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CONFORMAL INVARIANCE IN QUANTUM MECHANICS

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ABSTRACT

The properties of a field theory in one over-all time dimension, invariant under the full conformal group, are studied in detail. A compact operator, which is not the Hamiltonian, is diagonalized and used to solve the problem of motion, providing a discrete spectrum and normalizable eigenstates. The role of the physical parameters present in the model is discussed, mainly in connection with a semiclassical approximation.

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1. INTRODUCTION

Most quantum field theories, which are at present being used, contain only dimensionless coupling constants so that dilatation invariance is only broken by mass terms. This has led to much attention to limits in which such mass terms also tend to zero, either in terms of massless field theories or as special asymptotic limits of Feynman diagrams.

A special feature of massless field theories is that they exhibit an invariance group which is larger than Poincaré and which also contains the dilatation D and the conformal operator K_μ [1].

The simplest massless dilatation invariant Lagrangian for a scalar field ϕ has the general form

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g\,\phi^{\frac{2d}{d-2}} \tag{1.1}$$

where d is the total number of space-time dimensions.

A general study of the Lagrangian (1.1) for any value of the dimension number d has been carried out by means of a semi-classical approximation [2].

It is, however, important to fix our attention to the simplest, but far from trivial, example $d = 1$ which corresponds to a single physical operator $Q(t)$ depending only upon time. In this case an exact solution is available and many of the general features can be precisely tested.

The one-dimensional Lagrangian is:

$$L = \frac{1}{2}\left(\dot{Q}^2 - \frac{g}{Q^2}\right) \tag{1.2}$$

and represents the prototype of several singular wave equations leading to anomalous dimensions [3].

In this paper we shall investigate the consequences of the invariance of a system described by the Lagrangian (1.2) under the full conformal group. In this case we have to deal with three generators, H, D, K (H is the Hamiltonian) which obey the algebra

$$[H,D] = iH \qquad [K,D] = -iK$$
$$[H,K] = 2iD \tag{1.3}$$

These operators leave the action invariant and are hence constants of the motion; as a consequence, the problem of motion becomes purely group-theoretical and the problem is exactly soluble, both classically and quantum mechanically.

It is to be noted that any combination

$$G = \alpha H + \sigma D + \varkappa K \tag{1.3'}$$

of the three fundamental operators is a constant of the motion, not in the sense that it commutes with H [see Eqs. (1.3)] but in the more general sense

$$\frac{\partial G}{\partial t} + i[H, G] = 0 \,. \tag{1.4}$$

Equation (1.4) expresses the fact that the transformation generated by G leaves the action (but not the Lagrangian) invariant. This means that any of the operators G can be employed to study the time evolution of the state vector. Although the use of any of these operators is, in principle, on the same footing, there are strong differences on the practical side. Indeed only some of our conformal group operators have normalizable eigenvectors and can therefore be safely used.

In order to understand this classification of operators, we refer to the isomorphism of the conformal group with the $O(2,1)$ group of non-compact rotations [4]. If we define

$$\begin{aligned} R &= \frac{1}{2}\left(\frac{1}{a}K + aH\right) \\ S &= \frac{1}{2}\left(\frac{1}{a}K - aH\right), \end{aligned} \tag{1.5}$$

where a is any constant with dimension of length, we have from Eqs. (1.3) the explicit $O(2,1)$ algebra:

$$\begin{aligned} [D, R] &= iS \\ [S, R] &= -iD \\ [S, D] &= -iR \,. \end{aligned} \tag{1.6}$$

We note that R is the generator of a compact rotation, whereas S and D correspond to hyperbolic non-compact transformations. We shall see that the general operator G corresponds to a compact transformation if the determinant

$$\Delta = v^2 - 4uw > 0 . \qquad (1.6')$$

It will be shown that only compact operators of the kind of R can be safely used to solve the problem of motion. Their eigenstates are normalizable and their spectrum is discrete. It will also be seen that only these compact operators lead to time evolution laws which are acceptable in the full $-\infty < t < +\infty$ interval.

At this point a natural question is where the Hamiltonian stands in this classification. From Eqs. (1.5) one sees that H stands at the border between R and S and it can be obtained from either of them in the limit $a \to \infty$. In the framework of our conformal invariant theory, the spectrum of H is continuous and bounded from below and its lowest eigenstate is not normalizable. This circumstance is, of course, a reflection in our elementary framework of the well-known infra-red problem.

If we wish to use eigenstates of H, an elegant way of handling the infra-red problem is first to use eigenstates of R, and then set $a \to \infty$. On the other hand, an exciting possibility is that the operator R will have an important role of its own (see the preceding paper and the developments in the present one). For example, the form of its eigenfunctions will be the same at all times (apart from an over-all dilatation) and, if we start from appropriate initial conditions, the system remains for all times an eigenstate of R; then the constant a will acquire a fundamental meaning.

Leaving those delicate questions aside, we notice the important, particular role of the coupling constant g appearing in the Lagrangian (1.2). Whereas the kind of group is simply determined by the absence of any mass term or dimensional constant, the value of g will determine to what representation of the group the physical states belong. As a consequence we shall see that all important quantities, spectrum, transition matrix elements, will be determined by purely group theoretical formulae in which g appears only through the Casimir operator of the physical representation.

2. CONFORMAL TRANSFORMATION

As discussed in the Introduction, our considerations start from a single Hermitian field $Q(t)$ and a Lagrangian

$$L = \frac{1}{2}\left(\dot{Q}^2 - \frac{g}{Q^2}\right). \tag{2.1}$$

The equation of motion is

$$\ddot{Q} = \frac{g}{Q^3}. \tag{2.2}$$

The coupling constant g is dimensionless (ordinary dimensions are $[L] = [t^{-1}]$, $[Q] = [t^{1/2}]$) and for reasons which will appear clearer in the following we shall consider only the case $g > 0$.

The absence of dimensional constants implies that the action integral,

$$A = \int L(Q, \dot{Q})\, dt \tag{2.3}$$

has larger invariance properties than just the usual time translation invariance. Indeed such a theory is invariant under translations, dilatations and special conformal transformations. All these can be embodied in the single projective transformation defined as

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta} \equiv \omega[t], \tag{2.4}$$

where formally the real numbers α, β, γ, δ can be considered the elements of a real unimodular matrix

$$\omega = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \tag{2.5}$$

The transformation properties of the field $Q(t)$ are simply

$$Q'(t') = (\gamma t + \delta)^{-1} Q(t). \tag{2.6}$$

Correspondingly, we can define a unitary operator $U(\omega)$ [a representation of the group of the real unimodular matrices (2.6)] such that

$$Q(t') = U(\omega)\, Q(t)\, U^{-1}(\omega) . \tag{2.7}$$

Equations (2.6) and (2.7) can be combined in a single relation

$$U^{-1}(\omega)\, Q(t)\, U(\omega) = (\gamma t + \delta)\, Q(t') . \tag{2.8}$$

Finally, we have for the state vector

$$|\Psi(t')\rangle = U(\omega)\, |\Psi(t)\rangle .$$

Combined use of Eqs. (2.1), (2.3), (2.4) and (2.6) allows us to check that A is invariant. Let us stress that, while the action remains invariant, these transformations in general do <u>not</u> leave the Lagrangian invariant since the differential element dt changes. This point will be relevant to the future developments.

To keep in contact with the familiar field theoretical formalism, let us recall the transformations corresponding to particular one-parameter subgroups of ω.

Translations. The subgroup is that of the triangular matrices, $\gamma = 0$, $\alpha = 1$, $\beta = -\omega_-$

$$\omega = \begin{pmatrix} 1 & 0 \\ -\omega_- & 1 \end{pmatrix} = e^{-\omega_- \sigma_-} \tag{2.9}$$

We have

$$Q'(t') = Q(t)$$

and introducing the infinitesimal generator X_- through the definition

$$U(\omega) = e^{i\omega_- X_-} \tag{2.10}$$

one has the known formulae

$$\frac{\delta Q}{\delta \omega_-} \equiv i[X_-, Q] = \dot{Q}$$
$$\frac{\delta |\Psi\rangle}{\delta \omega_-} = i X_- |\Psi\rangle = -|\dot{\Psi}\rangle . \tag{2.11}$$

One can thus identify X_- with the Hamiltonian H:

$$H = X_- .$$

Dilatations. We consider now the subgroup of the diagonal matrices, $\beta = \gamma = 0$, $\alpha = 1/\delta = e^{-\omega_3/2}$:

$$\omega = \begin{pmatrix} e^{-\omega_3/2} & 0 \\ 0 & e^{\omega_3/2} \end{pmatrix} = e^{-\frac{1}{2}\omega_3 \sigma_3} . \tag{2.12}$$

Then

$$Q'(t') = e^{-\omega_3/2} Q(t) . \tag{2.13}$$

If we introduce the infinitesimal generator X_3

$$U(\omega) = e^{i\omega_3 X_3} \tag{2.14}$$

its action on operators and state vectors is

$$\frac{\delta Q}{\delta \omega_3} \equiv i[X_3, Q] = \left(t\frac{d}{dt} - \frac{1}{2}\right) Q ,$$
$$\frac{\delta}{\delta \omega_3} |\Psi\rangle \equiv i X_3 |\Psi\rangle = -t\frac{d}{dt} |\Psi\rangle . \tag{2.15}$$

We finally identify X_3 with the dilatation generator D:

$$X_3 = D \tag{2.16}$$

Conformal transformations. The last example is again a case of a triangular matrix, $\alpha = \delta = 1$, $\beta = 0$, $\gamma = \omega_+$:

$$\omega = \begin{pmatrix} 1 & \omega_+ \\ 0 & 1 \end{pmatrix} = e^{\omega_+ \sigma_+} \quad . \tag{2.17}$$

We have

$$Q'(t') = (1 + \omega_+ t)^{-1} Q(t) \tag{2.18}$$

and putting

$$U(\omega) = e^{i\omega_+ X_+} \tag{2.19}$$

it follows that

$$\frac{\delta Q}{\delta \omega_+} \equiv i[X_+, Q] = \left(t^2 \frac{d}{dt} - t\right) Q \,, \tag{2.20}$$

$$\frac{\delta}{\delta \omega_+} |\Psi\rangle \equiv i X_+ |\Psi\rangle = -t^2 \frac{d}{dt} |\Psi\rangle . \tag{2.21}$$

X_+ can be identified with the conformal generator K:

$$X_+ = K . \tag{2.22}$$

Looking at the simple representations (2.9), (2.12) and (2.17) we see that H, D and K verify the algebra of the matrices $i\sigma_-$, $i/2(\sigma_3)$, $-i\sigma_+$, i.e.,

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD \quad ^{*)} \tag{2.23}$$

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*) It should be recalled that the algebra (2.23) is only valid for conformal

Since H, K and D are Hermitian operators [see next formulae (2.30)] it is easy to check that the group in question is isomorphic to the O(2,1) group of "rotations" in a three-dimensional space with metric 1,-1,-1. If we choose the axes so that the metric is $g_{33} = 1$, $g_{11} = g_{22} = -1$, the generators of "rotations" in the planes 23, 31 and 12 indicated by L_1, L_2 and L_3 are given by

$$\begin{aligned} L_1 &= S = \frac{1}{2}\left(\frac{1}{a}K - aH\right) \\ L_2 &= D \\ L_3 &= R = \frac{1}{2}\left(\frac{1}{a}K + aH\right) . \end{aligned} \tag{2.24}$$

The constant a, with dimension of time, is needed because physically H and K have different dimensions. The constant a must be given, once and for all, and plays a fundamental role. Of course, any value can be chosen, but once it is chosen it will determine a number of things as we shall see in the rest of this investigation. The algebra is

$$[D,R] = iS, \quad [S,R] = -iD, \quad [S,D] = -iR . \tag{2.25}$$

The subgroup generated by R is hence compact (rotation group in a plane), while those generated by S and D are not, being of the "boost" type (see the Lorentz group).

Let us now consider the general form of the generator

$$G = uH + vD + wK . \tag{2.26}$$

From the commutators (2.23) it is easy to see that the determinant

$$\Delta = \begin{vmatrix} v & 2u \\ 2w & v \end{vmatrix} \tag{2.27}$$

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*) Footnote contd.
invariant theories. In the general case for instance, the first relation is substituted by

$$[H,D] = iH - i\frac{dD}{dt}$$

and analogous relations hold for the remaining ones.

is invariant with respect to the most general transformation

$$G \longrightarrow U^{-1} G U . \tag{2.28}$$

From this it follows that the operator G generates a true rotation for $\Delta < 0$ (generator R), whereas for $\Delta > 0$ (generator S) it gives rise to a hyperbolic transformation (analogous to a boost).

A final class is given by those generators whose discriminant vanishes, as it is the case for H and K. These "border" or "parabolic" operators generate "rotations" around the "light-like" axes.

Let us finally give the explicit expressions of the generators H, K and D in terms of the field operators $Q(t)$, $\dot{Q}(t)$. These formulae can be obtained expressing the variation δA of the action as the time derivative of the relevant generator, i.e.,

$$\delta A + \int dt \, \frac{df(Q)}{dt} = \int dt \, \frac{dG}{dt} = 0 . \tag{2.29}$$

It is then straightforward to find for the conserved generators the expressions

$$\begin{aligned} H &= \frac{1}{2}\left(\dot{Q}^2 + \frac{g}{Q^2}\right), \\ D &= tH - \frac{1}{4}\left(Q\dot{Q} + \dot{Q}Q\right), \\ K &= t^2 H - \frac{1}{2} t \left(Q\dot{Q} + \dot{Q}Q\right) + \frac{1}{2} Q^2, \end{aligned} \tag{2.30}$$

(we have adopted quantum symmetrization).

We can carry on the canonical quantization introducing the conjugate momentum P,

$$P \equiv \frac{\partial L}{\partial \dot{Q}} = \dot{Q} \tag{2.31}$$

and establishing the equal time commutator

$$\left[Q(t), P(t)\right] = i . \tag{2.32}$$

Using the representations (2.30) and the commutation relations (2.32) one can check again the correctness of the algebra (2.23). It is also convenient to give explicitly the expression of H, D and K at $t=0$: with $Q_0 = Q(0)$, $P_0 = P(0)$, one has

$$H = \frac{1}{2}\left(P_0^2 + \frac{g}{Q_0^2}\right),$$
$$D = -\frac{1}{4}\left(P_0 Q_0 + Q_0 P_0\right), \tag{2.33}$$
$$K = \frac{1}{2} Q_0^2 .$$

We next notice the existence among the generators of the following relation:

$$\frac{1}{2}(HK+KH) - D^2 = \frac{g}{4} - \frac{3}{16} . \tag{2.34}$$

Such a constraint eliminates the apparent problem of having too many constants of the motion (three) for a system with a two-dimensional phase space. The importance of the relation (2.34) and its group theoretical meaning will be discussed later.

Finally, Eqs. (2.30) can be used to get the formal solution of the equation of motion (2.2). From the above explicit expression for H, D, K, one easily obtains

$$\begin{aligned} Q^2(t) &= 2t^2 H - 4tD + 2K \\ &= (Q_0 + P_0 t)^2 + g t^2/Q_0^2 , \end{aligned} \tag{2.35}$$

which is the solution of motion since H, D and K are constants of the motion [connected by Eq. (2.34)].

It may be worth mentioning that by treating Eq. (2.35) classically, the discriminant of the equation $Q^2(t) = 0$ is

$$D^2 - HK = -g/4 \tag{2.36}$$

and the positivity requirement $g>0$ prevents the classical particle from going through the origin, as is obvious for the repulsive potential g/x^2.

3. EQUATIONS OF MOTION

We are dealing with a group of invariance of a system given by a general projective transformation over the time. It is almost evident that this fact opens the way to a very general description of the evolution of the system in time. It is the object of the present chapter to establish a general formalism displaying this very interesting unusual situation.

Let us consider the generator G introduced previously,

$$G = u H + v D + w K . \tag{3.1}$$

Its action on the field operator and on the state vector is given by

$$i\,[G, Q(t)] = f_G(t)\,\frac{dQ}{dt} - \frac{1}{2}\,\frac{df_G(t)}{dt}\,Q(t), \tag{3.2}$$

$$G\,|\Psi(t)\rangle = i\,f_G(t)\,\frac{d}{dt}\,|\Psi(t)\rangle, \tag{3.3}$$

where

$$f_G(t) = u + v t + w t^2 .$$

Equations (3.2) and (3.3) are greatly simplified if we introduce a new time variable by

$$d\tau_G = \frac{dt}{u + v t + w t^2} \tag{3.4}$$

and a new (rationalized) field $q(\tau)$, defined by

$$q(\tau) = \frac{Q(t)}{[u + v t + w t^2]^{1/2}} . \tag{3.5}$$

With these definitions, Eqs. (3.2) and (3.3) now read

$$i\,[G, q(\tau)] = \frac{dq}{d\tau} , \tag{3.6}$$

$$G\,|\Psi(\tau)\rangle = i\,\frac{d}{d\tau}\,|\Psi(\tau)\rangle \tag{3.7}$$

These equations can be formally integrated, yielding *)

$$q(\tau) = e^{iG(\tau-\tau_0)}\,q(\tau_0)\,e^{-iG(\tau-\tau_0)}, \tag{3.8}$$

$$|\Psi(\tau)\rangle = e^{-iG(\tau-\tau_0)}\,|\Psi(\tau_0)\rangle\,. \tag{3.9}$$

In particular, the state can be chosen to be an eigenstate of G at time $\tau = \tau_0$

$$G\,|\Psi(\tau_0)\rangle = G'\,|\Psi(\tau_0)\rangle \tag{3.10}$$

so that

$$|\Psi(\tau)\rangle = e^{-iG'(\tau-\tau_0)}\,|\Psi(\tau_0)\rangle \tag{3.11}$$

is the analogue of the stationary state, now "stationary" with respect to the new variable.

What we have developed up to this point has been rather formal and, in particular, we do not know whether Eqs. (3.8) and (3.9) are able to describe the evolution in time from $-\infty$ to $+\infty$ as one could wish. We must perform a closer examination of the variable τ just introduced. We have from (3.4)

$$\tau = \int_{t_0}^{t} \frac{dt'}{u + vt' + wt'^2} + \tau_0\,. \tag{3.12}$$

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*) Equations (3.6) and (3.7) can be regarded as the analogue of the equations of motion in the Heisenberg and in the Schrödinger picture. We shall see in the Appendix that, as a consequence, the time dependent Schrödinger equation separates in τ and $q(\tau)$ (apart from a phase factor).

The form of the integral depends on the zeros of the denominator which are controlled by the discriminant $\Delta = v^2 - 4uw$. The three separate cases $\Delta \gtrless 0$ correspond to the classification of generators of the previous chapter.

a) Non-compact operators with $\Delta > 0$. Then the roots $t_{1,2}$ of $f_G = 0$ are real and one has [normalizing, e.g., to $\tau(0) = 0$]

$$\tau = \frac{1}{\sqrt{\Delta}} \ln \frac{t - t_1}{t - t_2} \frac{t_2}{t_1} \quad . \tag{3.13}$$

In this case τ cannot be defined over the whole time interval $-\infty \leq t \leq +\infty$; such physically unacceptable characteristics are connected to the fact that the spectrum of these operators, as it will be seen, has unpleasant characteristics. This is the case for S and D (in the case of D one of the roots goes to ∞).

b) Compact operators with $\Delta < 0$. Then the roots of $f_G(t) = 0$ are complex conjugate and one has [again normalizing $\tau(0) = 0$].

$$\tau = \frac{4w}{\sqrt{|\Delta|}} \left\{ \operatorname{arctg} \frac{2wt + v}{|\Delta|} - \operatorname{arctg} \frac{v}{|\Delta|} \right\} . \tag{3.14}$$

One sees that the physical interval $-\infty \leq t \leq \infty$ can be swept without any singularity in the variable τ. Such pleasant characteristics are connected to the fact that operators of this class generate a compact rotation and to the fact, as we shall see, that the spectrum of these operators has rather appealing features. R is an operator of this class.

c) Parabolic operators. In this case the roots are coincident and we have

$$\tau = \frac{1}{w} \left(\frac{1}{t - t_0} + \frac{1}{t_0} \right) . \tag{3.15}$$

While t sweeps the physical region we have $-\infty \leq \tau \leq +\infty$, but there is one singular point in τ at finite t in general. K and H belong to this class. The spectrum will also reflect this situation.

We give hereafter a table of the variable τ relevant to the most important operators.

Operator	H	D	K	R	S
τ	t	$\ell n\ t$	$1/t$	$2\ \mathrm{arctg}(t/a)$	$\ell n(1-t/a)/(1+t/a)$

The next step in the reformulation of the theory is to build up a complete Lagrangian scheme in the new basis, expressing then the dynamical quantities in terms of the new Lagrangian. A simple calculation, based on Eq. (3.5), shows that the action takes the form

$$2A = \int d\tau \left\{ \dot{q}^2 - \frac{g}{q^2} + \left(\frac{v}{2} + wt\right)^2 q^2 + (v + 2wt)\, q\dot{q} \right\} =$$

$$= \int d\tau \left\{ \dot{q}^2 - \frac{g}{q^2} + \frac{\Delta}{4} q^2 + \frac{d}{d\tau}\left[\left(\frac{v}{2} + wt\right) q^2\right] \right\},$$

where, of course,

$$\dot{q} \equiv \frac{dq}{d\tau} .$$

Since a total τ derivative is irrelevant in the definition, the action can finally be rewritten as

$$A = \int d\tau_G\ L_G[q(\tau), \dot{q}(\tau)] \qquad (3.16)$$

with

$$L_G = \frac{1}{2}\left(\dot{q}^2 + \frac{\Delta}{4} q^2 - \frac{g}{q^2}\right). \qquad (3.17)$$

If we take L_G as the new Lagrangian and calculate the Hamiltonian, we have

$$H_G(q(\tau), \dot{q}(\tau)) \equiv \dot{q}\frac{\partial L_G}{\partial \dot{q}} - L_G = \\ = \frac{1}{2}\left(\dot{q}^2 + \frac{g}{q^2} - \frac{\Delta}{4}q^2\right). \tag{3.18}$$

We can check, using the previous results, that the foreseeable result

$$G(\varphi, \dot{\varphi}) = H_G(q, \dot{q}) \tag{3.19}$$

holds.

Let us also remark that the equal time commutation relations for Q, $\dot{Q}$ imply

$$[q(\tau), \dot{q}(\tau)] = i. \tag{3.20}$$

We now proceed to solve the equations of motion. Since the field depends only on time, it is possible to develop a wave formalism introducing a Schrodinger picture, by considering formally our original problem as that of a particle in a one-dimensional space x with repulsive potential g/x^2. While we are not too interested in the interpretation of $Q(t)$ as a coordinate of a particle, we are very much interested in the wave formalism which makes things much simpler. In such a framework, which will be developed in Appendix A, the possibility of writing the equation of motion in the form (3.6), (3.7) will correspond to the separation of the time-dependent Schrodinger equation in the variables τ, q (apart from a phase factor), as a consequence of invariance of the system under conformal transformations.

As further steps we therefore determine the time evolution of the wave function and the form of the eigenfunctions of G at a certain time. We therefore resort to the analogue of the Schrödinger picture and represent $q(0)$ and $\dot{q}(0)$ by

$$q(0) \to y \ , \quad \dot{q}(0) \longrightarrow -i \frac{d}{dy} \ . \tag{3.21}$$

Introducing the wave function

$$\psi(y,\tau) = \langle y | \psi(\tau) \rangle \tag{3.22}$$

and recalling Eq. (3.11), one gets immediately

$$i \frac{\partial \psi}{\partial \tau} = H_G \left(y, -i \frac{d}{dy} \right) \psi(y,\tau) \tag{3.23}$$

The relation between the wave function (3.22) and the usual solution of the temporal Schrödinger equation

$$i \frac{\partial}{\partial t} \psi = H \psi$$

is discussed in detail in Appendix A.

Finally, choosing the wave function to be an eigenfunction of G (i.e., H_G) one writes:

$$\psi_{G'}(\tau, y) = e^{-iG'\tau} \psi_{G'}^{(0)}(y) \ . \tag{3.24}$$

The eigenvalue equation

$$G \, \psi_{G'}^{(0)}(y) = G' \psi_{G'}^{(0)}(y) \tag{3.25}$$

[using the canonical form (3.18)] becomes

$$\left(-\frac{d^2}{dy^2} + \frac{g}{y^2} - \frac{\Delta}{4} y^2 \right) \psi_{G'}^{(0)}(y) = G' \psi_{G'}^{(0)}(y) \ . \tag{3.26}$$

We stress that the above equation, which formally looks like a Schrödinger time-independent equation with a "potential" $W(y) = (g/y^2) - \Delta/4(y^2)$, is

actually the eigenvalue equation for the operator $G(0)$ characterized by a certain Δ; y and $-i(d/dy)$ are the Schrödinger picture realizations of the relevant $q(t)$ and $\dot{q}(t)$ commutator algebra at $t=0$.

A glance at Eq. (3.23) shows that the spectrum of eigenvalues and the normalizability of eigenfunctions are determined by:

a) the behaviour at $y \to 0$ which depends on g;

b) the behaviour at $y \to \infty$ which depends on Δ.

The problem at the origin is, of course, not new and the sign of g plays a crucial role. Negative values of g correspond to infinite attraction into the origin and lead to solutions which are unacceptable on physical grounds.

The limiting case $g=0$ describes the usual no-interacting case and the wave function can be defined in the range $-\infty < x < \infty$ without any singularity at the origin.

Finally, for $g>0$ we have an infinite repulsive well which keeps the particle confined in the interval $0<x<\infty$, leading to wave functions vanishing outside of this interval. This indicates, as shown in more detail in the next Section, the radical difference between the $g=0$ and $g>0$ cases.

Let us now come to the new exciting feature of the large distance behaviour which depends on the sign of Δ, i.e., on the form of the operator G which is being diagonalized. If $\Delta=0$ the potential, monotonically decreasing, approaches a constant (zero) for large y, thus producing a continuous spectrum of eigenvalues limited below *). If $\Delta>0$ the "potential", also monotonically decreasing, is unbounded from below.

On the contrary, if $\Delta<0$ (and $g>0$), the "potential" $W(y)$ has a minimum at $y=y_0 \equiv \sqrt{2}(g/|\Delta|)^{1/4}$ and tends to $+\infty$ both for $y \to 0$ and for $y \to \infty$. Intuitively the eigenfunctions will be finite in norm and the eigenvalues discrete. It is not hard to obtain both these results explicitly by reducing Eq. (3.23) with $\Delta<0$ to a hydrogen atom type wave equation. The three curves of Fig. 1 show the typical behaviour of the function $W(y)$ for the cases $\Delta = \pm 1, 0$.

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*) This is, of course, the case of the familiar stationary Schrödinger equation with energy eigenfunctions

$$\psi_E(x) = \text{const.}\, x^{1/2} J_{2\pi_0 - 1}(x\sqrt{2E})$$

normalized in the continuum.

All this is not new from the mathematical point of view, and the above properties of the spectrum of the various operators are a consequence of working with the non-compact $O(2,1)$ group. What is new, however, is the message following from this discussion. In the framework of conformally invariant theories the spectrum and normalization properties naturally suggest the consideration of the physical states as eigenstates of R rather than of H (or D); in doing so, simple localization properties arise which are related to the presence of the fundamental constant a.

To conclude this Section, let us observe that Eq. (3.23), which is the main results of this Section, is indeed very well known in the framework of non-relativistic quantum mechanics where it is usually considered as the eigenvalue equation for a generalized harmonic oscillator problem. We stress that the physical meaning of Eq. (3.23) is very different in our framework. It follows from the diagonalization of an operator (which is <u>not</u> the Hamiltonian) in the framework of a conformal invariant Lagrangian which contains no length parameter.

Usually the confinement is due to a force whose origin is in a Lagrangian with dimensional constants. Here, on the contrary, it is generated by the choice of the operator to be taken diagonal, which contains a length and determines a different time evolution. Loosely speaking, the confinement usually depends on the Lagrangian, here it depends on the initial conditions (choice of eigenstates of R). Of course, the time evolution is quite different from a stationary one.

Although the mathematics associated with the solution of Eq. (3.23) is hardly new, we reproduce it in the next Section for sake of completeness.

4. THE SPECTRUM OF R

We shall be concerned here with the exercise of determining the eigenvalues and eigenstates of the operator R [5]. It will be useful to remember the explicit expression of the various operators using their expressions at $t=0$:

$$R = \frac{1}{4}\left(P_0^2 + Q_0^2 + \frac{a}{Q_0^2}\right),$$
$$S = \frac{1}{4}\left(-P_0^2 + Q_0^2 - \frac{a}{Q_0^2}\right),$$
$$D = -\frac{1}{4}\left(Q P_0 + P_0 Q_0\right). \qquad (4.1)$$

We have put $a = 1$ here and in the following, in order to simply the formulae.

We define the rising and lowering operators

$$L_{\pm} = S \pm i D \tag{4.2}$$

for which

$$[R, L_{\pm}] = \pm L_{\pm} ,$$
$$[L_{+}, L_{-}] = -2R . \tag{4.3}$$

We further introduce the Casimir operator which turns out to have the particularly simple c number expression of Eq. (2.38):

$$J^2 = R^2 - S^2 - D^2 = R^2 + R - L_- L_+ =$$
$$= \frac{1}{2}(HK + KH) - D^2 \tag{4.4}$$
$$= \frac{q}{4} - \frac{3}{16} .$$

Putting $J^2 = r_0(r_0 - 1)$ one finds

$$r_0 = \frac{1}{2}\left(1 \pm \sqrt{q + 1/4}\right) \tag{4.5}$$

and a further discussion will lead us to select the positive sign for the square root. It then follows from the algebra (4.3) that

$$L_{\pm} |r_0, r\rangle = C_{\pm}(r_0, r) \, |r_0, r \pm 1\rangle \tag{4.6}$$

implying that successive eigenvalues differ by unity. Using Eq. (4.4) one has further that

$$|C_{\pm}(r_0, r)|^2 = r(r \pm 1) - J^2 =$$
$$= r(r \pm 1) - r_0(r_0 - 1) \geq 0 \tag{4.7}$$

namely

$$r \geq r_0 \quad , \quad r \leq -r_0$$

We thus obtain the series of positive eigenvalues:

$$r_n = r_0 + n, \quad n = 0, 1, 2, \cdots \tag{4.8}$$

which reproduces one of the discrete dimensional representations of $O(2,1)$ bounded below by the value r_0.

Let us now discuss how discrimination between the two possible values for r_0 can be obtained by looking at the small distance behaviour of the wave function. Its explicit form will be obtained soon, but for the present considerations a glance at the differential equation is sufficient and we find that the lowest eigenfunction behaves as

$$\psi_0(y) \sim y^{2r_0 - 1/2} \quad , \quad y \to 0 . \tag{4.9}$$

The limitations on r_0 depend, of course, on the requirements we want to impose on the wave function. If we take into account the quantum mechanical nature of the problem, the presence of the infinitely repulsive impenetrable potential barrier at the origin and the ensuing confinement property of the wave function of being non-vanishing only on the positive x axis, imply the vanishing of both $\psi_0(y)$ and its first derivative as $y \to 0$. This requires $r_0 > 3/4$ which is only verified by the positive root choice, leading to the eigenvalue series (4.9).

For the zero coupling value $g = 0$, however, both solutions are acceptable and have to be retained as a consequence of the changed physical situation. Correspondingly, the two series of eigenvalues combine at $g = 0$ to form a single spectrum with spacing $\delta = \frac{1}{2}$ *).

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*) The case $g = 0$ can also be worked out directly, and it is interesting to mention that R can be put in the familiar form

$$R = \tfrac{1}{2}\left(a^+ a + \tfrac{1}{2}\right) , \qquad [a, a^+] = 1 .$$

The eigenvalues of the number operator $a^+ a$ are $n = 0, 1, 2, \ldots$, the over-all spectrum is simply $r_n = \frac{1}{2}(n + \frac{1}{2})$, and $L_\pm$ are squares:

$$L_+ = \tfrac{1}{2} a^{+2} , \quad L_- = \tfrac{1}{2} a^2 .$$

We further notice that for the lowest eigenvalue one has

$$L_- |r_0, r_0\rangle = 0 \tag{4.10}$$

and the higher eigenstates $|r_0, r_0 + n\rangle$ are built by successive applications of the operator L_+. Using Eqs. (4.7) and (4.8) we can now write, with the simplest choice of the phase factor,

$$L_\pm |r_0, r_m\rangle = C_\pm |r_0, r_m\rangle =$$
$$= \left[r_m(r_m \pm 1) - r_0(r_0 - 1)\right]^{1/2} |r_0, r_m \pm 1\rangle . \tag{4.11}$$

From this relation we can evaluate the matrix elements of the generators H, D and K. It is convenient to express them as

$$H = R - S = R - \frac{1}{2}(L_+ + L_-),$$
$$D = \frac{1}{2i}(L_+ - L_-), \tag{4.12}$$
$$K = R + S = R + \frac{1}{2}(L_+ + L_-).$$

One then finds, for instance,

$$H |r_m\rangle = r_m |r_m\rangle - \frac{1}{2}\left[r_m(r_m+1) - r_0(r_0-1)\right]^{1/2} |r_m+1\rangle$$
$$- \frac{1}{2}\left[r_m(r_m-1) - r_0(r_0-1)\right]^{1/2} |r_m-1\rangle \tag{4.13}$$

and analogous expressions for the other generators.

We finally turn to the explicit determination of the eigenfunctions $\psi_n(x)$ solution of Eq. (3.23) for R. Working in the {x} representation and using the form (4.1) of the various operators, we find that

$$L_\pm = S \pm iD = K - R \pm iD = K - r \pm iD \rightarrow \tag{4.14}$$
$$\rightarrow \frac{1}{2}x^2 - r \mp \frac{1}{2}\left(x\frac{d}{dx} + \frac{1}{2}\right)$$

(since $L_\pm$ acts on eigenstates of R, there is the obvious replacement $R \to r$). The lowest eigenfunction is determined by the requirement

$$L_- \psi_0(x) = 0 \tag{4.15}$$

i.e.,

$$\left\{ x\frac{d}{dx} + x^2 - \left(2r_0 - \frac{1}{2}\right) \right\} \psi_0(x) = 0 . \tag{4.16}$$

The solution is

$$\psi_0(x) = C_0\, e^{-x^2/2}\, x^{2r_0 - \frac{1}{2}} \quad , x \geq 0, \tag{4.17}$$

and one immediately determines C_0 by the normalization condition $\int_0^\infty |\psi_0(x)|^2\, dx = 1$:

$$C_0 = \left[2\,\Gamma(2r_0) \right]^{-1/2} . \tag{4.18}$$

Repeated application of the "creation" operator can now be used to generate the successive wave functions. The result is, taking into account the normalization and keeping track of the constant a,

$$\psi_n(x) = \left[\frac{\Gamma(n+1)}{2\,\Gamma(n+2r_0)} \right]^{1/2} x^{-1/2} \left(\frac{x^2}{a}\right)^{r_0} e^{-x^2/2a}\, L_n^{2r_0-1}\left(\frac{x^2}{a}\right), \tag{4.19}$$

where $L_n^{2r_0-1}$ is the associated Laguerre polynomial [6] (n is integer) *).

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*) The derivation is based on the simple identity

$$L_+^{(n)} \psi_n(x) = e^{x^2/2}\, x^{-2r_n + \frac{3}{2}} \frac{d}{dx^2}\, e^{-x^2/2}\, x^{2r_n + \frac{1}{2}}\, \psi_n(x) ,$$

where ψ_n is the n^{th} eigenfunction and

$$L_+^{(n)} = \frac{1}{2} x \frac{d}{dx} + r_n + \frac{1}{4} - \frac{x^2}{2} \quad , \quad r_n = r_0 + n .$$

These formulae give a first illustration about the way in which the two free parameters of the theory, a and g, act to give a respectable wave function. The role of a is to provide the large distance cut-off, while the coupling constant g which determines the small distance behaviour always appears through r_0, i.e., through the Casimir operator.

This concludes our discussion on the eigenvalues and eigenfunctions of the operator R. Once more we can remark how these nice properties, discrete spectrum and normalizability, point out the relevance of R to a description of the physical world. In particular, physical states should be considered as eigenstates of R, which immediately leads to the obvious question of the connection of such a description with the familiar language based on energy eigenfunctions. We shall examine this point in the next Section. It is important to notice that these properties are, of course, common to all compact operators G. Indeed, using the invariance properties of the quantity $\Delta = v^2 - 4uw$ [see Eqs. (3.32) and (2.33)], it is easy to see that the eigenvalues of G are given by:

$$G' = r_n |\Delta| . \tag{4.20}$$

To conclude we devote a few lines to an approximate treatment of the eigenvalue equation for R which is familiar to physicists. The eigenvalue equation is obtained from (3.23) with $u = a/2$, $v = 0$, $w = 1/2a$, $\Delta = -1$, $q(0) = Q(0)\sqrt{2/a}$ so that, writing $y = \sqrt{2/a}$, we have

$$\left(-\frac{d^2}{dx^2} + \frac{g}{x^2} + \frac{x^2}{a^2} \right) \psi = 4 r \psi . \tag{4.21}$$

Footnote contd.

Defining then (apart from an over-all constant)

$$\psi_n(x) = L_+^{(n-1)} \psi_{n-1}(x) = L_+^{(n-1)} \dots L_+^{(0)} \psi_0(x)$$

one is finally led to the result (4.22) via the Rodrigues representation for Laguerre polynomials

$$L_n^{2r_0-1}(x^2) = e^{x^2} (x^2)^{-2r_0+1} \left(\frac{d}{dx^2}\right)^n e^{-x^2} (x^2)^{2r_0-1+n} .$$

Since the "potential" $W(x) = (g/x^2) + (x^2/a^2)$ has a unique minimum at

$$x = x_0 \equiv (a^2 g)^{1/4}$$

we can expand

$$W(x) \simeq W(x_0) + \frac{1}{2}(x-x_0)^2 W''(0) = \frac{2g^{1/2}}{a} + \frac{4(x-x_0)^2}{a^2} . \qquad (4.22)$$

Introducing the variable $\xi = \sqrt{2/a}\,(x - x_0)$, the equation becomes then

$$\psi''(\xi) - \xi^2 \psi(\xi) + (2\nu - g^{1/2})\psi(\xi) = 0 \qquad (4.23)$$

which is of harmonic oscillator type, with a spectrum

$$\nu_n = n + \frac{1}{2} + \frac{1}{2} g^{1/2} . \qquad (4.24)$$

Using the "semi-classical" approximation (4.21), we clearly poorly represent the small distance behaviour of the exact theory. This means that the approximate expression (4.21) is reliable as long as $x_0 \equiv (ga^2)^{1/4}$ is far enough from the origin, which corresponds to a strong coupling limit $g \gg 1$. Indeed the eigenvalue law (4.23) follows directly from the exact expression (4.8) as $g \to \infty$. A more general semi-classical treatment in the framework of the Heisenberg representation will be presented in Section 8.

5. THE ENERGY REPRESENTATION

(i) In order to establish the energy content of the eigenfunctions of R, the direct way is to expand the relevant state in stationary eigenstates of H, namely

$$|\nu_n\rangle = \int dE\, c_n(E)\, |E\rangle \qquad (5.1)$$

and

$$c_n(E) = \langle E|\nu_n\rangle = \int dx\, \psi_E^*(x)\, \psi_{\nu_n}(x) . \qquad (5.2)$$

Explicit wave function calculations of $C_n(E)$ will be performed in Appendix A using the explicit form of the wave functions given in the previous Section. It is fruitful, however, to establish a general group theoretical framework where this, and similar problems, can be discussed avoiding the Schrödinger picture.

We first look for another simple realization of the $O(2,1)$ algebra and of the abstract basis of eigenvectors of the Casimir operator and of R, in the space of one-parameter functions. In the previous Section this has been done in the x space, which is quite natural when studying the stationary problem at $t=0$ from the point of view of quantum mechanics. On the other hand, since the actual parameter of our field theoretical model is the time t, it is preferable to use this variable in developing the formalism.

We thus look for a particular realization of the abstract operators $X_\alpha \to x_t^{(\alpha)}$ acting on time-dependent functions $\beta_n(t)$. We first demand the validity of the familiar realization

$$H \equiv X_- \longrightarrow x_t^{(-)} \equiv i\,\frac{d}{dt} \quad . \tag{5.3}$$

As far as the other generators are concerned, it is not hard to check that the requirements of verifying the $O(2,1)$ algebra and of reproducing the form of the Casimir operator $\frac{1}{2}(HK+KH) - D^2 = r_o(r_o-1)$ lead to the following simple representations

$$D \equiv X_3 \longrightarrow x_t^{(3)} = i\left(t\frac{d}{dt} + r_o\right)$$

$$K \equiv X_+ \longrightarrow x_t^{(+)} = i\left(t^2\frac{d}{dt} + 2r_o t\right) . \tag{5.4}$$

Let us notice the presence of the r_o term in this realization of the algebra. One might, in particular, think of $-r_o$ as of a scale dimension, interestingly bearing a dependence on the coupling constant g.

The basis functions $\beta_n(t)$ are then the eigenfunctions of the operator R, i.e.,

$$R\,\beta_n(t) \equiv \frac{i}{2}\left\{(1+t^2)\frac{d}{dt} + 2r_o t\right\}\beta_n(t) = r_n\,\beta_n(t) \tag{5.5}$$

and their form turns out to be

$$\beta_n(t) = (-1)^n \left\{ \frac{\Gamma(n+2r_0)}{\Gamma(n+1)} \right\}^{1/2} \left(\frac{1-it}{1+it} \right)^{r_n} \frac{1}{(1+t^2)^{r_0}} . \tag{5.6}$$

The over-all constant has been fixed, at least as far as the n dependence is concerned, by requiring the validity of the recurrence relations (4.11) and (4.13).

Thus the action of the abstract generators X_α, $\alpha = +,-,3$ on the states $|n>$ and of the generators $x_t^{(\alpha)}$ on the states $\beta_n(t)$ can be summarized in the compact form

$$\begin{aligned} X_\alpha |n> &= \gamma^\alpha_{nn'} |n'> , \\ x_t^{(\alpha)} \beta_n(t) &= \gamma^\alpha_{nn'} \beta_{n'}(t) \end{aligned} \tag{5.7}$$

The coefficients $\gamma^\alpha_{nn'}$ are given by

$$\begin{aligned} \gamma^-_{nn'} &= r_n \delta_{nn'} - \tfrac{1}{2} C_+(r_0, r_n) \delta_{n',n+1} - \tfrac{1}{2} C_-(r_0, r_n) \delta_{n',n-1} \\ \gamma^+_{nn'} &= r_n \delta_{nn'} + \tfrac{1}{2} C_+(r_0, r_n) \delta_{n',n+1} + \tfrac{1}{2} C_-(r_0, r_n) \delta_{n',n-1} \\ \gamma^3_{nn'} &= \frac{1}{2i} C_+(r_0, r_n) \delta_{n',n+1} - \frac{1}{2i} C_-(r_0, r_n) \delta_{n',n-1} \end{aligned} \tag{5.8}$$

where

$$C_\pm(r_0, r_n) = \left[r_n (r_n \pm 1) - r_0 (r_0 - 1) \right]^{1/2} . \tag{5.9}$$

It is useful to remark that the matrix γ^α is Hermitian, so that one has from (5.7)

$$x_t^{(\alpha)*} \beta_n^*(t) = \beta_{n'}^*(t) \gamma^\alpha_{n'n} . \tag{5.10}$$

The fundamental recurrence relations (5.7), (5.8) can be re-expressed in a more convenient form by introducing the following combination of the

eigenstates $|n>$:

$$|t> = \sum_n \beta_n^*(t)\, |n> \tag{5.11}$$

so that

$$\beta_n(t) = <t|n> . \tag{5.12}$$

For it one can easily establish the property

$$X_\alpha |t> = x_t^{(\alpha)\,*} |t> . \tag{5.13}$$

The continuous set of kets $|t>$, besides from being the basis for the realization (5.4) of the algebra, can very simply be connected to the eigenstates of all the operators relevant to the theory, and therefore constitute a powerful task for computation. The connection with eigenstates of R, Eq. (5.11), is used for its definition; in what follows we shall discuss the fundamental connection with eigenstates of the Hamiltonian.

Before doing this, it is important to notice that the functions $\beta_n(t)$ satisfy the following addition formula easily obtained by direct computation

$$\sum_n \beta_n(t_1)\beta_n^*(t_2) = \Gamma(2\lambda_0)\left[2i(t_1-t_2)\right]^{-2\lambda_0} . \tag{5.14}$$

More generally, one can view (5.14) as the scalar product

$$<t_1|t_2> = \Gamma(2\lambda_0)\ \left[2i(t_1-t_2)\right]^{-2\lambda_0} . \tag{5.14'}$$

This result has actually a deeper origin and could be derived as a direct consequence of the conformal transformation properties (5.13). It is indeed easy to check that $<t_1|t_2>$ obeys the set of the differential equations

$$\left\{ x_{t_1}^{(\alpha)} + x_{t_2}^{(\alpha)} \right\} <t_1|t_2> = 0 .$$

Similar relations will be considered in full detail for more complicated cases in Section 6.

(ii) Our next aim is to build energy eigenstates in terms of the basis kets $|n>$, what we expect to be accomplished by Fourier transforming the quantity $|t>$. Actually, it is more convenient to introduce the quantity

$$C_n(E) = 2^{\nu_0} E^{\frac{1}{2}-\nu_0} \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{iEt} \beta_n(t) . \tag{5.15}$$

In order to motivate the extra energy factor let us evaluate the completeness sum $\Sigma_n C_u(E) C_u^*(E')$. A simple calculation based on Eqs. (5.14) and (5.15) shows that the normalization condition

$$\sum_n C_n(E) C_n^*(E') = \delta(E-E') \tag{5.16}$$

is fulfilled.

It is now easy to establish the analogue of Eq. (5.7), namely

$$\chi_E^{(\alpha)} C_n(E) = \gamma^{\alpha}_{nn'} C_{n'}(E) . \tag{5.17}$$

In this basis the realizations of the generators take on the form

$$H \longrightarrow \chi_E^{(-)} \equiv E , \quad D \longrightarrow \chi_E^{(3)} \equiv -i\left(E\frac{d}{dE} + \frac{1}{2}\right) ,$$

$$K \longrightarrow \chi_E^{(+)} \equiv -E\frac{d^2}{dE^2} - \frac{d}{dE} + \left(\nu_0 - \frac{1}{2}\right)^2 \frac{1}{E} . \tag{5.18}$$

To conclude these preliminaries we have to evaluate the Fourier transform (5.15). This can be done directly, but it is perhaps more instructive to determine $C_n(E)$ from the eigenvalue equation

$$\mathcal{R}\, C_n(E) \equiv \frac{1}{2}\left\{E + \left(\nu_0 - \frac{1}{2}\right)^2 \frac{1}{E} - \frac{d}{dE} - E\frac{d^2}{dE^2}\right\} C_n(E) =$$

$$= \nu_n C_n(E) . \tag{5.19}$$

Putting

$$C_n(E) = 2^{r_0}\, E^{r_0-\frac{1}{2}}\, e^{-E}\, \varphi_n(E) \tag{5.20}$$

one finds that $\varphi_n(E)$ is a solution of the differential equation for Laguerre polynomials (n integer),

$$\eta\,\varphi_n'' + (2r_0-\eta)\varphi_n' + n\,\varphi_n = 0 \quad , \quad \eta = 2E, \tag{5.21}$$

so that

$$C_n(E) = 2^{r_0}\left[\frac{\Gamma(n+1)}{\Gamma(n+2r_0)}\right]^{1/2}\frac{(aE)^{r_0}}{E^{1/2}}\, e^{-aE}\, L_n^{2r_0-1}(2aE), \qquad E>0 \tag{5.22}$$

where the constant has been adjusted by the orthonormality condition

$$\int_0^\infty C_n(E)\, C_{n'}^*(E)\, dE = \delta_{nn'} \tag{5.23}$$

and we have explicitly inserted the dependence on a in order to have a clearer understanding of its role.

The final step is the definition of the energy eigenstates: the natural definition is

$$|E\rangle = 2^{r_0} E^{\frac{1}{2}-r_0} \int_{-\infty}^{+\infty} dt\, e^{-iEt}\, |t\rangle . \tag{5.24}$$

Equation (5.24) shows that one can extract from $|t\rangle$ the eigenstates of the Hamiltonian once the content in R is known. One can thus obtain through (5.15) the relation

$$|E\rangle = \sum_n C_n^*(E)\, |n\rangle . \tag{5.25}$$

The formalism developed above indeed guarantees that

$$H|E\rangle = E\,|E\rangle \tag{5.26}$$

and

$$\langle E'|E\rangle = \delta(E-E') \,, \qquad \int_0^\infty dE \; |E\rangle\langle E| = \mathbb{1} \,. \tag{5.27}$$

Finally, $C_n(E) = \langle E|n\rangle$, completely determined by Eq. (5.21), is the quantity we were looking for, and $|C_n(E)|^2 dE$ has the usual meaning of probability of measuring for the state $|n\rangle$ the energy in the interval E, $E + dE$.

(iii) We now perform a short discussion of the features of the quantity

$$\rho_m(E) = |C_m(E)|^2 \tag{5.28}$$

whose functional form, incidentally, is fairly similar to the space probability distribution of the hydrogen atom. As an orientation we first consider the case $n = 0$. The function $\rho_0(E)$ is shown in Fig. 2 for a couple of values, $r_0 = \frac{3}{2}$ and $r_0 = 3$. In general, $\rho_0(E)$ has a maximum at

$$E_0 = \left(r_0 - \frac{1}{2}\right)\frac{1}{a} \tag{5.29}$$

whose effective width *) is

$$\Gamma = 2\sqrt{E_0/a} = \frac{2}{a}\sqrt{r_0 - \frac{1}{2}} \,. \tag{5.30}$$

Expectation values can also be easily computed, and one finds

$$\langle H\rangle_0 = \frac{r_0}{a} \,, \tag{5.31}$$

$$\langle H^2\rangle_0 - \langle H\rangle_0^2 \equiv (\Delta E)^2 = \frac{r_0^2}{2a^2} \,. \tag{5.32}$$

---

*) i.e., defining it as for the classical resonance curve by

$$8/\Gamma^2 = -\left.\frac{\rho_0''(E)}{\rho_0(E)}\right|_{E=E_0} .$$

A few words of comment are appropriate. In order to have a non-trivial behaviour and to be confined in time, the system cannot be in an energy eigenstate. However, the probability distribution in energy, although continuous and nowhere vanishing, exhibits a clear maximum at E_o and, because of the exponential decrease, it is unlikely to have energies very far from the maximum. So the emerging picture is that the system practically behaves like a resonant state whose characteristics, position of the maximum, and width, depend both on a and on the coupling constant g (actually the latter dependence comes through r_o, i.e., the Casimir operator).

Looking at the role of the parameters, it is interesting to mention the two limits $a \to 0, \infty$. For $a \to 0$, $R \propto K$, $\Delta E \to \infty$ and the energy distribution is no longer peaked anywhere; for $a \to \infty$, $R \to H$, $\Delta E \to 0$, $E_o \to 0$ and $|C_o(E)|^2$ behaves as in the case of a stationary $E = 0$ eigenstate.

Similar features are exhibited by the higher distribution functions. Clearly the presence of a n^{th} order polynomial gives now rise to a series of peaks whose intensity increases *) with the position, i.e., the one located at the highest energy is the most important. In order to have a first indication on the position of the highest peak, one can use for large E the simple recipe of just retaining the highest power in the Laguerre polynomial. This gives

$$(E_o)_n \approx \frac{1}{a}\left(n + r_o - \frac{1}{2}\right) = \frac{1}{a}\left(r_n - \frac{1}{2}\right) . \qquad (5.33)$$

Thus the following general trend emerges from this discussion: the energy content of an eigenstate of R is represented by a finite set of resonant-like states.

We do not push further any more detailed analysis. One can easily compute expectation values which for some quantity can be done on a purely algebraic basis: for instance, for the Hamiltonian one finds [use Eq. (4.16)]

$$\langle H \rangle_n = \frac{r_n}{a} . \qquad (5.34)$$

---

*) This can be understood by noticing that $C_n(E)$ has practically the same form as the hydrogen atom wave function in co-ordinate space $[\ell(\ell+1) \to r_o(r_o - 1),\ 2aE \to r\sqrt{-E}]$. Then a look at the radial density curves plotted in a book of quantum mechanics provides the hint [7].

Similarly, combining Eqs. (2.35) and (4.15), one can evaluate at once the expectation value of $Q^2(t)$, namely

$$\langle m|Q^2(t)|m\rangle = 2a\pi_m\left(1+t^2/a^2\right) \tag{5.35}$$

It is interesting to interpret this relation, from the point of view of the (configuration) x space representation, as a formula for the square radius for the state n, pointing the way g and "a" determine a configuration space confinement.

If, interpreting for a moment this work as an elementary field theoretical toy model, we give to the lowest state the role of the vacuum, we are in the rather unconventional situation of having non-time translation invariant vacuum expectation values.

A spontaneous way out from this unpleasant situation which could induce energy non-conservation has been advanced in Ref. 2), Section 6. It is suggested that we are in presence of a statistical ensemble of an infinite number of "vacuum" states obtainable from each other by appropriate applications of the operator generating finite time translations. The fundamental quantity is the "energy spread" of the ground state wave function, which in this case is

$$C_0(E) = \frac{(2a)^{\pi_0}}{[\Gamma(2\pi_0)]^{1/2}}\, E^{\pi_0-1/2}\, e^{-aE} \tag{5.36}$$

If we apply a time translation $t \to t+h$, then the ground state gets transformed so that

$$C_0(E) \longrightarrow C_0(E)\, e^{-iEh} \tag{5.37}$$

Supposing that, as pointed out in Ref. 2), a statistical average over the ensemble of "vacua" be required, we are led to introduce the vacuum statistical matrix

$$\rho = \int_0^\infty dE\, |C_0(E)|^2\, |E\rangle\langle E| \tag{5.38}$$

or explicitly

$$\varrho(H) = \frac{2\pi}{\Gamma(2n_0)} (2a)^{2n_0} H^{2n_0-1} e^{-2aH} \theta(H) \qquad (5.39)$$

It is amusing to note that this is approximately equivalent to a canonical distribution with $1/2a$ as the temperature.

6. TRANSITION MATRIX ELEMENTS

This Section is devoted to the evaluation of the matrix elements of a "tensor" operator $B(t)$ corresponding to a well-defined transformation under $O(2,1)$ *), namely

$$[H, B(t)] = -i \frac{d}{dt} B(t),$$

$$[D, B(t)] = -i \left(t \frac{d}{dt} - \delta\right) B(t),$$

$$[K, B(t)] = -i \left(t^2 \frac{d}{dt} - 2t\delta\right) B(t).$$

So, let us introduce the matrix element

$$\langle m_2 | B(t) | m_1 \rangle = M^{(\delta)}_{m_2 m_1}(t). \qquad (6.1)$$

The following exercise can be regarded as the calculation of the amplitude for the transition $n_1 \to n_2$ due to the coupling of the operator $B(t)$ to an external field. If one pursues the analogy, the observable quantity will be finally obtained by folding the t dependence of $M^{(\delta)}_{n_1 n_2}(t)$ with the $e^{i\omega t}$ familiar factor coming from the external source. Thus one must investigate the properties of $M^{(\delta)}_{n_2 n_1}(t)$, and this discussion will generalize the considerations developed for in the previous Section.

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*) This is, of course, nothing but the Wigner-Eckart theorem for the $O(2,1)$ group. We consider useful to present a derivation in our framework, mainly in view of future developments.

Let us first indicate how to dispose of the t dependence of the matrix element (6.1). It is convenient to resort to the right set of variables, i.e., the evolution parameter τ and the reduced operator $b(\tau)$:

$$\tau = 2 \operatorname{arctg} t$$
$$b(\tau) = \frac{B(t)}{(1+t^2)^\delta} \tag{6.2}$$

One can then immediately check that

$$i[R, b(\tau)] = \frac{db(\tau)}{d\tau} \tag{6.3}$$

i.e.,

$$b(\tau) = e^{iR\tau} b(0) e^{-iR\tau} \; ; \quad b(0) = B(0). \tag{6.4}$$

Combining the various equations, one finally obtains

$$\langle m_2 | B(t) | m_1 \rangle = (1+t^2)^\delta \left(\frac{1-it}{1+it} \right)^{m_1 - m_2} \langle m_2 | B(0) | m_1 \rangle . \tag{6.5}$$

We now have to determine the quantity $\langle n_2 | B(0) | n_1 \rangle$. To this aim, it is fruitful to work in the t basis previously introduced; let us define the quantity (a "three-point function")

$$F(t; t_2, t_1) = \langle t_2 | B(t) | t_1 \rangle =$$
$$= \sum_{m_2 m_1} \beta_{m_2}(t_2) \beta^*_{m_1}(t_1) \langle m_2 | B(t) | m_1 \rangle . \tag{6.6}$$

The advantage is that the action of the generators X^α on the states $|t\rangle$ is particularly simple, allowing a direct determination of $F(t; t_2, t_1)$. Once the latter is known, $\langle n_2 | B(t) | n_1 \rangle$ is obtained without difficulties. We start from the simple formula

$$X^{(\alpha)*}_t(\delta) \, F(t; t_2, t_1) = \langle t_2 | [X^\alpha, B(t)] | t_1 \rangle \tag{6.7}$$

Expressing through Eq. (5.14) the action of X^α on the states $|t\rangle$, one finds the following set of conditions:

$$\left\{ \chi_{t}^{(\alpha)}(\delta) + \chi_{t_1}^{(\alpha)}(-\pi_0) + \chi_{t_2}^{(\alpha)}(-\pi_0) \right\} F(t; t_2, t_1) = 0 \qquad (6.8)$$

The explicit form of these relations is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) F(t; t_2, t_1) = 0 \qquad (6.9a)$$

$$\left(t\frac{\partial}{\partial t} + t_2 \frac{\partial}{\partial t_2} + t_1 \frac{\partial}{\partial t_1} - \delta + 2\pi_0 \right) F(t; t_2, t_1) = 0 \qquad (6.9b)$$

$$\left(t^2\frac{\partial}{\partial t} + t_2^2 \frac{\partial}{\partial t_2} + t_1^2 \frac{\partial}{\partial t_1} - 2\delta t + 2\pi_0(t_1+t_2) \right) F(t; t_2, t_1) = 0 \qquad (6.9c)$$

and they are only consequences of the properties of transformation of the external states and of the operator $B(t)$ under $O(2,1)$.

From Eq. (6.9a) we learn that $F = f(t - t_1, t - t_2, t_1 - t_2)$. Making the further ansatz $F = f(t - t_1)^{A1} (t - t_2)^{A2} (t - t_3)^{A3}$, consistency with (6.9c) or (6.9b) allows the immediate determination of the exponents and the final result turns out to be

$$F(t; t_2, t_1) = f\, i^{(2\pi_0 - \delta)} (t - t_1)^{\delta} (t - t_2)^{\delta} (t_1 - t_2)^{-\delta - 2\pi_0} \qquad (6.10)$$

where f is a real arbitrary constant. Apart from this, the solution is unique *).

To proceed further it is convenient to introduce the variables

$$z_i = \frac{1 - it_i}{1 + it_i} = e^{-i\theta_i} \qquad (6.11)$$

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*) The same technique can easily be applied to the quantity $F(t_1, t_2) = \Sigma_n \beta_n(t_1)\, \beta_n^*(t_2)$ already evaluated directly in (5.14), which can therefore be regarded as the analogue of a two-point function.

which, after a few manipulations, allow us to rewrite Eq. (6.10) in the form

$$\sum_{m_1 m_2} \langle m_2 | B(t) | m_1 \rangle z_2^{m_2} z_1^{-m_1} (-1)^{m_1+m_2} \left\{ \frac{m_1!\, m_2!}{\Gamma(2\pi_0+m_1)\Gamma(2\pi_0+m_2)} \right\}^{-\frac{1}{2}} \tag{6.12}$$

$$= f\, 2^{2\pi_0+\delta}\, z_1^{2\pi_0} \frac{(z-z_1)^{\delta}(z_2-z)^{\delta}(z_1-z_2)^{-\delta-2\pi_0}}{(1+z)^{2\delta}}$$

We first note that the t dependence of the matrix element $\langle n_2 | B(t) | n_1 \rangle$ given in Eq. (6.5) is correctly reproduced: putting $z_i \to z z_i$ one indeed selects the factor

$$z^{m_1-m_2} \left(\frac{z}{1+z}\right)^{2\delta} = \left(\frac{1-it}{1+it}\right)^{m_1-m_2} (1+t^2)^{\delta}$$

Thus from now on we can work at $t = 0 (z = 1)$. Furthermore, the quantity at the left-hand side is a double Taylor series around $z = 0$, $z = \infty$; the separation of the coefficient is then immediate and we obtain

$$\langle m_2 | B(0) | m_1 \rangle = \frac{f}{(2\pi i)^2} \left\{ \frac{m_1!\, m_2!}{\Gamma(2\pi_0+m_1)\Gamma(2\pi_0+m_2)} \right\}^{1/2}$$

$$(-1)^{m_1+m_2}\, 2^{2\pi_0-\delta} \oint_{C_0} dz_2 \oint_{C_1} dz_1 (1-z_1)^{\delta}(z_2-1)^{\delta} \cdot \tag{6.13}$$

$$\cdot (z_1-z_2)^{-\delta-2\pi_0}\, z_1^{2\pi_0+m_1-1}\, z_2^{-(m_2+1)} .$$

The integration contour is built by the suitable anticlockwise circuits C_0 enclosing $z_2 = 0$ and C_∞ enclosing $z_1 = \infty$. Equation (6.13) can be transformed to a more convenient form by an appropriate change of variable. Setting $z_1 = 1/y$, $z_2 = z$, we obtain

$$\langle m_2 | B(0) | m_1 \rangle = \frac{f}{(2\pi i)^2} \left\{ \frac{m_1!\, m_2!}{\Gamma(2r_0+m_1)\Gamma(2r_0+m_2)} \right\}^{1/2} (-1)^{m_1+m_2}$$

$$2^{2r_0-\delta} \oint_{c_0} \frac{dz}{z^{m_2+1}} \oint \frac{dy}{y^{m_1+1}} \cdot (1-yz)^{-\delta-2r_0} \cdot$$

$$\cdot (1-y)^{\delta} (z-1)^{\delta} \tag{6.13'}$$

where now both integrals are made around anticlockwise circuits enclosing $y = 0$, $z = 0$.

We can now conveniently express the arbitrary constant f in terms of the lowest matrix element $\langle 0 | B(0) | 0 \rangle$:

$$\langle 0 | B(0) | 0 \rangle = f\, 2^{2r_0-\delta} [\Gamma(2r_0)]^{-1}. \tag{6.14}$$

In our model, once $B(0)$ is known as a function of q, p, the calculation can be performed in the x representation using the lowest eigenfunction given in Section 4. The dependence on n_1, n_2 and δ is embodied in the double z_1, z_2 integral. We remark again the interesting fact that there is a "kinematical" dependence on the interaction through r_0 which is dictated by the transformation properties of the external states.

The evaluation of the integrals in (6.13) does not offer any essential difficulty, and a straightforward application of the Cauchy theorem produces the formula

$$\langle m_2 | B(0) | m_1 \rangle = \langle 0 | B(0) | 0 \rangle \left\{ \frac{m_1!\, m_2!\, \Gamma^2(2r_0)}{\Gamma(2r_0+m_1)\Gamma(2r_0+m_2)} \right\}^{1/2}$$

$$\sum_{k=0}^{\min[m_1,m_2]} (-1)^k \binom{-\delta-2r_0}{k} \binom{\delta}{m_1-k} \binom{\delta}{m_2-k}. \tag{6.15}$$

This formula takes a simple form when one of the states is the lowest one, $|n=0>$. Then

$$\frac{\langle 0|B(0)|m\rangle}{\langle 0|B(0)|0\rangle} = \left\{ \frac{m!}{2\pi_0 \dots (2\pi_0+m-1)} \right\}^{1/2} \tag{6.16}$$

This ratio is remarkably independent of the dimensionality of the operator B. This property is particularly interesting for the strong coupling limit $g \to \infty$: Eq. (6.16) shows that

$$\frac{\langle 0|B(0)|m\rangle}{\langle 0|B(0)|0\rangle} \sim \sqrt{m!}\; g^{-m/4} \tag{6.17}$$

We therefore see that any transition from the ground to the excited state is depressed, independently of the dimensionality of the operator B. The strong dominance of the diagonal matrix element $\langle 0|B(0)|0\rangle$ confirms the validity of the semiclassical approximation in the limit $g \to \infty$.

In turn, the matrix element for the transition $n_1 \to n_2$ induced by an interaction $\lambda B(t)\, a(t)$ [$a(t)$ is an external classical field] is proportional to the Fourier transform.

$$N^{(\delta)}_{m_2 m_1}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \langle m_2|B(t)|m_1\rangle =$$

$$= \langle m_2|B(0)|m_1\rangle \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, e^{i\omega t} (1+t^2)^{\delta} \left(\frac{1-it}{1+it}\right)^{m_1-m_2}$$

$$= \langle m_2|B(0)|m_1\rangle\, F^{(\delta)}_{\nu}(\omega) \tag{6.18}$$

where

$$\nu = m_1 - m_2$$

The evaluation of this Fourier transform can be performed either directly or using the method of the previous section consisting in transforming to the ω space the t differential equation satisfied by the

integrand. The result is $(\omega > 0)$

$$F_\nu^{(\delta)}(\omega) = \frac{2^{2\delta+1}}{\Gamma(\nu-\delta)} e^{-\omega} \psi(1+\delta-\nu, 2(1+\delta); 2\omega), \quad (6.19)$$

where ψ is the so-called Tricomi function [8] solution of the confluent hypergeometric equation. Its behaviour is $\psi \sim \omega^{-1-\delta-\nu}$ as $\omega \to \infty$ and $\psi \to$ const. as $\omega \to 0$, with the restriction $\delta < -\frac{1}{2}$ required by the existence of the Fourier integral for any ω. In the particular cases in which $1+\delta-\nu$ or $-\nu-\delta$ are negative integers, ψ is simply related to Laguerre polynomials [as is the case for the function $C_n(E)$ of Eq. (5.15)].

The investigation of the properties of the matrix element $N^{\delta}_{n_2 n_1}(\omega)$ embodied in (6.19) well deserves a separate detailed treatment stressing similarities with the energy amplitudes C_n (indeed, they are both determined by the same mathematics); this will hopefully be done elsewhere and here we limit ourselves to a discussion of a simple specific example.

Let us consider the simple form of interaction $B(t) = 1/Q^2$, i.e., $\delta = -1$. Then, using Eqs. (6.18), (6.19), we obtain

$$N_{01}^{(-1)}(\omega) = \omega e^{-\omega} \langle 0 | \frac{1}{Q^2} | 0 \rangle \frac{1}{\sqrt{2\pi_0}} \quad (6.20)$$

We note that the matrix element (6.20) has a maximum at $\omega = 1$ (actually $1/a$) in line with the behaviour one can expect from intuitive considerations.

The above results are only based on group theoretical properties and can be established for any operator with definite conformal transformation properties. The only unknown left is the vacuum expectation value $< 0|B|0 >$ [see Eq. (6.14)]. It is an important feature that, when the operator can be expressed in terms of powers of the field Q, this quantity can be obtained on fairly general grounds, using completeness and the structure of the spectrum of physical states.

As a first example we consider the interesting case of the "vacuum" expectation value of the field itself, $< 0|Q|0 >$. We start from the relation

$$Q^2(0) = 2K = 2R + S_+ + S_- \quad (6.21)$$

taken between the "vacuum" and evaluate the left-hand side inserting tne complete set of eigenstates of R. The relevant matrix elements can be evaluated using (6.14) [i.e., $< 0|Q|\, n >=< 0|Q|0 >(\frac{1}{n}) \{(n!\Gamma(2r_0))/ /(\Gamma(2r_0+n))\}^{\frac{1}{2}}$] and we obtain

$$2\pi_0 = |\langle 0|Q|0\rangle|^2 \frac{\Gamma(2\pi_0)}{4} \sum_m \frac{\Gamma^2(m-1/2)}{\Gamma(m+1)\Gamma(2\pi_0+m)} =$$

$$= |\langle 0|Q|0\rangle|^2 \frac{2\pi_0 \Gamma^2(2\pi_0)}{\Gamma^2(2\pi_0+1/2)} \qquad (6.22)$$

where the summation on the Γ functions has been performed using the Dugall's formula [8]. Hence,

$$|\langle 0|Q|0\rangle| = \frac{\Gamma(2\pi_0+1/2)}{\Gamma(2\pi_0)} \qquad (6.23)$$

In the specific case of $B = 1/Q^2$ we were studying before, it is enough to start from the identity

$$\langle 0|\frac{1}{Q^2} Q^2|0\rangle = 1$$

and to apply similar considerations to obtain

$$\langle 0|\frac{1}{Q^2}|0\rangle = \frac{1}{2\pi_0 - 1} . \qquad (6.24)$$

7. THE SEMICLASSICAL APPROXIMATION

As we have discussed in the final part of Section 4, a semiclassical approximation is strongly suggested for the eigenvalue equation of the operator R. While this is not particularly important since in this simple model we are in possession of exact solutions, it will be interesting to

spend some more time on this semiclassical approximation by giving the Heisenberg representation version; it can be useful in order to understand cases (in higher dimensions) where exact treatment is not available.

Let us recall the form of the Lagrangian

$$L = \frac{1}{2}\left(\dot{Q}^2 - \frac{g}{Q^2}\right), \tag{7.1}$$

together with the Heisenberg equation of motion

$$\frac{d^2Q}{dt^2} = \frac{g}{Q^3}. \tag{7.2}$$

We then introduce the general operator

$$G = u\,H + v\,D + w\,K \tag{7.3}$$

(u,v,w real) and its lowest eigenstate we shall simply indicate by $|>$. (It is tacitly assumed that such an eigenstate does indeed exist. In the following we shall see that the assumption is justified only if G is a compact generator.) Finally, we consider the expectation value of the field Q(t) in that state

$$C(t) = <|\,Q(t)\,|> . \tag{7.4}$$

In the framework of our "semiclassical" approximation, $g \to \infty$, it will be assumed that the off-diagonal matrix elements of Q(t) can be neglected. In that approximation C(t) will then satisfy the classical equation of motion

$$\frac{d^2C}{dt^2} = \frac{g}{C^3} \tag{7.5}$$

The time dependence of C(t) is, of course, completely fixed by the form of the operator G, and as a consequence of the commutation relations of G with Q one has for C(t) the differential equation

$$(u+vt+wt^2)\,\frac{dC}{dt} = \frac{1}{2}\,C(t)\,\frac{d}{dt}(u+vt+wt^2) \tag{7.6}$$

i.e.,

$$C(t) = c(\mu + \nu t + \omega t^2)^{1/2} . \tag{7.7}$$

Invariance of the theory under the conformal group guarantees that $C(t)$, as given in Eq. (7.7), can be a solution of the equation of motion (7.5). Substituting the form (7.7) in (7.5), we indeed see that the time dependence of both sides does perfectly match, leading to the numerical equation

$$\Delta = \nu^2 - 4\mu\omega = -4g/c^4 , \tag{7.8}$$

which fixes the value of the constant C.

It thus follows that if we wish the constant C to be real (as suggested by the Hermiticity of Q), for positive values of g we have to choose negative values of Δ, i.e., to use a compact operator G.

It is instructive to quote for further use the explicit form of the generators H, K, D, when expressed in terms of the classical function $C(t)$. One finds

$$H_{cl.} = \frac{c^2}{2}\omega , \quad D_{cl.} = -\frac{c^2}{4}\nu , \quad K_{cl.} = \frac{c^2}{2}\mu \tag{7.9}$$

The next step is to look for the first correction to the classical approximation and to write

$$Q(t) = C(t) + Q'(t) . \tag{7.10}$$

The equation of motion for $Q'(t)$ can be obtained in an easier and more transparent way by resorting to the general formalism developed in Section 3. We have seen, in particular, that in order to study the properties of the operator G, it is convenient to express the action in the form

$$A = \int d\tau \, L_c(q, \dot{q}) \tag{7.11}$$

where

$$d\tau = \frac{dt}{\mu + \nu t + \omega t^2} , \tag{7.12}$$

$$L_\tau (q, \dot{q}) = \frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 - W(q^2) , \tag{7.13}$$

$$W(q^2) = \frac{1}{2} \left(\frac{g}{q^2} - \frac{\Delta}{4} q^2 \right) , \tag{7.14}$$

and

$$q(\tau) = \frac{Q(t)}{(\mu + \nu t + \omega t^2)^{1/2}} \tag{7.15}$$

In terms of the new "co-ordinate" $q(\tau)$, Eq. (7.10) then simply reads

$$q(\tau) = C + q'(\tau) . \tag{7.16}$$

Furthermore, it is easy to get convinced that the potential $W(q^2)$ has an extremum at $q(\tau) = C$, where C is the quantity determined by Eq. (7.8). Expanding around that point, we obtain, at the lowest significant order in $q'(\tau)$,

$$W(q^2) \approx W(C) + (q-C)^2 \frac{1}{2} W''(C) \approx$$

$$\approx \frac{1}{2} \left[\left(-\frac{\Delta g}{4} \right)^{1/2} - \Delta q'^2 \right] . \tag{7.17}$$

A first important point exhibited by (7.17) is that the position $q(\tau) = C$ (i.e., $q' = 0$) is a minimum, corresponding to a stable equilibrium configuration, only when $\Delta < 0$. This clearly shows, once more, that we should only use compact operators [and, according to Eq. (7.8), positive values of g]. One can then establish, in this approximation, the equation of motion for the shifted field q', which reads

$$\frac{d^2}{d\tau^2} q'(\tau) = \Delta q'(\tau) \tag{7.18}$$

At the same time a comparison between Eq. (7.18) and the corresponding one derived in the case $g = 0$, namely

$$\frac{d^2}{d\tau^2} q(\tau) = \frac{\Delta}{4} q(\tau) , \tag{7.18'}$$

shows the fundamental difference between the two situations. In the free case, the exact solution of the equation of motion is

$$q(\tau) = c_1 e^{i\frac{\tau}{2}\sqrt{|\Delta|}} + c_2 e^{-i\frac{\tau}{2}\sqrt{|\Delta|}} . \tag{7.19}$$

On the other hand, for $g \neq 0$ we have the approximate semiclassical solution

$$q(\tau) \approx c + a_1 e^{i\tau\sqrt{|\Delta|}} + a_2 e^{-i\tau\sqrt{|\Delta|}} \tag{7.20}$$

We can thus conclude that

a) <u>for $g = 0$</u> there is no constant contribution to $q(\tau)$ and <u>the fundamental frequency is $\frac{1}{2}\sqrt{|\Delta|}$</u>. In quantum terms this means that diagonal matrix elements of $q(\tau)$ are vanishing and that the difference between two successive levels is

$$f_{m+1} - f_m = \frac{\sqrt{|\Delta|}}{2} . \tag{7.21}$$

b) <u>for $g \neq 0$</u> there is a large constant contribution to $q(\tau)$ and <u>the fundamental frequency is now $\sqrt{|\Delta|}$</u>. In quantum terms this means that there is a large non-vanishing diagonal matrix element of $q(\tau)$; at the same time <u>the level difference has now doubled</u> and has become

$$f_{m+1} - f_m = \sqrt{|\Delta|} . \tag{7.22}$$

It is amusing to see that many of the general features of the exact solution are already clearly exhibited in the semiclassical approximation. The reason for this is not hard to understand. Most of those features have an evident group theoretical origin. Now it is well known that the semiclassical approximation does not spoil order by order those group theoretical

properties. Only those predictions which depend, in an essential manner, on the form of the Casimir operator are modified in the classical limit. Indeed in the limit $g \to \infty$ use of the classical expressions (7.9) for the generators provides for the Casimir operator the result

$$J^2_{cl.} = K_{cl} H_{cl} - D^2_{cl} = -\frac{\Delta c^4}{16} = \frac{g}{4} \qquad (7.23)$$

which differs from the exact one (2.38) by the "quantum" factor -3/16.

8. CONCLUSIONS

This paper has been devoted to the simplest example of fully conformal invariant Lagrangian theory.

Beyond the expected result that the number of constants of the motion is more than enough to provide a complete solution of the dynamical problem, our solutions present a certain number of new physical features which could be a guidance towards the understanding of more realistic conformal invariant field theories.

As already mentioned, part of our work can be considered as an amusing exercise in the use of the O(2,1) group. The three O(2,1) constants of the motion are not independent, and the invariant Casimir operator is a simple function of the coupling constant, i.e., $\frac{1}{2}(HK + KH) - D^2 = r_o(r_o - 1) = g/4 - 3/16$. In other words, the only way g appears in all expressions is through r_o. The value of the coupling constant tells us which representation of O(2,1) we should use.

In this group theoretical frame it is easy to isolate the time dependence of the various modes, evaluate the eigenvalues of the different operators, and use the Wigner-Eckart theorem, in order to obtain the most general matrix elements.

An important feature which emerges from our calculations and which is due to the non-compact nature of O(2,1) is that not all the constants of the motion are on the same ground. We have seen that only compact operators like

$$R = \frac{1}{2}(aH + \frac{1}{a}K)$$

exhibit the pleasant features of having a discrete spectrum and fully normalizable states. The familiar Hamiltonian operator is on the border-line and shows a continuous spectrum with a non-normalizable lowest eigenfunction.

The difference is due to the presence of the length "a" that provides a sort of infra-red cut-off responsible for good behaviour at large distances and discrete spectrum. As a consequence, a strong indication from this work is that eigenstates of R and not of H should be used to solve the problem of motion and to characterize physical states.

It is interesting to note that the typical time dependence of the R eigenfunctions involves the factor $e^{in\tau}$, where $\tau = 2 \operatorname{arctg} t/a$, corresponding to a finite number of oscillations which take place in a lapse of time of the order of a.

If we consider this "one-dimensional" model as a specimen of elementary field theory, we are naturally led to the idea that the vacuum state is the lowest (normalizable) eigenstate of the operator R. This introduces a fundamental length which enters the theory not through the Lagrangian but through the specific form of the vacuum. The investigation of more realistic conformal invariant field theoretical models, based on the use of semiclassical approximations, fully confirms this point of view.

In the language of spontaneously broken symmetries, we can say that in the framework of fully conformal invariant Langrangians, invariance under transformations generated by D and by $S = \frac{1}{2}(\frac{1}{a}K - aH)$ is spontaneously broken. Of course, the problem arises of reconciling the motion of spontaneously broken translation invariance with experimentally verified energy momentum conservation. A reasonable recipe, based on a statistical interpretation, is proposed in Ref. 2). In our case the "vacuum" statistical matrix has the beautiful form given in Eq. (5.39).

Another very important topic is the role played by the coupling constant. From a purely group theoretical standpoint, one might argue that nothing significant happens since the coupling constant enters the results only through the value of the Casimir operator; however, a more careful analysis shows that this is not so. From a purely formal point of view, this depends on the non-linearity of the equation $r_o(r_o - 1) = g/4 - 3/16$ giving the lowest eigenvalue r_o of the series in terms of the Casimir operator (and of g). In the $g = 0$ case it turns out that both roots

$$r_o = \frac{1}{2}\left(1 + \sqrt{g + 1/4}\right)$$

$$\bar{r}_o = \frac{1}{2}\left(1 - \sqrt{g + 1/4}\right)$$

are acceptable, whereas for $g > 0$ only the higher root is acceptable. This shows that the level distance abruptly doubles as soon as g takes a non-vanishing value.

The cause for such a phenomenon can be understood by looking at the detailed form of the wave function $\psi(q)$. The g dependent term gives rise to an infinite repulsive well at the origin $(q=0)$. This means that, while for $g=0$ the wave function is extending between $-\infty<q<\infty$, in the $g>0$ case it is "confined" to positive values of q.

It is this "change of phase" which causes for $g>0$ a non-vanishing zero expectation value of the operator q.

Let us conclude this discussion with a few words about semiclassical approximations. It is easy to see that for large values of the coupling constant g all formulae simplify greatly. In particular, the diagonal matrix elements of the relevant operators have a strong tendency to dominate over the non-diagonal ones. This suggests that those large g results can be directly obtained by means of a semiclassical approximation. This is indeed the case: the key point in this approach is the existence of a non-vanishing "vacuum" expectation value of the field which obeys the classical Lagrange equations. Although one might question the usefulness of a semiclassical approach in a case where all exact answers are known, those approximate results are a useful link to more general Lagrangians where a semiclassical approach is the only one available.

In this paper we have mainly emphasized the fundamental and the formal aspects of an elementary model exhibiting conformal invariance. An entirely open problem is the investigation of the detailed physical aspects of an elementary theory in which the vacuum is the lowest eigenstate of R and contains a fundamental length. In particular, this model could lead to a first useful indication about the "observable" properties of such theories. This and other questions will, hopefully, be the object of further investigations.

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APPENDIX

SCHRÖDINGER FORMALISM

In this Appendix we shall treat the problem in the Schrödinger formalism. We introduce the representation of q and p of Eqs. (2.33)

$$q \longrightarrow x \,, \qquad p \longrightarrow -i\frac{d}{dx} \,. \tag{A.1}$$

The Hamiltonian is

$$H_S = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{g}{x^2}\right) . \tag{A.2}$$

The operators D_S and K_S are obtained in the same way from their classical expressions *):

$$D_S = D_{oS} + t\,H_S \,, \qquad K_S = K_{oS} + 2t\,D_{oS} + t^2 H_S \,, \tag{A.3}$$

where

$$D_{oS} = \frac{i}{2}\left(x\frac{d}{dx} + \frac{1}{2}\right) , \tag{A.4}$$

$$K_{oS} = \frac{1}{2}x^2 . \tag{A.5}$$

In general, given an observable

$$G = u\,H + v\,D + w\,K$$

*) Footnote on next page.

the operator G_s is given by

$$G_s(t) = (u + vt + wt^2)\, H_s + \\ + (v + 2wt)\, D_{os} + w\, K_{os} \quad . \tag{A.6}$$

For further use, we observe that for $t = t_o \equiv -v/2w$,

$$G_s(t_o) = -\frac{\Delta}{4w} H_s + w K_{os} \quad .$$

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*) Footnote from p. 48

The form of an operator in the Schrödinger representation can be obtained from the operator in the Heisenberg representation introducing the unitary transformation U such that

$$|\Psi_s(t)\rangle = U(t)\, |\Psi\rangle$$

with $|\psi_s(0)\rangle = |\psi\rangle$, i.e., $U(0) = 1$.

The Schrödinger representation operators are given by $O_s = U\, O\, U^{-1}$. In particular, $Q_s \equiv q = U\, Q(t)\, U^{-1}$ and $P_s \equiv p = U\, P(t)\, U^{-1}$. If an operator is explicitly function of time, we have

$$O_s(t) = U\, O(t, Q, P)\, U^{-1} = O(t, q, p),$$

hence it is still a function of t, with the values of Q and P taken at $t = 0$. It may be interesting to check the time independence of, say, D_s. We have

$$\dot{D}_s = \dot{D}_{os} + H_s$$

where

$$D_o = D(t=0, Q, P).$$

Now D_o is a Heisenberg operator not explicitly depending on time. We have

$$\frac{dD_o}{dt} = -i[D_o, H] = -H$$

Hence

$$\dot{D}_{os} = -H_s \qquad \text{and} \qquad \dot{D}_s = 0.$$

The fact that any operator G is a constant of the motion has the important consequence, in the Schrödinger representation, that if ψ is a solution of the time-dependent Schrödinger equation, $G_s\psi$ also is a solution of the same equation. In particular, this ensures that ψ can be an eigenfunction of G_s for all times. We then look for solutions of the Schrödinger equation

$$H\Psi(x,t) = i\frac{\partial}{\partial t}\Psi(x,t) \tag{A.7}$$

that are eigenstates of the operator G_s:

$$G_s\Psi_{G'}(x,t) = G'\Psi_{G'}(x,t). \tag{A.8}$$

Using Eqs. (A.7), (A.8) can be put in the form

$$\left\{ i(u+vt+wt^2)\frac{\partial}{\partial t} + \frac{i}{2}(v+2wt)x\frac{\partial}{\partial x} + \right.$$

$$\left. + \frac{i}{4}(v+2wt) + \frac{w}{2}x^2 - G' \right\}\Psi_{G'}(x,t) = 0. \tag{A.9}$$

In Eq. (A.9) the first two terms containing the partial derivatives correspond, as one sees easily, to a derivative with respect to t in which $y = x/(u+vt+wt^2)^{\frac{1}{2}}$ is kept fixed. Introducing the variable y in place of x, we get then

$$\left\{ i\frac{\partial}{\partial t} + \frac{w}{2}y^2 + \frac{1}{u+vt+wt^2}\left[\frac{i}{4}(v+2wt) - G'\right] \right\}\Psi_{G'}(y,t) = 0. \tag{A.10}$$

Equation (A.10) can be solved quite easily:

$$\Psi_{G'}(x,t) = k\, e^{i\frac{w}{2}y^2(t-t_0)}\, \frac{e^{-iG'\tau}}{(u+vt+wt^2)^{1/4}}\, A_{G'}(y) \tag{A.11}$$

where

$$\frac{d\tau}{dt} = (u+vt+wt^2)^{-1}$$

and $A_{G'}(y)$ is a function of y that will be determined by the request that Ψ be an eigenfunction of G_s. t_o is a constant whose value would simply affect the definition of $A_{G'}(y)$. We shall fix t_o to the value $t_o = -v/2w$ that will help in obtaining the simplest form for the eigenvalue equation (A.8). Indeed, let us consider (A.8), where now we use the form (A.2) for H_s; then t is a parameter and the differential equation must hold for any value of t. Let us choose $t = t_o$. Then the equation for $A_{G'}(y)$ ensuing from (A.8) is

$$\left(-\frac{d^2}{dy^2} + \frac{g}{y^2} - \frac{\Delta}{4} y^2\right) A_{G'}(y) = 2G' A_{G'}(y) \ . \tag{A.12}$$

This equation is exactly what one would get from the expression (3.18) of the text looking at the eigenvalue equation for the operator $H_G \equiv G$ in the representation *) where $q(0) \to y$, $\dot{q}(0) \to -i(d/dy)$, and indeed it coincides with Eq. (3.23).

In particular, if $G = R$, we have (with $G' = r$)

$$\Psi_r(x,t) = K \frac{e^{\frac{i}{2}\frac{x^2 t}{a^2+t^2}}}{(a^2+t^2)^{1/4}} \left(\frac{a-it}{a+it}\right)^{ir} A_r\left(\frac{x\sqrt{2a}}{\sqrt{a^2+t^2}}\right) \tag{A.13}$$

and since

$$\tau = 2 \operatorname{arctg} \frac{t}{a} = \frac{1}{i} \ln \frac{a+it}{a-it}$$

we have

$$\Psi_r(x,t) = K \frac{e^{\frac{i}{2}\frac{x^2 t}{a^2+t^2}}}{(a^2+t^2)^{1/4}} \ e^{-ir\tau} \ A_r\left(\frac{x\sqrt{2a}}{\sqrt{a^2+t^2}}\right)$$

where $A_r(y)$ obeys the equation

$$\left(-\frac{d^2}{dy^2} + \frac{g}{y^2} + \frac{1}{4} y^2\right) A_r(y) = 2r A_r(y) . \tag{A.14}$$

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*) Confusion must not be made between $q(t=0)$ and $q \equiv Q(t=0)$. They are proportional.

The solutions are given by

$$A_n(y) = y^{2r_0-\frac{1}{2}}\, e^{-\frac{y^2}{2}}\, L_n^{2r_0-1}(y^2) \tag{A.15}$$

where $L_n^{2r_0-1}$ is the Laguerre associated polynomial of degree n in the argument. The eigenvalue r_n is given by

$$r_n = r_0 + n \; ; \quad n = 0, 1, 2, \ldots$$

Hence the general solution is given by

$$\Psi_{r_n}(x,t) = \sqrt{2}\left[\frac{\Gamma(n+1)}{\Gamma(n+2r_0)}\right]^{1/2}\left(\frac{2a}{a^2+t^2}\right)^{1/4} \cdot$$

$$\cdot\left(\frac{a-it}{a+it}\right)^{r_n} y^{2r_0-\frac{1}{2}}\, e^{-\frac{y^2}{2}}\, L_n^{2r_0-1}(y^2) \tag{A.16}$$

where $y = x\sqrt{2a}/\sqrt{a^2+t^2}$ and the wave function has been normalized in the usual way, namely

$$\int_0^\infty dx\; \Psi^*_{r_n}(x,t)\, \Psi_{r_n}(x,t) = 1 \; .$$

It is interesting to obtain the coefficients in the expansion of ψ_{r_n} in eigenstates of the energy. We write

$$\Psi_{r_n}(x,t) = \int_0^\infty dE\, e^{-iEt}\, C_n(E)\, \Psi_E(x) \tag{A.17}$$

Introducing the function $\beta_n(t)$ defined in Eq. (5.6), we obtain

$$\psi_E(x) = x^{1/2}\, J_{2r_0-1}(x\sqrt{2E})$$

has been normalized so that

$$\int_0^\infty dx\; \psi_E(x)\, \psi_{E'}(x) = \delta(E-E') \; .$$

We then easily have

$$C_n(E) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi}\, e^{iEt}\, \frac{\Psi_{2n}(x,t)}{\psi_E(x)} \quad . \tag{A.19}$$

It is particularly simple to evaluate the right-hand side by taking the limit $x \to 0$. Using the explicit forms of $\psi_{r_n}(x,t)$ and $\psi_E(x)$ one gets

$$\lim_{x\to 0} \frac{\Psi_{2n}(x,t)}{\psi_E(x)} = 2^{2\nu_0} a^{\nu_0} \left\{ \frac{\Gamma(n+2\nu_0)}{\Gamma(n+1)} \right\}^{1/2} \cdot$$

$$\cdot E^{1/2-\nu_0} (a^2+t^2)^{-\nu_0} \left(\frac{a-it}{a+it} \right)^{2n} \tag{A.20}$$

where $B_u(t)$ is defined according to Eq. (5.3) of the text. We have

$$C_n(E) = (2a)^{\nu_0} E^{\frac{1}{2}-\nu_0} (-1)^n \int_{-\infty}^{+\infty} \frac{dt}{2\pi}\, e^{iEt}\, B_n(t) \, . \tag{A.21}$$

Equation (A.21) coincides with Eq. (5.15) in the text.

REFERENCES

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2) S. Fubini, "A new approach to conformally invariant field theories" (to be published).

3) For one of the first examples, see, e.g.,
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4) See for instance,
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7) For instance,
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8) See, e.g.,
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FIGURE CAPTIONS

Figure 1 Plot of the function $W(y) = g/y^2 - \Delta y^2$ for the cases $\Delta = \pm 1, 0$.

Figure 2 Plot of the function $\rho_o(E)$ for the values $r_o = \frac{3}{2}$ and $r_o = 3$. The energy is measured in units $1/a$.

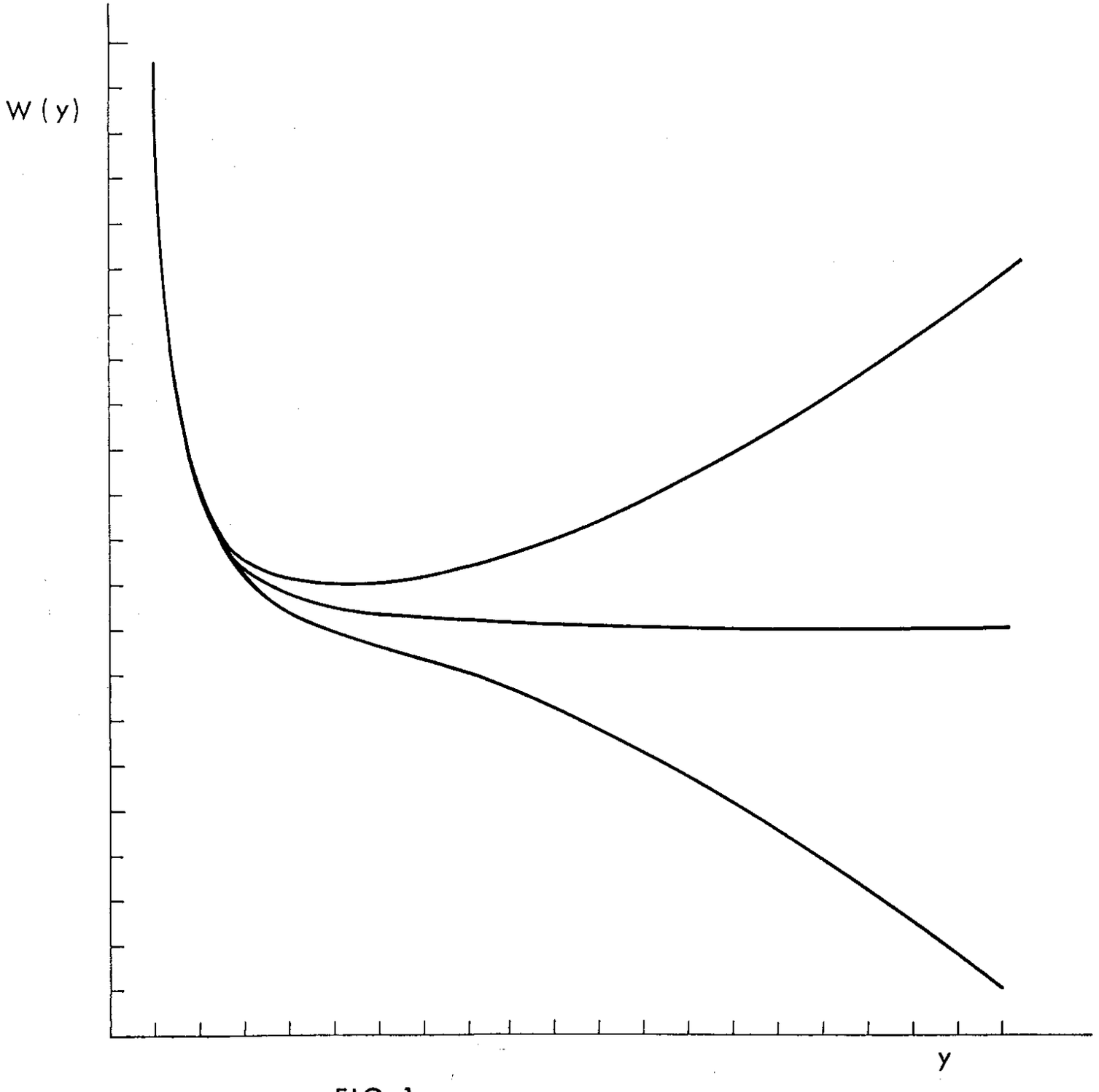

FIG. 1

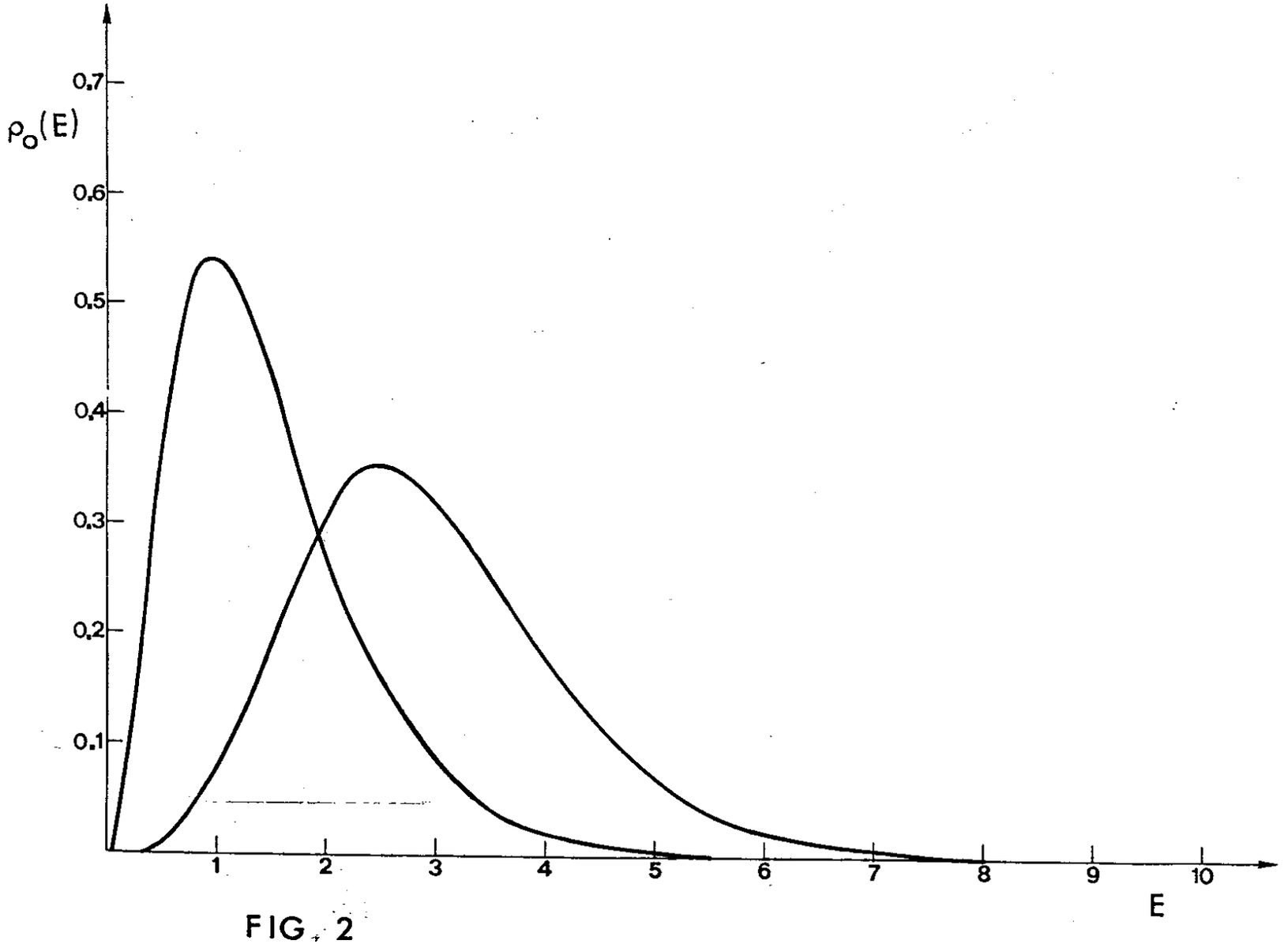

FIG. 2