Lie algebras, Representations, and Analytic Semigroups

through Dual Vector Fields

Philip Feinsilver

Southern Illinois University

Carbondale, Illinois USA

CIMPA-UNESCO-VENEZUELA School

Mérida, Venezuela

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Introduction

We want to present the basics of a new point of view in a variety of areas using the idea of *Dual Vector Fields*. These topics include operator calculus, representations of Lie algebras, analytic semigroups, and probability semigroups.

Ph. Feinsilver
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1 Coherent State Representations: Operators and Duality

Let us start with the basics of operators and duality with some examples relating to probability theory.

I. Simple Fock Spaces

We have a *vector space* $\mathcal{H}$ with a basis $\{\psi_n\}_{n\geq 0}$. Throughout, our scalars will be $\mathbf{C}$, the complex numbers, or alternatively, we restrict to $\mathbf{R}$, the real numbers.

The *Dirac notation* writes $\psi_n = |\, n\,\rangle$, called "ket", where the label n is the eigenvalue of an operator on $\mathcal{H}$. In this case, it is the *number operator*, $\mathcal{N}$, $\mathcal{N}\psi_n = n\,\psi_n$. In other words, $\mathcal{N}$ is diagonal in this basis with eigenvalues $\{0, 1, 2, \ldots\}$. In realizing these as functions, it is convenient to label them according to the number of underlying variables. For d variables, $\{\, x_1, \ldots, x_d\,\}$, we write the basis as $\psi_n = \psi_{n_1,\ldots,n_d} = |\, n_1, \ldots, n_d\,\rangle$, so that n denotes the corresponding multi-index $(n_1, \ldots, n_d)$, with number operators $\mathcal{N}_i\psi_n = n_i\,\psi_n$. Then $\mathcal{N} = \sum_i \mathcal{N}_i$ acts as $\mathcal{N}\psi_n = |n|\,\psi_n$, the total degree of ψ_n. The state $|\,0\,\rangle$ is called the *vacuum state*, is often denoted by Ω, and is mapped to the zero vector by all lowering operators.

1.1 RAISING AND LOWERING OPERATORS

For a single index, introduce *raising* and *lowering* operators, $\mathcal{R}$ and $\mathcal{V}$.

$$\mathcal{R}\,|\,n\,\rangle = |\,n+1\,\rangle, \qquad \mathcal{V}\,|\,n\,\rangle = n|\,n-1\,\rangle$$

Think of going from $x^n \to x^{n+1}$ by multiplying by x, and correspondingly from $x^n \to nx^{n-1}$ by differentiation. The specific operators analogous to differentiation are denoted by $\mathcal{V}$'s and referred to as *velocity operator*s as "lowering operator" refers more generally to any operator lowering the degree. For d variables, we have

$$\mathcal{R}_i\,|\,n\,\rangle = |\,n+\mathrm{e}_i\,\rangle = |\,n_1, \ldots, n_i+1, \ldots, n_d\,\rangle, \qquad \mathcal{V}_i\,|\,n\,\rangle = n_i|\,n-\mathrm{e}_i\,\rangle$$

where e_i is a vector of 0's except for a 1 in the i^{th} spot.

1.2 LIE ALGEBRAS

A *Lie algebra*, $\mathfrak{g}$, is an algebra where the multiplication, denoted by brackets $[a, b]$, satisfies $[a, a] = 0$ and the *Jacobi identity*

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$$

In our case, we will use the Lie product given by $[a, b] = ab - ba$, the *commutator* on an associative algebra.

A **representation** of $\mathfrak{g}$ is a realization of $\mathfrak{g}$ where the elements are given as linear maps on a vector space and the Lie product maps to the commutator. The action of a as a linear map on $\mathfrak{g}$ given by $b \to [a, b]$ is the *adjoint representation*, the mapping written as

$$(\mathrm{ad}\, a)(b) = [a, b]$$

Typically a Lie algebra is specified by prescribed commutation relations on a basis. Elements a and b *commute* if $[a, b] = 0$. Commutation relations between commuting elements are not explicitly indicated.

Throughout, we will use $\{\, \xi_1, \xi_2, \ldots, \xi_d \,\}$ as the basis for a d-dimensional Lie algebra. Then the Lie algebra is determined by the linear maps

$$(\mathrm{ad}\, \xi_k)(\xi_j) = [\xi_k, \xi_j] = \sum_i c^i_{kj} \xi_i$$

The coefficients c^i_{kj} are called the *structure constants* of the Lie algebra. These determine matrices of the adjoint representation, which we denote by $\check{\xi}_k$,

$$(\check{\xi}_k)_{ij} = c^i_{kj}$$

The fact that this is a representation follows from the Jacobi identity.

We work mainly with operators acting on polynomials and by extension to holomorphic functions defined in some given neighborhood of 0, which we call *locally holomorphic functions*. Alternatively, we can use formal power series. We refer to these three classes of objects as “suitable functions”.

The *Heisenberg-Weyl* algebra is given by the commutation rule

$$[\xi_3, \xi_1] = \xi_2$$

where it is implicit that ξ_2 is in the **center**, i.e., it commutes with ξ_1 and ξ_3. A matrix representation of the HW algebra is

$$\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the adjoint representation is different:

$$\check{\xi}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \check{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \check{\xi}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.3 REPRESENTATIONS OF HW

Now, notice that, for one variable, $\mathcal{R}$ and $\mathcal{V}$ acting on the vectors $|\,n\,\rangle$ satisfy $[\mathcal{V},\mathcal{R}] = I$, where I is the identity operator, i.e.,

$$(\mathcal{V}\mathcal{R} - \mathcal{R}\mathcal{V})|\,n\,\rangle = (n+1-n)|\,n\,\rangle = |\,n\,\rangle$$

And I commutes with all operators. So this is a representation of the HW algebra.

Remark. We will usually identify a multiple of the identity operator, say, cI, with the number c.

Let's use the realization of operators on polynomials as follows. We denote

X operator of multiplication by x, $\qquad$ D differentiation with respect to x

The basis is $|\,n\,\rangle = x^n$, with $|\,0\,\rangle = 1$. For polynomials in d variables, we have correspondingly X_i as multiplication by x_i and D_i partial differentiation with respect to x_i. Note the commutation relations

$$[D_j, X_i] = \delta_{ij}\, I$$

which prescribe the d-dimensional HW algebra. Any family of operators $\{\,\mathcal{R}_i, \mathcal{V}_j\,\}$ satisfying analogous commutation relations are called *boson operators* in quantum probability.

Note that any Lie algebra may be realized using first-order differential operators, **vector fields**, by the mapping,

$$\xi_i \leftrightarrow X_\lambda c_{i\mu}^{\lambda} D_\mu$$

called the Jordan map.

Notation. Our summation convention is: *Greek indices are always summed.*

When we have specific realizations of $\mathcal{R}$'s and $\mathcal{V}$'s acting on polynomials or spaces of functions, we denote the corresponding operators by R's and V's.

1.4 EXAMPLES IN PROBABILITY THEORY

Interesting examples are available from probability theory. We look at the *moment polynomials* arising from a distribution and we look at certain families of orthogonal polynomials for some probability distributions.

1.4.1 Gaussian

Let $p_t(dx) = \dfrac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}\, dx$ be the Gaussian density with mean zero, variance $t > 0$. Defining

$$h_n(x) = \int_{-\infty}^{\infty} (x+y)^n \; p_t(dy) = \int_{-\infty}^{\infty} (x + y\sqrt{t})^n p_1(dy) \tag{1.4.1.1}$$

we write this, using angle brackets to denote expected value, as

$$\langle (x + X_t)^n \rangle$$

where X_t is the corresponding Gaussian variable.

One sees that $V = D$, i.e., $Dh_n = n\, h_{n-1}$. The raising operator, R, is no longer X, but, in fact, is $R = X + tD$. This can be written as a **recurrence formula**. Another way to think of it as a realization of X in terms of R and V. From $X = R - tD = R - tV$ we have

$$x\, h_n = h_{n+1} - tn\, h_{n-1}$$

It turns out that a family of *Hermite polynomials* is orthogonal with respect to this distribution. They are given by

$$H_n(x) = \int_{-\infty}^{\infty} (x + iy)^n \; p_t(dy)$$

where $i = \sqrt{-1}$. From the second formulation in equation (1.4.1.1), we see that one has replaced $t \to -t$. Thus,

$$R = X - tD, \qquad V = D$$

for the Hermite polynomials. The recurrence is thus

$$x\, \psi_n = h_{n+1} + tn\, h_{n-1}$$

which is the **three-term recurrence** a family of orthogonal polynomials must satisfy.

Notice that $R^* = tV$, the operator adjoint to R with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(y) g(y)\, p_t(dy)$$

on polynomials or smooth functions with derivatives in $\mathrm{L}^2(\mathbf{R})$ of the corresponding Gaussian measure.

1.4.2 Poisson

Now consider the Poisson distribution, with

$$p_t(x) = e^{-t}\,\frac{t^x}{x!}$$

for integer $x \geq 0$. The *Poisson-Charlier polynomials* are orthogonal with respect to this Poisson distribution. They have generating function

$$G(v) = G(v; x, t) = (1 + v)^x\, e^{-vt} = \sum_{n\geq 0} \frac{v^n}{n!}\, P_n(x, t)$$

Verifying that $\langle G(v)G(w)\rangle$ is a function of vw alone shows that the polynomials P_n are indeed orthogonal. We have the *difference operator* expressed in terms of D by

$$(e^D - 1)f(x) = f(x + 1) - f(x)$$

on polynomials (in general, suitable functions). Notice the duality "multiplication by v" and the lowering operator $VP_n = nP_{n-1}$. Acting on G, we see that $V = e^D - 1$. The raising operator R is dual to differentiation with respect to v. In other words, the operators V, R are given by transferring the action of the HW representation "multiplication by v, differentiation with respect to v" via the generating function G to the sequence $\{\, P_n \,\}$. We must express the result of differentiating with respect to v in terms of X and D. Noting that

$$\frac{1}{1 + V} = e^{-D}$$

we find the HW representation

$$R = Xe^{-D} - tI, \qquad V = e^D - I$$

Solving, we find

$$X = (R + t)(1 + V) = t + R + RV + tV$$

Note that RV is the number operator. Thus the recurrence formula

$$xP_n = P_{n+1} + (n + t)P_n + ntP_{n-1}$$

1.4.3 Analytic HW realizations

To see why we expect that $[V, R] = I$ from the above formulas, we first note that for any polynomial $f(x)$, inductively it follows that $[V, f(R)] = f'(R)$ acting on kets. Dually, $[f(V), R] = f'(V)$. So the analogous formulas hold for all boson operators. These extend to suitable functions f. In particular, if $V(z)$ denotes a locally holomorphic function, such that $V(0) = 0$, $V'(0) \neq 0$, we define *canonical boson operators* associated to V by

$$R = XW(D), \qquad V = V(D)$$

where $W(D) = V'(D)^{-1}$, a notation to be used consistently throughout. The vacuum for the representation is the function equal to 1.

We always have the actions of R and V the same as the corresponding actions of the abstract operators $\mathcal{R}$ and $\mathcal{V}$, so we will use them interchangeably. The question is to express various operators in the representation of the Lie algebra in terms of R's and V's. In particular, of interest is the operator L, the adjoint of R, with respect to the inner product of the L^2 space of the underlying measure.

The properties of these operators are determined by the squared norms. Let $L\psi_n = b_n\psi_{n-1}$. Then the condition $L = R^*$ yields

$$\langle \psi_n, \psi_n \rangle = \gamma_n = b_n \gamma_{n-1}$$

With $\|\Omega\|^2 = 1$, $b_0 = 0$, we get

$$\gamma_n = b_1 b_2 \cdots b_n$$

From the Gaussian example, we have $R = X - tD$, $L = tD$. Thus, the squared norms are $\|H_n\|^2 = t^n\, n!$.

II. Coherent states and CSR's

The techniques we use are based on dualizing the action of operators through a generating function. For the HW algebra, the basic generating function is the exponential. Borrowing terminology from quantum physics, we call the generating function

$$\psi_v = e^{vR}\Omega$$

a *coherent state*. To get an inner product space, let

$$\langle \psi_n, \psi_m \rangle = \delta_{mn} n!$$

That is, $\gamma_n = n!$. In this way, we have $\langle R\psi_n, \psi_m \rangle = \langle \psi_n, V\psi_m \rangle$, i.e., R and V are adjoint with respect to the inner product.

Note that

$$V\psi_v = Ve^{vR}\Omega = e^{vR}V\Omega + [V, e^{vR}]\Omega = ve^{vR}\Omega = v\psi_v$$

Thus, on ψ_v we have

$$R\psi_v = \frac{\partial}{\partial v}\,\psi_v, \qquad V\psi_v = v\,\psi_v$$

rendering the duality $R \leftrightarrow$ differentiation, $V \leftrightarrow$ multiplication. Multiplying by $v^m/m!$ and summing gives $\langle \psi_n, \psi_v \rangle = v^n$. Multiplying by $w^n/n!$ and summing gives the inner product of coherent states

$$\langle \psi_w, \psi_v \rangle = \Upsilon_{wv} = e^{wv}$$

This function is called the *Leibniz function* as it embodies the Leibniz rule for differentiating the product of functions. It satisfies the partial differential equation

$$\frac{\partial \Upsilon}{\partial w} = \langle R\psi_w, \psi_v \rangle = v\,\Upsilon \tag{2.1}$$

which is another way of expressing that the adjoint of R on this space is V.

2.1 CSR'S FOR HW

The *coherent state representation* of an operator Q, say, is defined by

$$\langle Q\rangle_{wv} = \frac{\langle \psi_w, Q\psi_v\rangle}{\langle \psi_w, \psi_v\rangle}$$

For the Gaussian case, we have $R = X - tD$, $L = tD$. Proceeding as above, we have

$$\Upsilon_{wv} = \langle e^{wR}\Omega, e^{vR}\Omega\rangle = e^{twv}$$

Differentiating with respect to v yields the CSR of R, differentiating with respect to w yields the CSR of L. We find

$$\langle R\rangle_{wv} = tw, \qquad \langle L\rangle_{wv} = tv$$

The partial differential equation, cf. equation (2.1),

$$\frac{\partial\Upsilon}{\partial w} = tv\Upsilon$$

is another way to see that $L = tV$.

2.2 REPRESENTATIONS OF SL(2)

A second basic example is given by the Lie algebra sl(2) of 2×2 matrices with zero trace. The basis given by

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

satisfies the commutation relations

$$[\Delta, R] = \rho, \quad [\rho, R] = 2R, \quad [\Delta, \rho] = 2\Delta$$

An operator realization is given on functions of d variables:

$$\Delta = \tfrac{1}{2}\sum \frac{\partial^2}{\partial x_j^2}, \quad R = \tfrac{1}{2}\sum x_j^2, \quad \rho = \sum x_j \frac{\partial}{\partial x_j} + \frac{d}{2}$$

so that Δ is one-half times the Laplacian acting on radial functions and ρ is a variation on the number operator.

The basis vectors for the representation space are $\psi_n = R^n\Omega$, with $\Delta\Omega = 0$, $\rho\Omega = c\Omega$, for some scalar c. Inductively we find the commutation rule

$$[\Delta, R^n] = n(\rho + n - 1)R^{n-1}$$

Applying this to Ω yields

$$\Delta\psi_n = n(c + n - 1)\psi_{n-1}$$

Taking Δ as lowering operator L, we have $L = cV + RV^2$ and the squared norms

$$\|\psi_n\|^2 = n!(c)_n$$

Proceeding as before, we have, with $\psi_v = e^{vR}\Omega$,

$$\langle \psi_n, \psi_v \rangle = (c)_n v^n$$

Multiplying by $w^n/n!$ and summing gives the Leibniz function

$$\Upsilon_{wv} = (1 - wv)^{-c}$$

This satisfies the partial differential equation

$$\frac{\partial \Upsilon}{\partial w} = cv\Upsilon + v^2 \frac{\partial \Upsilon}{\partial v}$$

corresponding to the lowering operator $cV + RV^2$.

It is useful to denote the operators of the representation as $\hat{\Delta}, \hat{\rho}, \hat{R}$. So for the Lie algebra we have $\hat{R} = R$ and $\hat{\Delta} = cV + RV^2$. Then,

$$\hat{\rho} = [\hat{\Delta}, \hat{R}] = c + 2RV$$

We use this relation to find the CSR of $\hat{\rho}$, converting it via

$$(c + 2RV)\psi_v = (c + 2v\frac{\partial}{\partial v})\psi_v$$

to

$$\langle \rho \rangle_{wv} = c + \frac{2v}{\Upsilon} \frac{\partial \Upsilon}{\partial v}$$

Differentiating, we find directly the CSR's of $\hat{R}$ and $\hat{\Delta}$. Thus,

$$\langle R \rangle_{wv} = \frac{cw}{1 - wv}, \quad \langle \rho \rangle_{wv} = c\,\frac{1 + wv}{1 - wv}, \quad \langle L \rangle_{wv} = \frac{cv}{1 - wv}$$

III. Appell families

In one variable, Appell systems $\{h_n\}$ are typically defined by these properties:

$$\begin{aligned} h_n(x) &\text{ is a polynomial of degree } n, \quad n \geq 0 \\ Dh_n(x) &= n\, h_{n-1}(x) \end{aligned}$$

For $N \geq 1$, we have analogously

$$\begin{aligned} h_n(x) &\text{ is a polynomial of degree } n, \quad n \geq 0 \\ D_j h_n(x) &= n_j\, h_{n-\mathrm{e}_j}(x) \end{aligned} \tag{3.1}$$

where degree n means that the polynomial has top term x^n and other terms are of lower (total) degree. The condition on the degree is a non-degeneracy assumption that will become clear below.

Let $\{h_n(x)\}$ be an Appell system. Let

$$F(z,x) = \sum_{n\geq 0} z^n\, h_n(x)/n!$$

be a generating function for this system. The basic property (3.1) implies

$$\frac{\partial F}{\partial x_i} = z_i\, F \tag{3.2}$$

In general we have the form

$$F(z,x) = e^{z\cdot x}\, G(z)$$

The expansion $G(z) = \sum_{n\geq 0} z^n\, c_n/n!$ yields

$$h_n(x) = \sum_{m\geq 0} \binom{n}{m} c_m x^{n-m}$$

as a generic expression for Appell polynomials . The condition on the degree gives us $c_0 \neq 0$, i.e., $G(0) \neq 0$.

Next we notice that (3.2) may be read from right to left, i.e., multiplication by z_i acts as differentiation D_i. Now consider the action of $\partial/\partial z_j$:

$$\frac{\partial F}{\partial z_j} = \sum_{n\geq 0} z^n\, h_{n+\mathrm{e}_j}(x)/n!$$

i.e., $\partial/\partial z_j$ acts as a raising operator: $h_n \to h_{n+\mathrm{e}_j}$. With $G(0) \neq 0$ we can locally express $G(z) = e^{H(z)}$ so that F takes the form

$$F(z,x) = e^{z\cdot x + H(z)}$$

where we normalize by $G(0)=1$, $H(0)=0$. The operators D_j and $\partial/\partial z_j$ satisfy

$$D_j\,F = z_j\,F, \qquad \frac{\partial F}{\partial z_j} = \left(x_j + \frac{\partial H}{\partial z_j}\right)\,F$$

Thus, X_j denoting the operator of multiplication by x_j,

$$h_{n+\mathrm{e}_j} = \left(X_j + \frac{\partial H}{\partial D_j}\right)\,h_n$$

In summary,

3.1 Theorem. *For Appell systems, given $H(z)$ an arbitrary function holomorphic in a neighborhood of 0, the boson calculus is given by $R_i = X_i + \dfrac{\partial H}{\partial D_i}$, $V_i = D_i$, with states $|\,n\,\rangle = h_n$. The h_n have the generating function*

$$e^{z\cdot R}\,|\,0\,\rangle = e^{z\cdot x + H(z)} \;= \sum_{n\geq 0} \frac{z^n}{n!}\,h_n(x)$$

3.1 EVOLUTION EQUATION AND HAMILTONIAN FLOW

Now consider the evolution equation

$$\frac{\partial u}{\partial t} = H(D)u\,, \qquad u(x,0) = e^{z\cdot x}$$

with H locally holomorphic, as in the above discussion. We find

$$u(x,t) = e^{tH(D)}\,e^{z\cdot x} \;= e^{z\cdot x + tH(z)}$$

and expanding in powers of z, we have the Appell system

$$h_n(x,t) = e^{tH(D)}\,x^n$$

Note that in the previous section the t is absorbed into the H, alternatively, set to 1. The h_n satisfy $\partial u/\partial t = H(D)u$ with polynomial initial condition $u(x,0) = x^n$. Thus, we see Appell systems as *evolved powers.* The monomials x^n are built by successive multiplication by x_j, which we denote by the operators X_j: $X_j x^n = x^{n+\mathrm{e}_j}$. Here we conjugate by the flow e^{tH} :

$$h_{n+\mathrm{e}_j} = \left(e^{tH}\,X_j e^{-tH}\,\right)e^{tH}\,x^n = e^{tH}\,x^{n+\mathrm{e}_j}$$

I.e., the raising operator is given by $R = e^{tH} X e^{-tH}$. By the holomorphic operator calculus we have $[e^{tH}, X_j] = t(\partial H/\partial D_j)e^{tH}$, so that

$$R_j = X_j + t\frac{\partial H}{\partial D_j}$$

as we have seen previously (for $t = 1$).

The mapping $(X, D) \to (R, V)$ is given by the Heisenberg-Hamiltonian flow

$$R = e^{tH} X e^{-tH}\ , \qquad V = e^{tH} D e^{-tH}$$

which induces an automorphism of the entire Heisenberg-Weyl algebra. As t varies, writing $X(t)$ for R, we have the *Heisenberg-Hamiltonian equations of motion* (suppressing subscripts)

$$\dot{X} = [H, X] = \frac{\partial H}{\partial D}\,, \qquad \dot{D} = [H, D] = -\frac{\partial H}{\partial X}$$

where in the case $H = H(D)$, D remains constant so that $V = D$.

3.2 STOCHASTIC FORMULATION

Suppose that H comes from a family of probability measures p_t with corresponding random variables X_t by Fourier-Laplace transform as follows:

$$\langle e^{z\cdot X_t}\rangle = \int e^{z\cdot x}\, p_t(dx) = e^{tH(z)} \tag{3.2.1}$$

with $H(0) = 0$ here corresponding to the fact that the measures integrate to 1. Then

$$e^{z\cdot x+tH(z)} = \int e^{z\cdot(x+u)}\, p_t(du)$$

and

$$h_n(x,t) = \int (x+u)^n p_t(du) = \langle (x + X_t)^n\rangle \tag{3.2.2}$$

are *moment polynomials.*

3.2.1 Proposition. *In the stochastic case,*

$$h_n(x,t) = \sum_{m\geq 0} \binom{n}{m} \mu_m(t) x^{n-m}$$

where $\mu_m(t)$ are moments of the probability measure p_t.

Proof: Expand out equation (3.2.2). ■

The probability measures satisfying eq. (3.2.1) form a convolution family: $p_t * p_s = p_{t+s}$, with the X_t a corresponding stochastic process. In this sense, we see the $h_n(x,t)$ as averages of the evolution of the functions x^n along the paths of the stochastic process X_t.

Remark. Unless the measures p_t are *infinitely divisible,* one will not be able to take t to be a continuous variable. But in any case, we *always* have Appell systems, analytic in t. What can be guaranteed is that if $e^{H(z)} = \int e^{z\cdot x}\, p(dx)$ then this extends to the discrete-parameter process for integer-valued $t \geq 0$. For other values of t, the corresponding measures will not necessarily be probability measures, i.e., positivity may not hold.

3.3 CANONICAL SYSTEMS

The principal feature of (X, D) in the construction of Appell systems is that they are boson variables. We can make Appell systems starting from any canonical pair (Y, V), $Y = XW$, and evolve under the Heisenberg-Hamiltonian flow

$$\dot{Y} = [H, Y]\,, \qquad \dot{V} = [H, V]$$

For $H = H(D)$, $V = V(D)$ is invariant, while, writing $H' = (\partial H/\partial D_1, \ldots, \partial H/\partial D_N)$,

$$\begin{aligned} R = Y(t) &= e^{tH}\, X e^{-tH}\, W(D) = (X + tH')W \\ &= Y + tH'W \end{aligned} \tag{3.3.1}$$

The canonical Appell system is thus $h_n(x,t) = e^{tH}\, y_n(x)$.

3.3.1 Theorem. *For canonical Appell systems, we have:*

1. *The generating function*

$$e^{v\cdot R}\,|\,0\,\rangle = e^{x\cdot U(v) + tH(U(v))} = \sum_{n\geq 0} \frac{v^n}{n!}\, h_n(x,t)$$

2. *The relation*

$$e^{V(z)\cdot R}\,|\,0\,\rangle = e^{z\cdot x + tH(z)}$$

3. *The form of* X

$$X = RV' - tH'$$

Proof: The first relation comes by applying $e^{tH(D)}$ to the generating function for the y_n

$$e^{tH}\, e^{x\cdot U(v)} = e^{x\cdot U(v) + tH(U(v))}$$

on the one hand, which is then the generating function for $h_n(x,t) = e^{tH}\, y_n(x)$. Relation #2 follows from #1 by replacing $v = V(z)$. For #3, recall eq. (3.3.1),

$$R = (X + tH')W = (X + tH')(V')^{-1}$$

Solving for X yields the result. ■

IV. References

The main reference for these notes is the 3-volume work [13], [12], [11].

Overall references:

The operator calculus approach we use is very close in spirit to that developed by Rota [21] under the general term “umbral calculus, ” see [7].

Lie algebras and Lie groups in general: [15], [5], and [6].

Mathematical physics: [16] and [2].

Group theory and special functions: [17] is a must.

Coherent states: [19] gives a general approach

The coherent state representation is based on Berezin quantization, see [4].

Connections of our work to that of Hua [14] merits exploration.

2 Lie Algebras: Representations and Groups

I. Enveloping algebras

We are given a Lie algebra $\mathfrak{g}$ with basis $\{\,\xi_1,\ldots,\xi_d\,\}$. Originally we considered representations on the space generated by raising operators R_i, which commute. More generally, we can take the algebra generated by all of the elements ξ_i and find the action of $\mathfrak{g}$ on that space. The Poincaré-Birkhoff-Witt theorem says that there is an associative algebra with basis vectors given by ordered monomials

$$|\,n\,\rangle = \xi^n = \xi_1^{n_1}\cdots\xi_d^{n_d}$$

on which $\mathfrak{g}$ acts. If there are no other relations imposed besides the defining commutation relations of the algebra, this yields the *universal enveloping algebra*, $\mathcal{U}(\mathfrak{g})$. For any representation of $\mathfrak{g}$ as linear maps, the algebra they generate is the associated enveloping algebra. In general, the corresponding monomials will no longer be linearly independent.

Take, for example, the HW algebra with basis $\{Q,H,P\}$, where we may have, e.g., $Q=X$, $P=tD$, $H=tI$. Then, for $\mathcal{U}(\mathfrak{g}))$, we have the basis elements

$$|\,l,m,n\,\rangle = Q^l H^m P^n$$

By induction, we find $[P,Q^l]=lQ^{l-1}H$. Thus, the representation

$$\begin{aligned}
\hat{Q}|\,l,m,n\,\rangle &= |\,l+1,m,n\,\rangle \\
\hat{H}|\,l,m,n\,\rangle &= |\,l,m+1,n\,\rangle \\
\hat{P}|\,l,m,n\,\rangle &= |\,l,m,n+1\,\rangle + l|\,l-1,m+1,n\,\rangle
\end{aligned} \tag{1.1}$$

We now want to use duality techniques to see how multiplication by the basis elements ξ_i on $\mathcal{U}(\mathfrak{g})$ looks. Form the generating function

$$g(A) = \sum_{n\geq 0}\frac{A^n}{n!}\,\xi^n = \sum_{n_1,n_2,\ldots,n_d}\frac{(A_1\xi_1)^{n_1}}{n_1!}\cdots\frac{(A_d\xi_d)^{n_d}}{n_d!} = e^{A_1\xi_1}\,\cdots e^{A_d\xi_d} \tag{1.2}$$

This is an element of the group $\mathcal{G}$ generated by $\mathfrak{g}$, as it is a product of the one-parameter subgroups generated by the basis elements.

The *group law* is written in terms of the variables A as

$$g(A)g(A') = g(A\odot A')$$

Example. For the HW group we have, using the 3×3 matrix representation

$$e^{A_1\xi_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & A_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{A_2\xi_2} = \begin{pmatrix} 1 & 0 & A_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{A_3\xi_3} = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying these gives $g(A) = \begin{pmatrix} 1 & A_3 & A_2 \\ 0 & 1 & A_1 \\ 0 & 0 & 1 \end{pmatrix}$. Multiplying

$$g(A)g(B) = \begin{pmatrix} 1 & A_3 + B_3 & A_2 + B_2 + A_3B_1 \\ 0 & 1 & A_1 + B_1 \\ 0 & 0 & 1 \end{pmatrix}$$

which, comparing with the form of $g(A)$ yields the group law

$$\begin{aligned} (A \odot B)_1 &= A_1 + B_1 \\ (A \odot B)_2 &= A_2 + B_2 + A_3B_1 \\ (A \odot B)_3 &= A_3 + B_3 \end{aligned}$$

We introduce the boson operators R_i, V_j acting on the basis as

$$R_i|\, n\,\rangle = |\, n + \mathrm{e}_i\,\rangle, \qquad V_i|\, n\,\rangle = n_i|\, n - \mathrm{e}_i\,\rangle$$

The idea is to express the elements of $\mathfrak{g}$ in terms of R's and V's.

In this Chapter, X will denote a general element of $\mathfrak{g}$, with coefficients $\{\alpha_i\}$,

$$X = \alpha_\mu \xi_\mu$$

The operator of multiplication by x we will identify with x.

1.1 ADJOINT REPRESENTATION OF THE GROUP

The adjoint representation of the algebra extends to the group by exponentiating the corresponding matrices. Let Y and X denote any two elements of the Lie algebra. Then

$$e^{AY} X e^{-AY} = e^{A\,\mathrm{ad}\,Y} X$$

To see this, observe that $u = e^{AY} X e^{-AY}$ satisfies

$$\frac{\partial u}{\partial A} = Yu - uY = (\mathrm{ad}\,Y)u$$

with initial condition $u(0) = X$. We can express this as the series expansion

$$e^{AY} X e^{-AY} = X + \sum_{n\geq 1} \frac{A^n (\operatorname{ad} Y)^n}{n!} X$$

If X is written as a linear combination of basis elements ξ_i, then we get an expression of the form $C_\mu(A)\xi_\mu$. Specifically, to extend the adjoint representation to the group, we have functions $C^i_{kj}(A)$ determined by

$$\begin{aligned} e^{A\xi_k} \xi_j e^{-A\xi_k} &= C^1_{kj}(A)\xi_1 + C^2_{kj}(A)\xi_2 + \cdots + C^d_{kj}(A)\xi_d \\ &= C^\mu_{kj}(A)\xi_\mu \end{aligned}$$

The functions $C^i_{kj}(A)$ extend the structure constants of the Lie algebra to the adjoint group. We denote the corresponding matrices by $\check{C}_k(A)$, so that

$$(\check{C}_k(A))_{ij} = C^i_{kj}(A)$$

Note that $\check{C}_k(0)$ is the identity matrix for every k.

Example. For the HW algebra, we have $(\operatorname{ad} P)(Q) = H$, $(\operatorname{ad} P)^2(Q) = [P, H] = 0$. Thus,

$$e^{AP} Q e^{-AP} = Q + AH$$

For any suitable f,

$$e^{AP} f(Q) e^{-AP} = f(Q + AH)$$

Acting on the vacuum with $P\Omega = 0$, $H\Omega = 1$, $Q\Omega = x$, we get P acting as a translation operator

$$e^{AP} f(x) = f(x + A)$$

since Q and H commute, we may iteratively calculate $e^{AP} Q^n \Omega = (x + A)^n \Omega$.

Example. Introduce the *affine algebra*, aff(2), having basis elements ξ_1, ξ_2 satisfying commutation relation $[\xi_2, \xi_1] = \xi_1$. For example, we may take $\xi_1 = x$, multiplication by x, $\xi_2 = xD$, the number operator. We have

$$e^{A\xi_2} \xi_1 e^{-A\xi_2} = \xi_1 + A\xi_1 + \frac{A^2}{2}\xi_1 + \cdots = e^A \xi_1$$

I.e., we have the formula

$$e^{A\,xD} x\, e^{-A\,xD} = e^A x$$

Raising both sides to the n^{th} power, we have, for suitable functions f,

$$e^{A\,xD} f(x) e^{-A\,xD} = f(e^A x)$$

Applying this to the vacuum, 1, we get the action

$$e^{A\,xD} f(x) = f(e^A x)$$

Denoting $\lambda = e^A$ shows that xD generates the dilation group

$$\lambda^{xD} f(x) = f(\lambda x)$$

Example. For sl(2), we have $[\Delta, R] = \rho$, so $(\operatorname{ad}\Delta)^2(R) = 2\Delta$. Thus,

$$e^{A\Delta} R e^{-A\Delta} = R + A\rho + A^2\Delta$$

On the vacuum with $\Delta\Omega = 0$, $R\Omega = x$, $\rho\Omega = c\Omega$, we get

$$e^{A\Delta} f(x) = f(R + A\rho + A^2\Delta)\Omega$$

The action is not immediate as the elements do not commute. This is one of the motivations behind the splitting technique developed in this Chapter.

Example. Method of characteristics. An important case is the flow of a vector field. Write $X = \pi_\mu(x)\dfrac{\partial}{\partial x_\mu}$, where $\pi_i(x)$ are locally analytic functions. Note that $X1 = 0$. Let

$$x_i(t) = e^{tX} x_i\, e^{-tX}$$

Then for suitable functions f,

$$f(x(t)) = e^{tX} f(x) e^{-tX}$$

Thus the solution to

$$\frac{\partial u}{\partial t} = Xu, \qquad\qquad u(0) = f(x)$$

is given by

$$u = e^{tX} f(x) = f(x(t))1$$

Observe that, in fact,

$$(\operatorname{ad} X) f(x) = [X, f(x)] = \pi_\mu(x)[D_\mu, f(x)] = X f(x)$$

is a function, i.e., no derivative operators are involved. Iterating, we get

$$f(x(t)) = e^{tX} f(x) = f(x(t))1$$

as a function of x and t. And

$$\dot{x}_i(t) = e^{tX}[X, x_i]e^{-tX} = e^{tX}\pi_i(x)e^{-tX} = \pi_i(x(t))$$

holds for $x_i(t)$ as functions of x and t. These equations

$$\dot{x} = \pi(x)$$

are the *characteristic equations* for the flow generated by X. They are solved with initial conditions $x_i(0) = x_i$.

II. Dual Representations

Now we will find realizations of the Lie algebra as vector fields acting on functions of the coordinates A_i.

2.1 PI-MATRICES

To get the action of $\mathfrak{g}$ on the enveloping algebra, we define left and right multiplication operators acting on the generating function, group element, g according to

$$\xi_i g = \xi_i^{\ddagger} g, \qquad g\xi_i = \xi_i^* g$$

where now $\xi_i^{\ddagger}$ and ξ_i^* are to be expressed in terms of the variables A_i and the corresponding partial differentiation operators ∂_i.

One approach is to start with the action of the operators ∂_i on g. We think of ξ and $\xi^{\ddagger}$ as row vectors with components ξ_i and $\xi_i^{\ddagger}$ respectively. Then, we can write

$$e^{A\xi_i}\xi_j e^{-A\xi_i} = \xi_\mu C_{ij}^{\mu}(A) = (\xi \check{C}_i(A))_j$$

First, $\partial_1 g = \xi_1^{\ddagger}$. Next,

$$\begin{aligned}
\partial_2 g &= e^{A_1\xi_1}\xi_2 e^{A_2\xi_2}\cdots e^{A_d\xi_d} \\
&= e^{A_1\xi_1}\xi_2 e^{-A_1\xi_1} e^{A_1\xi_1} e^{A_2\xi_2}\cdots e^{A_d\xi_d} \\
&= (\xi^{\ddagger}\check{C}_1(A_1))_2\, g
\end{aligned}$$

For ∂_3 we find

$$\partial_3 g = (\xi^{\ddagger}\check{C}_1(A_1)\check{C}_2(A_2))_3$$

And so on. We write

$$\Pi^{\ddagger}_{1i}(A) = \delta_{i1},\, \Pi^{\ddagger}_{2i}(A) = \check{C}_1(A_1)_{i2},\, \Pi^{\ddagger}_{3i}(A) = (\check{C}_1(A_1)\check{C}_2(A_2))_{i3}\, \ldots$$

Generally,

$$\begin{aligned}
\partial_i &= (\xi^{\ddagger}\check{C}_1(A_1)\check{C}_2(A_2)\check{C}_3(A_3)\ldots\check{C}_{k-1}(A_{k-1}))_i \\
&= \Pi^{\ddagger}_{i\mu}(A)\xi^{\ddagger}_\mu
\end{aligned}$$

We can write these in terms of column vectors $\partial = (\partial_1, \partial_2, \ldots, \partial_d)$ and $\xi^{\ddagger}$ as

$$\partial = \Pi^{\ddagger}(A)\xi^{\ddagger}$$

Recalling that all of the $\check{C}$'s are the identity at $A = 0$, there is a neighborhood of the identity of the group where $\Pi^{\ddagger}(A)$ is invertible. The inverse of $\Pi^{\ddagger}(A)$ is the *pi-matrix* $\pi^{\ddagger}(A)$. We thus have the *left-dual vector fields*

$$\xi_i^{\ddagger} = \pi^{\ddagger}(A)_{i\mu}\partial_\mu$$

Similarly, we can convert ∂_i in terms of multiplying g on the right by ξ_i's , progressively pulling across the exponentials ending with $e^{A_d\xi_d}$, converting the adjoint actions into matrices $\check{C}_j(-A_j)$. We see that there is the pi-matrix $\pi^*(A)$ defining the *right-dual vector fields*

$$\xi_i^* = \pi^*(A)_{i\mu}\partial_\mu$$

The right dual mapping $\xi \to \xi^*$ gives a Lie homomorphism, i.e., $[\xi_i, \xi_j]^* = [\xi_i^*, \xi_j^*]$, while the action on the left reverses the order of operations, giving a Lie antihomomorphism $[\xi_i, \xi_j]^{\ddagger} = [\xi_j^{\ddagger}, \xi_i^{\ddagger}]$. An important feature is that the left and right actions commute. Thus, as vector fields, **every $\xi_i^{\ddagger}$ commutes with every ξ_j^*.**

2.2 SPLITTING LEMMA

As a vector space with basis $\{\xi_1, \ldots, \xi_d\}$, a typical element of $\mathfrak{g}$ has the form $X = \alpha_\mu\xi_\mu$. The α_i are called *coordinates of the first kind.* The variables $\{ A_i \}$ are *coordinates of the second kind.*

For the one-parameter subgroup generated by X we have

$$\begin{aligned} e^{tX} &= e^{A_1(t)\xi_1}e^{A_2(t)\xi_2}\cdots e^{A_d(t)\xi_d} \\ &= g(A(t)) \end{aligned}$$

When $t = 1$, we have the coordinate mapping

$$\alpha \leftrightarrow A$$

corresponding to the relation

$$e^X = g(A) = e^{A_1(\alpha)\xi_1}e^{A_2(\alpha)\xi_2}\cdots e^{A_d(\alpha)\xi_d}$$

Writing the group element $g(A)$ in terms of coordinates of the second kind, we have effectively factorized, "split", the exponential into a product of one-parameter subgroups. Thus the lemma relating the two types of coordinates is called the splitting lemma.

In fact, the factorization corresponds to the right and left dual vector fields and the flow of the group (composition) law. To see this, consider the left dual:

$$X\,g(A) = X^{\ddagger}g(A) = \alpha_\lambda\pi^{\ddagger}_{\lambda\mu}\partial_\mu\,g(A)$$

Denote $\alpha = (\alpha_1, \dots, \alpha_d)$, so that $t\alpha = (t\alpha_1, \dots, t\alpha_d)$ for a real parameter t. Note that $A(t) = A(t\alpha)$ as $X \to tX$ maps $\alpha \to t\alpha$.

Now let $x(t) = A(t\alpha) \odot A$ denote the 'flow of the group law', for t in some neighborhood of 0. And $g(x(t)) = g(A(t\alpha))g(A) = e^{tX} g(A) = e^{x_1(t)\xi_1} e^{x_2(t)\xi_2} \dots e^{x_d(t)\xi_d}$. Let's differentiate with respect to t.

$$\frac{d}{dt} g(x(t)) = e^{tX} X\, g(A) = X g(x(t)) = X^{\ddagger} g(x(t))$$

since X and $X^{\ddagger}$ commute. So, noting that $x(0) = A$, we have

$$g(x(t)) = e^{tX^{\ddagger}} g(A) = g(A(t\alpha) \odot A)$$

The characteristics for the flow generated by $X^{\ddagger}$ are given by

$$\dot{x}_i = \alpha_\lambda \pi^{\ddagger}_{\lambda i}(x)$$

A similar argument, writing $g(x(t)) = g(A)e^{tX}$, yields the corresponding result for $x(t) = A \odot A(t\alpha)$. So,

2.2.1 Lemma. Flow of the group
Let $X = \alpha_\mu \xi_\mu$. Let $A(\alpha)$ be the map of coordinates determined by

$$\exp(X) = g(A) = e^{A_1(\alpha)\xi_1} \cdots e^{A_d(\alpha)\xi_d}$$

Let $\odot$ denote the group law: $g(A)g(B) = g(A \odot B)$.

1. *Let $A(t) = A(t\alpha) \odot A$. Then $A(t)$ satisfies the equations $\dot{A}_j = \alpha_\lambda \pi^{\ddagger}_{\lambda j}(A)$, with initial condition $A(0) = A$.*

2. *Let $A(t) = A \odot A(t\alpha)$. Then $A(t)$ satisfies the equations $\dot{A}_j = \alpha_\lambda \pi^{*}_{\lambda j}(A)$, with initial condition $A(0) = A$.*

We may reformulate this in terms of vector fields:

2.2.2 Corollary.
1. *The integral curves of the vector field $X^{\ddagger} = \alpha_\lambda \pi^{\ddagger}_{\lambda\mu}(A)\partial_\mu$ are of the form $A(t\alpha) \odot A$.*

2. *The integral curves of the vector field $X^{*} = \alpha_\lambda \pi^{*}_{\lambda\mu}(A)\partial_\mu$ are of the form $A \odot A(t\alpha)$.*

Now follows

2.2.3 Splitting Lemma.

Let $X = \alpha_\mu \xi_\mu$. Consider the factorization

$$\exp(X) = g(A) = e^{A_1(\alpha)\xi_1} \cdots e^{A_d(\alpha)\xi_d}$$

Let $\tilde{\pi}$ denote the coefficient matrix (pi-matrix) of either the left or the right dual representation, Then the coordinate map $\alpha \to (A_1(\alpha), \ldots, A_d(\alpha))$ is determined as follows. Solve the differential equations

$$\dot{A}_j = \alpha_\lambda \tilde{\pi}_{\lambda j}(A), \qquad j = 1, \ldots, d$$

for A_i as functions of t with the initial conditions $A_1(0) = \cdots = A_d(0) = 0$. Then $A_i(\alpha) = A_i(t)\big|_{t=1}$, for $1 \le i \le d$.

Proof: For $\tilde{\pi} = \pi^\ddagger$, we have $A(1) = A(\alpha) \odot A$. With the initial variables $A_i = 0$, $1 \le i \le d$, we have $x(1) = A(\alpha)$ as required. Note that for $\tilde{\pi} = \pi^*$, the zero initial conditions yield the same result.

Since the flows with zero initial conditions are identical, we have the interesting

2.2.4 Corollary. *For the coordinate map $\alpha \to A$ of coordinates of the first kind to coordinates of the second kind, we have the identity*

$$\alpha_\lambda \pi^\ddagger_{\lambda j}(A(\alpha)) = \alpha_\lambda \pi^*_{\lambda j}(A(\alpha))$$

for $1 \le j \le d$.

Remark. Taking transposes, this may be reformulated as $\breve{\pi}(A(\alpha))\alpha = \alpha$, where $\breve{\pi}$ is the group element formed by exponentiating the adjoint representation. I.e., this shows invariance of the α's under the adjoint group.

The splitting lemma makes it expedient to find the pi-matrices. Here's the procedure:

1. Write $X = \alpha_\mu \xi_\mu$.
2. Calculate $g(A)$. Formally differentiate with respect to t.
3. Equate the result of step 2 with $Xg(A)$. Solve for $\dot{A}_i$.
4. Express the formulas for $\dot{A}_i$ as $\alpha_\mu \pi^\ddagger_\mu(A)$.
5. Similarly, use $g(A)X$ to find $\pi^*(A)$.

Example. For HW, we get $X = \begin{pmatrix} 0 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$. From our result for $g(A)$, we find

$$\dot{g} = \begin{pmatrix} 0 & \dot{A}_3 & \dot{A}_2 \\ 0 & 0 & \dot{A}_1 \\ 0 & 0 & 0 \end{pmatrix} = Xg = \begin{pmatrix} 0 & \alpha_3 & \alpha_2 + A_1\alpha_3 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$\begin{aligned} \dot{A}_1 &= \alpha_1 \\ \dot{A}_2 &= \alpha_2 + A_1\alpha_3 \\ \dot{A}_3 &= \alpha_3 \end{aligned}$$

We read off

$$\pi^{\ddagger}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{pmatrix}$$

Similarly, we find

$$\pi^{*}(A) = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solving the above equations for the left flow with initial conditions $A(0) = A$ we get

$$\begin{aligned} A_1(t) &= A_1 + \alpha_1 t \\ A_2(t) &= A_2 + \alpha_2 t + A_1\alpha_3 t + \alpha_1\alpha_3 t^2/2 \\ A_3(t) &= A_3 + \alpha_3 t \end{aligned}$$

With $t = 1$, $A = 0$, this gives the coordinate map $\alpha \to A$. Then at $t = 1$ we can verify that $A(1) = A(\alpha) \odot A$. Similar properties hold for the right flow.

Example. A matrix realization of aff(2) is given by

$$X = \begin{pmatrix} \alpha_2 & \alpha_1 \\ 0 & 0 \end{pmatrix}$$

The corresponding group element is

$$g(A) = \begin{pmatrix} e^{A_2} & A_1 \\ 0 & 1 \end{pmatrix}$$

The group law is

$$\begin{aligned} (A \odot B)_1 &= A_1 + B_1 e^{A_2} \\ (A \odot B)_2 &= A_2 + B_2 \end{aligned}$$

Equating $\dot{g} = Xg$ and $\dot{g} = gX$ we find the pi-matrices

$$\pi^{\ddagger} = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}$$

and

$$\pi^{*} = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

For the left flow, we have $\dot{A}_1 = \alpha_1 + \alpha_2 A_1$, $\dot{A}_2 = \alpha_2$ with solution

$$A_1(t) = A_1 e^{\alpha_2 t} + \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2 t} - 1\right), \qquad A_2(t) = A_2 + \alpha_2 t$$

For the right flow, we have $\dot{A}_1 = \alpha_1 e^{A_2}$, $\dot{A}_2 = \alpha_2$ with solution

$$A_1(t) = A_1 + \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2 t} - 1\right)e^{A_2}, \qquad A_2(t) = A_2 + \alpha_2 t$$

Now, setting $t = 1$ yields $A(\alpha) \odot A$ and $A \odot A(\alpha)$. Further, setting $A = 0$, we have the coordinate map

$$A_1(\alpha) = \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2} - 1\right), \qquad A_2(\alpha) = \alpha_2$$

And from this we can check consistency with the flow of the group.

2.3 DOUBLE DUAL

The right dual vector fields ξ_i^* give a Lie homomorphism. To get a Lie homomorphism from the left dual, we must dualize it. I.e., we rewrite the left dual in terms of boson operators R's and V's, exchanging $A \leftrightarrow V$, $\partial \leftrightarrow R$, ordering with all R's on the left. Thus, we let

$$\hat{\xi}_i = R_\mu \pi^{\ddagger}_{i\mu}(V)$$

This is the original action of multiplication by ξ_i in terms of R and V acting on the basis $|\, n\,\rangle$. So **we have calculated the action by multiplication of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$**.

We make the further observation that since R and V are boson variables, we may conveniently replace them by $R \to x$, $V \to D$ to get a realization of $\mathfrak{g}$ in terms of operators acting on functions of x.

Examples will show how this works.

Example. HW. Let's find ξ^*, $\xi^{\ddagger}$, and $\hat{\xi}$. First, let's go back to equation (1.1) and predict the double dual. We see that

$$\hat{Q} = R_1, \quad \hat{H} = R_2, \quad \hat{P} = R_3 + R_2 V_1$$

Now let's use the pi-matrices found previously.

$$\xi_1^* = \partial_1 + A_3\partial_2, \quad \xi_2^* = \partial_2, \quad \xi_3^* = \partial_3$$

And

$$\xi_1^\ddagger = \partial_1, \quad \xi_2^\ddagger = \partial_2, \quad \xi_3^\ddagger = A_1\partial_2 + \partial_3$$

which gives the double dual

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2, \quad \hat{\xi}_3 = R_2V_1 + R_3$$

We may write the double dual in terms of (x, D) as

$$\hat{\xi}_1 = x_1, \quad \hat{\xi}_2 = x_2, \quad \hat{\xi}_3 = x_2D_1 + x_3$$

Example. Affine. Using our previously found pi-matrices we have

$$\xi_1^* = e^{A_2}\partial_1, \quad \xi_2^* = \partial_2$$

And

$$\xi_1^\ddagger = \partial_1, \quad \xi_2^\ddagger = A_1\partial_1 + \partial_2$$

which gives the double dual

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_1V_1 + R_2$$

which we may write as

$$\hat{\xi}_1 = x_1, \quad \hat{\xi}_2 = x_1D_1 + x_2$$

which recovers our original formulation of aff(2) if we ignore x_2.

III. Matrix elements

Exponentiating the representation of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ we get a representation of $\mathcal{G}$ on $\mathcal{U}(\mathfrak{g})$. We define the matrix elements of the representation on $\mathcal{U}(\mathfrak{g})$ by

$$g(A)|\,n\,\rangle = \sum_m \left\langle {m \atop n} \right\rangle_A |\,m\,\rangle$$

These matrix elements are types of special functions and typically can be expressed in terms of generalized hypergeometric functions.

The following proposition gives a useful formula for calculating the matrix elements.

3.1 Principal formula.

With the standard basis $c_m(A) = A^m/m! = (A_1^{m_1}/m_1!)\cdots(A_d^{m_d}/m_d!)$ for polynomials in A, the matrix elements are given by

$$\left\langle {m \atop n} \right\rangle_A = (\xi^*)^n A^m/m!$$

where $(\xi^*)^n = (\xi_1^*)^{n_1}\cdots(\xi_d^*)^{n_d}$, basis monomials in terms of the right dual representation.

Proof: Write the product of group elements $g(A)$ and $g(B)$ as

$$\begin{aligned} g(A)g(B) &= g(A,\xi)\sum_n c_n(B)|\,n\,\rangle \\ &= \sum_n c_n(B)g(A)|\,n\,\rangle \\ &= \sum_{m,n} c_n(B)\left\langle {m \atop n} \right\rangle_A |\,m\,\rangle, \end{aligned}$$

since the A's and B's commute. On the other hand, pulling exponentials in B across $g(A)$ one at a time reconstitutes the group element $g(B)$ with ξ replaced by ξ^*. Denoting this by $g(B)^*$ we have

$$\begin{aligned} g(A)g(B) &= g(B)^* g(A) \\ &= \sum_{n,m} c_n(B)(\xi^*)^n c_m(A)|\,m\,\rangle \end{aligned}$$

Comparing these two expressions leads to the desired formula.

Example. An immediate consequence of this formula is that the right dual pi-matrices are matrix elements for transitions between basis elements. I.e.,

$$\pi^*_{ij} = \left\langle {\mathrm{e}_j \atop \mathrm{e}_i} \right\rangle$$

Proof: This follows from the principal formula thus

$$\left\langle {\mathrm{e}_j \atop \mathrm{e}_i} \right\rangle = \xi^*_i A_j = \pi^*_{i\lambda}\partial_\lambda A_j = \pi^*_{ij}$$

We mention some of the many interesting relations for the matrix elements that can now be deduced from the group law and the relations of the operators ξ^*. This approach to special functions is in the spirit of now classic work of Vilenkin, see Klimyk & Vilenkin [17].

3.1 ADDITION THEOREMS

Writing the group law (as in the above proof)

$$g(A)g(B) = \sum_{m,n} c_n(B)\left\langle {m \atop n} \right\rangle_A |\,m\,\rangle$$

and as

$$g(A \odot B) = \sum_m c_m(A \odot B)|\,m\,\rangle,$$

we read off the transformation formula

$$c_m(A \odot B) = \sum_n \left\langle {m \atop n} \right\rangle_A c_n(B)$$

that is, the coefficients c_n transform as a vector for the representation. Similarly,

$$g(A)g(B)|\, n \,\rangle = g(A \odot B)|\, n \,\rangle$$

yields the addition theorem

$$\left\langle {m \atop n} \right\rangle_{A \odot B} = \left\langle {m \atop \lambda} \right\rangle_A \left\langle {\lambda \atop n} \right\rangle_B$$

where in the implied summation λ is a multi-index. So these are indeed a matrix representation of the group acting on $\mathcal{U}(\mathfrak{g})$.

3.2 DIFFERENTIAL RECURRENCE RELATIONS

Define the matrix elements of left multiplication by ξ_i on $|\, n \,\rangle$ by

$$\xi_i |\, n \,\rangle = \sum_r M_{rn}(\xi_i) |\, r \,\rangle$$

Since the right dual representation gives a homomorphism of Lie algebras, we have

$$\begin{aligned} \xi_i^* \left\langle {m \atop n} \right\rangle_A &= \xi_i^* (\xi^*)^n c_m(A) \\ &= \sum_r M_{rn}(\xi_i)(\xi^*)^r c_m(A) \\ &= \sum_r \left\langle {m \atop r} \right\rangle_A M_{rn}(\xi_i) \end{aligned}$$

Now, recall that this action is the same as the double dual $\hat{\xi}_i = R_\mu \pi^\ddagger_{i\mu}(V)$ acting on the n-indices. In other words,

$$\xi_i^* \left\langle {m \atop n} \right\rangle_A = \hat{\xi}_i \left\langle {m \atop \mathbf{n}} \right\rangle_A$$

the boldface indicating that the multi-index n is varied.

Example. For the affine group, the principal formula gives the matrix elements

$$\left\langle {m_1, m_2 \atop n_1, n_2} \right\rangle_{A_1, A_2} = (\xi_1^*)^{n_1} (\xi_2^*)^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!)$$

Introduce the difference indices $\Delta = m - n = (m_1 - n_1, m_2 - n_2)$. Using the right dual we find

$$\begin{aligned}\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2} &= (e^{A_2}\partial_1)^{n_1}(\partial_2)^{n_2}(A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!) \\ &= e^{n_1 A_2}\frac{A_1^{\Delta_1}}{\Delta_1!}\,\frac{A_2^{\Delta_2}}{\Delta_2!}\end{aligned}$$

Bringing in the double dual, $\hat{\xi}_1 = R_1$, $\hat{\xi}_2 = R_2 + R_1 V_1$, we find the following differential recurrence relations:

$$\begin{aligned}(e^{A_2}\partial_1)\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2} &= \left\langle \begin{matrix} m_1, m_2 \\ n_1+1, n_2 \end{matrix} \right\rangle_{A_1,A_2} \\ \partial_2\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2} &= \left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2+1 \end{matrix} \right\rangle_{A_1,A_2} + n_1\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2}\end{aligned}$$

More generally, one finds rather involved types of functions of generalized hypergeometric type. Our approach provides a canonical formalism for expressing and discovering the properties these matrix elements satisfy as special functions. One can develop our approach further and find "pure" recurrence relations, not involving derivatives, that generalize the well-known 'contiguous relations' satisfied by classical hypergeometric functions.

IV. References

Enveloping algebras: [8]

Splitting techniques: [24] and [25] are originals

Mathematical physics: the left and right dual representations are used by Tomé [23] to develop path integrals on groups

3 Dual Vector Fields

I. DVFs

As in the double dual, in (x, D) variables, we have operators of the form

$$x_\mu W_\mu(D)$$

where W_i are suitable functions of D_i. We write optionally $a_\mu b_\mu$ as $a \cdot b$. The basic fact is the following primitive version of the Fourier transform:

$$x \cdot W(D)\, e^{A \cdot x} = W(A) \cdot \partial_A\, e^{A \cdot x} = x \cdot W(A)\, e^{A \cdot x} \tag{1.1}$$

where the components of ∂_A are $\partial_i = \partial/\partial A_i$.

What this does is exchange the operators in the (x, D) variables with corresponding operators in (A, ∂_A) variables. Thus each vector field has its dual and vice versa.

We abbreviate "dual vector field" by "dvf". Since the dvf's are our main interest, we write

$$Y_i = x_\mu W_{\mu i}(D), \qquad \tilde{Y}_i = W_{\mu i}(A)\partial_\mu$$

Til now we have been considering vector fields realizing a Lie algebra. Now we will be interested in families of commuting vector fields. We indicate a standard construction.

When discussing functions of D, we use the variable z for complex variables defining the function involved. For an operator $f(D)$, the function $f(z)$ is referred to as its *symbol*. The expression "canonical" refers to operators satisfying the boson commutation relations.

Start with a function $V(z) = (V_1(z), \ldots, V_N(z))$ holomorphic in a neighborhood of 0, with $V(0) = 0$, and the Jacobian matrix $V' = \left(\dfrac{\partial V_i}{\partial z_j}\right)$ nonsingular at 0.

$U(v)$ denotes the functional inverse of V, i.e., $z_j = U_j(V(z))$.

We call the V_i "canonical coordinates" or canonical functions. The associated dvf's are called the "canonical variables" and are given by

$$Y_j = x_\lambda W_{\lambda j}(D)$$

where $W(z) = V'(z)^{-1}$ is the matrix inverse to $V'(z)$.

1.1 Proposition. *The commutation relations*

$$[V_i(D), Y_j] = \delta_{ij} I$$

hold.

Proof: We have

$$[V_i(D), x_\lambda] W_{\lambda j} = (V')_{i\lambda} W_{\lambda j} = \delta_{ij} I$$

■

As the $V_i(D)$ mutually commute, to check that we indeed have a boson calculus, we need the commutativity of the Y's.

1.2 Proposition. *The variables $Y_1, \ldots, Y_N$ commute.*

Proof: Denoting differentiation by a comma followed by the appropriate subscript, we have

$$\begin{aligned}[Y_i, Y_j] &= [x_\lambda W_{\lambda i}, x_\mu W_{\mu j}] = x_\lambda W_{\lambda i,\mu} W_{\mu j} - x_\mu W_{\mu j,\lambda} W_{\lambda i} \\ &= x_\varepsilon W_{\varepsilon i,\mu} W_{\mu j} - x_\varepsilon W_{\varepsilon j,\mu} W_{\mu i}\end{aligned}$$

I.e., we need to show that $W_{ki,\mu} W_{\mu j} = W_{kj,\mu} W_{\mu i}$, that the expression $W_{ki,\mu} W_{\mu j}$ is symmetric in ij. Recall that if W depends on a parameter, t, say, then $W = Z^{-1}$ satisfies $\dot{W} = -W \dot{Z} W$. Thus, the relation $W = (V')^{-1}$ yields the matrix equation

$$\frac{\partial W}{\partial z_i} = -W \frac{\partial V'}{\partial z_i} W$$

which gives

$$\begin{aligned}\frac{\partial W_{ki}}{\partial z_\mu} W_{\mu j} &= -W_{k\varepsilon} \left(\frac{\partial V'}{\partial z_\mu} \right)_{\varepsilon\lambda} W_{\lambda i} W_{\mu j} \\ &= -W_{k\varepsilon} \frac{\partial^2 V_\varepsilon}{\partial z_\mu \partial z_\lambda} W_{\lambda i} W_{\mu j}\end{aligned}$$

so that the required symmetry follows from equality of the mixed partials of V. ■

From now on, W will refer to the inverse Jacobian of a given function V.

Example. If V is a linear mapping, $V_i(z) = S_{i\lambda} z_\lambda$, where S is an invertible constant matrix, with inverse T, we have $V' = S$. Thus

$$V_i(D) = S_{i\lambda} D_\lambda, \qquad Y_i = x_\mu T_{\mu i}$$

The inverse function is $U_i(v) = T_{i\lambda} v_\lambda$.

Example. As for the Poisson case, with $V(z) = e^z - 1$, we have $W(z) = e^{-z}$ and $U(v) = \log(1 + v)$.

II. Flow of a dual vector field

We would like to calculate the solution to

$$\frac{\partial u}{\partial t} = Yu, \qquad u(0) = f(x)$$

for the dvf $Y = v_\lambda Y_\lambda = v_\lambda x_\mu W_{\mu\lambda}(D)$.

Using equation (1.1), we have

$$e^{tY} e^{A\cdot x} = e^{t\tilde{Y}} e^{A\cdot x}$$

We know how to calculate the flow of the vector field $\tilde{Y}$ by the characteristic equations

$$\dot{A}_i = v_\lambda W_{i\lambda}(A)$$

Multiplying both sides by $V'(A)$ we get

$$\dot{A}_\mu V'_{k\mu} = v_k$$

Now the left-hand side is an exact derivative. I.e.,

$$\frac{d}{dt} V_k(A(t)) = v_k$$

Integrating, with initial conditions $A(0) = A$, we get

$$V(A(t)) = V(A) + tv$$

Solving, we have

$$A(t) = U(tv + V(A))$$

Thus,

$$e^{tY} e^{A\cdot x} = e^{t\tilde{Y}} e^{A\cdot x} = e^{x\cdot U(tv+V(A))} \tag{2.1}$$

This is our main formula. A main corollary is the action of the dvf Y on the vacuum function equal to 1. We get this by setting $A = 0$:

$$e^{tY} 1 = e^{x\cdot U(tv)}$$

For $d = 1$, we get

2.1 Main Formula.

For a canonical function $V(z)$, with $W(z) = 1/V'(z)$, $U(V(z)) = z$, let $Y = xW(D)$ be the associated canonical variable. Then we have

$$e^{vY} e^{Ax} = e^{xU(v+V(A))}$$

And, in particular,

$$e^{vY} 1 = e^{xU(v)}$$

III. Canonical polynomials

Now we get the basis for the vector space:

$$y_n(x) = Y^n 1$$

where the vacuum is the constant function equal to 1. These are polynomials in x, the associated *canonical polynomials*. They satisfy

$$Y y_n(x) = y_{n+1}(x), V(D) y_n(x) = n y_{n-1}(x)$$

providing a representation of the HW algebra.

We have the expansion

$$e^{v \cdot Y} 1 = e^{x \cdot U(v)} = \sum_{n \geq 0} \frac{v^n}{n!} \, y_n(x)$$

Example. With $V(z) = e^z - 1$, we have $Y = xe^{-D}$. Since $e^{-D} f(x) = f(x-1)$, we have

$$y_n(x) = Y^n 1 = xe^{-D} y_{n-1}(x) = x(x-1)\cdots(x-n+1) = x^{(n)}$$

the n^{th} factorial power. With $U(v) = \log(1+v)$, the expansion is

$$(1+v)^x = \sum_{n \geq 0} \frac{v^n}{n!} \, x^{(n)}$$

the standard binomial theorem.

3.1 CANONICAL POLYNOMIALS AND RANDOM WALKS

There is an interesting connection with random walks in the case when $W(z)$ is the moment generating function for a probability distribution. Let, in general,

$$W(z) = \sum_{n \geq 0} \frac{z^n}{n!} \, \mu_n$$

where, in the probabilistic case, μ_n would be the moments of a probability distribution.

In any case, define the generalized moments

$$\langle\!\langle X^n \rangle\!\rangle = \mu_n$$

and probabilistic case:

$$\langle X^n \rangle = \mu_n$$

For an analytic function f, expand

$$f(x+X)=\sum_{n=0}^{\infty}\frac{X^n}{n!}\,f^{(n)}(x)$$

where here X denotes a virtual or actual random variable.

Taking (generalized) expected value, we have the action of the operator $W(D)$ as a formal convolution operator

$$W(D)\,f(x)=\sum_{n=0}^{\infty}\frac{\mu_n}{n!}\,f^{(n)}(x)=\langle\!\langle f(x+X)\rangle\!\rangle$$

We extend the generalized averaging to several variables by taking them to be effectively independent:

$$\langle\!\langle X_1^{n_1}X_2^{n_2}\dots X_m^{n_m}\rangle\!\rangle=\mu_{n_1}\mu_{n_2}\cdots\mu_{n_m}$$

Then we have

3.1.1 Random walk formula.

The basic polynomials are given in the form of generalized factorials by

$$y_n(x)=\langle\!\langle x(x+X_1)(x+X_1+X_2)\cdots(x+X_1+X_2+\cdots+X_{n-1})\rangle\!\rangle$$

In the probabilistic case, we denote the random walk generated by the underlying distribution by $S_n=X_1+X_2+\cdots+X_n$, where the X_i are independent, identically distributed random variables with moment generating function equal to W. With $S_0=x$, the corresponding expectation value is denoted by $\langle\cdot\rangle_x$. Then the formula yields

$$y_n=\langle S_0S_1S_2\cdots S_{n-1}\rangle_x$$

Note that this is the product of consecutive variables of the random walk.

In the probabilistic case, write

$$W(D)=\int e^{uD}\,p(du)$$

Then

$$(xW(D))^n=x\int e^{u_1D}\,p(du_1)\cdots x\int e^{u_nD}\,p(du_n)$$

With $e^{uD}f(x)e^{-uD}=f(x+u)$, we get

$$(xW(D))^n=\int x(x+u_1)(x+u_1+u_2)\cdots(x+u_1+\cdots+u_{n-1})\cdot\exp\big((\sum_{j=1}^{n}u_j)D\big)\,p(du_1)\cdots p(du_n)$$

This is a formula for the operator Y^n. I.e.,

$$Y^n = \langle S_0 S_1 S_2 \cdots S_{n-1} e^{S_n D} \rangle_x$$

Applying this to the constant function 1 yields the formula stated above.

We thus have

$$e^{xU(v)} = 1 + x \sum_{n=0}^{\infty} \frac{v^n}{n!} \langle \prod_{j=1}^{n-1} (x + S_j) \rangle_0$$

Example. Exponential random walk and Bessel polynomials

With $W(z) = (1 - qz)^{-1}$, an exponential distribution with mean q, we get

$$V = z - qz^2/2\,, \qquad U = \frac{1 - \sqrt{1 - 2qv}}{q}$$

Thus, with T_1, T_2,…,T_n,… independent exponentials with mean q we have

$$\langle T_1(T_1 + T_2) \cdots (T_1 + T_2 + \cdots + T_n) \rangle = n! \binom{2n}{n} \left(\frac{q}{2}\right)^n$$

Now, scaling out q, consider $V = z - z^2/2$, $U = 1 - \sqrt{1 - 2v}$. From the classical theory of random walks we have

$$\frac{(1 - \sqrt{1 - 2v})^n}{\sqrt{1 - 2v}} = \sum_{p \geq 0} \frac{v^{n+p}}{2^p} \binom{n + 2p}{p}$$

This gives the expansion of

$$\frac{1}{\sqrt{1 - 2v}} e^{x(1 - \sqrt{1 - 2v})}$$

which is the generating function for Bessel polynomials $\theta_n(x)$. Differentiating $e^{x(1-\sqrt{1-2v})}$ with respect to v and integrating back we find

$$e^{x(1 - \sqrt{1 - 2v})} = 1 + \sum_{n \geq 1} \frac{x^n}{n!} \sum_{p \geq 0} \frac{n}{n + p} \frac{v^{n+p}}{2^p} \binom{n + 2p - 1}{p}$$

Thus, we have

$$y_n(x) = \sum_p \binom{n + 2p - 1}{p} 2^p (\tfrac{1}{2})_p \, x^{n-p}$$

Example. Cayley example

With $V(z) = z\, e^{-z}$, we get $W(z) = e^z (1 - z)^{-1}$, so that the corresponding probability distribution is an exponential with mean 1 shifted by 1. Checking that

$$y_n(x) = x(x + n)^{n-1}$$

we find

$$n^{n-1} = \langle (1 + T_1)(2 + T_1 + T_2) \cdots (n - 1 + T_1 + T_2 + \cdots + T_{n-1}) \rangle$$

We can expand $e^{xU(v)}$ in powers of x

$$e^{xU(v)} = \sum_{n\geq 0} \frac{x^n}{n!}\,(U(v))^n$$

Another way to think of this is by applying the operator $g(D)$ to the expansion in powers of v and evaluating at $x = 0$

$$g(U(v)) = \sum_{n\geq 0} \frac{v^n}{n!}\,g(D)y_n(0)$$

which is the Taylor expansion of the composition of g with $U(v)$. This is an approach to inversion alternative to Lagrange's method. In particular, the expansion of $U(v)$ itself is the coefficient of x in the expansion of $e^{xU(v)}$ in powers of v:

$$U(v) = \sum_{n\geq 0} \frac{v^n}{n!}\,y_n'(0)$$

In the random walk formulation, we thus have

$$U(v) = \sum_{n=1}^{\infty} \frac{v^n}{n!}\,\langle \prod_{j=1}^{n-1} S_j\rangle_0$$

Example. Given an analytic moment generating function $W(z)$, we can form

$$V(z) = \int_0^z \frac{du}{W(u)}$$

And the inverse of V is given by the above formula. In particular, if $V'(x)$ is a density function, we have the expansion for the inverse distribution function.

Example. Gaussian random walk

With $W(z) = e^{z^2/2}$, we get V as the distribution function of a standard Gaussian, modulo a factor of $\sqrt{2\pi}$. Thus, we have the expansion of the inverse Gaussian distribution in terms of (i) the values $y_n'(0)$ or (ii) in terms of the Gaussian random walk.

3.2.1 Dual approach

Going back to equation (2.1), we have, for $d = 1$,

$$e^{vY} e^{Ax} = e^{v\tilde{Y}} e^{Ax} = e^{xU(v+V(A))} = \sum_{m\geq 0} \frac{(v+V(A))^m}{m!}\, y_m(x)$$

Thus differentiating n times with respect to v and letting $v = A = 0$

$$y_n(x) = (W(A)\partial_A)^n e^{Ax}\Bigg|_{A=0}$$

We see that differentiating with respect to v is the same as acting with Y so y_m shifts up every time. Differentiating n times with respect to v and letting $v = 0$ yields

$$(\tilde{Y})^n e^{Ax} = \sum_{m\geq 0} \frac{V(A)^m}{m!}\, y_{m+n}(x)$$

which gives the action of $\tilde{Y}^n$ on the exponential. Applying $g(D)$ yields

$$(\tilde{Y})^n g(A) e^{Ax} = \sum_{m\geq 0} \frac{V(A)^m}{m!}\, g(D) y_{m+n}(x)$$

We want to let $x = 0$. Note that $\tilde{Y}$ obeys the Leibniz rule, just as ∂_A itself does. I.e., $\tilde{Y}$ is a *derivation*. If we apply $\tilde{Y}$ to e^{Ax} it will bring down a factor of x. That is, the surviving terms involve $\tilde{Y}$ applied only to g. We have thus

$$(\tilde{Y})^n g(A) = \sum_{m\geq 0} \frac{V(A)^m}{m!}\, g(D) y_{m+n}(0)$$

which gives the action of $(\tilde{Y})^n$ on an arbitrary function g.

We can start directly from equation (2.1), applying $g(D)$ to get

$$e^{v\tilde{Y}} g(A) e^{Ax} = g(U(v+V(A))) e^{xU(v+V(A))}$$

This illustrates the general fact that the exponential of a derivation is a homomorphism.

First let $x = 0$, then let $A = 0$ to get

$$e^{v\tilde{Y}} g(0) = g(U(v)))$$

I.e.,

$$g(U(v)) = \sum_{n\geq 0} \frac{v^n}{n!}\, \tilde{Y}^n g(0)$$

(formula suggested by D. Dominici)

In particular, let $g(D) = D$. Then $\tilde{Y}A = W(A)$, so, for $n \geq 1$,

$$(\tilde{Y})^n A\Big|_{A=0} = (\tilde{Y})^{n-1} W(0)$$

gives the coefficient of $v^n/n!$ in the expansion of $U(v)$.

Example. For $V(z) = 1 - e^{-z}$, we have $\tilde{Y} = e^A \partial_A$. With $U(v) = -\log(1-v)$, we get

$$U(v)^m = \sum_{n\geq 0} \frac{v^n}{n!} (e^A \partial_A)^n A^m \Big|_{A=0}$$

On the other hand, we have

$$y_n(x) = (xe^D)^n 1 = x(x+1)\cdots(x+n-1) = (x)_n = \sum_k S_{nk} x^k$$

where S_{nk} are the absolute values of Stirling numbers of the first kind. Hence,

$$D^m y_n(0) = m!\, S_{nm}$$

And we get

$$(-\log(1-v))^m = \sum_{n\geq 0} \frac{v^n}{n!} (e^A \partial_A)^n A^m \Big|_{A=0} = \sum_{n\geq 0} \frac{v^n}{n!}\, m!\, S_{nm}$$

another variation on the binomial theorem as seen by expanding $(1-v)^{-x}$.

Finally, observe that our approach applies equally well in d variables, with n as multi-index and $Y^n = Y_1^{n_1} \cdots Y_d^{n_d}$, as it is based on equation (2.1) which holds in all dimensions. It is essential that $Y = v_\lambda Y_\lambda$, where Y_i generate an abelian algebra.

IV. References

Our main reference is [10].

Closely related to the material of this chapter is Winkel's [26] and [27].

[22] develops several formulas for the y_n in the framework of umbral calculus, including an equivalent to our formulation as generalized expectation of the product along a random walk. Also see [20].

4 Polynomials

We discuss two more special types of families of polynomials to go with the Appell families already introduced: orthogonal families, and canonical families associated to Lie algebras.

I. Orthogonal families

Orthogonal polynomials (in one variable) may be described in terms of Fourier-Laplace transforms as follows. Given a measure $p(dx)$, the functions $\phi_n(x)$ are orthogonal to all polynomials of degree less than n if and only if

$$\int_{-\infty}^{\infty} e^{sx}\,\phi_n(x)\,p(dx) = V_n(s)$$

such that $V_n(s)$ has a zero of order n at $s = 0$. The proof is immediate from

$$\int_{-\infty}^{\infty} x^k \phi_n(x)\,p(dx) = \left(\frac{d}{ds}\right)^k \bigg|_0 V_n(s)$$

Thus, if the $\phi_n(x)$ are polynomials, they form a sequence of orthogonal polynomials.

1.1 ORTHOGONALITY AND CONVOLUTION

We sketch a 'group theory' construction, using Fourier-Laplace transform and convolution that builds an orthogonal system from a given one. It is closely related to the reduction of the tensor product of two copies of the given L^2 space as in the construction of Clebsch-Gordan coefficients.

Remark. In this section, unless otherwise indicated, we will discuss the $N = 1$-dimensional case, for convenience. The constructions indicated hold for $N > 1$ as well, appropriately modified.

1.1.1 Convolutions and Orthogonal functions

Start with a family of functions, *kernels*,

$$K(x, z, A)$$

where A indicates some parameters, that form a group under convolution

$$\int_{-\infty}^{\infty} K(x-y, z, A)K(y, z', A')\,dy = K(x, z+z', A'')$$

(The integration here can be replaced analogously by a summation.) This means that the Fourier-Laplace transforms form a multiplicative family. Let

$$\hat{K}(s,z,A) = \int_{-\infty}^{\infty} e^{sy}\, K(y,z,A)\, dy$$

Then

$$\hat{K}(s,z,A) \times \hat{K}(s,z,' A') = \hat{K}(s,z+z',A'')$$

Form the product

$$K(x-y,-z,A)K(y,z,A')$$

This integrates to $K(x,0,A'')$ which is independent of z. This is the generating function for the orthogonal functions we are looking for:

$$K(x-y,-z,A)K(y,z,A') = \sum z^n H_n(x,y;A,A') \tag{1.1.1.1}$$

By construction, the integral

$$\int_{-\infty}^{\infty} H_n(x,y;A,A')\, dy = 0$$

for every $n > 0$. To get orthogonality of H_n with respect to all polynomials of degree less than n, consider

$$\sum z^n \int_{-\infty}^{\infty} y^k H_n(x,y;A,A')\, dy = \int_{-\infty}^{\infty} y^k K(x-y,-z,A)K(y,z,A')\, dy$$

where the terms of the summation must vanish for $k < n$. I.e., this must reduce to a polynomial in z of degree k. Or one can take the transform

$$\int_{-\infty}^{\infty} e^{sy}\, K(x-y,-z,A)K(y,z,A')\, dy$$

which has to be of the form such that the powers of z have factors depending on s so that each degree in z has a factor with a zero of at least that order in s, as observed in the remarks above.

1.1.2 Probabilities and means

Here is a general construction of kernels. Take any probability distributions whose means form an additive group. Suppose that they have densities. Then the kernels are of the form $K(x,z,A)$ where z is the mean, and A, e.g., is the variance, or other parameters determining the distribution. One example is provided by the Gaussian distributions:

$$K(x,z,A) = \frac{e^{-(x-z)^2/(2A)}}{\sqrt{2\pi A}}$$

Since means and variances are additive, you have a convolution family as required. In general, it may not always be possible to parametrize the family in terms of the means.

1.1.3 Bernoulli systems

Here we make a definition that applies for $N \geq 1$.

1.1.3.1 Definition. A *Bernoulli system* is a canonical Appell system such that the basis $\psi_n = R^n\Omega$ is orthogonal.

For $N = 1$, we have the binomial distributions, corresponding to Bernoulli trials, hence the name. We renormalize ψ_n and define a new generating function.

1.1.3.2 Definition. Define the basis $\phi_n = n! \times \psi_n/\gamma_n$, where $\gamma_n = \langle\psi_n, \psi_n\rangle$ are the squared norms of the ψ_n. The generating function ω^t is defined as

$$\omega^t(y,x) = \sum_{n\geq 0} \frac{y^n}{n!}\,\phi_n \tag{1.1.3.1}$$

Now we have an important property of ω^t.

1.1.3.3 Proposition. *Consider a Bernoulli system, in $N \geq 1$ dimensions, with canonical operator V and Hamiltonian H. I.e.,*

$$e^{z_\mu x_\mu - tH(z)} = \sum_{n\geq 0} \frac{V(z)^n}{n!}\,\psi_n$$

Let the basis ϕ_n and the function ω^t be as above. Then we have the Fourier-Laplace transform

$$\int e^{\zeta y}\,\omega^t(z,y)\,p_t(dy) = e^{zV(\zeta)+tH(\zeta)}$$

Proof: The integral on the left-hand side is the inner product

$$\langle e^{\zeta X}\,\Omega, \omega^t(z,X)\Omega\rangle = e^{tH(\zeta)}\,\langle e^{V(\zeta)R}\,\Omega, \omega^t(z,X)\Omega\rangle$$

By orthogonality, and the definition of ω^t, eq. (1.1.3.1), the inner product reduces to

$$\sum_{n\geq 0} \frac{z^n V(\zeta)^n n!\,\gamma_n}{n!\,n!\,\gamma_n} = e^{zV(\zeta)}$$

as required. ∎

Now go back to the case $N = 1$. Expanding in powers of z yields the relation

$$\int_{-\infty}^{\infty} e^{sy}\,\phi_n(y)\,p_t(dy) = V(s)^n\,e^{tH(s)}$$

so that $V(0) = 0$ is all we need to conclude that the ϕ_n are an orthogonal family. We take t as our parameter A and

$$K(x, z, A) = \omega^A(z, x) p_A(x) \tag{1.1.3.2}$$

writing $p_t(dx) = p_t(x)dx$ in the sense of distributions in the case of discrete spectrum (e.g., the Poisson case). In the case when $\omega^A(z, x) \geq 0$, these are a family of probability measures as noted in example 1, with mean $z + \mu t$, and variance $z + \sigma^2 t$, where μ and σ^2 are the mean and variance respectively of p_1.

We thus have from the basic construction, eq. (1.1.1.1),

$$K(x - y, -z, A)K(y, z, B) = \omega^A(-z, x - y)\omega^B(z, y)\, p_A(x - y)p_B(y) \tag{1.1.3.3}$$

Substituting in the expansions of the ω's, equation (1.1.3.1), yields

$$\sum_{n\geq 0} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k(x - y, A)\phi_{n-k}(y, B)\, p_A(x - y)p_B(y)$$

Thus, the functions $H_n(x, y; A, B)$ take the form

$$H_n(x, y; A, B) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k(x - y, A)\phi_{n-k}(y, B)\, p_A(x - y)p_B(y)$$

with corresponding orthogonal polynomials

$$\phi_n(x, y; A, B) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k(x - y, A)\phi_{n-k}(y, B)$$

and measure of orthogonality $p_A(x - y)p_B(y)$. (Proof of orthogonality is based on an addition formula for $V(s)$.)

The convolution property of the family p_t shows that

$$\int_{-\infty}^{\infty} p_A(x - y)p_B(y)\, dy = p_{A+B}(x)$$

and thus, that we can normalize to give a probability measure of the form

$$p_A(x - y)p_B(y)/p_{A+B}(x)$$

For the Meixner classes, i.e., the Bernoulli systems in one variable corresponding to sl(2), we have the corresponding classes generated as follows:

Gaussian $\longrightarrow$ Gaussian
Poisson $\longrightarrow$ Krawtchouk
Laguerre $\longrightarrow$ Jacobi
Binomial (3 types) $\longrightarrow$ Hahn (3 types)

Observe that for the binomial types, this is essentially the construction of Clebsch-Gordan coefficients for (real forms of) sl(2). This construction works for the multinomial case as well.

1.1.4 Associativity construction

Corresponding to associativity of the convolution family, we form

$$K(x-y,-z,A+A')K(y,z,A'')\,, \quad \text{and} \quad K(x-y,-z,A)K(y,z,A'+A'')$$

These both integrate to $K(x,0,A+A'+A'')$. The corresponding $H_n(x,y;A+A',A'')$, $H_n(x,y;A,A'+A'')$ provide two orthogonal families for $L^2(dy)$. The question is to find the unitary transformation between the two bases, analogous to the construction of Racah coefficients.

For Bernoulli systems, denote the squared norms

$$\gamma_n(A,B)=\int_{-\infty}^{\infty}\phi_n(x,y;A,B)^2\,p_A(x-y)p_B(y)\,dy$$

Then we have the generating function for the unitary matrix U_{mn} connecting the combined systems corresponding to $A+B+C=(A+B)+C=A+(B+C)$, via equations (1.1.3.2), (1.1.3.3),

$$\begin{aligned}\int_{-\infty}^{\infty}\omega^A(-z,x-y)\omega^{B+C}(z,y)\omega^{A+B}(-w,x-y)\omega^C(w,y)&\\ \times\sqrt{p_A(x-y)p_{B+C}(y)p_{A+B}(x-y)p_C(y)}\,dy&\\ =\sum_{m,n}z^m w^n\sqrt{\gamma_m(A,B+C)}U_{mn}\sqrt{\gamma_n(A+B,C)}&\end{aligned}$$

For the binomial distributions, these will yield the usual Racah coefficients and connections with Wilson polynomials.

II. Appell states

This provides a unified picture of classes of orthogonal polynomials connected with integral transforms. The Fourier-Laplace transform leads to the Meixner classes which give the Bernoulli systems in one dimension.

2.1 DEFINITION OF GENERAL APPELL STATES

Given a probability measure, $p(dx)$, we take a family of square-integrable functions $F(s,x)$. As usual, we denote integration with respect to p as the expected value, and we set, for the given family $F(s,\cdot)$

$$M(s)=\langle F(s,X)\rangle$$

2.1.1 Definition. The *Appell states* with respect to the measure p and the family F are the functions

$$\Psi_s(x) = F(s,x)/M(s)$$

That is, the Ψ_s are the functions F normalized to have unit expectation. The term *states* comes from physics denoting a function of unit norm in L^2 of p. The idea is that the state is a line or ray in the vector space, an equivalence class of functions up to multiplication by scalars. Note here that the Ψ_s remain invariant if the functions F are multiplied by scalars.

Typical choices of the family F are $F(s,x) = e^{sx}$, corresponding to Fourier-Laplace transforms, and $F(s,x) = (1-sx)^{-1}$, corresponding to Stieltjes transforms. More precisely, the Stieltjes transforms arise from the family $1/(s-x)$. If X is an element in a Lie algebra and ψ_0 is an element in a representation space of the algebra, the states corresponding to $e^{sX}\,\psi_0$ are *generalized coherent states*. The coherent state representation is thus in the spirit of the Appell transform for the family e^{sx} .

The Appell states are used to define transforms of operators acting on functions of x, typically L^2 of p.

2.1.2 Definition. The *Appell transform* of an operator Q is the function of a, b given by

$$\langle Q\rangle_{ab} = \frac{\langle \Psi_a, Q\Psi_b\rangle}{\langle \Psi_a, \Psi_b\rangle}$$

Thus, these are the normalized matrix elements of the operator Q with respect to the Appell states.

2.2 APPELL STATES AND ORTHOGONAL POLYNOMIALS

The main feature is that the family $F(s,x)$ has the property of being the eigenfunctions of an operator X_s acting on functions of x:

$$X_s\,F(s,x) = x\,F(s,x)$$

Denote the family of orthogonal polynomials with respect to p by $\{\,\phi_n\,\}$ with squared norms $\gamma_n = \|\phi_n\|^2$. We define the transforms

$$\langle \phi_n, \Psi_s\rangle = V_n(s) \tag{2.2.1}$$

Thus, we have the expansion (in general, under assumption of completeness of the ϕ_n)

$$\Psi_s = \sum_{n\geq 0} V_n(s)\phi_n(x)/\gamma_n$$

In terms of the family F, we have

$$F(s,x) = M(s) \sum_{n\geq 0} V_n(s)\phi_n(x)/\gamma_n$$

The orthogonal polynomials satisfy a three-term recurrence of the form

$$x\phi_n = c_n\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1}$$

with initial conditions $\phi_{-1} = 0$, $\phi_0 = 1$. Observe that the recurrence relation implies $\phi_1(x) = (x - a_0)/c_0$. We have

2.2.1 Theorem. *Let $F(s,x)$ satisfy $F(0,x) = 1$, $X_s F(s,x) = xF(s,x)$. Then*

$$M(s)^{-1}X_s(M(s)V_n(s)) = c_nV_{n+1} + a_nV_n + b_nV_{n-1}$$

with the initial conditions $V_0 = 1$, $V_1 = c_0^{-1}(M^{-1}X_sM - a_0)$.

Proof: With $\phi_0 = 1$, setting $n = 0$ in eq. (2.2.1) yields $V_0 = 1$. For general n, write eq. (2.2.1) in the form

$$\langle \phi_n, F(s,X)\rangle = M(s)V_n(s)$$

and apply X_s to get

$$\langle \phi_n, X\,F(s,X)\rangle = X_s(M(s)V_n(s))$$

Dividing out $M(s)$ yields

$$\langle \phi_n, X\,\Psi_s\rangle = M(s)^{-1}X_s(M(s)V_n(s))$$

Now use the recurrence formula for ψ_n on the left-hand side and apply eq. (2.2.1) to get the result. For $n = 0$, this procedure yields

$$\langle X\,\Psi_s\rangle = M(s)^{-1}X_sM(s)$$

But eq. (2.2.1) for $n = 1$ says $\langle \phi_1, \Psi_s\rangle = V_1$. Writing $X = c_0\phi_1 + a_0$ and taking inner products with Ψ_s thus gives

$$M(s)^{-1}X_sM(s) = c_0V_1(s) + a_0 \tag{2.2.2}$$

and the result follows. ∎

We illustrate for the Meixner case.

2.2.1 Fourier-Laplace transforms and Meixner systems

Taking the family $F(s,x) = e^{sx}$, we have $M(s) = \langle e^{sX} \rangle$ is the usual moment generating function or Fourier-Laplace transform. The operator X_s here is d/ds, the first derivative operator. We have

2.2.1.1 Theorem. *The exponential function e^{sx} has the expansion in orthogonal polynomials*

$$e^{sx} = M(s) \sum_{n\geq 0} V_n(s)\phi_n(x)/\gamma_n$$

where the coefficients V_n, $n \geq 1$, satisfy the recurrence formula

$$V_n' + c_0 V_1 V_n = c_n V_{n+1} + (a_n - a_0)V_n + b_n V_{n-1}$$

with $V_0(s) = 1$ and

$$V_1(s) = c_0^{-1}\left(\frac{M'(s)}{M(s)} - a_0\right)$$

Proof: In the proof of Theorem 2.2.1, eq. (2.2.2), yields, with $X_s = d/ds$,

$$M'(s)/M(s) = c_0 V_1(s) + a_0$$

This gives the formula for V_1. As well, for the left-hand side of the recurrence, we have

$$M(s)^{-1} X_s(M(s)V_n(s)) = (M'(s)/M(s))V_n(s) + V_n'(s)$$

and substituting back in the expression in terms of V_1, the result follows. ■

Meixner systems arise when we have the special form

$$V_n(s) = V(s)^n$$

where, in particular, $V_1(s) = V(s)$. From the above theorem we have

2.2.1.2 Theorem. *For Meixner systems we have the expansion*

$$e^{sx} = M(s) \sum_{n\geq 0} V(s)^n \phi_n(x)/\gamma_n$$

where

$$V(s) = c_0^{-1}\left(\frac{M'(s)}{M(s)} - a_0\right)$$

satisfies the Riccati differential equation

$$V' = \gamma + 2\alpha V + \beta V^2$$

and the recurrence formula for the orthogonal polynomials is of the form

$$x\phi_n = (c_0 + \beta n)\phi_{n+1} + (a_0 + 2\alpha n)\phi_n + \gamma n \phi_{n-1}$$

Proof: Substituting $V_n = V^n$ into the recurrence given by the theorem above yields

$$nV^{n-1}V' + c_0V^{n+1} = c_nV^{n+1} + (a_n - a_0)V^n + b_nV^{n-1}$$

Dividing out V^{n-1}, rewrite this in the form

$$V' = \frac{c_n - c_0}{n}V^2 + \frac{a_n - a_0}{n}V + \frac{b_n}{n}$$

Since these coefficients are independent of n, we set

$$c_n = c_0 + n\beta, \quad a_n = 2n\alpha + a_0, \quad b_n = n\gamma$$

and the result follows. ∎

We will look in more detail at these systems in the next section.

2.3 MEIXNER CLASSES

The Meixner polynomials are special families of orthogonal polynomials closely related to operator calculus and Lie algebras. We present the basic facts concerning Meixner polynomials and their connection with operator calculus.

2.3.1 Meixner polynomials and operator calculus

The Meixner polynomials are orthogonal polynomials such that V is expressed by an analytic function of $D = d/dx$, where x is the variable in which the polynomials are given. After suitable normalizations, one finds six families of orthogonal polynomials as follows with the corresponding functions $V(z)$ and $H(z)$ as for canonical Appell systems.

2.3.1.1 Proposition. *For the Meixner classes of polynomials the V and H operators take the form:*

$$\begin{array}{rll}
\text{Meixner} & V(z) = \dfrac{\tanh qz}{q - \alpha\tanh qz} & H(z) = -\dfrac{\alpha}{\beta}z - \log\dfrac{qV(z)}{\sinh qz} \\
\text{Meixner} - \text{Pollaczek} & V(z) = \tan z & H(z) = \log\sec z \\
\text{Krawtchouk} & V(z) = \tanh z & H(z) = \log\cosh z \\
\text{Charlier} & V(z) = e^z - 1 & H(z) = e^z - 1 - z \\
\text{Laguerre} & V(z) = z/(1-z) & H(z) = -\log(1-z) - z \\
\text{Hermite} & V(z) = z & H(z) = z^2/2
\end{array}$$

where, for the general case, α, β are given parameters and $q^2 = \alpha^2 - \beta$.

(Note the normalizations $V(0) = H'(0) = 0$, $V'(0) = 1$.)

We will see how these arise by specialization from families of canonical polynomials arising via Lie algebras. They come from some basic Lie algebras, namely, sl(2), HW, and **osc**.

III. Canonical polynomials from Lie algebras

Let's recall the left dual and double dual representation for a Lie algebra. The left dual form of X, $X^{\ddagger} = \alpha_\mu \xi^{\ddagger}_\mu$, generates the flow of the group law

$$\exp(tX^{\ddagger}) f(A) = f(A(\alpha t) \odot A)$$

Setting $t = 1$ we have

$$e^{X^{\ddagger}} f(A) = f(A(\alpha) \odot A)$$

Let $\hat{X} = \alpha_\mu \hat{\xi}_\mu$ be the double dual realization of X. In terms of (x, D) variables, it is the dvf to $X^{\ddagger}$. Therefore, we get

$$e^{\hat{X}} e^{ax} = e^{(A(\alpha)\odot a)x}$$

Compare with

$$e^{\alpha_\mu Y_\mu} e^{ax} = e^{x_\mu U_\mu (V(a)+\alpha)}$$

our main formula for dvf's.

Setting $a = 0$ yields the main

3.1 Theorem. *Acting on the vacuum state 1, the group elements generated by the double dual $\hat{X}$ and the canonical variable $\alpha_\mu Y_\mu$ give the same result*

$$e^{\hat{X}} 1 = \exp(x \cdot A(\alpha)) = e^{\alpha_\mu Y_\mu} 1 = \exp(x \cdot U(\alpha))$$

under the correspondence of the momentum variables with the coordinates

$$D \leftrightarrow A\,, \qquad V \leftrightarrow \alpha$$

I.e., the canonical operators Y_i are given as $x_\mu W_{\mu i}(D)$ where W is the inverse Jacobian matrix of the coordinate map $A \to \alpha$, equivalently, the Jacobian matrix of the coordinate map $\alpha \to A$ expressed in the A variables, then replacing every A_i by the corresponding partial differentiation operator D_i.

So for any Lie algebra, with a specified basis, we have the *Lie canonical system* of polynomials $\{y_n\}$

$$e^{x\cdot A(\alpha)} = \sum_{n\geq 0} \frac{\alpha^n}{n!}\, y_n(x)$$

The idea is to find representations where a family of commuting self-adjoint operators occur as elements of $\mathfrak{g}$. Then these are the quantum observables for the system.

Here we show how this works for one observable that we interpret as a generalized position coordinate. We want the Lie algebra to be a *symmetric Lie algebra* where we have sets of raising operators $\mathcal{P}$ and lowering operators $\mathcal{L}$ in one-to-one correspondence such that $\mathfrak{g}$ has a direct sum decomposition of the form

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{P}$$

with the relations

$$[\mathcal{L}, \mathcal{P}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{L}] \subset \mathcal{L}, \qquad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}$$

We assume that $\mathcal{L}$ and $\mathcal{P}$ are abelian subalgebras that generate $\mathfrak{g}$ as a Lie algebra. This splitting is called a *Cartan decomposition*. If we have an inner product where elements R_i and L_i are in correspondence as operators adjoint to each other, $L_i^* = R_i$, $\mathfrak{g}$ becomes a "Lie*-algebra". And we can construct self-adjoint operators of the general form $X_i = R_i + K_i + L_i$. An interesting problem is to find such operators explicitly that generate an abelian algebra.

We will study three basic algebras to illustrate how it goes.

Example. HW. We have the coordinate map

$$A_1 = \alpha_1, \qquad A_2 = \alpha_2 + \tfrac{1}{2}\alpha_1\alpha_3, \qquad A_3 = \alpha_3$$

Thus, we have from the double dual

$$\exp(\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3(R_3 + R_2 V_1))\,1 = \exp(\alpha_1 R_1 + (\alpha_2 + \tfrac{1}{2}\alpha_1\alpha_3)R_2 + \alpha_3 R_3)\,1$$

Note that R_3 and $\alpha_2 R_2$ drop out. Setting $R_2 = t$, $\alpha_1 = \alpha_3 = z$, we get, using $R = R_1$ as our raising operator,

$$\exp(z(R + tV))\,1 = e^{zR + z^2 t/2}\,1$$

We have $X = R + tV$ as our quantum observable with spectral variable x. Thus, with $v = z$,

$$e^{vR}1 = e^{vx - v^2 t/2}$$

the generating function for the Hermite polynomials for the corresponding Gaussian distribution. We have recovered our example of Chapter 1.

Example. sl(2). First, set $\delta = \sqrt{\alpha_2^2 - \alpha_1\alpha_3}$. Then we have the coordinate map

$$A_1 = \frac{\alpha_1\,\tanh\delta}{\delta - \alpha_2\tanh\delta}, \qquad A_2 = \log\frac{\delta\,\mathrm{sech}\,\delta}{\delta - \alpha_2\tanh\delta}, \qquad A_3 = \frac{\alpha_3\,\tanh\delta}{\delta - \alpha_2\tanh\delta}$$

The double dual is

$$\hat{\xi}_1 = R_1, \qquad \hat{\xi}_2 = R_2 + 2R_1V_1, \qquad \hat{\xi}_3 = R_3 e^{2V_2} + R_2V_1 + R_1V_1^2$$

Now take $\alpha_1 \to z$, $\alpha_2 \to \alpha z$, $\alpha_3 \to \beta z$, and $\delta \to qz$, $q^2 = \alpha^2 - \beta$. Noting that R_3 drops out, send $R_2 \to t$, and use $R = R_1$ as our raising operator to yield

$$e^{zX}1 = \left(\frac{q\,\operatorname{sech} qz}{q - \alpha \tanh qz}\right)^t \exp\left(\frac{\tanh qz}{q - \alpha \tanh qz}\, R\right) 1$$

where

$$X = R + \alpha t + 2\alpha RV + \beta(tV + RV^2)$$

is our quantum random variable. With spectral variable x, this is of the form

$$e^{zx} = e^{tH(z)} e^{V(z)R} 1$$

and solving for $e^{vR}1$ gives the generating function for the corresponding class of polynomials in general Bernoulli form. Various specializations lead to the Meixner classes for Bernoulli, negative binomial and continuous binomial (hyperbolic) distributions. The gamma/exponential family is an interesting limiting case where $q \to 0$. We get, then, with $\beta = \alpha^2$,

$$e^{zX}1 = (1 - \alpha z)^{-t} \exp\left(R\, \frac{z}{1 - \alpha z}\right) 1$$

and solving for $z = U(v)$ yields the generating function for Laguerre polynomials in an appropriate normalization.

Example. For the oscillator algebra we have the coordinate map

$$A_1 = \frac{\alpha_1}{\alpha_4}\,(e^{\alpha_4} - 1), \quad A_2 = \alpha_2 + \frac{\alpha_1\alpha_3}{\alpha_4^2}\,(e^{\alpha_4} - 1 - \alpha_4), \quad A_3 = \frac{\alpha_3}{\alpha_4}\,(1 - e^{-\alpha_4}), \quad A_4 = \alpha_4$$

The double dual is

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2, \quad \hat{\xi}_3 = R_3 + R_2V_1, \quad \hat{\xi}_4 = R_4 + R_1V_1 - R_3V_3,$$

We take $\alpha_4 \to \alpha z$, $\alpha_1 \to z$, $\alpha_3 \to \beta z$, with R_4 dropping out and get, setting $R_2 = t$,

$$e^{zX+zY}1 = \exp(R_1(e^{\alpha z} - 1)/\alpha) \exp(\beta t(e^{\alpha z} - 1 - \alpha z)/\alpha^2) \exp(R_3\beta(1 - e^{-\alpha z})/\alpha)\, 1$$

where

$$X = R_1 + \alpha R_1V_1 + \beta tV_1, \qquad Y = \beta R_3 - \alpha R_3V_3$$

The R_3 term gives an independent aff(2). The X term gives, with $R \to R_1$,

$$e^{vR}1 = (1 + \alpha v)^{x/\alpha + \beta t/\alpha^2} \exp(-v\beta t/\alpha)$$

which is the generating function for Poisson-Charlier polynomials for a scaled Poisson process with drift.

Finally, observe that in each case, the formula for X in terms of R and V gives the three-term recurrence relation for the corresponding orthogonal polynomials.

IV. References

The Lie algebra approach in its initial development was presented in [9].

The original article of Meixner is [18].

Askey-Wilson polynomials give an umbrella approach from the classical analysis point of view, see [1].

5 Jacobians

I. Adjoint group

Recalling the pi-matrices:
we denote the transpose of $\pi^{\ddagger}$ by $\hat{\pi}$ and the transpose of π^* by $\hat{\pi}^*$.

We will calculate the exponential of the adjoint representation in terms of the π matrices. First we remark that the exponential of the adjoint representation connects the right and left duals:

$$g\xi_j = g\xi_j g^{-1} g = \xi_j^* g = Ad_g(\xi_j)g = Ad_g(\xi_j^{\ddagger})g \tag{1.1}$$

where Ad_g denotes the exponential of the adjoint representation, conjugation by g. It is given explicitly as the matrix $g(A, \check{\xi})$ (acting on the row vector formed from the basis vectors). Now solving for ∂_i shows that

1.1 Proposition. *The left and right duals are related by*

$$\xi^* = \pi^* {\pi^{\ddagger}}^{-1} \xi^{\ddagger}$$

where ξ^* (resp. $\xi^{\ddagger}$) denotes the column array with entries ξ_i^* (resp. $\xi_i^{\ddagger}$).

1.2 Definition. Define the matrix

$$\check{\pi} = g(A, \check{\xi})$$

the exponential of the adjoint representation.

Now for the main result of this section.

1.3 Theorem. *The exponential of the adjoint representation, $g(A, \check{\xi})$, is given by*

$$\check{\pi} = \hat{\pi}^{-1} \hat{\pi}^*$$

Proof: From equation (1.1), we have the matrix of Ad_g, acting on the basis, given by the transpose of $g(A, \check{\xi})$. I.e.,

$$\xi^* = \check{\pi}^\dagger \xi^\ddagger \tag{1.2}$$

Comparing with Proposition 1.1, we have $\check{\pi}^\dagger = \pi^* {\pi^\ddagger}^{-1}$. And the result follows by definition of the matrices $\hat{\pi}$, $\hat{\pi}^*$ as transposes. ■

Example. For the affine group we have

$$\check{\xi}_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \check{\xi}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Calculating exponentials gives

$$e^{A_1\check{\xi}_1} e^{A_2\check{\xi}_2} = \begin{pmatrix} 1 & -A_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{A_2} & -A_1 \\ 0 & 1 \end{pmatrix}$$

Recalling the pi-matrices

$$\pi^\ddagger = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}, \qquad \pi^* = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

it is readily checked that this is the transpose of $\pi^* {\pi^\ddagger}^{-1}$.

II. Jacobians of the group law

Now we will see the pi-matrices arising as Jacobians of the group law, showing yet another aspect of their nature.

Write $g = g(A)g(B) = g(C)$, with $C = A \odot B$. Differentiating with respect to A_i we get

$$\Pi^\ddagger_{i\mu}(A)\, \xi^\ddagger_\mu(A) g = \frac{\partial g}{\partial C_\lambda} \frac{\partial C_\lambda}{\partial A_i}$$

And

$$\Pi^*_{i\mu}(B)\, \xi^*_\mu(B) g = \frac{\partial g}{\partial C_\lambda} \frac{\partial C_\lambda}{\partial B_i}$$

since we can differentiate and pull the ξ_j's through without interference to the corresponding side.

In other words, $\Pi^\ddagger_{i\mu}(A)\, \xi^\ddagger_\mu(A) g = \Pi^\ddagger_{i\mu}(A)\, \xi_\mu g$ and $\Pi^*_{i\mu}(B)\, \xi^*_\mu(B) g = \Pi^*_{i\mu}(B)\, g\xi_\mu$. And we have

$$\frac{\partial g}{\partial C_i} = \Pi^\ddagger_{i\mu}(C)\, \xi^\ddagger_\mu(C) g = \Pi^\ddagger_{i\mu}(C)\, \xi_\mu g = \Pi^*_{i\mu}(C)\, \xi^*_\mu(C) g = \Pi^*_{i\mu}(C)\, g\xi_\mu$$

Solving, writing $C = A \odot B$, we find

$$\begin{aligned}
\frac{\partial(A \odot B)}{\partial A} &= \hat{\pi}(A \odot B)\,\hat{\pi}^{-1}(A) \\
\frac{\partial(A \odot B)}{\partial B} &= \hat{\pi}^*(A \odot B)\,\hat{\pi}^{*-1}(B)
\end{aligned}$$

Letting $A = 0$ in the first equation, $B = 0$ in the second yields

$$\begin{aligned}
\left.\frac{\partial(A \odot B)}{\partial A}\right|_{A=0} &= \hat{\pi}(B) \\
\left.\frac{\partial(A \odot B)}{\partial B}\right|_{B=0} &= \hat{\pi}^*(A)
\end{aligned}$$

We know that

$$\breve{\pi}(A) = \hat{\pi}^{-1}(A)\hat{\pi}^*(A)$$

So, for example,

$$\frac{\partial B}{\partial(B \odot A)} \left.\frac{\partial(A \odot B)}{\partial B}\right|_{B=0} = \breve{\pi}(A)$$

Example. Recall the HW group law

$$\begin{aligned}
C_1 &= A_1 + B_1 \\
C_2 &= A_2 + B_2 + A_3 B_1 \\
C_3 &= A_3 + B_3
\end{aligned}$$

and the pi-matrices

$$\pi^{\ddagger}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{pmatrix}, \qquad \pi^*(A) = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculating the Jacobians, we find

$$\frac{\partial C_i}{\partial A_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & B_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\partial C_i}{\partial B_j} = \begin{pmatrix} 1 & 0 & 0 \\ A_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that in this case the pi-matrices are a representation of the group, i.e.

$$\hat{\pi}(A \odot B) = \hat{\pi}(A)\hat{\pi}(B), \qquad \pi^*(A \odot B) = \pi^*(A)\pi^*(B)$$

so we get the pi-matrices regardless of evaluations at 0.

Example. For aff(2) we have

$$\begin{aligned} C_1 &= A_1 + B_1 e^{A_2} \\ C_2 &= A_2 + B_2 \end{aligned}$$

and the pi-matrices

$$\pi^\ddagger(A) = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}, \qquad \pi^*(A) = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

Calculating the Jacobians, we find

$$\frac{\partial C_i}{\partial A_j} = \begin{pmatrix} 1 & B_1 e^{A_2} \\ 0 & 1 \end{pmatrix}, \qquad \frac{\partial C_i}{\partial B_j} = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

We readily verify the corresponding relations.

III. Three classes of operators for any Lie algebra

Here is an outline of the elements of the theory relating to LCS — Lie canonical systems.

1. Start with a given basis for the Lie algebra.

2. Find the coordinate map via the characteristic equations for the left dual flow: $\dot{A} = \alpha\pi^\ddagger(A)$. With initial conditions $A(0) = A$, this yields $A(\alpha t) \odot A$, and hence the map $\alpha \to A$, evaluating at $A = 0$, $t = 1$.

3. Interpret A as momentum variables, α as canonical momenta.
Dual variables are x to A, Y to α.

4. Jacobians:

   (i) $\dfrac{\partial A}{\partial \alpha}$, expressed in terms of A is used for the *raising operators* Y.

   (ii) $\dfrac{\partial \alpha}{\partial A}$ in terms of α computed as the algebraic inverse is used to express the variables x in terms of raising and lowering operators. The x variables in that form are the *recursion operators*.

5. Generic formulae $Y = xW(D) = xU'(V(D))$, $x = YV'(D) = YU'(V)^{-1}$ become

$$Y = x\,A'(\alpha(A))\Big|_{A\to D}, \qquad x = Y\,A'(\alpha)^{-1}\Big|_{\substack{Y\to R\\ \alpha\to V}}$$

6. Canonical polynomials $y_n(x) = Y^n 1$. Abstract raising and lowering operators on the basis y_n are

$$R_i\, y_n = Y_i\, y_n = y_{n+\mathrm{e}_i}$$
$$V_i\, y_n = n_i\, y_{n-\mathrm{e}_i}$$

Acting on the basis y_n, x's yield recursion formulas.
Basic expressions are (row vector times matrix) :

$$Y = x\, A'(\alpha(D))$$
$$x = R\, A'(V)^{-1}$$

with $D = (D_1, \ldots, D_N)$ operators of partial differentiation with respect to x-variables.

7. We can include the change-of-variables in $\xi^{\ddagger}$ yielding the general

$$\hat{\xi}_i = x_\nu W_{\nu\lambda}(D)\pi^{\ddagger}_{i\lambda}(V(D))$$

with

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{x_\mu U_\mu(A(\alpha))}$$

8. In particular,

$$\hat{\xi}_i = x_\nu A'(D)^{-1}_{\nu\lambda}\, \pi^{\ddagger}_{i\lambda}(A(D))$$

yields

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{\alpha_\mu x_\mu}\ .$$

Since the coherent state is the same as for an abelian algebra, we call these the *ACS operators.*

3.1 JACOBIAN OF THE COORDINATE MAP

To get the canonical variables requires the Jacobian of the map $\alpha \to A$. Since one has the differential equations for A, namely the characteristic equations $\dot{A} = \alpha\pi(A)$, one would think it possible to find $A'(\alpha) = \partial A/\partial\alpha$ directly in terms of the π-matrices. This turns out to be the case and is the subject of an interesting theorem stated without proof.

3.1.1 Theorem.

Let $J = \partial A/\partial\alpha$ denote the Jacobian of the coordinate map $\alpha \to A$. Then

$$J(\alpha) = \hat{\pi}(A(\alpha)) \int_0^1 \check{\pi}(A(s))\, ds$$

Alternatively, we have

$$J(\alpha) = \hat{\pi}^*(A(\alpha)) \int_0^1 \check{\pi}^{-1}(A(s))\, ds$$

Note that in our application to dual vector fields, we really want J as a function of A. The factor outside the integral is naturally given in terms of the A_i, but the integral is evaluated by expressing $A(s)$ in terms of the α_i scaled by s. On the other hand, J^{-1}, expressed in terms of α immediately gives us what we need for the corresponding recursion operators.

3.2 CANONICAL VARIABLES IN THE NONABELIAN CASE

Let's see how points #7 and #8 work. We combine the two fundamental constructions: i.e., we use canonical variables in the Lie case.

3.2.1 Proposition. *Let* $Y = x_\nu \alpha_\mu \pi^\ddagger_{\mu\lambda}(V(D))W_{\nu\lambda}(D)$. *Then*

$$e^{tY} e^{a\cdot x} = \exp\Big(x \cdot U\Big(A(\alpha t) \odot V(a)\Big)\Big)$$

Proof: Acting on $e^{a\cdot x}$ we have

$$Ye^{a\cdot x} = x_\nu \alpha_\mu \pi^\ddagger_{\mu\lambda}(V(a))W_{\nu\lambda}(a)e^{a\cdot x} = \alpha_\mu \pi^\ddagger_{\mu\lambda}(V(a))W_{\nu\lambda}(a)\,\frac{\partial}{\partial a_\nu}\,e^{a\cdot x}$$

This latter is a vector field in the a-variables. The characteristic equations are

$$\dot{a}_i = \alpha_\mu \pi^\ddagger_{\mu\lambda}(V(a))W_{i\lambda}(a)$$

Multiplying by $V'(a)$ yields

$$V'(a)_{k\lambda}\dot{a}_\lambda = \alpha_\mu \pi^\ddagger_{\mu k}(V(a))$$

Now the left-hand side is an exact derivative, $\dfrac{d}{dt}V(a(t))$. So these are characteristic equations for the left dual flow in the V-variables. Integrating, we have, with $A(\alpha)$ denoting the coordinate map on the group,

$$V(a(t)) = A(t\alpha) \odot V(a)$$

In other words,

$$a(t) = U(A(t\alpha) \odot V(a))$$

which gives the stated result. ■

And

3.2.2 Theorem. *To the vector fields*

$$\xi_i^\ddagger(V(x)) = \pi_{i\lambda}^\ddagger(V(x))W_{\nu\lambda}(x)D_\nu$$

correspond the dvf's

$$\hat{\xi}_i = x_\nu W_{\nu\lambda}(D)\pi_{i\lambda}^\ddagger(V(D))$$

And with $\hat{X} = \alpha_\mu \hat{\xi}_\mu$,

$$e^{\hat{X}}1 = e^{x\cdot U(A(\alpha))}$$

Note that the $\hat{\xi}_i$ are the double dual in the canonical variables (Y, V).

Now choose U and A to be inverse maps, i.e., $V(z) = A(z)$. Then we have the nonabelian Lie algebra yielding the same result on the vacuum state, 1, as the abelian one, namely

$$\exp(\hat{X})1 = \exp(\alpha \cdot x)\,.$$

How do these work for some Lie algebras of particular interest?

Example. Let's look at HW in detail.

We have the coordinate map

$$A_1 = \alpha_1, \qquad A_2 = \alpha_2 + \alpha_1\alpha_3/2, \qquad A_3 = \alpha_3$$

The Jacobians are

$$\frac{\partial A}{\partial \alpha} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_3/2 & 1 & \alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial \alpha}{\partial A} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_3/2 & 1 & -\alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

with the latter calculated as $\left(\frac{\partial A}{\partial \alpha}\right)^{-1}$. In terms of A,

$$\frac{\partial A}{\partial \alpha}(A) = \begin{pmatrix} 1 & 0 & 0 \\ A_3/2 & 1 & A_1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Contracting with x and replacing A by D yields the raising operators

$$Y_1 = x_1 + \tfrac{1}{2}x_2 D_3\,, \quad Y_2 = x_2\,, \quad Y_3 = x_3 + \tfrac{1}{2}x_2 D_1$$

These are commuting variables. The basic expansion is

$$e^{\alpha_\mu Y_\mu}1 = e^{\alpha_1 x_1} e^{x_2(\alpha_2+\alpha_1\alpha_3/2)} e^{\alpha_3 x_3} = \sum_{n\geq 0} \frac{\alpha^n}{n!}\, y_n(x)$$

Contracting with R and replacing α by V in $\partial\alpha/\partial A$ yields the x-variables as recursion operators

$$x_1 = R_1 - R_2V_3/2\,, \quad x_2 = R_2\,, \quad x_3 = -R_2V_1/2 + R_3$$

On the basis y_n, we thus have

$$\begin{aligned} x_1\, y_n &= y_{n+\mathrm{e}_1} - \tfrac{1}{2}\, n_3\, y_{n+\mathrm{e}_2-\mathrm{e}_3} \\ x_2\, y_n &= y_{n+\mathrm{e}_2} \\ x_3\, y_n &= y_{n+\mathrm{e}_3} - \tfrac{1}{2}\, n_1\, y_{n-\mathrm{e}_1+\mathrm{e}_2} \end{aligned}$$

Finally, replacing R by x, V by D and contracting with the transpose of $\pi^{\ddagger}(A(D))$, yields the ACS representation of the Lie algebra

$$\hat{\xi}_1 = x_1 - \tfrac{1}{2}x_2D_3\,, \quad \hat{\xi}_2 = x_2\,, \quad \hat{\xi}_3 = x_3 + \tfrac{1}{2}x_2D_1$$

which satisfy the commutation relations for the Heisenberg algebra while satisfying

$$\exp{(\alpha_\mu\hat{\xi}_\mu)}1 = \exp{\alpha_\mu x_\mu}$$

Example. aff(2)

We have the coordinate map

$$A_1(\alpha) = \frac{\alpha_1}{\alpha_2}\,(e^{\alpha_2}-1)\,, \qquad A_2(\alpha) = \alpha_2$$

and

$$\pi^{\ddagger} = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}$$

The Jacobians are

$$\frac{\partial A}{\partial \alpha} = \begin{pmatrix} \alpha_1\,(e^{\alpha_2}-1)/\alpha_2 & -\alpha_1\,(e^{\alpha_2}-1-\alpha_2)/\alpha_2^2 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{\partial \alpha}{\partial A} = \begin{pmatrix} \dfrac{\alpha_2}{e^{\alpha_2}-1} & \dfrac{\alpha_1}{\alpha_2} - \alpha_1\,\dfrac{1}{1-e^{-\alpha_2}} \\ 0 & 1 \end{pmatrix}$$

from these the raising operators, recursion operators and ACS representation are found as prescribed.

IV. Conclusion

There are many points for continued study. By specializing the coordinates one can find certain elements of the Lie algebra that generate classically interesting polynomials, such as Hermite polynomials via the Heisenberg algebra. In any case, the polynomials found in the approach indicated here have particular structure depending on their associated Lie algebra. Exactly how these, the polynomials and the structure of the Lie algebra, are related in some deeper way has not been clarified.

Another source of interest is, of course, the Jacobians. One can look at Jacobians of the form $\dfrac{\partial A(t)}{\partial A(s)}$, for $s < t$. As the Jacobians form a multiplicative family along paths, there are some possibilities for interesting dynamical systems, or perhaps, matrix-valued stochastic processes.

Generally speaking, it looks challenging and interesting to get some detailed information for classes of higher-dimensional Lie algebras. Certain classes of Lie algebras, such as symmetric Lie algebras, may allow for general structural results.

V. References

The idea of quantum observables as we use it appears in Berceanu-Gheorghe [3].

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