Theory of Cosmological Perturbations

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- Characterizing perturbations
- Why do we need inflation?
- Generation of primordial fluctuations
- How robust are predictions from inflation?

Characterizing perturbations

Homogeneous Universe:

Metric:

$$ds^2 = a^2(\eta)(d\eta^2 - \gamma_{ik}dx^i dx^k) \;\; = {}^0g_{\alpha\beta}dx^\alpha dx^\beta$$

Matter:

$$\varepsilon(x,\eta) = \varepsilon_0(\eta) \quad - \quad \textit{energy density}$$

$$p(\varepsilon) \quad - \quad \textit{pressure}$$

Inhomogeneities:

Metric:

$$g_{\alpha\beta} = {}^0\, g_{\alpha\beta} + \delta g_{\alpha\beta}(x,\eta),$$

$$\delta g_{\alpha\beta}dx^\alpha dx^\beta = 2a^2[\phi d\eta^2 - B_{|i}dx^i d\eta + (\psi\gamma_{ik} - E_{|ik})dx^i dx^k]$$

Matter:

$$\varepsilon = \varepsilon_0(\eta) + \delta\varepsilon(x,\eta),$$

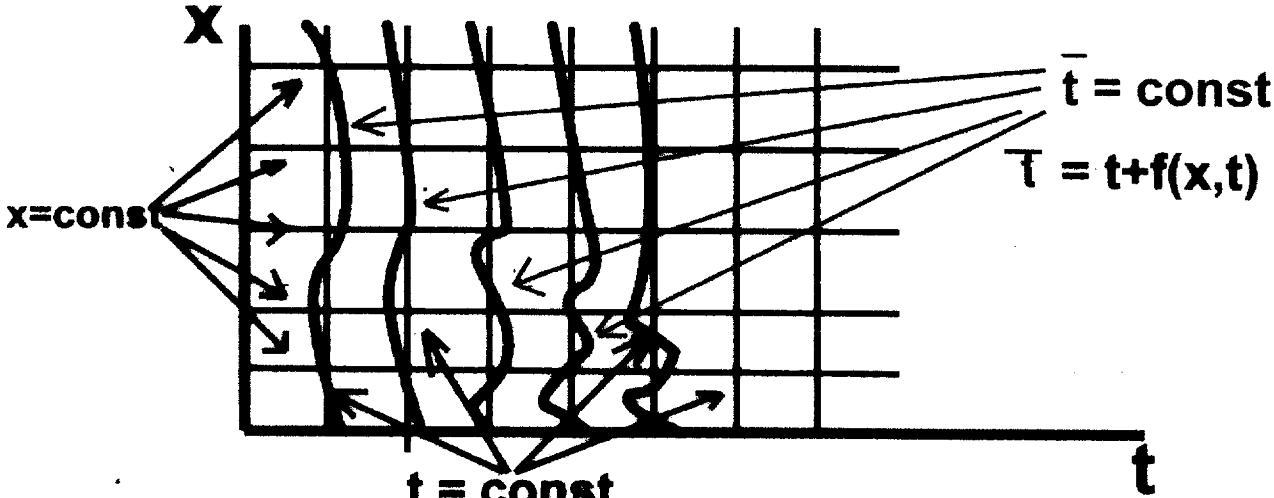
or in the case of scalar field

$$\varphi = \varphi_0(\eta) + \delta\varphi(x,\eta)$$

Gauge transformations and fictitious perturbations

Example:

Let us consider the coordinate transformation: $x \to x$ and $t = \int a d\eta \to \bar{t} = t + f(x,t)$ in unperturbed homogeneous Universe:



Then $\varepsilon(t) = \varepsilon(\bar{t} - f(x,t)) \approx \varepsilon(\bar{t}) - \frac{\partial \varepsilon}{\partial t} f.$

The Universe is homogeneous, but in the new coordinate system it looks like inhomogeneous: $\delta\varepsilon(x,t) = -\frac{\partial \varepsilon}{\partial t} f.$

These inhomogeneities are *fictitious*.

Gauge transformations in general:

Under coordinate transformation:

$$x^{\alpha} \Rightarrow \bar{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}$$

one has

$$\delta g \Rightarrow \bar{\delta g} = \delta g - L_{\xi} {}^{0}g$$

Resolution of the problem of fictitious modes:

1. use gauge invariant variables or
2. completely fix coordinate system and do not allow any coordinate transformations

Gauge invariant variables:

Build out of the metric perturbations: ϕ, ψ, B, E and/or matter variables

(e.g. $\delta\varphi$) formal "4 vector" X^{α} which transforms as $X^{\alpha} \Rightarrow \bar{X}^{\alpha} = X^{\alpha} + \xi^{\alpha}$

Then, for the perturbations of any tensor: $q = {}^{0}q + \delta q$, the variables

$$\delta Q = \delta q + L_X {}^{0}q$$

are gauge invariant

Example: $X^{\alpha} = [(B - E'), E_{|i}]$, $(') = \frac{\partial}{\partial \eta}$

Basic gauge invariant variables:

- Gravitational potential

$$\Phi = \phi + (1/a)[(B - E')a]'$$

Potential Φ is directly related with the temperature fluctuations of CMB in big scales as $\frac{\delta T}{T} = \frac{1}{3}\Phi$ and enters the metric in the newtonian gauge as

$$ds^2 = a^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\gamma_{ik}dx^i dx^k]$$

- Canonical quantization variable

$$v = a\delta\varphi + z\psi$$

where $\delta\varphi$ are the fluctuations of scalar field or velocity potential and

$$z = a\left(1 + \frac{p}{\varepsilon}\right)^{1/2}$$

Equations:

Considering plane wave perturbation $\Phi, v \propto \exp(ikx)$ and introducing new "rescaled" variables $u = \frac{ak}{\sqrt{\varepsilon}}\Phi$ and $\zeta = \frac{v}{z}$, we can derive from Einstein equations the following equations for ζ and u :

$$\zeta' = -(k/z^2)u,$$
$$u' = kz^2\zeta$$

Duality: $\zeta \Rightarrow u,\ u \Rightarrow -\zeta,\ z \Rightarrow 1/z.$

Matching conditions: When the equation of state $p(\varepsilon)$ changes sharply ζ and u are continuous

Solutions:

In the scales bigger than the horizon scale the "conserved quantity" ζ is about gravitational potential Φ : $\Phi \approx O(1)\zeta$ if $\varepsilon + p \sim \varepsilon$.

- For the perturbations with the scale smaller than curvature scale ($\lambda_{ph} = a/k < H^{-1}, H = \frac{\dot{a}}{a}$ is the Hubble constant):

$$\zeta \propto e^{\pm ik\eta}/z = e^{\pm ik\eta}/a\left(1 + \frac{p}{\varepsilon}\right)^{1/2}$$

- When the perturbation scale is bigger than the curvature scale ($\lambda_{ph} < H^{-1}$)

$$\zeta \propto const$$

These statements are true for an arbitrary equation of state $p = p(\varepsilon)$

Why do we need inflation?

Initial conditions at $t_i \sim 10^{43}\,\mathrm{sec}$:

- Homogeneity problem:

$$\left(\frac{\textit{size of the Universe}}{\textit{size of causal region}}\right)_{t_i} \geq \frac{\dot{a}_i}{\dot{a}_0}$$

where $\dot{a}_i = \left(\frac{da}{dt}\right)_i$ and $\dot{a}_0$ are the initial and current "speed" of expansion

- Initial velocities (flatness) problem:

For the given matter distribution the "allowed mistake" in the initial velocities:

$$\frac{\Delta \dot{a}_i}{\dot{a}_i} \leq \left(\frac{\dot{a}_0}{\dot{a}_i}\right)^2$$

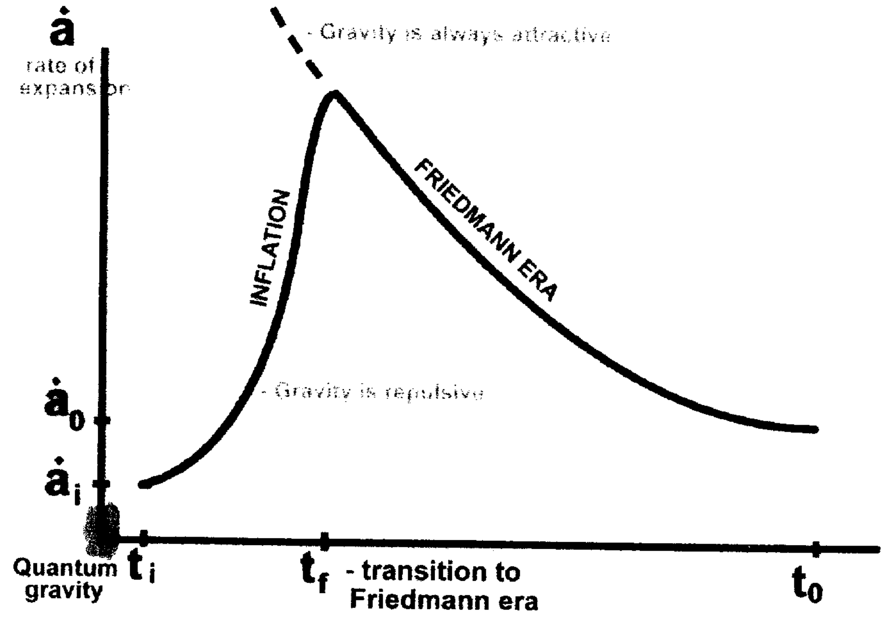
If $p = \varepsilon/3$, then $a \propto \sqrt{t}$ and $\dot{a}_i/\dot{a}_0 > 10^{30}$.

Root of the problem:

Gravity is *attractive* force$\Rightarrow$ $\dot{a}_i \gg \dot{a}_0$

Idea of inflation:

Gravity acts as *repulsive* force during some period of the universe evolution when we have accelerated expansion and $\dot{a}(t)$ is increasing.



$\dot{a}_i < \dot{a}_0 \Rightarrow$ no problems anymore with causality and "fine tuning" of initial velocities.

-On inflation $\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \Rightarrow |\dot{H}| < H^2$

The duration of inflation estimated as $t_f \sim (H^2/\dot{H})_i H^{-1} \sim \frac{2}{3}\left(\frac{\varepsilon}{\varepsilon + p}\right)_i H^{-1}$ should be longer than $70H^{-1} \Rightarrow (\varepsilon + p)_i < \frac{2}{3}\frac{1}{70}\varepsilon_i$

- Inhomogeneities problem:

Generic initial conditions: $\delta\varepsilon/\varepsilon \sim O(1)$

Inflation should:

- eliminates initial inhomogeneities $\Rightarrow$

$\dot{a}_i \ll \dot{a}_0$. Since $(\Omega_0 - 1) = (\Omega_i - 1)\left(\frac{\dot{a}_i}{\dot{a}_0}\right)$,

one gets ***prediction*** $\Omega_0 = 1$!

- generate new inhomogeneities (10^{-5})

Generation of primordial fluctuations

- Uncertainty principle $\Rightarrow$ *inevitable* quantum fluctuations of the fields in the vacuum

Example: quantum fluctuations of the metric in Minkowski space

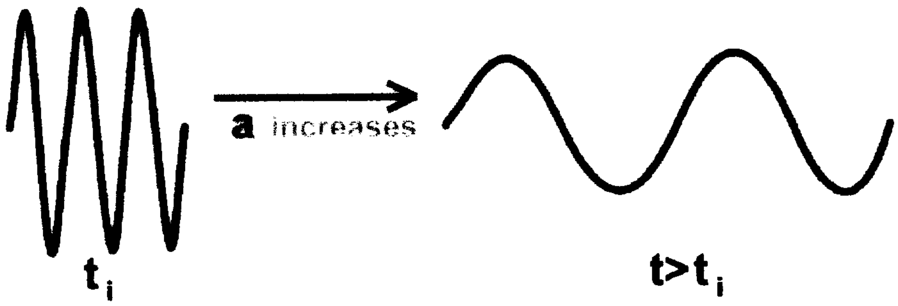
$$h_\lambda \approx \frac{10^{-33}cm}{\lambda}$$

These fluctuations are smaller than 10^{-58} in the appropriate galactic scales today

- Quantum metric fluctuations are big enough (10^{-5}) only in the scales close to the Planck scale $(10^{-33}cm)$. They should be transferred to the galactic scales $(10^{25}cm)$ as a result of the expansion of the Universe without loosing in amplitude.

-To characterize the fluctuations we use canonical variable $v = a(\delta\varphi + z\psi), z = a(1+\frac{p}{\varepsilon})^{\frac{1}{2}}$ and the "conserved quantity" $\zeta = v/z$ which at hydrodynamical stage when $p+\varepsilon \sim \varepsilon$ is about gravitational potential: $\Phi = O(1)\zeta$ (reminder: $\frac{\delta T}{T} = \frac{1}{3}\Phi$ in COBE scales)

For the plane wave perturbation with comoving wavenumber k , the physical wave length $\lambda_{ph} = a/k$ increases $\propto a$



$H^{-1} = \frac{a}{\dot{a}}$ - curvature scale

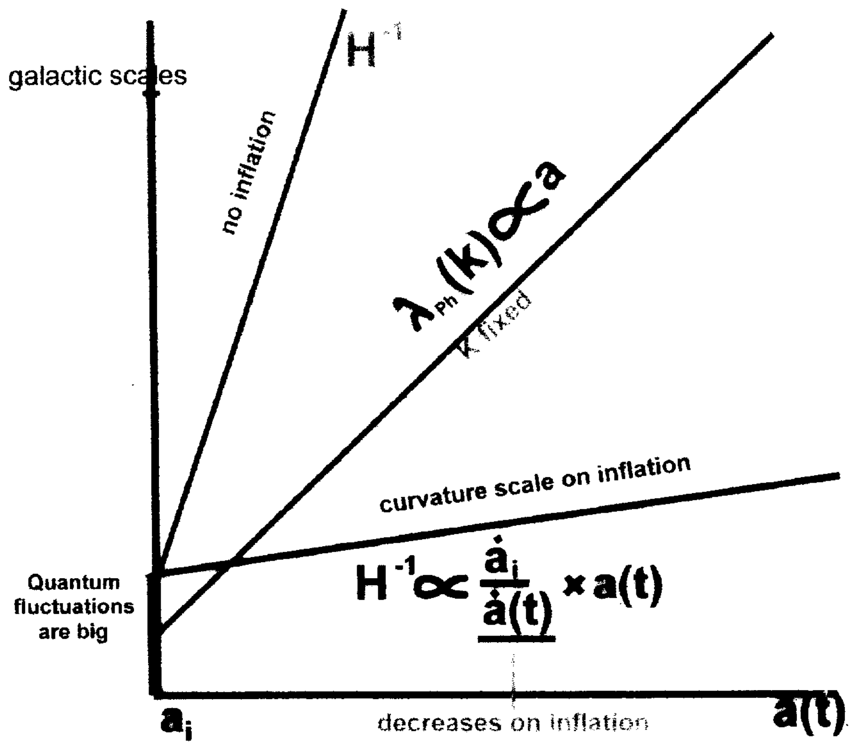
The evolution of the amplitude of the perturbation with given k crucially depends on how big is the physical scale of perturbation λ_{ph} compared to the curvature scale H^{-1} :

- When $\lambda_{ph} < H^{-1}$, the amplitude of ζ decays as 1/z.

v is canonical variable$\Rightarrow$ quantum fluctuations have the amplitude $v_k \sim \frac{1}{\sqrt{k}}$ in the scales $< H^{-1}$. Therefore,

$$\zeta_k = \frac{v_k}{z} \approx \frac{1}{\sqrt{k}} \frac{1}{a\left(1+\frac{p}{\varepsilon}\right)^{1/2}}$$

- When $\lambda_{ph} > H^{-1}$, the amplitude of ζ stays constant irrespectively on the equation of state.

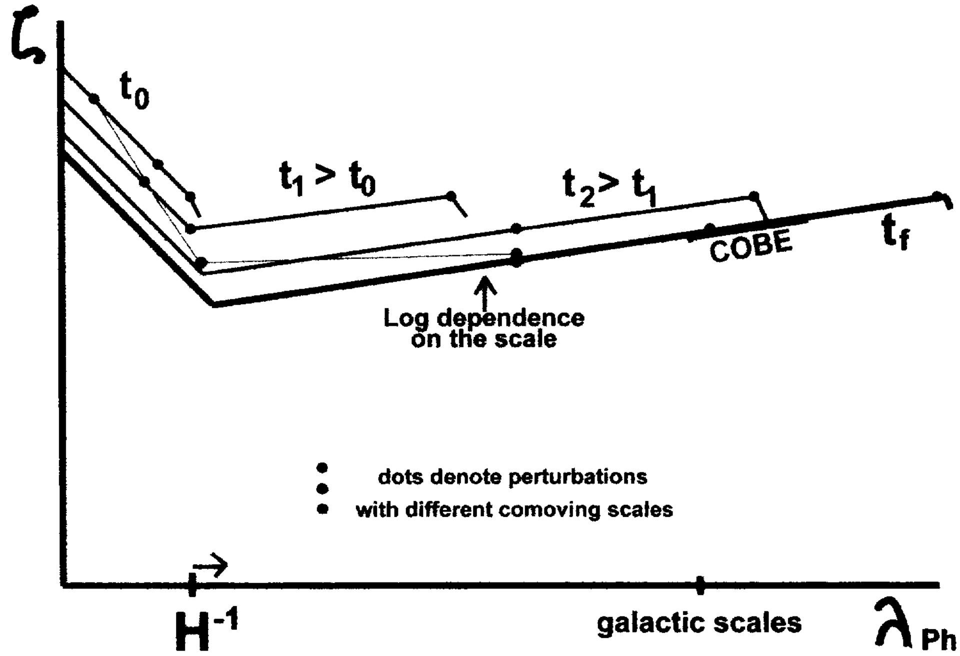


No Inflation- no chance to get big fluctuations in galactic scales! Curvature scale increases faster than a and the amplitude of perturbations which were originally big always decays ($\propto 1/a$)

Inflation- Curvature scale on inflation increases not so fast as $\lambda_{ph} = a/k$. Quantum fluctuations in comoving scale k become frozen when $\lambda_{ph} = a/k \sim H^{-1}$ with amplitude

$$\zeta_\lambda \equiv \sqrt{\zeta_k^2 k^3} \approx \left(\frac{\varepsilon}{1 + \frac{p}{\varepsilon}} \right)^{\frac{1}{2}}_{k \sim Ha}$$

Spectrum:



-Spectral index:

$$n_s - 1 = \frac{d\ln\zeta}{d\ln k} = -3\left(1+\frac{p}{\varepsilon}\right)_{k\sim Ha} + 3\frac{d}{d\ln\varepsilon}\left(1+\frac{p}{\varepsilon}\right)_{k\sim Ha}$$

Generically $(1+\frac{p}{\varepsilon})_{gal} \sim \frac{2}{3}\frac{1}{50}$ and second term is negative $\Rightarrow n_s \leq 0.96$

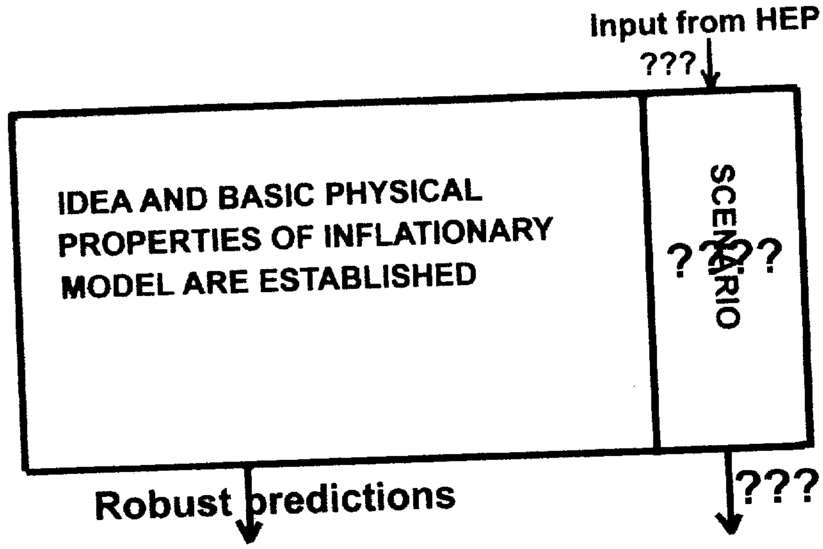
Example: Scalar field with potential $\frac{1}{2}m^2\varphi^2$

$$\Phi \sim \zeta \sim \frac{m}{m_{pl}}\ln\left(\frac{\lambda}{\lambda_\gamma}\right)$$

Required $m \sim 10^{13} Gev$

Gravity waves have an amplitude $h \sim \varepsilon^{1/2}_{k\sim Ha}$, that is they are smaller by factor $(1+\frac{p}{\varepsilon})^{-1/2}_{k\sim Ha}$.

How robust are predictions of inflation?



- *Spatially flat Universe* $\Omega_{total} = 1 \pm 10^{-5}$
- *Nearly scale-invariant spectrum* ($n_S \leq 0.96$). *Perturbations are Gaussian*

- Energy scale of inflation→prediction of the amplitude of perturbations, concrete n_S

- Transition from inflation to Friedmann, reheating mechanism

-The origin of small number 10^{-5}, characterizing perturbations

- Is is possible to get $\Omega < 1$? Yes.

Cost: Two inflationary stages. Second one which takes place "inside the bubble" the kills the prediction $\Omega = 1$ should have *fine tuned* duration. Ω exponentially depends on the duration.

- Is it possible to get non-flat spectrum, nonadiabatic, nongaussian perturbations? Yes.

Cost: -In "one fluid" model with *specially designed* peculiar behavior of the equation of state (potential) one can get peculiarities in the spectrum of (gaussian, adiabatic) perturbations. To put them in the interesting scale one should *fine tune* time when it happened. *Exponential* dependence on parameters!

- Two (or more) "fluids" models: $\varepsilon = \varepsilon_1 + \varepsilon_2$, $p = p_1 + p_2$. *Fine tuned parameters* to have both of the component of the matter to be relevant simultaneously. Possible to have few inflationary stages and then one can get:

1. peculiarities in the spectrum (picks, valley)

2. spectrum with $n_s \geq 1$

3. even nongaussian isocurvature perturbations.