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Quantum algebraic description of vibrational and transitional nuclear spectra

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Abstract

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A physically motivated extension of the $SU_q(2)$ model of rotational nuclear spectra is introduced, which is applicable in the vibrational and transitional regions as well. The deformation parameter is related to the centrifugal stretching effect, while the new parameter c allows the spectrum to be an expansion in terms of $J(J+c)$ instead of $J(J+1)$, thus describing nuclear anharmonicities in a way similar to the Interacting Boson Model and the Generalized Variable Moment of Inertia model.

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Quantum algebras (also called quantum groups) [1] and their applications in physics are recently attracting much attention. One of the first applications of quantum algebras in physics came with the realization that rotational spectra of deformed nuclei [2], superdeformed nuclei [3] and diatomic molecules [4-6] can be accurately described by the quantum algebra $SU_q(2)$. The correspondence between the $SU_q(2)$ expression and the Variable Moment of Inertia (VMI) model [7] has also been established [8]. It has been shown that in both cases the energy is described by a series in powers of $J(J+1)$, the deformation parameter q of the algebra been connected to the softness parameter of the VMI model. When the deformation parameter q is set equal to 1, the usual Lie algebra SU(2) is obtained.

It is not surprising that the applicability of the $SU_q(2)$ formalism is limited to the rotational region (where the ratio $R_4 = E(4)/E(2)$ obtains values between 3.0 and 3.33), since it is based on a deformation of the rotation algebra. For describing nuclear spectra in the vibrational ($2.0 \leq R_4 \leq 2.4$) and transitional ($2.4 \leq R_4 \leq 3.0$) regions it is clear that an extension of the model is needed. In order to be guided towards such an extension, we briefly review the existing experience of other successful models:

i) In the rotational (SU(3)) limit of the Interacting Boson Model (IBM) [9] the spectrum is described by a $J(J+1)$ expression, while in the vibrational (U(5)) and transitional (O(6)) limits expressions of the form $J(J+c)$ with $c > 1$ appear. In the U(5) limit, in particular, the energy levels are given by

$$E(N, n_d, v, n_\Delta, J, M_J) = E_0 + \epsilon n_d + \alpha n_d(n_d+4) + \beta 2v(v+3) + \gamma 2J(J+1), \quad (1)$$

where the usual notation is used. The ground state band, in particular, is characterized by quantum numbers $n_d = 0, 1, 2, \ldots$, $v = n_d$, $n_\Delta = 0$, $J = 2n_d$, so that the energy expression for it reads

$$E(J) = E_0 + \frac{\epsilon}{2}J + \frac{\alpha}{4}J(J+8) + \frac{\beta}{2}J(J+6) + 2\gamma J(J+1). \quad (2)$$

In the O(6) limit the energy is given by

$$E(N,\sigma,\tau,\nu_\Delta,J,M_J) = E_0 + \beta 2\tau(\tau+3) + \gamma 2J(J+1) + \eta 2\sigma(\sigma+4), \qquad (3)$$

with the usual notation used. The ground state band is characterized by the quantum numbers $\sigma = N$, $\tau = 0, 1, 2, \ldots$, $\nu_\Delta = 0$, $J = 2\tau$, so that the relevant energy expression takes the form

$$E(J) = E_0 + \frac{\beta}{2}J(J+6) + \gamma 2J(J+1) + \eta 2N(N+4). \qquad (4)$$

The message from eqs (2) and (4) is that nuclear anharmonicities are described by expressions in which J and J^2 appear with different coefficients, and not with the same coefficient as in $J(J+1)$. The earliest introduction of this idea is in fact the Ejiri [10] formula $E(J) = kJ(J+1) + aJ$, which has been subsequently justified microscopically in [11].

ii) The two-parameter VMI model is known to continue giving good fits in the transitional and even in the vibrational region. In these regions, however, the accuracy of the model is substantially improved by adding a third parameter, which essentially allows for treating J and J^2 with a different coefficient [12,13].

In the usual VMI model [7] the levels of the ground state band are given by

$$E(J) = \frac{J(J+1)}{2\Theta(J)} + \frac{1}{2}C(\Theta(J) - \Theta_0)^2, \qquad (5)$$

where C and Θ_0 are the two parameters of the model, the latter being the ground state moment of inertia. The moment of inertia for each J is determined by minimizing the energy with respect to the moment of inertia for constant J. This procedure leads to a cubic equation with only one real root. The energy can be written as an expansion in powers of $J(J+1)$ as [8]

$$E(J) = \frac{1}{2\Theta_0}(J(J+1) - \frac{\sigma}{2}(J(J+1))^2 + \sigma^2(J(J+1))^3 - 3\sigma^3(J(J+1))^4 + \ldots), \qquad (6)$$

where the softness parameter $\sigma = 1/(2C\Theta_0^3)$ is serving as the small parameter of the expansion.

One of the (essentially equivalent) three-parameter extensions of the model, which give improved fits of vibrational and transitional spectra, is the generalized VMI (GVMI) model [12], in which the energy levels can be written as

$$E(J) = \frac{J(J+x')}{2\Phi'(J)} + \frac{1}{2}k'(\Phi'(J) - \Phi'_0)^2, \qquad (7)$$

where $x' = x^{-1} - 2$. It is clear that for $x = 1/3$ the GVMI reduces to the usual VMI, while for transitional and vibrational nuclei x obtains lower values [12], so that x' becomes greater than 1. The moment of inertia for each J is still determined by minimizing the energy with respect to the moment of inertia for constant J, while the expansion of the energy turns out to be the same as in eq. (6) with the substitutions $J(J+1) \to J(J+x')$, $\Theta_0 \to \Phi'_0$, $\sigma \to \sigma' = 1/(2k'(\Phi'_0)^3)$. We remark that an expansion in terms of $J(J+x')$ is obtained, as compared to an expansion in terms of $J(J+1)$ in the case of the usual VMI. The physical content of the parameters is clear: the centrifugal stretching effect is accounted for by the softness parameter σ', as in the case of the usual VMI, while anharmonicities, important in the vibrational region, are introduced by $x' > 1$. Since centrifugal stretching and anharmonicities are two effects of different origins, it is reasonable to describe them by two different parameters.

In the case of the $SU_q(2)$ model the energy spectrum is given by the eigenvalues of the second order Casimir operator of the algebra, i.e.

$$E(J) = \frac{1}{2I}[J]_q[J+1]_q, \qquad (8)$$

where q-numbers (with $q = e^{i\tau}$) are defined as $[x]_q = \frac{\sin(\tau x)}{\sin\tau}$. In this case the energy can be expanded as [8]

$$E(J) = \frac{1}{2I}\frac{1}{(j_0(\tau))^2}(j_0(\tau)J(J+1) - \tau j_1(\tau)(J(J+1))^2 + \frac{2}{3}\tau^2 j_2(\tau)(J(J+1))^3$$

$$-\frac{1}{3}\tau^3 j_3(\tau)(J(J+1))^4 + \frac{2}{15}\tau^4 j_4(\tau)(J(J+1))^5 - \ldots), \quad (9)$$

where $j_n(\tau)$ are the spherical Bessel functions of the first kind. It has been proved [14] that eq. (8) gives better fits to rotational nuclear spectra than the first order expansion, containing only the first two terms of eq. (9), both formulae containing two free parameters each.

The evidence coming from the IBM and the generalized VMI model described above, suggests a model in which the spectrum is given by

$$E(J) = \frac{1}{2I}[J]_q[J+c]_q, \quad (10)$$

which contains 3 parameters: the moment of inertia I, the deformation parameter q and the new parameter c, which is expected to be 1 in the rotational limit and larger than 1 in the vibrational and transitional regions. This expansion of this energy expression looks like eq. (9) with $J(J+1)$ replaced by $J(J+c)$.

It is expected that the deformation parameter τ, which plays the role of the small parameter in the expansion, as the softness parameter does in the case of the VMI, will describe the centrifugal stretching effect, while the parameter c will correspond to the anharmonicity effects. These expectations are corroborated from least square fits of the experimental data. In Tables 1, 2, 3 some examples of vibrational, transitional (or near-transitional) and rotational nuclei of the rare earth region are given correspondingly. The following comments can be made:

i) The anharmonicity parameter c is clearly decreasing with increasing R_4, i.e. with increasing collectivity. It obtains high values (8-18) in the vibrational region, while in the rotational region it stays close to 1. (It should be noted that by fixing $c = 1$ in the rotational region the fits are only very slightly changed, as expected.) In the transitional region its values are close to 3.

ii) The deformation parameter τ, which corresponds to the centrifugal stretching, is known from the $SU_q(2)$ model to obtain values close to 0.3-0.4 in the rotational region, a

fact also seen here. The same range of values appears in the vibrational region as well, while in the transitional region τ reaches values as high as 0.6. It is not unreasonable for this parameter, which is connected to the softness of the nucleus, to obtain its highest values in the region of γ-soft nuclei.

iii) It is worth remarking that eq. (10) coincides for $q = 1$ and c=integer=N with the eigenvalues of the Casimir operator of the algebra SO(N+2) in completely symmetric states [16]. In the rotational region the fits gave N=1, which corresponds to SO(3), as expected, while in the transitional region the fits gave approximately N=3, which corresponds to SO(5), which is a subalgebra contained in both the U(5) and O(6) limits of the IBM.

iv) It is also worth remarking that a special case of the expression of eq. (10) occurs in the q-deformed version of the O(6) limit of the Interacting Boson Model [17-19]. The construction of the q-deformed version of IBM has been achieved by studying the chain of subalgebras

$$\mathrm{SU}^{sd}(1,1) \otimes \mathrm{SO}(6) \supset \mathrm{SU}^{d}(1,1) \otimes \mathrm{SO}(5) \supset \mathrm{SO}(3), \quad (11)$$

where SU^{sd}(1,1) is the algebra closed by the pair operators formed out of the s and d bosons, while SU^{d}(1,1) is the algebra closed by the pair operators formed out of d bosons alone. The irreps of SU^{sd}(1,1) are characterized by the same quantum numbers as the irreps of O(6) in the O(6) chain of the IBM, while the irreps of SU^{d}(1,1) are characterized by the same quantum numbers as the irreps of O(5) in the O(6) limit of IBM. (This fact is based on the concept of complementary subalgebras, introduced by Moshinsky and Quesne [20].) Therefore in the Hamiltonian one can use the Casimir operators of the SU^{sd}(1,1), SU^{d}(1,1) and SU(2) subalgebras (the deformed versions of which are well known [21-23]) instead of the Casimir operators of O(6), O(5), O(3) respectively. Keeping the same notation as in eq. (3) the final result reads

$$E(N,\sigma,\tau,\nu_\Delta,J,M_J) = E_0 + \beta 8\left[\frac{\tau}{2}\right]_q\left[\frac{\tau+3}{2}\right]_q + \gamma 2[J]_q[J+1]_q + \eta 8\left[\frac{\sigma}{2}\right]_q\left[\frac{\sigma+4}{2}\right]_q. \quad (12)$$

For the ground state band then the analog of eq. (4) is

$$E(J) = E_0 + \beta' 8[J]_{q^{1/4}}[J+6]_{q^{1/4}} + \gamma 2[J]_q[J+1]_q + \eta' 8[N]_{q^{1/2}}[N+4]_{q^{1/2}}, \qquad (13)$$

where the identities

$$\left[\frac{x}{2}\right]_q = [x]_{q^{1/2}}(q^{1/2}+q^{-1/2})^{-1}, \qquad \left[\frac{x}{4}\right]_q = [x]_{q^{1/4}}(q^{1/2}+q^{-1/2})^{-1}(q^{1/4}+q^{-1/4})^{-1}, \quad (14)$$

have been used and β', η' are related to β, η and q in an obvious way. We remark that the Casimir operator of $SU_q^d(1,1)$, which is complementary to O(5) in the undeformed case, leads to a term of the form $[J]_{q'}[J+6]_{q'}$ with $q' = q^{1/4}$.

v) The $SU_q(2)$ symmetry is known to make specific predictions for the deviation of the behaviour of the B(E2) values from the rigid rotator model [24]. It will be interesting to connect the spectrum of eq. (10) to some deformed symmetry, at least for special values of c, and examine the implications of such a symmetry for the B(E2) values. Such a study in the framework of the q-deformed version of the O(6) limit of IBM, mentioned in iv), is also of interest.

vi) It is worth noticing that an expansion in terms of $J(J+c)$ can also be obtained from a generalized oscillator [25] with a structure function $F(J) = [J(J+c)]_Q$, with $[x]_Q = (Q^x - 1)/(Q-1)$ and $Q = e^T$, with T real. This is similar to the oscillator successfully used in [26] for the description of vibrational spectra of diatomic molecules. It can also be considered as a deformation of the oscillator corresponding to the Morse potential [27].

In conclusion, we have introduced an extension of the $SU_q(2)$ model of rotational nuclear spectra, which is applicable in the vibrational and transitional regions as well. This extension is in agreement with the Interacting Boson Model and the Generalized Variable Moment of Inertia model. In addition to the overall scale parameter, the model contains two parameters, one related to the centrifugal stretching and another related to nuclear anharmonicities. In the rotational region the model coincides with the usual $SU_q(2)$

model, while in the transitional region an approximate SO(5) symmetry is seen. These results give additional motivation in pursueing the construction of a deformed version of the Interacting Boson Model [17-19].

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Table Captions

Table 1 Experimental (exp) and theoretical (th) spectra for ground state bands in the vibrational region. Experimental data are taken from [15], while the theoretical results are obtained from eq. (10) with the parameter values reported in the lower part of the table. The quality of the fits is measured by $x^2 = \sum_{J_{min}}^{J_{max}} (E_J(exp) - E_J(th))^2$. Energies and $1/2I$ are measured in MeV, while x^2 is measured in MeV^2. τ, c, R_4 are dimensionless quantities.

Table 2 Same as Table 1, but in the transitional (or near-transitional) region.

Table 3 Same as Table 1, but in the rotational region.

Table 1

J	^{150}Sm	^{150}Sm	^{152}Gd	^{152}Gd	^{154}Dy	^{154}Dy	^{156}Er	^{156}Er
	exp	th	exp	th	exp	th	exp	th
2	0.3340	0.3399	0.3443	0.3506	0.3346	0.3392	0.3445	0.3421
4	0.7734	0.7704	0.7554	0.7611	0.7467	0.7486	0.7969	0.7955
6	1.2788	1.2757	1.2273	1.2262	1.2237	1.2195	1.3403	1.3426
8	1.8371	1.8374	1.7467	1.7399	1.7473	1.7421	1.9587	1.9616
10	2.4320	2.4350	2.3004	2.2955	2.3043	2.3055	2.6329	2.6283
12	3.0480	3.0464	2.8837	2.8858	2.8930	2.8978	3.3146	3.3163
14			3.4991	3.5031	3.5092	3.5067		
16			4.1426	4.1395				
$1/2I$		0.0147		0.0093		0.0111		0.0171
τ		0.0479		0.0285		0.0362		0.0497
c		10.3		18.0		14.2		8.45
$10^3 x^2$		0.065		0.174		0.100		0.046
R_4	2.316		2.194		2.232		2.315	

Table 2

J	^{152}Sm	^{152}Sm	^{154}Gd	^{154}Gd	^{156}Dy	^{156}Dy	^{158}Er	^{158}Er
	exp	th	exp	th	exp	th	exp	th
2	0.1218	0.1311	0.1231	0.1316	0.1378	0.1475	0.1922	0.1953
4	0.3665	0.3677	0.3710	0.3720	0.4041	0.4059	0.5272	0.5244
6	0.7069	0.7015	0.7177	0.7124	0.7703	0.7650	0.9706	0.9669
8	1.1254	1.1208	1.1445	1.1401	1.2157	1.2111	1.4943	1.4958
10	1.6093	1.6110	1.6372	1.6391	1.7250	1.7267	2.0741	2.0788
12	2.1489	2.1549	2.1850	2.1909	2.2859	2.2920	2.6829	2.6802
14	2.7363	2.7335	2.7780	2.7749	2.8878	2.8849		
$1/2I$		0.0139		0.0144		0.0148		0.0187
τ		0.0468		0.0483		0.0493		0.0620
c		2.76		2.62		3.05		3.34
10^3x^2		0.186		0.169		0.195		0.064
R_4	3.009		3.015		2.932		2.743	

Table 3

J	^{154}Sm	^{154}Sm	^{158}Gd	^{158}Gd	^{162}Dy	^{162}Dy	^{166}Er	^{166}Er
	exp	th	exp	th	exp	th	exp	th
2	0.0820	0.0837	0.0795	0.0804	0.0807	0.0839	0.0806	0.0833
4	0.2667	0.2666	0.2614	0.2613	0.2657	0.2677	0.2650	0.2668
6	0.5443	0.5425	0.5389	0.5382	0.5485	0.5477	0.5454	0.5449
8	0.9031	0.9026	0.9044	0.9044	0.9213	0.9186	0.9112	0.9089
10	1.3333	1.3350	1.3505	1.3508	1.3752	1.3731	1.3495	1.3475
12	1.8262	1.8255	1.8667	1.8666	1.9030	1.9026	1.8465	1.8472
14					2.4940	2.4967	2.3893	2.3924
16					3.1430	3.1439	2.9685	2.9662
18					3.8330	3.8318		
$1/2I$		0.0128		0.0128		0.0127		0.0129
τ		0.0453		0.0392		0.0348		0.0441
c		1.29		1.14		1.31		1.24
10^3x^2		0.010		0.001		0.036		0.036
R_4	3.253		3.288		3.294		3.289	