+ip)(s-ip)(s+ip)(s-ip)\Big\}$$

where $\underset{c}{\mathrm{tr}}$ and $\underset{f}{\mathrm{tr}}$ denote the traces over colour and flavour indices respectively. The symbols $F^R_{\mu\nu}$, $F^L_{\mu\nu}$ stand for the field strengths associated with the right- and left-handed combinations $F^R_\mu = v_\mu + a_\mu$, $F^L_\mu = v_\mu - a_\mu$ of the external fields

$$F^R_{\mu\nu} = \partial_\mu F^R_\nu - \partial_\nu F^R_\mu - i\left[F^R_\mu, F^R_\nu\right] \tag{A.20}$$

and the covariant derivatives of s and p are defined by

$$\nabla_\mu s = \partial_\mu s - i\,[v_\mu, s] + \{a_\mu, p\}$$
$$\nabla_\mu p = \partial_\mu p - i\,[v_\mu, p] - \{a_\mu, s\} \qquad \text{(A.21)}$$

The first term in (A.19) is the familiar contribution of the quark loop to the β-function of QCD. The remaining terms are interaction independent and are invariant unter local $U(N_f)xU(N_f)$-transformations. (Note that we are considering the integrand of the Feynman path integral over the gluon field. The renormalization of the scalar and pseudoscalar densities comes from loops involving gluons. In the language used here these loops arise from the finite part of ln det D upon integration over the gluon field variables.)

The trace of $\beta\gamma_5 H_2$ which determines the transformation properties of det D under local chiral transformations may also be worked out in a straightforward manner. One obtains two categories of contributions: terms which contain the tensor $\varepsilon_{\alpha\beta\gamma\delta}$ and terms which do not. The second category is unessential in the following sense. One may modify the Green's functions of the theory by adding a polynomial in the external fields and their derivatives to the generating functional:

$$\bar{Z} = Z + \int dx\, P(v, a, s, p)$$

This operation only changes the Green's functions by contact terms (no change unless all arguments coincide). Equivalently, one may replace ln det D_μ by

$$\ln \overline{\det}\, D = \ln \det_\mu D - \int dx\, P(v, a, s, p) \qquad \text{(A.22)}$$

without changing the content of the theory. If P is invariant under the transformations generated by the vector currents, but is not invariant under chiral transformations then the transformation law of ln $\overline{\det}\, D_\mu$ picks up a contribution from P which is linear in β. It turns out that one may absorb all ε-independent terms in tr $\beta\gamma_5 H_2$ by such a redefinition of the Green's functions. The transformation law of the determinant then simplifies to

$$\delta \ln \overline{\det} D = \frac{i}{(4\pi)^2} \int dx \left[\chi \, \underset{f}{\mathrm{tr}} \beta + N_c \, \underset{f}{\mathrm{tr}} (\beta \Omega) \right]$$

$$\chi = \varepsilon_{\alpha\beta\mu\nu} \, \underset{c}{\mathrm{tr}} \, G_{\alpha\beta} G_{\mu\nu}$$

(A.23)

$$\Omega = \varepsilon_{\alpha\beta\mu\nu} \Big[v_{\alpha\beta} v_{\mu\nu} + \tfrac{4}{3} \nabla_\alpha a_\beta \nabla_\mu a_\nu + \tfrac{2i}{3} \{ v_{\alpha\beta}, a_\mu a_\nu \}$$

$$+ \tfrac{8i}{3} a_\mu v_{\alpha\beta} a_\nu + \tfrac{4}{3} a_\alpha a_\beta a_\mu a_\nu \Big]$$

where

$$v_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha - i [v_\alpha , v_\beta] \qquad \text{(A.24)}$$

is the field strength associated with v_α and $\nabla_\alpha a_\beta$ stands for

$$\nabla_\alpha a_\beta = \partial_\alpha a_\beta - i [v_\alpha , a_\beta] \qquad \text{(A.25)}$$

The external scalar and pseudoscalar fields do not contribute to the anomaly. The term proportional to χ is the familiar anomaly in the divergence of the flavour singlet axial current (Adler 1969; Bell and Jackiw 1969; Jackiw and Johnson 1969; Adler and Bardeen 1969). The remaining terms were first given by Bardeen (1969). If we restrict the flavour transformations to the subgroup $SU(N_f)xSU(N_f)xU(1)$ by putting tr $\beta = 0$ the transformation law of $\ln \overline{\det} D$ becomes interaction independent. Adler's nonrenormalization theorem (Adler 1969) asserts that this remains true to all orders in the strong coupling constant. The transformation law of the Euclidean generating functional under $SU(N_f)xSU(N_f)xU(1)$ is therefore known explicitly:

$$\delta \overline{Z} = \frac{-i}{(4\pi)^2} N_c \int dx \, \underset{f}{\mathrm{tr}} (\beta \Omega) \qquad \text{(A.26)}$$

It is not difficult to extend this framework to include the Ward identities involving the flavour singlet axial current: one simply includes in the generating functional a contribution proportional to the winding number density of the gluon field. With this extension the change of the generating functional under an arbitrary $U(N_f)\times U(N_f)$-transformation may be given in an interaction independent form (see section 2).

We close this appendix with the following observation concerning the chiral structure of the anomalies. The transformation law (A.23) hides a property of the determinant which is very easily established at the formal level: since the gauge fields conserve helicity the determinant of the massless Dirac operator is the product of the determinant of the right-handed components with the determinant of the left-handed components. The terms that break the invariance of the determinant with respect to local chiral transformations are independent of the external scalar and pseudoscalar fields; one should therefore expect that the contributions to $\delta \ln \det D$ split into a sum of two terms, one coming from the right-handed components, one from the left-handed components. This property may indeed be explicitly exhibited as follows: the quantity $\hat{\det} D$ defined by

$$\ln \hat{\det} D = \ln \overline{\det} D - \int dx\, Q \tag{A.27}$$

$$Q = (4\pi)^{-2} \frac{2}{3} \varepsilon_{\alpha\beta\mu\nu} \operatorname{tr}\left[i \hat{v}_{\mu\nu} \{\hat{v}_\alpha, a_\beta\} - a_\alpha a_\beta a_\mu \hat{v}_\nu - \hat{v}_\alpha \hat{v}_\beta \hat{v}_\mu a_\nu \right]$$

which differs from $\overline{\det} D$ by an irrelevant local renormalization of the type (A.22) obeys a transformation law of the form

$$\begin{aligned} \delta \ln \hat{\det} D = {} & \frac{i}{(4\pi)^2} \int dx \operatorname{tr}\left[(\alpha + \beta) A(\hat{F}^R) \right] \\ & + \frac{i}{(4\pi)^2} \int dx \operatorname{tr}\left[(\alpha - \beta) A(\hat{F}^L) \right] \end{aligned} \tag{A.28}$$

where $\hat{F}^R_\mu$ and $\hat{F}^L_\mu$ are the gauge potentials seen by the right- and the left-handed components respectively:

$$\hat{F}^{R}_{\mu}{}^{\!\!L} = (\hat{v}_\mu \pm a_\mu)$$

($\hat{v}_\mu$ includes the gluon field). In this form the explicit expression for the anomaly becomes

$$A(F) = \frac{1}{6}\,\varepsilon_{\alpha\beta\mu\nu}\Big[F_{\alpha\beta}F_{\mu\nu} + i\{F_{\alpha\beta}, F_\mu F_\nu\} + i\,F_\alpha F_{\mu\nu}F_\beta$$
$$- 2F_\alpha F_\beta F_\mu F_\nu\Big] \qquad \text{(A.29)}$$

(The price to pay is that the quantity $\hat{\det} D$ is not invariant under the transformations generated by the vector currents.)

Appendix B: Renormalizable σ-model

It is instructive to compare the general low energy representation of the Green's functions given in the first part of this paper with the low energy structure of a specific renormalizable model. An obvious candidate is the linear O(4) σ-model, a well known example of a theory that leads to spontaneous breakdown of O(4) $\simeq$ SU(2)xSU(2) to O(3) $\simeq$ SU(2). It turns out that the one loop corrections can as easily be worked out in the O(N) model with arbitrary N. We analyze the low energy structure of this slightly more general version of the O(4) model and set N = 4 only at the end of this section in order to compare with our general SU(2)xSU(2) results.

To analyze the Green's functions we couple the N-component field ϕ^A to a set of external fields of spin zero and spin one

$$\mathcal{L}_\sigma = \frac{1}{2}\nabla_\mu\phi^A\nabla^\mu\phi^A + \frac{1}{2}m^2\phi^A\phi^A - \frac{g}{4}(\phi^A\phi^A)^2$$
$$+ f^A\phi^A + h\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} \qquad \text{(B.1)}$$

(Note that in our metric $m^2 > 0$ corresponds to the spontaneously broken phase.) The external vector and axial vector fields are contained in the covariant derivative

$$\nabla_\mu\phi^A = \partial_\mu\phi^A + F_\mu^{AB}\phi^B$$
$$F_\mu^{0i} = a_\mu^i \qquad \text{(B.2)}$$
$$F_\mu^{ik} = -\varepsilon^{ikl}v_\mu^l$$

as well as in the c-number term tr $F_{\mu\nu} F^{\mu\nu}$ which represents the trace over the square of the field strength tensor $F^{AB}_{\mu\nu}$ associated with the nonabelian field F^A_μ. In the one loop approximation the generating functional becomes

$$Z_\sigma = \int dx \mathcal{L}_\sigma + \frac{i}{2} \ln \det D$$

where D is the differential operator

$$(y, Dy) = \int dx \left\{ -\nabla_\mu y^T \nabla^\mu y + (-m^2 + g \phi_0^T \phi_0) y^T y \right.$$
$$\left. + 2g (\phi_0^T y)^2 \right\} \tag{B.3}$$

and ϕ_0^A is the classical solution of the equation of motion in the presence of the external fields f^A and F^{AB}_μ.

The d-dimensional determinant of D contains a pole at d = 4

$$\frac{i}{2} \ln \det D = -\frac{1}{(4\pi)^2} \frac{1}{d-4} \int d^d x \; \mathrm{tr} \left\{ \frac{1}{12} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sigma^2 \right\} + O(1)$$
$$\sigma^{AB} = (-m^2 + g \phi_0^T \phi_0) \delta^{AB} + 2g \phi_0^A \phi_0^B \tag{B.4}$$

This pole is removed by the following renormalization of g, m, h

$$g = g_r \left\{ 1 - 2(N+8) g_r \lambda_0 \right\} + O(g_r^3)$$

$$m^2 = \frac{M_r^2}{2} \left\{ 1 - 2(N+2)(\lambda_0 + \gamma) g_r \right\} + O(g_r^2)$$

$$h = h_r + \frac{1}{12} \lambda_0 + O(g_r) \tag{B.5}$$

$$\lambda_0 = \frac{1}{(4\pi)^2} M_r^{d-4} \left\{ \frac{1}{d-4} - \frac{1}{2} \left(\ln 4\pi + \Gamma'(1) + 1 \right) \right\}$$

where g_r, M_r and h_r are finite but arbitrary otherwise. We have included in the renormalization of m^2 the finite piece γ for later convenience, see the remark after equation (B.11). With these substitutions the effective one loop action is finite at d = 4.

We wish to expand this action in powers of the momenta. (This amounts to consider the model in the limit where m^2 is large.) We generalize known techniques (Akhoury and York-Peng Yao 1982) to allow for the presence of external vector and axial vector fields. We first parametrize the classical solution ϕ_o^A by

$$\phi_o^A = \frac{m}{\sqrt{g}} R U^A \quad , \quad U^T U = 1 \tag{B.6}$$

(One might be inclined to first integrate over the fluctuations of the radial variable R, and to take the large m-limit before integration over the fluctuations in the unit vector U^A. This procedure does however not produce the correct large m-limit for one loop graphs which involve both internal pion- and internal σ-lines. Regularizing in d dimensions the virtual pion momenta occurring in these graphs are of order m and the interchange of the limit $m \to \infty$ with the momentum integration is therefore not permitted.) We next write the fluctuations around ϕ_o^A as follows:

$$\phi^A = \phi_o^A + y^A$$
$$y^A = \xi U^A + \sum_{i=1}^{N-1} \varepsilon_i^A \eta^i$$

with

$$\sum_{i=1}^{N-1} \varepsilon_i^A \varepsilon_i^B = 1 - U^A U^B \,, \quad \varepsilon_i^T \varepsilon_j = \delta_{ij} \,, \quad \varepsilon_i^T U = 0 \tag{B.7}$$

The differential operator D defined in (B.3) then acts in the flavor space ξ, η^i through a matrix of the form

$$\left(\begin{array}{c|c} \Box + 2m^2 + O(1) & O(1) \\ \hline O(1) & O(1) \end{array}\right) \tag{B.8}$$

where O(1) denote differential operators which tend to finite limits as $m \to \infty$. To diagonalize D we introduce new differential operators d, δ, $\hat{D}_\mu$, $\hat{D}$:

$$
\begin{aligned}
d &\doteq \partial^\mu \partial_\mu - f_\mu^{\,i} f^{\mu i} + m^2(3R^2-1) \\
\delta\eta &\doteq f_\mu^{\,i} (\hat{D}^\mu)^{ik} \eta^k + \partial_\mu (f_i^{\,\mu} \eta^i) \\
(\hat{D}_\mu)^{ik} \eta^k &\doteq \partial_\mu \eta^i + \varepsilon_i^{\,T} \nabla_\mu \varepsilon_k \eta^k \\
(\eta, \delta^T \xi) &\doteq (\delta\eta, \xi) \\
\hat{D} &\doteq \hat{D}_\mu \hat{D}^\mu - f_\mu^{\,i} f^{\mu k} + m^2 (R^2-1)\delta^{ik} \\
f_\mu^{\,i} &\doteq U^T \nabla_\mu \varepsilon_i
\end{aligned}
\tag{B.9}
$$

Then one finds with $\hat{\xi} \doteq \xi + d^{-1}\delta\eta$

$$
(y, Dy) = (\hat{\xi}, d\hat{\xi}) + (\eta, \hat{D}\eta) - (\eta, \delta^T d^{-1} \delta\eta) \tag{B.10}
$$

Note that (for large m) $\hat{D}$ is the differential operator of the nonlinear σ-model (compare section 7). Hence, we write

$$
\ln\det D = \ln\det \hat{D} + \Delta
$$

$$
\Delta = \ln\det(1 - \hat{D}^{-1}\delta^T d^{-1}\delta) + \ln\det d
$$

and it remains to expand Δ for large m^2. Writing

$$
d = \Box + 2m^2 + \sigma_1 = d_0 + \sigma_1
$$

it turns out that nonvanishing contribtuions are produced only by the terms

$$
\ln\det d = \ln\det d_0 + \operatorname{tr} d_0^{-1}\sigma_1 - \frac{1}{2}\operatorname{tr}[(d_0^{-1}\sigma_1)^2] + \dots
$$

$$
\operatorname{tr}\ln(1 - \hat{D}^{-1}\delta^T d^{-1}\delta) = -\operatorname{tr}\hat{D}^{-1}\delta^T d^{-1}\delta - \frac{1}{2}\operatorname{tr}[(\hat{D}^{-1}\delta^T d^{-1}\delta)^2] + \dots
$$

where the dots denote contributions which vanish for large values of m^2. We then find that the low energy expansion of the generating functional up to and including terms of order p^4 indeed is of the general form given in section 7 (eq. 7.12). The (unrenormalized) constants ℓ_1 - ℓ_7, h_1 - h_3, the external fields χ^A and the pion decay constant F are given, for any N, by

$$
\begin{aligned}
F^2 &= \frac{M_r^2}{2g_r} \; ; \; \chi^A = \frac{1}{F} f^A \left\{ 1 - \frac{g_r}{32\pi^2} + O(g_r^2) \right\} \\
\ell_1 &= \frac{1}{4g} - \frac{17}{3}\lambda_0 - \frac{1}{(4\pi)^2}\frac{35}{36} \\
\ell_2 &= \frac{2}{3}\lambda_0 - \frac{1}{(4\pi)^2}\frac{11}{18} \\
\ell_3 &= -\ell_4 + \frac{1}{4g} - \frac{9}{2}\lambda_0 - \frac{1}{(4\pi)^2}\frac{11}{6} \\
\ell_4 &= \frac{1}{2g} - 10\lambda_0 - \frac{1}{(4\pi)^2}\cdot 3 \\
\ell_5 &= -\frac{1}{6}\lambda_0 - \frac{1}{(4\pi)^2}\frac{1}{72} \\
\ell_6 &= -\frac{1}{3}\lambda_0 + \frac{1}{(4\pi)^2}\frac{11}{36} \\
\ell_7 &= 0 \\
h_1 &= \ell_4 + \frac{1}{(4\pi)^2}\frac{1}{12} \\
h_2 &= h \\
h_3 &= 0 \\
\gamma &= \frac{1}{32\pi^2}\cdot\frac{1}{(N+2)}
\end{aligned}
\qquad (B.11)
$$

Remarks: i) The constant γ which defines the renormalized mass has been fixed such that $F^2 = M_r^2/2g_r$. ii) The infinities in ℓ_i and h_i are cancelled by the contributions from the loop integrals over the pion momenta of the nonlinear σ-model ($\ln \det \hat{D}$). iii) Since the model is renormalizable, all constants that appear in the general low energy expansion are fixed in terms of the three σ-model parameters g_r, M_r and h_r.

Let us now consider the case N = 4. The scale independent quantities $\bar{\ell}_i$ and $\bar{h}_i$, the external field χ^A and the pion decay constant F are found to be (see also (Bessis and Zinn - Justin 1972))

$$
\begin{aligned}
F^2 &= \frac{M_r^2}{2g_r} \; ; \; \chi^A = \frac{f^A}{F}\left\{1 - \frac{g_r}{32\pi^2} + O(g_r^2)\right\} \\
\bar{\ell}_1 &= \frac{24\pi^2}{g_r} - \ln\frac{M_\pi^2}{M_r^2} - \frac{35}{6} \\
\bar{\ell}_2 &= -\ln\frac{M_\pi^2}{M_r^2} - \frac{11}{6} \\
\bar{\ell}_3 &= \frac{1}{3}\left(2\bar{\ell}_1 + \bar{\ell}_2 - \frac{1}{2}\right) \\
\bar{\ell}_4 &= \frac{1}{3}\left(\bar{\ell}_1 + 2\bar{\ell}_2 + \frac{1}{2}\right) \\
\bar{\ell}_5 &= \bar{\ell}_2 + 2 \\
\bar{\ell}_6 &= \bar{\ell}_2 \\
\bar{h}_1 &= \frac{1}{3}\left(\bar{\ell}_1 + 2\bar{\ell}_2 + \frac{3}{4}\right) \\
\bar{h}_2 &= 384\pi^2 h_r - \ln\frac{M_\pi^2}{M_r^2}
\end{aligned}
\qquad (B.12)
$$

The pion mass does not occur in the σ-model Lagrangian (B.1); it shows up in the above expressions, because the constants $\bar{\ell}_i$, $\bar{h}_i$ are by definition normalized at running mass M_π. Note, in particular, that these expressions for the low energy constants $\bar{\ell}_1, \ldots, \bar{\ell}_6$ and $\bar{h}_1$, $\bar{h}_2$ do contain the proper chiral logarithms. (The model does not allow for isospin breaking - the constants ℓ_7 and h_3 vanish.)

On the leading level of the low energy expansion it is impossible to distinguish between different models - the leading low energy properties of different models are the same, as long as the values of F_π and M_π agree. This is not any more the case at first nonleading order: different models in general lead to different values of the low energy constants $\bar{\ell}_i$ and $\bar{h}_i$. In the case of the σ-model these constants are fixed by a single new parameter, the mass M_r. The model clearly fails to reproduce the observed low energy structure: the relations (B.12) among the constants $\bar{\ell}_i, \ldots, \bar{\ell}_6$ are not consistent with the observed values.

(Note that the relation $g_r = M_r^2/2F^2$ requires the mass of the σ-particle to be rather low for a first order perturbation theory in g to make sense. In order for the coefficient in front of the logarithm in the relation between the bare and renormalized coupling constants not to exceed unity the mass M_r must be below 400 MeV. A firm believer in the σ-model may therefore argue that perturbation theory is not adequate to analyze the properties of this model and insist that more work is needed to dispose of it. In the present context we merely use the model as a mathematical illustration of the general low energy structure - the model does obviously not qualify as a realistic alternative to QCD.)

Appendix C: The ρ

The low energy representations given in this paper are perfectly consistent with the presence of resonances. To explicitly show how the excited $q\bar{q}$ states manifest themselves in the low energy structure of the Green's functions we consider the ρ-meson and construct an effective Lagrangian which contains the ρ degrees of freedom and is consistent with chiral symmetry. The standard description of the ρ in terms of a vector field $\rho_\mu^i(x)$ with a $\rho\pi\pi$ coupling of the form $\varepsilon_{ik\ell}\rho_\mu^i\, \phi^K \partial^\mu \phi^\ell$ breaks chiral symmetry. We instead describe the degrees of freedom of the ρ in terms of an antisymmetric tensor field $\rho_{\mu\nu}^i(x)$ which transforms according to the nonlinear realization $D^{(1)}$ of SU(2)xSU(2). (For an alternative scheme see (Weinberg 1968).) The kinetic part of the Lagrangian is given by

$$\mathcal{L}_0^\rho = -\frac{1}{2} D^\lambda \rho^i_{\lambda\mu} D_\nu \rho^{i\nu\mu} + \frac{1}{4} M_\rho^2 \rho^i_{\mu\nu} \rho^{i\mu\nu} \tag{C.1}$$

where the covariant derivative D_μ is defined in (7.7). To lowest order in the external fields the ρ-propagator associated with this Lagrangian is

$$i\int dx\, e^{ip(x-y)} \langle 0|T\, \rho^i_{\mu\nu}(x)\, \rho^k_{\alpha\beta}(y)|0\rangle = \delta^{ik} \Delta_{\mu\nu\alpha\beta}$$

$$\Delta_{\mu\nu\alpha\beta} = -M_\rho^{-2}\left[g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha} + (M_\rho^2 - p^2)^{-1}(g_{\mu\alpha}p_\nu p_\beta \right.$$

$$\left. - g_{\mu\beta}p_\nu p_\alpha - g_{\nu\alpha}p_\mu p_\beta + g_{\nu\beta}p_\mu p_\alpha)\right] \tag{C.2}$$

corresponding to the normalization

$$\langle 0|\rho^i_{\mu\nu}|\rho^k, p\rangle = (-i)M_\rho^{-1}(p_\mu \varepsilon_\nu - p_\nu \varepsilon_\mu)\delta^{ik} \qquad \text{(C.3)}$$

(The wave equation associated with the Lagrangian (C.1) implies that only the longitudinal components $\sim p^\mu \rho_{\mu\nu}$ of the field oscillate, the transverse components are frozen.)

At order p^2 chiral symmetry and G-parity permit the following couplings linear in the ρ-field:

$$\mathcal{L}_1^\rho = \epsilon_{ABCD}\,\rho^A_{\mu\nu}\,U^B\{f\,\nabla^\mu U^C \nabla^\nu U^D + \frac{1}{4}F_\rho F^{\mu\nu CD}\} \qquad \text{(C.4)}$$

where the field $\rho^A_{\mu\nu}$ is obtained from $\rho^i_{\mu\nu}$ by performing the change of basis which takes the nonlinear representation $D^{(1)}$ into the linear representation $D^{(1/2,1/2)}$ (see section 7):

$$\rho^A_{\mu\nu} = \sum_i \varepsilon^{Ai}\,\rho^i_{\mu\nu} .$$

The quantity $F^{\mu\nu}$ denotes the nonabelian field strength associated with the external vector and axial vector fields. The constant F_ρ measures the matrix element of the vector current:

$$\langle 0|V^i_\mu|\rho^k, p\rangle = \delta^{ik}\varepsilon_\mu F_\rho M_\rho \qquad \text{(C.5)}$$

and the constant f determines the strength of the $\rho\pi\pi$ coupling; in terms of f the width of the ρ is given by

$$\Gamma_\rho = \frac{1}{48\pi}\,\frac{f^2}{F_\pi^4}\,(M_\rho^2 - 4M_\pi^2)^{3/2} \qquad \text{(C.6)}$$

In the Green's functions associated with the quark currents the ρ only appears virtually. To lowest order in f and F_ρ the contributions arise from exchange graphs which are easily evaluated by solving the classical equations of motion of the Lagrangian $L_0^\rho + L_1^\rho$. The contribution of these graphs to the generating functional Z is given by

$$Z^\rho = f^2\,\nabla_\mu U^A \nabla_\nu U^B \Delta^{\mu\nu\rho\sigma} \nabla_\rho U^A \nabla_\sigma U^B$$
$$+ \frac{f F_\rho}{2} \nabla_\mu U^A \nabla_\nu U^B \Delta^{\mu\nu\rho\sigma} F^{AB}_{\rho\sigma}$$
$$+ \frac{F_\rho^2}{16} F^{AB}_{\mu\nu} \Delta^{\mu\nu\rho\sigma} F^{AB}_{\rho\sigma} \tag{C.7}$$
$$- \frac{F_\rho^2}{8} F^{AB}_{\mu\nu} U^B \Delta^{\mu\nu\rho\sigma} F^{AC}_{\rho\sigma} U^C$$

In particular, the ρ-contribution to the vector form factor reads

$$F_V^\rho(t) = \frac{f F_\rho}{F_\pi^2} \frac{t}{M_\rho^2 - t} \tag{C.8}$$

and the contribution to the ππ scattering amplitude becomes

$$A^\rho(s,t,u) = \frac{f^2}{F_\pi^4 M_\rho^2}\Big[-2(s-2M_\pi^2)^2 + \frac{1}{2}\{s^2 + (t-u)^2\} + \frac{t^2}{M_\rho^2 - t}(s-u) + \frac{u^2}{M_\rho^2 - u}(s-t)\Big] \tag{C.9}$$

In the low energy region $p^2 \ll M_\rho^2$ the presence of the ρ only manifests itself indirectly, through a contribution to the low energy constants $\ell_1, \ldots, h_3$. The ρ-contribution to the form factor $F_V(t)$ e.g. reduces to a term linear in t. Comparison with (15.3) shows that this contribution merely renormalizes the constant $\bar{\ell}_6$. Likewise the low energy limit of the expression (C.9) for the ρ-contribution to the scattering amplitude is a polynomial of order p^4 which may be absorbed in the constants $\bar{\ell}_1$ and $\bar{\ell}_2$. More generally, since for $p^2 \ll M_\rho^2$ the propagator $\Delta_{\mu\nu\rho\sigma}$ becomes momentum independent, the quantity Z^ρ reduces to a local expression of the form given in section 5, as it should. The change in the value of the low energy constants produced by the ρ is easily obtained by comparing

(C.7) with (5.5):

$$\begin{aligned}
&\bar{\ell}_1^{\rho} = -96\pi^2 f^2/M_\rho^2 && \bar{\ell}_6^{\rho} = 96\pi^2 f F_\rho/M_\rho^2 \\
&\bar{\ell}_2^{\rho} = 48\pi^2 f^2/M_\rho^2 && \ell_7^{\rho} = 0 . \\
&\bar{\ell}_3^{\rho} = 0 && \bar{h}_1^{\rho} = 0 \\
&\bar{\ell}_4^{\rho} = 0 && \bar{h}_2^{\rho} = 48\pi^2 F_\rho^2/M_\rho^2 \\
&\bar{\ell}_5^{\rho} = 48\pi^2 F_\rho^2/M_\rho^2 && h_3^{\rho} = 0
\end{aligned} \tag{C.10}$$

To evaluate these contributions numerically, we use the experimental value F_ρ = 144 MeV and determine the coupling constant f from the experimental width of the ρ which implies f = 69 MeV. Note that the experimental vector form factor is quite well reproduced by the vector meson dominance formula

$$1 + F_V^{\rho}(t) = M_\rho^2/(M_\rho^2 - t) \tag{C.11}$$

This relation requires f to be positive (with $f = F_\pi^2/F_\rho$ = 60 MeV, in good agreement with the value given above). With these values for F_ρ and f we obtain

$$\begin{aligned}
&\bar{\ell}_1^{\rho} = -8 \quad , \quad \bar{\ell}_2^{\rho} = 4 \quad , \quad \bar{\ell}_3^{\rho} = 0 \\
&\bar{\ell}_4^{\rho} = 0 \quad , \quad \bar{\ell}_5^{\rho} = 17 \quad , \quad \bar{\ell}_6^{\rho} = 16
\end{aligned} \tag{C.12}$$

A comparison with the observed values of the low energy constants given in section 19 shows that the ρ is indeed the most important low energy phenomenon apart from the poles and cuts due to the Goldstone bosons. In fact one obtains a good estimate for the low energy constants $\ell_1, \ldots, \ell_6$ if one assumes that the running constants at a scale of the order of M_ρ are given by the ρ-contribution alone:

$$\bar{\ell}_i = \bar{\ell}_i^{\rho} + \ln \frac{M_\rho^2}{M_\pi^2} \tag{C.13}$$

(There is no particular reason for choosing M_ρ as the scale of the chiral logarithms. One could just as well take μ = 1 GeV or μ = 500 MeV - this changes the values of the low energy constants by less than one unit.) The prescription (C.13) fixes all threshold parameters discussed in this paper (except for isospin breaking effects related to ℓ_7) in a parameter free manner, with F_π, M_π, F_ρ, M_ρ and Γ_ρ as experimental inputs. The resulting predictions are shown in Table III. Note in particular, that the D-waves are well reproduced. Although the contributions of the ρ are given by corrections of order M_π^2 they do affect the values of the threshold parameters significantly. This can be seen directly in the I = 1 amplitude $T^1(s, t)$: in the low energy limit $p^2 \ll M_\rho^2$ the ρ-contribution modifies the lowest order expression $(t-u)/F_\pi^2$ for this amplitude as follows:

$$T^1(s,t) = \frac{t-u}{F_\pi^2}\left\{1 + 3s\frac{f^2}{F_\pi^2 M_\rho^2}\right\} \qquad \text{(C.14)}$$

At threshold, $s = 4\,M_\pi^2$, the ρ-contribution amounts to a correction of the form $(1 + 7\,M_\pi^2/M_\rho^2)$ which increases the soft pion value for a_1^1 by more than 20% - the ρ-contribution is at the origin of the difference between the experimental value of a_1^1 and the soft pion result. Note that the ρ pole generates sizeable contributions to the higher partial waves. Although the ρ-contribution to a_3^1 e.g. is algebraically of relative order M_π^2 compared to the leading term given by the low energy theorem (18.7), it increases the value of this term by about 60%. (The leading term amounts to $a_3^1 = 1.9 \cdot 10^{-5}$, the ρ contribution increases this to $3.1 \cdot 10^{-5}$. The experimental value is $(6 \pm 2) \cdot 10^{-5}$.)

Equation	Leading contribution	$1+\frac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi$	Soft pions
(20.1)	a_1^1	1.12 ± 0.11	1.28 ± 0.07
(20.2)	b_0^0	1.13 ± 0.19	1.40 ± 0.17
(20.3)	b_0^2	1.18 ± 0.10	0.92 ± 0.09
(20.4)	$2\,a_0^0 - 5\,a_0^2$	1.10 ± 0.05	1.15 ± 0.05
Mean value		1.12 ± 0.04	
Prediction		1.10 ± 0.02	1

Table I: Scattering lengths interpreted as measurements of the scalar radius of the pion.

	Value	Obtained from	Equation
$\bar{\ell}_1$	-2.3 ± 3.7	D-wave scattering lengths	(19.3)
$\bar{\ell}_2$	6.0 ± 1.3		
$\bar{\ell}_3$	2.9 ± 2.4	SU(3) mass formulae	(19.8)
$\bar{\ell}_4$	4.6 ± 0.9	$K \to \pi e \nu$ or F_K/F_π	(19.12)
$\bar{\ell}_5$	13.9 ± 1.3	$\pi \to e \nu \gamma$	(19.20)
$\bar{\ell}_6$	16.5 ± 1.1	$\langle r^2 \rangle^\pi_V$	(19.16)
ℓ_7	$O(5 \cdot 10^{-3})$	$\pi^0 - \eta$ mixing	(12.9) (19.21)

Table II : Values of the low energy coupling constants.

	Experiment	$\pi + \rho$
a_0^0	0.26 ± 0.05	0.20
b_0^0	0.25 ± 0.03	0.24
$2\,a_0^0 - 5\,a_0^2$	0.614 ± 0.028	0.60
a_0^2	$-\,0.082 \pm 0.008$	$-\,0.069$
a_1^1	0.038 ± 0.002	0.038
b_1^1		0.0069
a_2^0	$(17 \pm 3)\ 10^{-4}$	$20 \cdot 10^{-4}$
a_2^2	$(1.3 \pm 3)\ 10^{-4}$	$0.5 \cdot 10^{-4}$
$\langle r^2 \rangle_S^\pi$	$0.60 \pm 0.15\ \mathrm{fm}^2$	$0.40\ \mathrm{fm}^2$
$\langle r^2 \rangle_V^\pi$	$0.439 \pm 0.03\ \mathrm{fm}^2$	$0.52\ \mathrm{fm}^2$
γ	0.44 ± 0.12	$-\,0.11$

Table III: The third column gives the values of various physical quantities in case the running coupling constants at scale M_ρ are assumed to be given by the ρ-contribution alone.

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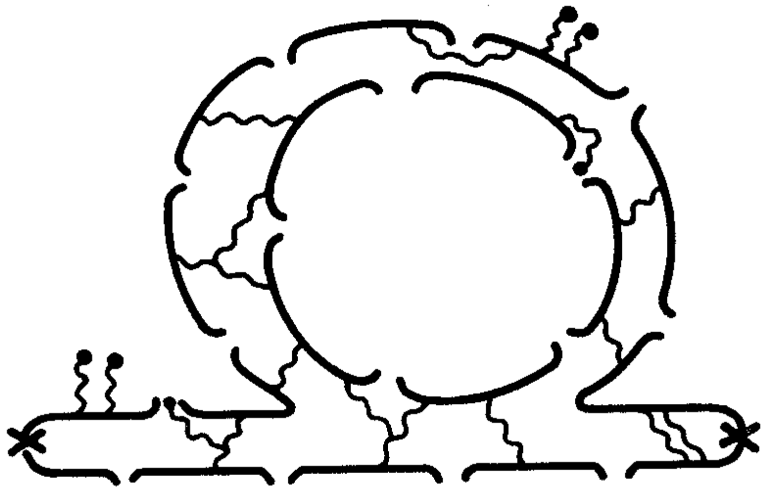

 $\langle 0|\bar{q}q|0\rangle$

 $\langle 0|G_{\mu\nu}G^{\mu\nu}|0\rangle$

 $\langle 0|\bar{q}G_{\mu\nu}\sigma^{\mu\nu}q|0\rangle$

Figure caption

A typical graph which contributes to the Green's function $\langle 0|TA^{i}_{\mu}(x)A^{k}_{\nu}(y)|0\rangle$.