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A TWO-DIMENSIONAL MODEL FOR MESONS

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ABSTRACT

A recently proposed gauge theory for strong interactions, in which the set of planar diagrams play a dominant role, is considered in one space and one time dimension. In this case, the planar diagrams can be reduced to self-energy and ladder diagrams, and they can be summed. The gauge field interactions resemble those of the quantized dual string, and the physical mass spectrum consists of a nearly straight "Regge trajectory".

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It has widely been speculated that a quantized non-Abelian gauge field without Higgs fields, provides for the force that keeps the quarks inseparably together [1-4]. Due to the infra-red instability of the system, the gauge field flux lines should squeeze together to form a structure resembling the quantized dual string.

If all this is true, then the strong interactions will undoubtedly be by far the most complicated force in nature. It may therefore be of help that an amusingly simple model exists which exhibits the most remarkable feature of such a theory: the infinite potential well. In the model there is only one space, and one time dimension. There is a local gauge group $U(N)$, of which the parameter N is so large that the perturbation expansion with respect to $1/N$ is reasonable.

Our Lagrangian is, like in Ref. 4),

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu i}{}^{j} G_{\mu\nu j}{}^{i} - \bar{q}^{ai}(\gamma_\mu D_\mu + m_{(a)})q^{a}_{i} \, , \tag{1}$$

where

$$G_{\mu\nu i}{}^{j} = \partial_\mu A_i{}^j{}_\nu - \partial_\nu A_i{}^j{}_\mu + g[A_\mu, A_\nu]_i{}^j \; ; \tag{2.a}$$

$$D_\mu q^a{}_i = \partial_\mu q^a{}_i + g A_i{}^j{}_\mu q^a{}_j \; ; \tag{2.b}$$

$$A_i{}^j{}_\mu(x) = - A^{*}_j{}^i{}_\mu(x) \; ; \tag{2.c}$$

$$q^1 = p \quad ; \quad q^2 = n \quad ; \quad q^3 = \lambda \, . \tag{2.c}$$

The Lorentz indices μ, ν, can take the two values 0 and 1. It will be convenient to use light cone coordinates. For upper indices :

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^1 \pm x^0) \, , \tag{3.a}$$

and for lower indices

$$\begin{aligned} p_\pm &= \frac{1}{\sqrt{2}}(p_1 \pm p_0) \, , \\ A_\pm &= \frac{1}{\sqrt{2}}(A_1 \pm A_0) \, , \text{ etc.} \end{aligned} \tag{3.b}$$

where

$$p_1 = p^1 \, , \quad p_0 = -p^0 \, .$$

Our summation convention will then be as follows,

$$x_\mu p^\mu = x^\mu p^\mu = x_\mu p_\mu = \qquad (4)$$
$$x_+ p^+ + x_- p^- = x^+ p^- + x^- p^+ = x_+ p_- + x_- p_+ .$$

The model becomes particularly simple if we impose the light cone gauge condition :

$$A_- = A^+ = 0 . \qquad (5)$$

In that gauge we have

$$G_{+-} = - \partial_- A_+ , \qquad (6)$$

and

$$\mathcal{L} = - \tfrac{1}{2} \operatorname{Tr} (\partial_- A_+)^2 - \bar{q}^a (\gamma\partial + m_{(a)} + g\gamma_- A_+) q^a . \qquad (7)$$

There is no ghost in this gauge. If we take x^+ as our time direction, then we notice that the field A_+ is not an independent dynamical variable because it has no time derivative in the Lagrangian. But it does provide for a (non-local) Coulomb force between the Fermions.

The Feynman rules are given in Fig. 1 [using the notation of Ref. 4)].

The algebra for the γ matrices is

$$\gamma_-^2 = \gamma_+^2 = 0 , \qquad (8.a)$$

$$\gamma_+\gamma_- + \gamma_-\gamma_+ = 2 \qquad (8.b)$$

Since the only vertex in the model is proportional to γ_- and $\gamma_-^2 = 0$, only that part of the quark propagator that is proportional to γ_+ will contribute. As a consequence we can eliminate the γ matrices from our Feynman rules (see the right-hand side of Fig. 1).

We now consider the limit $N \to \infty$; g^2N fixed, which corresponds to taking only the planar diagrams with no Fermion loops [4)]. They are of the type of Fig. 2. All gauge field lines must be between the Fermion lines and may not cross each other.

They are so much simpler than the diagrams of Ref. 4), because the gauge fields do not interact with themselves. We have nothing but ladder diagrams with self-energy insertions for the Fermions. Let us first concentrate on these self-energy parts. Let $i\Gamma(k)$ stand for the sum of the irreducible self-energy parts (after having eliminated the γ matrices). The dressed propagator is

$$\frac{-ik_-}{m^2 + 2k_+k_- - k_-\Gamma(k) - i\varepsilon} \qquad (9)$$

Since the gauge field lines must all be at one side of the Fermion line, we have a simple bootstrap equation (see Fig. 3)

$$i\Gamma(p) = \frac{4g^2}{(2\pi)^2 i}\int dk_+ dk_- \frac{1}{k_-^2}\,\frac{-i(k_- + p_-)}{\left(m^2 + [2(k_+ + p_+) - \Gamma(k+p)](k_- + p_-) - i\varepsilon\right)} \qquad (10)$$

Observe that we can shift $k_+ + p_+ \to k_+$, so $\Gamma(p)$ must be independent of p_+, and

$$\Gamma(p_-) = \frac{ig^2}{\pi^2}\int \frac{dk_-(k_- + p_-)}{k_-^2}\int \frac{dk_+}{m^2 - (k_- + p_-)\Gamma(k_- + p_-) + 2(k_- + p_-)k_+ - i\varepsilon} \qquad (11)$$

Let us consider the last integral in (11). It is ultra-violet divergent, but as it is well known, this is only a consequence of our rather singular gauge condition, Eq. (5). Fortunately, the divergence is only logarithmic (we work in two dimensions), and a symmetric ultra-violet cut-off removes the infinity. But then the integral over k_+ is independent of Γ. It is

$$\frac{\pi i}{2|k_- + p_-|} .$$

so,

$$\Gamma(p_-) = -\frac{g^2}{2\pi}\int \frac{dk_-}{k_-^2}\,\mathrm{sgn}(k_- + p_-) . \qquad (12)$$

This integral is infra-red divergent. How should we make the infra-red cut-off ? One can think of putting the system in a large but finite box, or turning off the interactions at large distances, or simply drill a hole in momentum space around $k_- = 0$. We shall take $\lambda < |k_1| < \infty$ as our integration region and postpone the limit $\lambda \to 0$ until it makes sense. We shall not try to justify this procedure here, except for the remark that our final result will be completely independent of λ, so even if a more thorough discussion would necessitate a more complicated momentum cut-off, this would in general make no difference for our final result.

We find from (12), that

$$\Gamma(p) = \Gamma(p_-) = -\frac{g^2}{\pi}\left(\frac{\mathrm{sgn}(p_-)}{\lambda} - \frac{1}{p_-}\right) , \qquad (13)$$

and the dressed propagator is

$$\frac{-ik_-}{m^2 - \frac{g^2}{\pi} + 2k_+k_- + \frac{g^2}{\pi}\frac{|k_-|}{\lambda} - i\varepsilon} \qquad (14)$$

Now because of the infra-red divergence, the pole of this propagator is shifted towards $k_+ \to \infty$ and we conclude that there is no physical single quark state. This will be confirmed by our study of the ladder diagrams, of which the spectrum has no continuum corresponding to a state with two free quarks.

The ladder diagrams satisfy a Bethe-Salpeter equation, depicted in Fig. 4. Let $\psi(p,r)$ stand for an arbitrary blob out of which comes a quark with mass m_1 and momentum p, and an antiquark with mass m_2 and momentum $r - p$. Such a blob satisfies an inhomogeneous bootstrap equation. We are particularly interested in the homogeneous part of this equation, which governs the spectrum of two-particle states:

$$\psi(p,r) = -\frac{4g^2}{(2\pi)^2 i}(p_- - r_-)p_-\left[M_2^2 + 2(p_+ - r_+)(p_- - r_-) + \frac{g^2}{\pi\lambda}|p_- - r_-| - i\varepsilon\right]^{-1}$$

$$\left[M_1^2 + 2p_+p_- + \frac{g^2}{\pi\lambda}|p_-| - i\varepsilon\right]^{-1} \iint \frac{\psi(p+k,r)}{k_-^2}\, dk_+ dk_- , \qquad (15)$$

where

$$M_i^2 = m_i^2 + \frac{g^2}{\pi} \quad . \tag{16}$$

writing

$$\varphi(p_-, r) = \int \psi(p_+, p_-, r)\, dp_+ \tag{17}$$

we have for φ,

$$\varphi(p_-, r) = -\frac{g^2}{(2\pi)^2 i}\left\{\int dp_+\left[p_+ - r_+ + \frac{M_2^2}{2(p_- - r_-)} + \left(\frac{g^2}{2\pi\lambda} - i\varepsilon\right)\mathrm{sgn}(p_- - r_-)\right]^{-1}\right.$$
$$\left.\left[p_+ + \frac{M_1^2}{2p_-} + \left(\frac{g^2}{2\pi\lambda} - i\varepsilon\right)\mathrm{sgn}(p_-)\right]^{-1}\right\}\int \frac{\varphi(p_- + k_-, r)}{k_-^2}\, dk_- \quad . \tag{18}$$

One integral has been separated. This was possible because the Coulomb force is instantaneous. The p_+ integral is only non-zero if the integration path is <u>between</u> the poles, that is,

$$\mathrm{sgn}(p_- - r_-) = -\mathrm{sgn}(p_-) \quad , \tag{19}$$

and can easily be performed. Thus, if we take $r_- > 0$, then

$$\varphi(p_-, r) =$$
$$\frac{g^2}{2\pi}\theta(p_-)\theta(r_- - p_-)\left[\frac{M_1^2}{2p_-} + \frac{M_2^2}{2(r_- - p_-)} + \frac{g^2}{\pi\lambda} + r_+\right]^{-1}\int \frac{\varphi(p_- + k_-, r)}{k_-^2}\, dk_- \quad . \tag{20}$$

The integral in Eq. (20) is again infra-red divergent. Using the same cut-off as before, we find

$$\int \frac{\varphi(p_- + k_-, r)}{k_-^2}\, dk_- = \frac{2}{\lambda}\phi(p_-) + \mathcal{P}\int \frac{\varphi(p_- + k_-, r)}{k_-^2}\, dk_- \quad , \tag{21}$$

where the principal value integral is defined as

$$\mathcal{P}\int \frac{\varphi(k_-)\, dk_-}{k_-^2} = \frac{1}{2}\int \frac{\varphi(k_- + i\varepsilon)\, dk_-}{(k_- + i\varepsilon)^2} + \frac{1}{2}\int \frac{\varphi(k_- - i\varepsilon)\, dk_-}{(k_- - i\varepsilon)^2} \quad , \tag{22}$$

and is always finite.

Substituting (21) into (20) we find

$$-r_+ \varphi(p_-, r) =$$
$$\left(\frac{M_1^2}{2p_-} + \frac{M_2^2}{2(r_- - p_-)}\right)\varphi(p_-, r) - \frac{g^2}{2\pi} P\int_{-p_-}^{r_- - p_-} \frac{\varphi(p_- + k_-, r)}{k_-^2}\, dk_- \quad . \tag{23}$$

The infra-red cut-off dependence has disappeared ! In fact, we have here the exact form of the Hamiltonian discussed in Ref. 4). Let us introduce dimensionless units :

$$\alpha_{1,2} = \frac{\pi M_{1,2}^2}{g^2} = \frac{\pi m_{1,2}^2}{g^2} - 1 \quad ;$$
$$-2r_+ r_- = \frac{g^2}{\pi}\mu^2 \quad ; \qquad p_-/r_- = x \; , \tag{24}$$

μ is the mass of the two-particle state in units of $g/\sqrt{\pi}$.

Now we have the equation

$$\mu^2 \varphi(x) = \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right)\varphi(x) - P\int_0^1 \frac{\varphi(y)}{(y-x)^2}\, dy \quad . \tag{25}$$

We were unable to solve this equation analytically. But much can be said, in particular about the spectrum. First, one must settle the boundary condition. At the boundary $x = 0$ the solutions $\varphi(x)$ may behave like $x^{\pm\beta_1}$, with

$$\pi\beta_1 \cot g\, \pi\beta_1 + \alpha_1 = 0 \; , \tag{26}$$

but only in the Hilbert space of functions that vanish at the boundary the Hamiltonian [the right-hand side of (25)] is Hermitean :

$$(\psi, H\varphi) = \int\left(\frac{\alpha_1+1}{x} + \frac{\alpha_2+1}{1-x}\right)\varphi(x)\psi^*(x) + \frac{1}{2}\int_0^1 dx \int_0^1 dy \frac{(\varphi(x)-\varphi(y))(\psi^*(x)-\psi^*(y))}{(x-y)^2} \; . \tag{27}$$

In particular, the "eigenstate"

$$\varphi(x) = \left(\frac{x}{1-x}\right)^{\beta_1}$$

in the case $\alpha_1 = \alpha_2$, is not orthogonal to the ground state that does satisfy $\varphi(0) = \varphi(1) = 0$.

Also, from (27) it can be shown that the eigenstates φ^k with $\varphi^k(0) = \varphi^k(1) = 0$ form a complete set. We conclude that this is the correct boundary condition *). A rough approximation for the eigenstates φ^k is the following. The integral in (25) gives its main contribution if y is close to x. For a periodic function we have

$$\mathcal{P}\int_0^1 \frac{e^{i\omega y}}{(y-x)^2}\, dy \simeq \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{i\omega y}}{(y-x)^2}\, dy = -\pi|\omega|\, e^{i\omega x} .$$

The boundary condition is $\varphi(0) = \varphi(1) = 0$. So if $\alpha_1, \alpha_2 \simeq 0$ then the eigenfunctions can be approximated by

$$\varphi^k(x) \simeq \sin k\pi x , \qquad k = 1, 2, \cdots \tag{28}$$

with eigenvalues

$$\mu^2_{(k)} \simeq \pi^2 k . \tag{29}$$

This is a straight "Regge trajectory", and there is no continuum in the spectrum ! The approximation is valid for large k, so (29) will determine the asymptotic form of the trajectories whereas deviations from the straight line are expected near the origin as a consequence of the finiteness of the region of integration, and the contribution of the mass terms.

Further, one can easily deduce from (27), that the system has only positive eigenvalues if $\alpha_1, \alpha_2 > -1$. For $\alpha_1 = \alpha_2 = -1$ there is one eigenstate with eigenvalue zero ($\varphi = 1$). Evidently, tachyonic bound states only emerge if one or more of the original quarks were tachyons [see Eq. (24)]. A zero mass bound state occurs if both quarks have mass zero.

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*) This will certainly not be the last word on the boundary condition. For a more thorough study we would have to consider the unitarity condition for the interactions proportional to 1/N. That is beyond the aim of this paper.

The physical interpretation is clear. The Coulomb force in a one-dimensional world has the form

$$V \propto |x_1 - x_2| ,$$

which gives rise to an insurmountable potential well. Single quarks have no finite dressed propagators because they cannot be produced. Only colourless states can escape the Coulomb potential and are therefore free of infra-red ambiguities. Our result is completely different from the exact solution of two dimensional massless quantum electrodynamics [5),2)], which should correspond to $N = 1$ in our case. The perturbation expansion with respect to $1/N$ is then evidently not a good approximation ; in two dimensional massless Q.E.D. the spectrum consists of only one massive particle with the quantum numbers of the photon.

In order to check our ideas on the solutions of Eq. (25), we devised a computer program that generates accurately the first 40 or so eigenvalues μ^2. We used a set of trial functions of the type $x^{\beta_1}(1-x)^{2-\beta_1}$; $(1-x)^{\beta_2} x^{2-\beta_2}$ and $\sin k\pi x$. The accuracy is typically of the order of 6 decimal places for the lowest eigenvalues, decreasing to 4 for the 40th eigenvalue, and less beyond the 40th.

A certain W.K.B. approximation that yields the formula

$$\mu^2(n) \underset{k\to\infty}{\longrightarrow} \pi^2 n + (\alpha_1+\alpha_2)\log n + C^{st}(\alpha_1,\alpha_2) , \qquad n = 0,1,\ldots \tag{30}$$

was confirmed qualitatively (the constant in front of the logarithm could not be checked accurately).

In Fig. 5 we show the mass spectra for mesons built from equal mass quarks. In the case $m_q = m_{\bar{q}} = 1$ (or $\alpha_1 = \alpha_2 = 0$) the straight line is approached rapidly, and the constant in Eq. (30) is likely to be exactly $3\pi^2/4$.

In Fig. 6 we give some results for quarks with differenct masses. The mass difference for the nonets built from two triplets are shown in two cases :

a) $m_1 = 0$; $m_2 = 0.200$; $m_3 = 0.400$

and

b) $m_1 = 0.80$; $m_2 = 1.00$; $m_3 = 1.20$, in units of $g/\sqrt{\pi}$. The higher states seem to spread logarithmically, in accordance to Eq. (30). But, contrary to Eq. (30), it is rather the average mass, than the average squared mass of the quarks that determines the mass of the lower bound states.

Comparing our model with the real world we find two basic flaws. First there are no transverse motions, and hence there exists nothing like angular momentum, nor particles such as photons. Secondly, at $N = 3$ there exist also other colourless states : the baryons, built from three quarks or three antiquarks. In the $1/N$ expansion, they do not turn up. To determine their spectra one must use different approximation methods and we expect those calculations to become very tedious and the results difficult to interpret. The unitarity problem for finite N will also be tricky.

Details on our numbers and computer calculations can be obtained from the author or G. Komen, presently at CERN.

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FIGURE CAPTIONS

Fig. 1 Planar Feynman rules in the light cone gauge.

Fig. 2 Large diagram a and b must have opposite U(N) charge, but need not be each other's antiparticle.

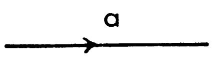
Fig. 3 Equations for the planar self-energy blob.

Fig. 4 Eq. (15).

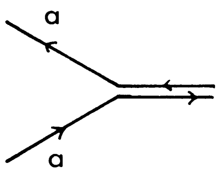
Fig. 5 "Regge trajectories" for mesons built from a quark-antiquark pair with equal mass, m, varying from 0 to 2.11 in units of $g/\sqrt{\pi}$. The squared mass of the bound states is in units g^2/π.

Fig. 6 Meson nonets built from quark triplets. The picture is to be interpreted just as the previous figure, but in order to get a better display of the mass differences the members of one nonet have been separated vertically, and the nth excited state has been shifted to the left by an amount $n\pi^2$. In case a) the masses of the triplet are $m_1 = 0.00$; $m_2 = 0.20$; $m_3 = 0.40$ and in case b) $m_1 = 0.80$; $m_2 = 1.00$; $m_3 = 1.20$. Again the unit of mass is $g/\sqrt{\pi}$.

$$\frac{1}{k_-^2} \quad \rightarrow \quad \frac{1}{k_-^2}$$



$$\frac{m_a - i\gamma_- k_+ - i\gamma_+ k_-}{m_a^2 + 2k_+k_- - i\varepsilon} \quad \rightarrow \quad \frac{-ik_-}{m_a^2 + 2k_+k_- - i\varepsilon}$$



$$-g\gamma_- \quad \rightarrow \quad -2g$$

Fig. 1

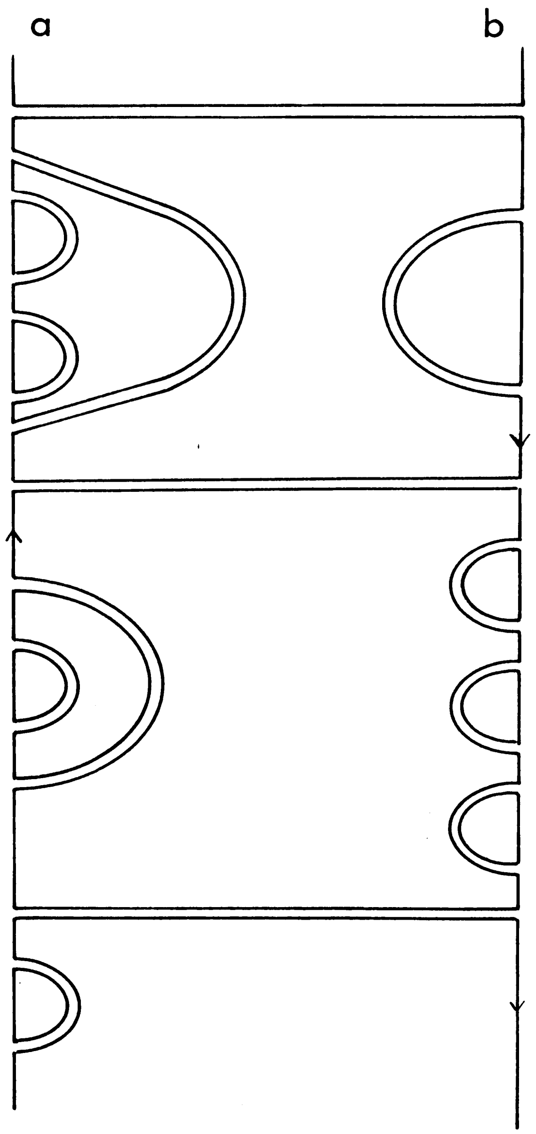


Fig. 2

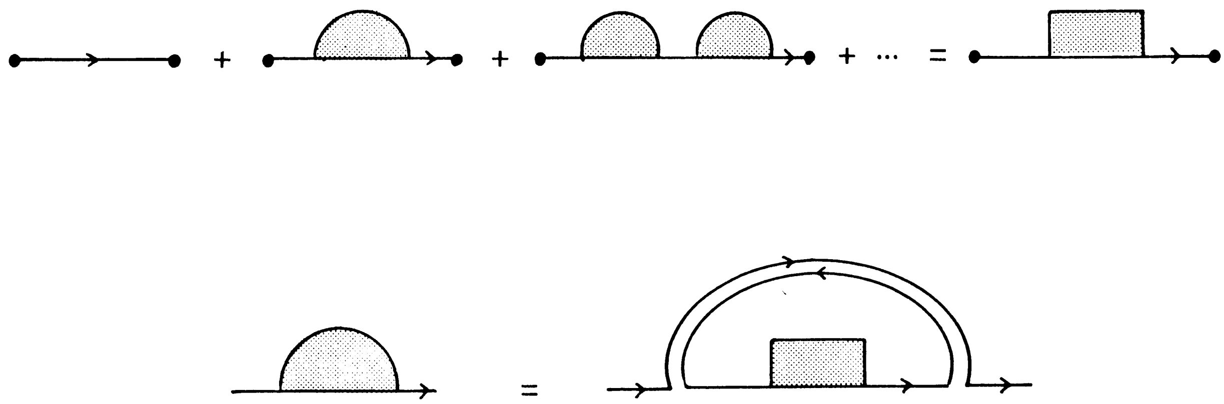


Fig. 3

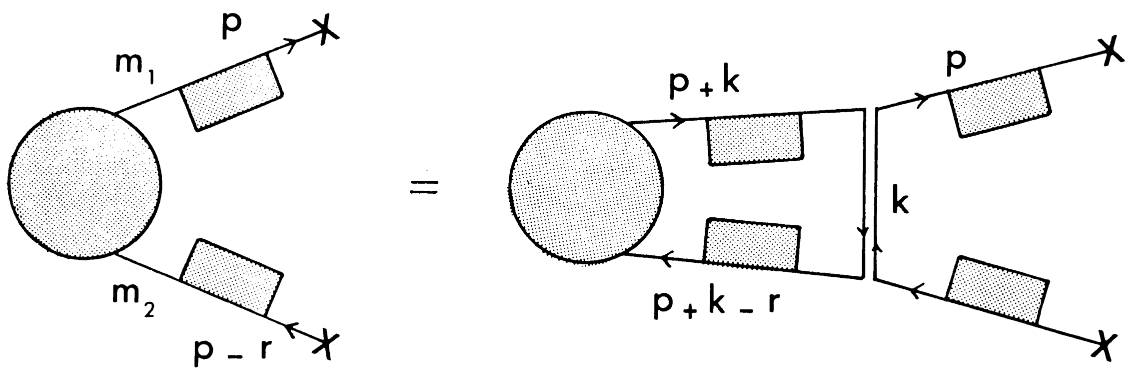


Fig. 4

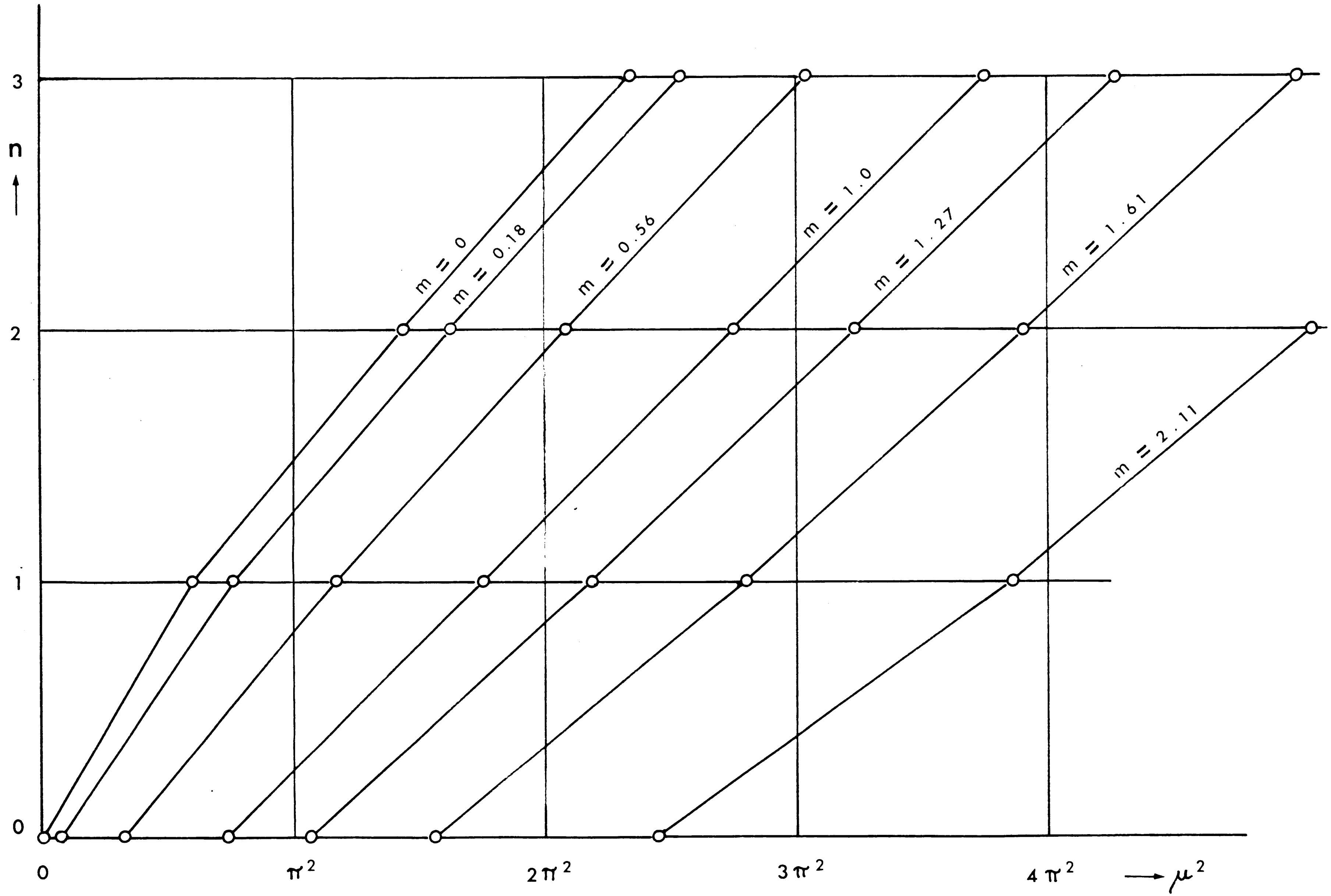


Fig. 5

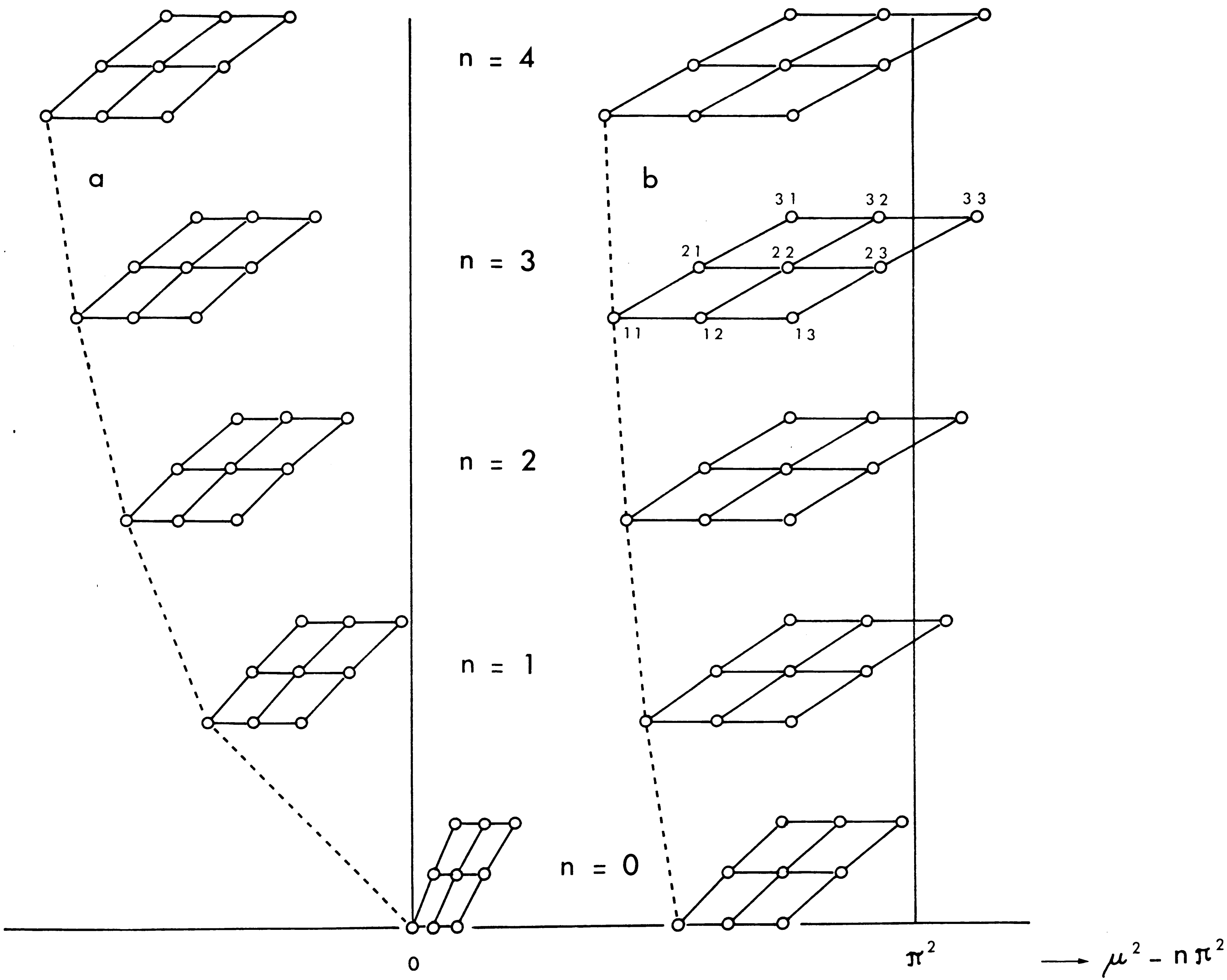


Fig. 6