CERN-TH.4493/86

LATTICE ACTION SUM RULES

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A B S T R A C T

We develop identities which relate the "energy" in the colour fields to the mass of a system using lattice gauge theory techniques. Applications are made to glueballs, static quark potentials and glue-lump states. The colour electric and magnetic components are extracted and discussed.

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CERN-TH.4493/86

July 1986

1. INTRODUCTION

In considering a hadronic state such as a glueball, it is natural to ask about how the energy of the state is built up from the colour fields which compose it. In lattice gauge theory calculations one can extract the mass (or total energy) in a well established way from the long-time correlations of operators which have an overlap with the state. One can also attempt to study the construction of the state by probing it with local (e.g. plaquette) operators - as has been done for the colour field configuration in the static source potential[1,2,3]. This is an important field of study - it will provide answers to such questions as the width of the flux tube joining static colour sources, the size of a glueball, the electric/magnetic nature of the colour fields,etc.

In a classical Minkowski approach, one would be able to identify the total energy in the colour fields with the mass of the state directly for a glueball. In a field theoretic approach, only energy differences are meaningful. Furthermore, the lattice gauge theory approach has only been successful in extracting spectrum values in Euclidian space-time. Our aim is to derive identities which relate the 'energy' in the colour fields as measured by lattice techniques with the total 'energy' of the system as measured also by lattice techniques. These identities can then be used as a check of lattice Monte Carlo calculations. They are also powerful enough to allow some conclusions to be drawn without relying on Monte Carlo results. For instance, the colour electric and colour magnetic contributions to a glueball are found to be of equal strength. For the interquark potential, one can likewise obtain the relative strength of the electric and magnetic components - finding the electric component to be somewhat stronger because of the self-energy of the static sources.

The main technique used to derive the identities is to consider derivatives with respect to β of expressions involving transfer matrix eigenvalues. In this respect, the method is similar to that used to discuss the finite size dependence of the average plaquette action[4]. Another element we need is the treatment[5] of asymmetric lattices with space separation a_s and time separation a_t. Our first application is to glueball observables. Then we consider the self-energy components arising from static sources. This enables us to extend our analysis to potentials,etc. We conclude with a discussion of the consequences of our results on the energy density of colour fields.

2. GLUEBALL STATES

Glueball states can be studied in lattice gauge theory by considering correlations of two Wilson loops at separation Δt. Then on a Euclidian lattice of sufficiently large size in the t-direction this correlation decreases as $\exp(-m\,\Delta t)$ so that at large Δt the lightest glueball (with non-zero coupling to the Wilson loops used) is dominant. For convenience a combination G of Wilson loops is chosen so that it excites glueball states in a specific representation of O_h and a sum over all spatial sites at a fixed t-value is used to give a zero-momentum state. Thus if Δt = na, one measures

$$\langle G(0)G(na)\rangle = \frac{\int e^{\beta S}\ G(0)G(na)}{\int e^{\beta S}}$$

$$= \Sigma_\alpha c_\alpha^2\ \ \lambda_\alpha^n \quad = \quad \Sigma_\alpha c_\alpha^2\ \ e^{-naE_\alpha} \tag{2.1}$$

where

$$S = \Sigma\, \frac{1}{N}\, \mathrm{ReTr}(U_\Box) = \Sigma\, \Box \qquad \text{in } SU(N).$$

Here λ_α are eigenvalues of the transfer matrix and correspond to glueball states with energies E_α. The Wilson loop combinations G acting on the vacuum can be decomposed in these eigenmodes as

$$G(0)|0\rangle = \Sigma_\alpha\ c_\alpha\ |\alpha\rangle \tag{2.2}$$

where we assume for simplicity of presentation that $c_0 = 0$, namely the vacuum mode is orthogonal to G. Now as $n \to \infty$, only the lowest energy glueball with $c_\alpha \neq 0$ will contribute, assuming a discrete spectrum for the lowest lying glueballs, so

$$\underset{n \to \infty}{Lt}\ \langle G(0)G(na)\rangle = \langle 0|G|1\rangle\ \lambda_1^n\ \langle 1|G|0\rangle = c_1^2\ e^{-naE_1} \tag{2.3}$$

This is how E_1 may be determined in principle.

To study the coupling of glueballs to plaquettes, consider the derivative of Eq.2.1 with respect to β

$$\frac{\int e^{\beta S}\ S\ G(0)G(na)}{\int e^{\beta S}} - \frac{\int e^{\beta S}\ G(0)G(na)\ \int e^{\beta S}\ S}{\int e^{\beta S}\qquad \int e^{\beta S}} \tag{2.4}$$

$$= \Sigma_\alpha\ 2c_\alpha\ \frac{\partial c_\alpha}{\partial \beta}\ e^{-naE_\alpha} - nc_\alpha^2\ \frac{\partial(aE_\alpha)}{\partial \beta}\ e^{-naE_\alpha}$$

Now the dominant terms for large n are those like ne^{-naE_1}. Again assuming a discrete spectrum, one can neglect other terms as $n \to \infty$. So one has

$$-nc_1^2\ \frac{\partial(aE_1)}{\partial\beta} e^{-naE_1} = \langle G(0)\ S\ G(na)\rangle - \langle G(0)G(na)\rangle\ \langle S\rangle \tag{2.5}$$

Now S refers to the total action, so that there are contributions from every plaquette, including those not between 0 and na. However, for such contributions there will be a cancellation between the two terms on the right hand side of Eq.2.5. To see this consider the plaquettes at m > n. Then

$$\begin{aligned}&\langle G(0)G(na)\square(ma)\rangle - \langle G(0)G(na)\rangle\ \langle\square(ma)\rangle = \\ &\Sigma_{\alpha,\mu}\langle 0|G(0)|\mu\rangle\ e^{-naE_\mu}\ \langle\mu|G(na)|\alpha\rangle\ e^{-(m-n)aE_\alpha}\ \langle\alpha|\square(ma)|0\rangle \\ &- \Sigma_\mu\langle 0|G(0)|\mu\rangle\ e^{-naE_\mu}\ \langle\mu|G(na)|0\rangle\quad \langle 0|\square(ma)|0\rangle\end{aligned} \tag{2.6}$$

Then the term with $\alpha = 0$ cancels in Eq. 2.6, while the terms with

$\alpha = 1,..$ are small unless $m \approx n$ and are not comparable to the leading term of size ne^{-naE_1} for large n. The terms in Eq. 2.5 with $0 < m < n$ yield

$$\Sigma_{\alpha,\mu}\langle 0|G(0)|\alpha\rangle\ e^{-maE_\alpha}\ \langle\alpha|\Box(ma)|\mu\rangle\ e^{-(n-m)E_\mu}\ \langle\mu|G(na)|0\rangle$$
$$-\Sigma_\alpha\ \langle 0|G(0)|\alpha\rangle\ e^{-naE_\alpha}\ \langle\alpha|G(na)|0\rangle\ \langle 0|\Box(ma)|0\rangle \qquad (2.7)$$

which can be simplified by noting that at large n the dominant term (of order e^{-naE_1}) involves the $\alpha = \mu = 1$ eigenstate $|1\rangle$ only. So Eq. 2.5 reduces at large n to (this amounts to picking the terms extensive in n)

$$-\frac{\partial(aE_1)}{\partial\beta} = \langle 1|\Sigma\ \Box|1\rangle - \langle 0|\Sigma\ \Box|0\rangle \qquad (2.8)$$

where $|0\rangle$ is the vacuum eigenstate and $\Sigma\ \Box$ refers to the sum over plaquettes at one time-value (i.e. $6L_xL_yL_z$ plaquettes on a $L_x\times L_y\times L_z\times\infty$ lattice). We shall refer to these plaquettes as magnetic $\Box_s$ if they are purely spatial and electric $\Box_t$ if they have time-directed links. For electric plaquettes, the notation of Eq. 2.8 implies

$$\langle 1|\Box_t|1\rangle \equiv \frac{\langle 1;t{=}0|\Box_t|1;t{=}a\rangle}{\langle 1;t{=}0|1;t{=}a\rangle} = \langle 1;t{=}0|\Box_t|1;t{=}a\rangle e^{aE_1} \qquad (2.9)$$

which is the combination that arises in the derivation. This quantity (and $\langle 1|\Box_s|1\rangle$ which only involves one t-value) are both measurable in lattice Monte Carlo simulations by evaluating the expectation value of plaquettes in a glueball state - that is $\langle G(0)\Box(ma)G(na)\rangle\ /\ \langle G(0)G(na)\rangle$ for large n ,m and n-m. They are also of relevance to finite size effects - see ref(4).

The derivation of Eq. 2.8 that we have given is in terms of the lattice observables which allow the quantities on either side of the equation to be determined in practice. A more formal proof is possible and this is presented in the Appendix. There also we give some identities for quantities such as $\langle 1|\Sigma\ \Box|0\rangle$.

Now as the continuum limit is approached, on a lattice of sufficient spatial extent, E_1 will be independent of β and a if the state has zero momentum. Then, writing m_1 for E_1,

$$\langle 1|\Sigma\Box|1\rangle - \langle 0|\Sigma\Box|0\rangle = -m_1 a \frac{\partial \ln a}{\partial\beta} \approx \frac{m_1 a}{4Nb_o} \qquad (2.10)$$

where $b_o = 11N/(48\pi^2)$ in SU(N). This identity is of the form we seek - it relates the difference of the action in the colour fields of a glueball and of the vacuum to the glueball mass. To express it in terms of colour fields directly, we use the naive classical limit ($a \to 0$) expansions

$$\Sigma_1^6 \beta\Box = 6\beta - \tfrac{1}{4}a^4 F^c_{\mu\nu}F^c_{\mu\nu} + O(a^6) \qquad (2.11)$$

where the sum is over the six orientations of a plaquette. Then Eq. 2.10 becomes

$$\Sigma_1^{L^3} \tfrac{1}{4}a^3 (F^c_{\mu\nu}\ F^c_{\mu\nu})_{1-0} = m_1 \frac{\partial \ln a}{\partial \ln\beta} = m_1 \frac{g}{2\beta_g} \approx -\frac{m_1\beta}{4Nb_o} \qquad (2.12)$$

where $\beta = 2N/g^2$, β_g is the beta-function, and the suffix 1-0 implies the difference of the expectation value in the glueball state $|1\rangle$ and in the vacuum. The left hand side is just the classical (Euclidian) expression for the action in the colour electric and magnetic fields. One sees that in the continuum limit ($\beta \to \infty$) the right hand side becomes much larger than m_1 so that the action of the colour fields is much larger than the total energy m_1.

This identity Eq. 2.12 is similar to the trace anomaly[6] for the energy-momentum tensor $\Theta^\mu{}_\mu$. For a pure gauge theory this is

$$\frac{g}{2\beta_g}\ \Theta^\mu{}_\mu = \tfrac{1}{4}\ F^c_{\mu\nu}F^c_{\mu\nu} \qquad (2.13)$$

Thus the integral over the colour fields is related to the energy in the same way as for our glueball relation Eq. 2.12. The difference is that unlike the trace anomaly expression which is divergent, our expression is a difference of energies and is finite at finite g. Somewhat similar considerations have been used by Svetitsky et al in discussing[7] the energy difference (latent heat) between confined and deconfined phases in lattice gauge theory.

<u>Asymmetric lattices</u> can be considered so allowing the colour fields to be resolved into electric and magnetic components. The basic formalism is that of Karsch[5] who considers an action $\beta_s S_s + \beta_t S_t$ generalizing β S. This leads to a lattice with time-spacing a_t and spatial-spacing a_s (= a) with asymmetry $\xi = a_s/a_t$. The relationship between the couplings β_s and β_t and the coupling β_E of the equivalent symmetric action is given by[5]

$$\beta_s \xi = \beta_E(a) + 2N\ c_s(\xi) + O(\beta_E^{-1})$$
$$\beta_t \xi^{-1} = \beta_E(a) + 2N\ c_t(\xi) + O(\beta_E^{-1}) \qquad (2.14)$$

The renormalization scale on a lattice with asymmetry ξ is related to the equivalent symmetric one by

$$\Lambda(\xi) = \Lambda_E\ e^{-(c_s + c_t)/4b_0} \qquad (2.15)$$

and we shall need the identity for the ξ-derivatives which ensures symmetry at $\xi = 1$, namely $c_s' + c_t' = b_0$.

The generalization of the procedure used above is to consider derivatives with respect to β_s or β_t. Thus Eq.2.8 becomes

$$-\frac{\partial(a_t E_1)}{\partial\beta_t} = \langle 1|\Sigma\Box_t|1\rangle - \langle 0|\Sigma\Box_t|0\rangle$$
$$-\frac{\partial(a_t E_1)}{\partial\beta_s} = \langle 1|\Sigma\Box_s|1\rangle - \langle 0|\Sigma\Box_s|0\rangle \qquad (2.16)$$

where the sum is over all electric or magnetic plaquettes at one time-value (i.e. $3L_xL_yL_z$ plaquettes). As the continuum limit is approached, E_1 will depend on β_s and β_t only through the ξ-dependence of the renormalization scale $\Lambda(\xi)$ as given in Eq. 2.15. Then using the derivatives evaluated from Eq. 2.14 (with symmetry restored after taking the derivative, so $\beta_s=\beta_t=\beta$ and $a_s=a_t=a$)

$$\frac{\partial \ln \Lambda(\xi)a_t}{\partial\beta_t} = \frac{1}{2}\frac{\partial \ln a}{\partial\beta} - \frac{1}{2\beta} + O(\beta^{-2}) \qquad (2.17)$$

$$\frac{\partial ln\ \Lambda(\xi)a_t}{\partial\beta_s} = \frac{1}{2}\ \frac{\partial lna}{\partial\beta} + \frac{1}{2\beta} + O(\beta^{-2}) \tag{2.18}$$

This then gives the ratios

$$\frac{\langle\Sigma\Box_t\rangle_{1-0}}{\langle\Sigma\Box_s\rangle_{1-0}} = \frac{\Sigma\langle\mathcal{E}^2\rangle_{1-0}}{\Sigma\langle\mathcal{B}^2\rangle_{1-0}} = 1 + \frac{8Nb_0}{\beta} + O(\beta^{-2}) \tag{2.19}$$

Here we define $\mathcal{E}^2 = F^c_{4i}F^c_{4i}$, etc. So at large β the electric and magnetic contributions are of equal magnitude. As well as the Euclidian action $\Sigma(\mathcal{E}^2+\mathcal{B}^2)/2$, it is of interest to evaluate the combination $\Sigma(-\mathcal{E}^2+\mathcal{B}^2)/2$ which corresponds to the Minkowski energy since $\mathcal{E}$ in Euclidian space has one time component which introduces a factor of i. This latter combination is from Eqs 2.12 and 2.19

$$\Sigma\,\frac{1}{2}\ a^3\ (-\mathcal{E}^2 + \mathcal{B}^2)_{1-0} = m_1 + O(\beta^{-1}) \tag{2.20}$$

This is the classical Minkowski-space result that the energy in the colour fields should sum to the rest mass m_1. It is very reassuring to recover it as a cross check on this approach which has used Euclidian lattice regularization. In the limit as a_t becomes small, the transfer matrix of the Euclidian formalism is closely related to the Hamiltonian which generates infinitesimal time translations. In our derivation, we have related $\mathcal{E}^2$ and $\mathcal{B}^2$ to $\Box_t$ and $\Box_s$ respectively, and this is only valid at small a. It may be possible to find improved lattice observables which are closer to $\mathcal{E}^2$ and $\mathcal{B}^2$ for larger a, one proposal is given in ref(3).

In summary, we have derived **exact** lattice identities (Eqs. 2.8 and 2.16) using the assumption that a discrete lowest excited glueball energy level exists. These are applicable to zero momentum glueballs of any J^{PC}. The combination of these exact identities and of results valid as $\beta \to \infty$ gives the overall picture of the electric and magnetic field energy in a glueball. We find (in Minkowski space convention) the electric component to be large and positive while the magnetic component is negative and cancels leaving the rest energy m_1.

3. STATIC SOURCES - POTENTIALS

The potential energy between static colour sources at spatial separation R can also be studied using transfer matrix methods[8]. The static sources are represented by Wilson lines in the time direction and the transfer matrix eigenstates can be decomposed as a sum over paths (directed link products including disconnected loops in principle) at one time-value which start at one source and end at the anti-source. Let $|R,\alpha\rangle$ label the eigenstate α with sources at separation R, the corresponding eigenvalue is $\lambda_\alpha(R)$, where this is equal to $\exp(-E_\alpha(R)a)$ with E_α the energy of this eigenmode. To extract these eigenvalues one can consider Wilson loops composed of a path (or a linear combination of paths) $P_R(0)$ from 0 to R at time t=0 and another similar path $P_R(na)$ at t=na joined to the former by the straight source lines. It is convenient[8] to classify these paths P_R using the discrete group D_{4h}. Then such a path P_R acting on the vacuum can be expanded

$$P_R|0\rangle = \Sigma_\alpha\, d_\alpha\, |R,\alpha\rangle \tag{3.1}$$

As in the glueball case, we assume a discrete spectrum for the lowest such eigenmodes. Then the Wilson loop referred to above can be used to extract the lowest energy $E_0(R)$, since at large n

$$\langle W_L\rangle = \langle P_R(0)\ \mathcal{T}\ P_R(na)\rangle = \Sigma_\alpha\, d_\alpha^2\, \lambda_\alpha^n(R) \underset{n\to\infty}{\Rightarrow} d_0^2\, \lambda_0^n = d_0^2\, e^{-naE_0(R)} \tag{3.2}$$

where $\mathcal{T}$ represents the source lines in the Wilson loop. Again as in the glueball case, we now consider the β-derivative of this identity. The left hand side yields

$$\langle P_R(0)\ \mathcal{T}\ P_R(na)\ S\rangle - \langle P_R(0)\ \mathcal{T}\ P_R(na)\rangle\ \langle S\rangle \tag{3.3}$$

where S implies a sum over all plaquettes and again we distinguish the case when this plaquette is between 0 and n or not. For $0 < n < m$, this yields

$$\langle 0|P_R|R,\mu\rangle\ \lambda_\mu^n(R)\ \langle R,\mu|P_R|\alpha\rangle\ \lambda_\alpha^{m-n}\ \langle\alpha|\Box(ma)|0\rangle$$
$$-\langle 0|P_R|R,\mu\rangle\ \lambda_\mu^n(R)\ \langle R,\mu|P_R|0\rangle\ \langle 0|\Box(ma)|0\rangle \qquad (3.4)$$

which cancels for $\alpha = 0$ (the vacuum), so there is only a contribution of size e^{-naE_o} when m is close (of order the inverse of the glueball mass) to n. The contribution from terms with $0 < m < n$ is of order ne^{-naE_o} since approximately all n values of m give contributions

$$\langle 0|P_R|R,\mu\rangle\ \lambda_\mu^m(R)\ \langle R,\mu|\Box(ma)|R,\nu\rangle\ \lambda_\nu^{n-m}(R)\ \langle R,\nu|P_R|0\rangle$$
$$-\langle 0|P_R|R,\mu\rangle\ \lambda_\mu^n(R)\ \langle R,\mu|P_R|0\rangle\ \langle 0|\Box(ma)|0\rangle \qquad (3.5)$$

where at large n , $\mu = \nu = 0$ gives the dominant piece. Then, for $n \to \infty$, equating these terms of size ne^{-naE_o} to those from the β-derivative of the right hand side of Eq. 3.2, gives

$$-\frac{\partial\ aE_o(R)}{\partial\ \beta} = \langle R,0|\Sigma\Box|R,0\rangle - \langle 0|\Sigma\Box|0\rangle = (\Sigma\Box)_{R-o} \qquad (3.6)$$

where the sum is over all plaquettes at one time value as before. The quantity on the right of Eq. 3.6 is directly measurable in principle by lattice simulation from $\langle W_L\ \Box\rangle/\langle W_L\rangle$ where the Wilson loop W_L is that introduced above and one measures the plaquette average (at $t = ma$ with $0 \ll m \ll n$) in the presence of this Wilson loop compared to the plaquette average without the loop. This is just the quantity evaluated hitherto[1,2,3] in the study of the energy distribution of the gluon colour flux between static sources. What is new is the exact identity relating the β-derivative of the potential to these plaquette action shifts summed over all space.

This β-derivative is less straightforward than in the glueball case because of the self-energy of the static sources. Thus the lattice energy $E_o(R)$ is related to the continuum potential $V(R)$ by

$$E_o(R)a = V(R)a + F(a)a \qquad (3.7)$$

where $F(a)$ is the lattice source self-energy which is independent of R. The self-energy can be eliminated by considering $E_o(R_1) - E_o(R_2)$ and if R_1 and R_2 are kept constant at physical values, one can evaluate the β-derivative of Eq. 3.6 since V(R) is physical and so independent of β. Alternatively one can try and deduce the β-dependence of the self-energy term as will be discussed next.

Just as for the glueball case, the spatial ($\Box_s$) and temporal ($\Box_t$) plaquettes can be studied by taking derivatives with respect to β_s and β_t. This gives exact identities

$$-\frac{\partial\ a_t E_o(R)}{\partial\ \beta_t} = (\Sigma\Box_t)_{R-O} \qquad (3.8)$$

$$-\frac{\partial\ a_t E_o(R)}{\partial\ \beta_s} = (\Sigma\Box_s)_{R-O} \qquad (3.9)$$

Now as $\beta \to \infty$, one can again use weak coupling perturbation theory arguments to extract the electric and magnetic field contributions and thence check that the total (Minkowski space) energy in the fields matches the potential energy $E_o(R)$. The component V(R) will satisfy this check just as for the glueball case since it is a physical measurable. Let the self-energy component be $F(a,\xi)$ on a lattice with asymmetry ξ. Then the energy conservation constraint requires at $\xi = 1$ that

$$a_t F(a,\xi) \approx -\frac{\partial\ a_t F(a,\xi)}{\partial\ \xi} + N b_o \frac{\partial\ a_t F(a,\xi)}{\partial\ \beta} \qquad (3.10)$$

One expects this self-energy term $a_t F(a,\xi)$ to be a short distance effect so that its dependence can be estimated from one (lattice) gluon exchange. This behaves like $a_t(g^2/a_s) \sim \xi^{-1}\beta^{-1}$ so that it becomes relatively more important at large β where a is tiny - as expected since it diverges in the continuum limit. Now, Eq. 3.10 is consistent with a parametrization of $a_t F(a,\xi)$ as $f(\beta)/\xi$ which is also in accord with the one gluon exchange estimate. Then with this form for the self-energy component,

$$E_o(R) = V(R) + f(\beta)/a \qquad (3.11)$$

and one can express the electric and magnetic contributions to the energy as

$$\sum \tfrac{1}{2}a^3(\mathcal{E}^2)_{R-O}=V(R)\ \Big(\frac{\beta}{2}\ \frac{\partial\, \ln a}{\partial\, \beta} - \frac{1}{2} + O(\beta^{-1})\Big) + \frac{\beta}{2a}\,\frac{\partial f}{\partial \beta} - \frac{f}{2a}\big(1 + O(\beta^{-1})\big)$$

$$\sum \tfrac{1}{2}a^3(\mathcal{B}^2)_{R-O}=V(R)\ \Big(\frac{\beta}{2}\ \frac{\partial\, \ln a}{\partial\, \beta} + \frac{1}{2} + O(\beta^{-1})\Big) + \frac{\beta}{2a}\,\frac{\partial f}{\partial \beta} + \frac{f}{2a}\big(1 + O(\beta^{-1})\big) \qquad (3.12)$$

Thus the sums over space of the electric and magnetic field energies corresponding to the continuum potential V(R) are enhanced by a factor of $-\beta/(8Nb_o) \mp 0.5$ which makes them equal at large β. Whereas, with the estimated behaviour of $f(\beta)$ of $\sim \beta^{-1}$, one sees that the self-energy component is not enhanced and lies entirely in the electric field.

We have derived exact lattice sum rules (Eqs. 3.8 and 3.9) which constrain the sums over all space of plaquette averages in the presence of a potential. Now the spatial dependence of such plaquette averages has been studied to try and determine the extent of the flux tube between static sources[1,2,3]. Our identities serve as a stringent cross check on such determinations. Furthermore, we have clarified the link between these action sum rules and the approximate (in Euclidian space) energy conservation requirements. This leads to predictions for the ratio of total colour electric (Euclidian sign convention) to colour magnetic energy in the flux tube of $1 + 8Nb_o/\beta$ or approximately 1.28 at $\beta = 6.0$ in SU(3). This ratio refers to the continuum piece V(R) which can be extracted by considering differences of R-values on a lattice. Some evidence for such a ratio has been found in Monte Carlo studies[3].

The spatial distribution of these colour fields is of course not obtainable from our identities. However, one would expect the self-energy component which is R-independent to be localized close to the sources, to be a sum of terms symmetric about each source, and we estimate that it is predominantly in the colour electric field. Some evidence for such a contribution is seen in ref(2). In general it is clear that a study of the $\mathcal{E}^2$ and $\mathcal{B}^2$ distributions can help in separating the self-energy contribution.

A direct lattice study of the (Minkowski space) energy density $(-\mathcal{E}^2 + \mathcal{B}^2)/2$ associated with the potential energy V(R) will be difficult, both because of the strong cancellation between the two terms and because of the need to remove the self-energy component which grows at small a. A study of the action density $(\mathcal{E}^2 + \mathcal{B}^2)/2$ is relatively easier and can give indications about the spatial distribution of the fields. In this case a nice cross check is provided by our exact lattice sum rules. As well as the usual potential our methods apply equally to excited potentials (e.g. in other representations of D_{4h}) which correspond to gluonic excitations relevant to hybrid mesons[8].

An <u>adjoint static source</u> can be considered in close analogy to the case of a fundamental source and anti-source just considered. In this case the physical interpretation of the eigenstates of the transfer matrix is as glue-lump states[9]. These are like a glueball except that one gluon is static. This facilitates the study of the spatial distribution of the colour field since the state is not translation invariant (as the glueball itself is in our approach). Such a study is currently underway[10].

4. CONCLUSIONS

We have derived exact identities for a Euclidian lattice regularized theory. These identities are similar to the conformal trace anomaly, but they are defined for the difference between a hadronic system and the vacuum. On a lattice, our identities take the form of sum rules relating the sum of the action over all space to the energy of the system itself. As well as these action sum rules one also has, at small lattice spacing a, approximate energy conservation relating the energy in the colour fields over all space to the energy of the system itself. We have shown that these two conditions are consistent and yield the following picture for the Minkowski space electric and magnetic field energy densities $-\mathcal{E}^2$ and $\mathcal{B}^2$ which build up a physical state of energy E

$$\Sigma \frac{1}{2} a^3 \left(-\mathcal{E}^2\right)_{E-O} = E \frac{1}{2} \left(- \frac{\partial \ln a}{\partial \ln \beta} + 1 + O(\beta^{-1}) \right)$$

$$\Sigma \frac{1}{2} a^3 \left(\mathcal{B}^2\right)_{E-O} = E \frac{1}{2} \left(+ \frac{\partial \ln a}{\partial \ln \beta} + 1 + O(\beta^{-1}) \right)$$

Since $\partial \ln a/\partial \ln \beta$ is about -7 for $\beta = 6.0$ with SU(3), this implies that there is a substantial cancellation of electric and magnetic energy, and the magnetic energy is lower in the state E than in the vacuum. As well as applying to glueballs, we extend these relations to the potential energy of static sources. Again the continuum component of the gluon flux around such static sources obeys the same relations as above. Lattice simulations[1,2,3] of this flux distribution are consistent with our results but have yet to achieve the required precision to allow the stringent test that our sum rules provide.

ACKNOWLEDGEMENTS

I wish to thank Paul Rakow, Peter Hasenfratz, Michael Stone, and Ian Jorysz for helpful discussions.

APPENDIX

Consider an eigenstate $|\alpha\rangle$ of the transfer matrix $\mathcal{T}$ so that

$$\lambda_\alpha\ |\alpha\rangle = \mathcal{T}\ |\alpha\rangle \equiv \int e^{\beta\Sigma\Box}\ |\alpha\rangle \tag{A.1}$$

where $\Sigma\Box$ is the sum over space of $\frac{1}{2}\Box_s(t=a) + \Box_t(t=0-a) + \frac{1}{2}\Box_s(t=0)$, the plaquettes with links at two adjacent time values. Now λ_α and $|\alpha\rangle$ both depend on β, but the eigenstates are orthonormal for any β. Then the derivative with respect to β yields

$$\begin{aligned} \lambda_\alpha' &= \frac{\partial}{\partial\beta} \langle\alpha|\ \mathcal{T}\ |\alpha\rangle = \langle\alpha|\ \mathcal{T}\ \Sigma\Box\ |\alpha\rangle \\ &= \lambda_\alpha\ (\ \langle\alpha|\Sigma\Box_s|\alpha\rangle + \langle\alpha|\mathcal{T}\ \Sigma\Box_t|\alpha\rangle/\langle\alpha|\mathcal{T}|\alpha\rangle\ ) \\ &\equiv \lambda_\alpha\ (\ \langle\alpha|\Sigma\Box_s|\alpha\rangle + \langle\alpha|\Sigma\Box_t|\alpha\rangle\ ) \end{aligned} \tag{A.2}$$

where we have used the definition of Eq. 2.9. Then since $\lambda_\alpha = \lambda_o \exp(-aE_\alpha)$, one obtains Eq. 2.8.

An identity for matrix elements such as $\langle 0|\ \mathcal{T}\ \Sigma\Box|\alpha\rangle$ can be obtained by taking the β-derivative of the orthogonality condition $\langle 0|\ \mathcal{T}\ |\alpha\rangle = 0$ for $\alpha \neq 0$. this gives

$$\frac{\partial\langle 0|}{\partial\beta}\ \mathcal{T}\ |\alpha\rangle + \langle 0|\ \mathcal{T}\ \Sigma\Box|\alpha\rangle + \langle 0|\ \mathcal{T}\ \frac{\partial|\alpha\rangle}{\partial\beta} = 0 \tag{A.3}$$

now since $\langle 0|\alpha\rangle = 0$, one has

$$\langle 0|\ \frac{\partial|\alpha\rangle}{\partial\beta} + \frac{\partial\langle 0|}{\partial\beta}\ |\alpha\rangle = 0 \tag{A.4}$$

so $$\langle 0|\ \mathcal{T}\ \Sigma\Box|\alpha\rangle = -(\lambda_o-\lambda_\alpha)\ \langle 0|\ \frac{\partial|\alpha\rangle}{\partial\beta} \tag{A.5}$$

One can evaluate the matrix element on the right in a path basis

$$|\alpha\rangle = \textstyle\sum_i\ d_i(\beta)\ P_i\ \ |0\rangle \tag{A.6}$$

where P_i are Wilson loops (closed for a glueball) acting on the vacuum. Then

$$\frac{\partial|\alpha\rangle}{\partial\beta} = \textstyle\sum_i\ d_i'\ P_i\ |0\rangle + \sum_i\ d_i\ P_i\ \frac{\partial|0\rangle}{\partial\beta} \tag{A.7}$$

so $$\langle 0|\ \frac{\partial|\alpha\rangle}{\partial\beta} = \frac{1}{2}\textstyle\sum_i\ d_i'\ \ \langle 0|P_i|0\rangle \tag{A.8}$$

because of Eq. A.4. Thus Eq. A.5 can be rewritten as

$$\langle 0|\Sigma\Box_s|\alpha\rangle\tfrac{1}{2}(\lambda_o+\lambda_\alpha) + \langle 0|\Sigma\Box_t|\alpha\rangle\sqrt{\lambda_o\lambda_\alpha} = -(\lambda_o-\lambda_\alpha)\tfrac{1}{2}\textstyle\sum_i d_i' \langle 0|P_i|0\rangle$$

where we have defined $\langle 0|\Box_t|\alpha\rangle$ so that it is to be evaluated at the time value of the centre of the plaquette. Introducing the energy E_α this can be written

$$\langle 0|\Sigma\Box_s|\alpha\rangle/\tanh(aE_\alpha) + \langle 0|\Sigma\Box_t|\alpha\rangle/\sinh(aE_\alpha) = -\textstyle\sum_i d_i' \langle 0|P_i|0\rangle \qquad (A.10)$$

Equation A.10 relates the plaquette overlaps of a glueball state to the β-dependence of the path coefficients. Such relations can be extended to β_s and β_t derivatives easily. One possible application is to the 0^{++} glueball where lattice measurements[11] show that $\langle 0|\Box_s|G\rangle = \langle 0|\Box_t|G\rangle$. This could be explained if d_i for this state had the same dependence on β_s as on β_t ,that is it was independent of the asymmetry ξ.

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