Competitive Information Design for Pandora's Box

Bolin Ding* Yiding Feng† Chien-Ju Ho ‡ Wei Tang § Haifeng Xu ¶

Abstract

We study a natural competitive-information-design variant for the Pandora's Box problem [31], where each box is associated with a strategic information sender who can design what information about the box's prize value to be revealed to the agent when she inspects the box. This variant with strategic boxes is motivated by a wide range of real-world economic applications for Pandora's box. The main contributions of this article are two-fold: (1) we study informational properties of Pandora's Box by analyzing how a box's partial information revelation affects the search agent's optimal decisions; and (2) we fully characterize the pure symmetric equilibrium for the boxes' competitive information revelation, which reveals various insights regarding information competition and the resultant agent utility at equilibrium.

1 Introduction

The Pandora's Box problem, as formalized in the seminal work of Weitzman [31], is a foundational framework for studying how the cost of acquiring information affects the adaptive decisions about what information to acquire — the obtained information from the past will affect whether additional information is needed, and if so which information to acquire next. Specifically, the Pandora's Box problem is described as follows. An agent is presented with n boxes; each contains an unknown random prize. The value of the prize inside each box is independently sampled from its distribution. While the agent knows each box's prize distribution, he does not know its realized value. Nevertheless, the agent can open any box (in any order) to learn its *realized* prize value but suffers an associated opportunity cost for opening the box. The agent can stop at any time and claim one prize from some opened box, upon which the game terminates. The agent's goal is to maximize the expected prize value minus the total box-opening costs. This basic model finds applications in numerous economic applications and thus, unsurprisingly, has been extensively studied in the economics, operations research, and computer science literature. For example, in house hunting, a home buyer incurs cost to search for information about each potential house (e.g., attending its open house) and, at some point, decide to purchase one of the searched house and terminate the procedure. Similarly, many online customers spend time on free trials to obtain information about different digital services and, at some point, decide to subscribe to some tried service.

A surprisingly simple and elegant policy provided by Weitzman [31] has been shown to be optimal for the Pandora's Box problem, despite its seemingly complex sequential decision process. Specifically, Weitzman [31] defines certain *reservation value* for each box, which is determined by both the box's prize distribution and opening cost. The optimal policy simply sorts boxes in decreasing order of their reservation values, and then open boxes in this order until the thus-far maximum realized prize value exceeds the next box's reservation value. The agent then terminates the search by selecting that maximum realized prize.

An important assumption of the classic Pandora's Box problem — which is the one we intend to relax in this work — is that each box is an inanimate object and, once opened, will fully disclose its realized prize to the agent. Yet this may not be the case in many real-world applications where boxes often correspond to real strategic agents who may have incentives to selectively disclose information for their own interest [2, 11, 29]. This is usually the case when information is not controlled by nature but by humans or algorithms. The following are two of many such examples.

Example 1.1. (Open Houses in Housing Markets) *During open houses, many house sellers typically would design events to highlight their house qualities and these event schedules will be sent to potential buyers. This*

*bolin.ding@alibaba-inc.com, Alibaba Group
†yidingfeng@microsoft.com, Microsoft Research New England
‡chienju.ho@wustl.edu, Washington University in St. Louis
§wt2359@columbia.edu, Columbia University
¶haifengxu@uchicago.edu, University of Chicago

corresponds to the boxes' design and commitment to an information disclosure policy. Informed with these policies (i.e., learning what he expects to see), a buyer will decide which open houses to visit in what sequence, and during this process the buyer may make a purchase decision (i.e., stop searching). In this example, it is costly for a buyer to obtain the information from any box due to the time spent to travel and visit. Moreover, the seller usually selectively disclose information in order to maximize the chance of sale. Built upon Weitzman's elegant solution to the classic Pandora's box for the buyer's search, our work studies the house sellers' competitive information design problem and how sellers' revealed information affects the agent's total utility.

EXAMPLE 1.2. (FREE TRIALS OF DIGITAL SERVICES) *Consider online services like Youtube Music, Spotify, and Amazon Music. To attract users for subscription, these services often offer free trials (e.g., a one-month free trial with access to a limited set of functionalities of the service) before the user picks one service to subscribe. These free trials, including the functionalities included in this period, can be seen as a committed information revelation policy designed by the service provider. The user needs to pay search costs (i.e., time spent to explore) to obtain the information. Moreover, these information policies are usually not full-information revealing due to limited trial periods and limited functionality access. In contrast to the fully observable prize value in classic Pandora's Box, the user here can only form a posterior belief about the service quality before choosing a subscription.*

Motivated by real-world applications like the above, this paper studies a natural information design variant of the celebrated Pandora's Box problem by viewing each box as an economic agent with its own actions and incentives. We assume that, before the agent opens any box, each box commits to an information revelation policy — a.k.a., a signaling scheme which stochastically maps the underlying prize to a random *signal* — to selectively disclose information about the prize. Afterwards, the agent engages in a costly search across boxes, i.e., solving a standard Pandora's Box problem, in order to collect the most-rewarding prize in expectation. Notably, after opening any box, the agent now is only able to observe a realized signal that carries partial information about the underlying prize value, but cannot directly observe the prize value. We study a model where there are n symmetric boxes, competing with each other for being selected by the agent. The agent is assumed to initially hold the same common prior belief H about each boxes' prize distribution, i.e., there is no ex-ante asymmetry among prizes.

We assume boxes are decentralized (e.g., corresponding to different product sellers). Each box can choose any signaling scheme to strategically reveal information about his own prize. This gives rise to a natural competitive information design problem in the Pandora's Box with many senders, e.g., the boxes. Assuming a risk-neutral agent by convention, the agent is only concerned with the expected prize value upon seeing any signal after opening a box. Consequently, each box's information design problem boils down to choosing a distribution over the expected prize value, each conditioned on a signal, that respects the Bayes' plausibility constraint [25]. It is well-known that in this case the strategy of each box is precisely a *mean-preserving spread* (henceforth MPS) G_i of the prior prize distribution H [6, 10]. Given all boxes' strategy profile $\{G_i\}_{i\in[n]}$, the agent conducts a costly search cross $\{G_i\}_{i\in[n]}$ to learn the corresponding prize values. Naturally, we assume the agent performs the optimal search policy as prescribed by Weitzman [31]. Our focus is to study the game among the senders' competitive information design. This leads to a Stackelberg game with multiple leaders (i.e., the boxes) and a single follower (the agent). [1]

1.1 Our Contribution Our contributions are two-fold: (1) we study informational properties of Pandora's Box by analyzing how a box's partial information revelation affects the agent's optimal decisions and utilities; and (2) we fully characterize the pure symmetric equilibrium for the boxes' competitive information revelation, and reveals various insights regarding information competition and the resultant agent payoff at equilibrium.

Informational Properties of Pandora's Box and the Agent's Payoff. Our first main result shows that whenever a box uses a strategy that is more informative, the agent obtains a weakly higher expected payoff.[2] While this might appear obvious at first, a closer look reveals it is not a-priori clear at all that more information from *any* box would always benefit the agent. Recall that the agent's optimal inspection strategy depends on the order of reservation values of boxes' strategies. To prove the above result, we first show that the reservation value is always weakly larger if the corresponding strategy is more informative. Now suppose a box with very bad expected prize value chooses to disclose more information, this box's reservation value will also increase and

[1]Note that once boxes' choices $\{G_i\}_{i\in[n]}$ are determined, it is a subgame perfect equilibrium for the agent to use the optimal inspection strategy.

[2]We note that this result holds generally with the need of assuming symmetry on boxes' prior prize distribution.

thus it will be inspected early. However, it is not clear whether inspecting such a "bad" box earlier by lowering the priority of other possibly better boxes will always benefit the agent. Our main result gives an affirmative answer. Our proof heavily hinges on various properties of MPS in order to argue that the benefit of getting more information from any box can offset the possible harm of lowering the priority of other boxes.

A natural corollary of the above result in our competitive information design environment is that, when all boxes fully reveal the information about their prizes, the agent obtains the highest expected payoff. Nevertheless, we strengthen this observation by showing that the agent can derive the highest expected payoff as long as each box use a strategy which reveals full information whenever the value of the prize is *below* its reservation value. We refer to this class of strategies as *essentially full information strategy.* We provide necessary and sufficient conditions on when this strategy is the equilibrium strategy next.

Equilibrium Characterizations. Our second main result is to identify a necessary and sufficient condition for the existence of a pure symmetric Nash equilibrium. Moreover, if a pure symmetric equilibrium exists, our result provides a straightforward, and also computationally tractable, way to identify the equilibrium strategy. Specifically, we show that a pure symmetric equilibrium strategy G, if exists, must be fully characterized by the following three conditions:

(i) **Maximum reservation value**: strategy G must have maximum reservation value.

(ii) **G's shape below reservation value**: function G^{n-1} is convex over its support, and linear whenever the strategy G does not equal to the prior H.

(iii) **No deviation incentive**: there exists a reservation value σ^* such that deviating to a strategy that has this reservation value σ^* is not profitable.

We prove that the first two conditions above can already uniquely pin down a strategy as an equilibrium candidate. Core to our characterization is the third condition which verifies whether this strategy candidate is indeed an equilibrium or not. The verification in condition (iii), including the reservation value σ^*, has a closed form and can be easily computed given the structure of the identified strategy G from conditions (i) and (ii).

Next we describe additional insights conveyed by the above main result and discuss how the competition and the agent's cost affect the boxes' equilibrium strategy. Utilizing our conditions above, we can show that essentially full information strategy is the equilibrium strategy if and only if function H^{n-1} is convex in $[0, \sigma_H]$. Build upon this result, we are able to show that the essentially full information strategy is more likely to become the equilibrium strategy when increasing the competition (i.e., increasing the number of boxes) or increasing the cost. The former is because, intuitively, increasing competition "convexifies" the shape of function H^{n-1} and makes the condition more likely to be satisfied. The later is because the cost affects the reservation value σ_H and thus the structure of (possible) equilibrium strategy G. First, we can see that the essentially full information strategy is the equilibrium strategy under a larger cost if it is already the equilibrium strategy under a smaller cost. This is due to the monotonicity of reservation value σ_H over the cost, i.e., a larger cost leads to a smaller σ_H. Second, as the cost goes to 0, the above characterized behavior of G below its reservation value in condition (ii) spans to the whole interval $[0, 1]$.[3] Third, the cost also plays a role in condition (iii) as it determines the choice of reservation value σ^*.

We highlight two predominant challenges in deriving our main result on equilibrium characterizations, followed by our approaches to tackle these challenges. First, to see whether a strategy profile $(G, \ldots, G)$ is an equilibrium, we need to argue that no box has a profitable deviation under this strategy profile. A box's best response problem can be formulated as a linear program, after fixing all other boxes' strategies to be G. Prior works [5, 24] have investigated a special case of our setting where there is no cost and the agent observes all realized prizes. They have utilized this linear program approach to demonstrate that the box's best response strategy is indeed G itself if G is a certain equilibrium strategy candidate. Note that in their setting, no matter what the response strategy is, the box's expected payoff when realizing prize with value $x \in [0, 1]$ has a succinct and well-structured form: $G(x)^{n-1}$. However, in our setting, different strategies have different reservation values, which impact the order of the agent inspecting the box, and thus making the box's payoff function different and more complex. Consequently, there is no single linear program that can characterize a box's best response problem. Instead, for each possible

[3] To ease exposition consider that the value of prize is in $[0, 1]$.

reservation value σ, we consider a corresponding linear program which characterizes the best response strategy subject to a constraint that it has the same reservation value σ (requiring a strategy to have a reservation value σ can be formulated as a linear constraint). We then prove that the optimal objective value of the linear program, as a function of the given reservation value σ, is a single-peaked function with the peak achieved at some σ^*.

Second, for any reservation value σ, solving its corresponding linear program (i.e., the program to solve a box's best response problem) is highly non-trivial. Let F denote the response strategy used by the box and all other boxes use the strategy G. There are two major constraints in this program: one constraint accounts for the feasibility of the strategy F, i.e., H is an MPS of F; and the other accounts for the reservation value constraint as it requires that the reservation value of strategy F equals to σ. Dworczak and Martini [17] developped an optimality verification technique based on strong duality for the special case with only the first constraint (later employed by [24]). Unfortunately, this technique does not directly apply to our more general case in presence of the second constraint as well. To overcome this barrier, we generalize the approach in [17] to account for the additional constraint and characterize corresponding optimal dual solution (of a new format). This then allows us to verify the optimality of certain desired information structure based on the complementary slackness.

1.2 Related Work This paper is built on the seminar work of Pandora's Box by Weitzman [31]. The Pandora's Box problem has been extensively studied in computer science [8, 13, 14, 27], economics [15, 30], and operation research [1, 20, 22].

Our paper studies an information design variant of Pandora's Box, with the formulation following the Bayesian persuasion setup by Kamenica and Gentzkow [26], who studies the game between a single sender (a.k.a., box) and a receiver (a.k.a., agent). Their work has inspired an active line of research in information design games [e.g., see the recent surveys by 7, 16, 25]. Our work extends this line of research by exploring the competition in information design in the setup of Pandora's Box and discusses the role of competition and inspection cost on the agent's payoff and boxes' information strategies in equilibrium. This paper relates closely to the works in the multiple sender Bayesian Persuasion literature [18, 19, 21], especially those examining situations with ex ante symmetric information among multiple senders [4, 5, 12, 24]. Our model is similar to theirs since each box is associated with a sender who provides information only about his own prize. Our work differs from theirs in that they assume there is no inspection cost for the agent, and the agent can simply observe all realized values and then select a best one, while we consider the setting in which the agent needs to pay cost to acquire information. The introduction of the cost makes the analysis significantly more complex. Another related line of works is the (competitive) information design problem in searching market [3, 23, 32]. Our work differs from this literature in that we consider the setting where the agent uses the optimal inspection strategy, while in their setting, the agent uses a random searching strategy. Another closely related work is by Au [3], who studies the same agent model as ours, but they only address a simplified setting where the prize is binary, while ours addresses the continuously distributed prizes.

Lastly, we mention recent technical developments on using the duality theory to characterize the optimal persuasion scheme in information design. In particular, Dworczak and Martini [17], Kolotilin [28] study the sender's problem on how to optimize the sender's (indirect) payoff as a function of expected value (state) he induces, subject to the feasible information strategy constraint. Our work differs from theirs as we study the equilibrium in a strategic environment. Moreover, though we can write the box's payoff as a function of the expected prize value, this payoff function further depends on the reservation value of the box's strategy (and other boxes' reservation values), and thus, their results does not apply directly. Instead, we extend their results to account for the additional reservation value constraint, and use the extended results to characterize the optimal dual (primal) solution.

2 A Model of Competitive Information Design for Pandora's Box

In this section, we first revisit the formulation of the classic Pandora's Box problem, and then formally introduce our setting as its natural variant with competitive information design.

2.1 The Pandora's Box Problem In the Pandora's Box problem, a risk-neutral agent is presented with a set of n boxes. Each box $i \in [n]$ contains a prize of value $x_i \in [0, 1]$. The value x_i is distributed according to a distribution G_i, independent of the values of other boxes. For each box i, the agent does not know the value x_i but knows the value distribution G_i. Moreover, the agent can pay a cost c_i to inspect box i and observe the value x_i.

The agent can choose to inspect any number of boxes in any order and take one of the values from the inspected boxes. The goal of the agent is to maximize the value from the chosen box minus the total cost for inspecting boxes.

The agent's strategy π is a rule that determines adaptively, at any time $t \geq 0$, whether to terminate the inspection and, if not, which box to inspect next. The strategy also determines which box to select after the inspection ends. Following the terminology in Kleinberg et al. [27], given a strategy π, let $\mathbb{I}_i$ denote the indicator for whether box i is inspected and $\mathbb{A}_i$ denote the indicator for whether box i is chosen according to π. The agent's goal is to choose a strategy π which maximizes the following expected payoff

$$\mathbb{E}\left[\sum_i [\mathbb{A}_i x_i - \mathbb{I}_i c_i]\right] . \tag{2.1}$$

Importantly, the agent can *only* claim one prize but must pay for all inspection costs.

To describe the agent's optimal inspection strategy, we utilize the notion of *reservation value* [31]. This notion is critical for our analysis and is formally defined as follows:

Definition 2.1. (Reservation Value – Weitzman [31]) *For any distribution $G \in \Delta([0,1])$, the value σ_G satisfying $\sigma_G = \sup\{\sigma : \mathbb{E}_{x\sim G}[\max\{x-\sigma, 0\}] = c\}$ is referred to as the* reservation value.

With the notion of reservation value, the agent's optimal inspection strategy can be characterized by the simple procedure below.

Theorem 2.1. (Weitzman [31]) *Given the boxes' value distributions $(G_1, \ldots, G_n)$, the agent's optimal inspection strategy runs as follows: the agent inspects each box in order of decreasing σ_{G_i}, stopping when the largest observed value x_{i^*} exceeds all uninspected $\sigma_{G_{-i}}$ and claims box i^*'s value.*

2.2 Pandora's Box with Competitive Information Design In this paper, we consider a natural competitive information design variant of the Pandora's Box problem. Specifically, each box is associated with a strategic sender[4] who can design what information about the prize value the agent will see when he inspects the box. Similar to the classic problem, the agent does not know the values in boxes but holds some prior beliefs about the distribution of each value x_i. However, different from the classic problem, when the agent pays a cost c_i to inspect box i, he does not directly observe the value x_i. Instead, he observes some information signal, designed by the sender of box i, that is related to the prize x_i. Following the literature in information design, this can be formalized as follows: each sender i can choose a signaling scheme $\{\Phi_i(q|x), \mathcal{M}_i\}$, where $\mathcal{M}_i$ is a signal space and $\Phi_i(q|x) \in [0,1]$ specifies the conditional distribution of signal $q \in \mathcal{M}_i$ when the value x is realized. The senders' signaling schemes $\{\Phi_i(q|x), \mathcal{M}_i\}_{i\in[n]}$ are known to the agent in advance. When the agent inspects box i, he observes a realized signal q drawn according to the conditional distribution Φ_i and forms a posterior distribution about the underlying value x_i. Since the agent is risk neutral, only the conditional expected value $\mathbb{E}[x_i \mid q]$ matters for the agent's decision. The agent's goal is to determine a strategy π to inspect boxes to maximize her expected payoff in (2.1). In our setting, each box i (a.k.a., sender i) is competing with each other for the final selection from the agent. Specifically, the payoff of each sender i can be expressed as

$$\mathbf{1}\{\mathbb{A}_i = 1\} . \tag{2.2}$$

Namely, a sender obtains payoff 1 if he is selected and payoff 0 if he is not selected. We note that our results in this paper can be generalized to the setting where each sender i gets different $r_i \geq 0$ payoff if he is finally chosen by the agent.

In the paper, we assume that senders are ex ante symmetric, in the sense that the prior distribution for the values and the costs for inspection among all senders are the same. In particular, let $c \equiv c_i, \forall i$ denote the common inspection cost, and $H \in \Delta([0,1])$ denote the common prior distribution over the values, which has mean $\lambda = \mathbb{E}_{x\sim H}[x]$ and continuously differentiable density. [5]

[4] In the following discussion, we interchangeably use "box" and "sender".

[5] Our results can be readily generalized to an arbitrary interval $[a, b]$. To simplify the presentation, in this paper, we restrict our attention to the interval $[0, 1]$.

Senders' strategies. Recall that upon seeing a signal q from sender i's signaling scheme $\{\Phi_i(q|x), \mathcal{M}_i\}$, given prior H, the agent forms a posterior belief about sender i's value x_i, i.e., a sender's signaling scheme begets a distribution over posterior distributions of this sender's value. Since the risk-neutral agent's strategy only depends on her posterior means of senders' values, each sender's payoff depends only on the mean of the agent's posterior induced by the sender's signal and the *means* of the posterior beliefs induced by other senders' signals (instead of the detailed characteristics of the distributions). We can represent a sender's information strategy by a distribution over posterior means. A natural next question is which distributions over posterior means can indeed be implemented by some signaling schemes given prior H. This question can be answered using the notion of mean-preserving spread (MPS), which characterizes *feasible* distributions to represent senders' information strategies.

DEFINITION 2.2. (MEAN-PRESERVING SPREAD) *A distribution $H \in \Delta([0,1])$ is a Mean-preserving Spread (MPS) of a distribution $G \in \Delta([0,1])$, represented as $H \succeq G$, if and only if for all $\sigma \in [0,1]$:*

$$\int_{\sigma}^{1} G(x)\ \mathrm{d}x \geq \int_{\sigma}^{1} H(x)\ \mathrm{d}x\ , \tag{2.3}$$

where the inequality holds as equality for $\sigma = 0$.

It turns out that a distribution G over posterior means can be induced by some signaling scheme from prior H if and only if H is an MPS of G.

LEMMA 2.1. (AUMANN ET AL. [6], BLACKWELL AND GIRSHICK [10]) *There exists a signaling scheme that induces the distribution G over posterior means if and only if $H \succeq G$.*

With Lemma 2.1, we can without loss of generality assume that each sender i's strategy[6] is to directly choose a distribution $G_i \in \Delta([0,1])$ that satisfies $G_i : H \succeq G_i$, without the need of concerning the design of the underlying signaling scheme $\{\Phi_i(q|x), \mathcal{M}_i\}$. In the following discussion, we directly refer to G_i as sender i's strategy. Moreover, following Blackwell's ordering of informativeness [9], we say a strategy G' is *more informative* than G if G' is an MPS of G, i.e., $G' \succeq G$.

To illustrate the connection between the signaling schemes and the distributions of posterior means, consider the following two simple strategies. (1) No information strategy – in this strategy, the signal is completely uninformative (i.e., the distribution $\Phi_i(q|x)$ of q does not depend on the realized value x). Therefore, the distribution of posterior means G_i is a single point mass at the prior mean λ. (2) Full information strategy – in this strategy, the signal perfectly reveals sender's value to the agent (e.g., $\Phi_i(q \equiv x|x) = 1$ for every realized $x \in [0,1]$), and thus, the posterior mean distribution G_i coincides with the prior distribution H.

Solution concept. The timing of our competitive information design game can be detailed as follows: First, each sender commits to a strategy (a.k.a., a signaling scheme). Second, the agent observes all senders' strategies, and uses an inspection strategy π to determine how to inspect and when to terminate the inspection. Finally, the agent chooses the sender that has the maximum value among all inspected senders. When the agent is indifferent between multiple senders, he chooses one of them uniformly at random.

Note that in this game, after the senders determine their strategies, it is the agent's best response to use the optimal strategy as characterized by Theorem 2.1. Therefore, in this paper, we assume that the agent is always using the optimal inspection strategy, and our discussion focuses on the senders' game of competitive information design: choosing information strategies to maximizing the chance of being selected. Following the earlier works [18, 19], throughout the paper, we focus on the solution concept of pure-strategy equilibria. Similar to previous works, when analyzing the equilibrium strategy, we assume that senders in our setting are *symmetric* ex ante, our analysis in this part will thus focus on symmetric equilibria [5, 24]. We aim to investigate how the competition and agent's inspection cost affect the senders' information strategies at equilibrium,[7] and how the information strategy at equilibrium affects the ability of the agent to take her optimal decision.

[6] We will use sender's strategy synonymously with sender's information strategy.
[7] We will use equilibrium synonymously with pure symmetric equilibrium.

3 Informational Properties of Pandora's Box

In this section, we investigate how senders' strategies affect the agent's payoff under optimal inspection strategy. To this end, we provide several properties about the reservation values which will be useful for our later equilibrium analysis in Section 4. While reservation values have been well-studied in the Pandora's Box problem, to our knowledge, the informational properties we present in this section are not known before.

Below we present the main result in this section. Intuitively, the result shows that the agent obtains a higher payoff whenever a box becomes more informative.

THEOREM 3.1. *For any two sets of value distributions $(G_1, \dots, G_i, \dots G_n)$ and $(G_1, \dots, G'_i, \dots G_n)$ that only differ in the distribution of box i, if $G'_i \succeq G_i$, the agent obtains a weakly higher expected payoff under $(G_1, \dots, G'_i, \dots G_n)$ with optimal inspection strategy.*

Importantly, it is worth noting that the above results do not require the assumption of symmetric prior prize distribution. With the results in Theorem 3.1, an important corollary is that, when all senders are performing full information strategy, i.e., $G_i = H$ for all i, the agent obtains the highest payoff. Below we demonstrate a stronger version of the above claim. We define the following *essentially* full information strategy which fully reveals information whenever the value is no larger than its reservation value.

DEFINITION 3.1. (ESSENTIALLY FULL INFORMATION STRATEGY) *A strategy $G : H \succeq G$ is essentially full information strategy if G satisfies that $G(x) = H(x), \forall x \in [0, \sigma_H]$, where σ_H is the reservation value of the prior H.*

We can show that, for the agent to achieve the highest payoff, it suffices that all senders use essentially full information strategy.

COROLLARY 3.1. *Let G be an essentially full information strategy. The agent obtains the highest expected payoff $\sigma_H - \int_0^{\sigma_H} H(x)^n dx$ under $(G, \dots, G)$ among all possible (symmetric or asymmetric) strategy profiles.*

The basic intuition behind the above Corollary 3.1 is that in Pandora's Box, when the agent uses the optimal inspection strategy, after he inspects sender i, as long as the mean of the posterior for sender i after inspection is higher than its reservation value, the agent will take the same action: stop inspection and choose sender i. This observation demonstrates that, the distribution above the reservation value for the sender's strategy does not change the agent's decisions and payoffs.

Note that since the agent chooses exactly one sender at the end, the total payoff to all senders is 1 no matter what the agent's inspection strategy is and what the senders' strategies are. Therefore, when all senders use essentially full information strategy, it not only maximizes the agent's payoff, it also achieves the maximum social welfare. Given this desired property for essentially full information strategy, in Section 4, we characterize the sufficient and necessary condition for all senders to use essentially full information strategy (see Corollary 4.1).

Additional useful properties. Before presenting the proof of Theorem 3.1, we describe a few other informational properties of Pandora's Box. First, as discussed earlier, we say a distribution G' is *more informative* than G if G' is an MPS of G, i.e., $G' \succeq G$. This partial order of informativeness is from Blackwell's information theorem [9]. En route to proving Theorem 3.1, we also show the following total order on the reservation values induced by information strategies.

PROPOSITION 3.1. *For any cost $c \geq 0$ and two distributions G' and G, if $G' \succeq G$, $\sigma_{G'} \geq \sigma_G$.*

That is, a more informative sender strategy leads to a higher reservation value. Since the agent inspects the senders in an decreasing order of their reservation values, the proposition confirms the intuition that the agent would first inspect the sender who uses more informative strategy. Below we give the lower and upper bounds of the reservation values for any feasible sender's strategy G given prior H. Moreover, we provide conditions on when the sender's strategy G has the lowest or highest reservation value, corresponding to the most uninformative or most informative strategy.

COROLLARY 3.2. *Given the prior H and the cost $c \geq 0$, for any strategy G that satisfies $H \succeq G$, we have $\lambda - c \leq \sigma_G \leq \sigma_H$. Moreover,*

- *$\sigma_G = \lambda - c$ if and only if G has no support over $[0, \lambda - c]$;*
- *$\sigma_G = \sigma_H$ if and only if H is an MPS of G over the interval[8] $[0, \sigma_H]$, denoted by $H \succeq_{[0,\sigma_H]} G$.*

The above corollary characterizes the sender's strategies that reach the lowest and highest reservation values. We should expect when the sender uses no (full) information strategy, the strategy should lead to the lowest (highest) reservation value. As a sanity check, when the sender uses no information strategy, the corresponding G contains a single point mass at λ, and it is easy to see that corresponding reservation value is $\lambda - c$. When the sender uses full information strategy, i.e., the corresponding G equals to the prior, the reservation value is σ_H. We conclude this section with two additional properties which will be useful for our later analysis: Lemma 3.1 provides an alternative definition of the reservation value, and Lemma 3.2 is a natural implication of the above Corollary 3.2. The proofs of these two properties are provided in Appendix A.

LEMMA 3.1. *For any G with mean λ and for any $c \geq 0$, $\sigma_G = \sigma$ if and only if $\int_0^\sigma G(x)dx = \sigma - (\lambda - c)$.*

LEMMA 3.2. *For any H, a strategy $G : H \succeq G$ satisfying $\sigma_G = \sigma_H$ must have $G(\sigma_H) = H(\sigma_H)$.*

3.1 Proof of Theorem 3.1 We first provide an overview of our proof. In the agent's optimal inspection strategy (as specified in Theorem 2.1), both the selection rule and the stopping rule depend on the reservation value. To see how the agent's payoff changes if one sender uses a different strategy, one needs to understand how the reservation value ties with sender's strategy. Therefore, our first step for proving Theorem 3.1 is to show that the reservation value is always weakly larger if the strategy is more informative (see Proposition 3.1). With this characterization, armed with an already known result which shows the expected payoff of any agent's inspection policy is bounded above by the expectation of highest "capped" reservation value (see Lemma 3.3), we can then prove Theorem 3.1. Below we first provide the proof of Proposition 3.1. We will then utilize it to prove the theorem.

Proof of Proposition 3.1. When $c = 0$, we have $\sigma_G = \sigma_{G'} = +\infty$. Below we prove the result for $c > 0$. From Lemma 3.1, we know

$$\begin{aligned}\sigma_{G'} - \sigma_G = \int_0^{\sigma_{G'}} G'(x)dx - \int_0^{\sigma_G} G(x)dx &\overset{(a)}{=} \int_{\sigma_G}^1 G(x)dx - \int_{\sigma_{G'}}^1 G'(x)dx \\ &\overset{(b)}{\geq} \int_{\sigma_G}^1 G(x)dx - \int_{\sigma_{G'}}^1 G(x)dx\ ,\end{aligned}$$

where (a) is due to $\int_0^1 G(x)dx = \int_0^1 H(x) = 1 - \lambda$, (b) is due to Definition 2.2. Now suppose $\sigma_{G'} < \sigma_G$,

$$\sigma_G - \sigma_{G'} \leq \int_{\sigma_{G'}}^{\sigma_G} G(x)dx \overset{(a)}{\leq} \sigma_G - \sigma_{G'},$$

where (a) holds only when $G(x) = 1, \forall x \in [\sigma_{G'}, \sigma_G]$. However we note that it cannot be $G(\sigma_{G'}) = 1$ when $\sigma_{G'} < \sigma_G$. Suppose $G(\sigma_{G'}) = 1$ when $\sigma_{G'} < \sigma_G$, then we have $G(\sigma_G) = 1$ and $\mathbb{E}_{x\sim G}[(x - \sigma_G)_+] = 0 \neq c$. As a result, when $G(\sigma_{G'}) < 1$, we have $\sigma_G - \sigma_{G'} \leq \int_{\sigma_{G'}}^{\sigma_G} G(x)dx < \sigma_G - \sigma_{G'}$, which contradicts itself. Thus, we must have $\sigma_{G'} \geq \sigma_G$. □

Proof of Theorem 3.1. To prove Theorem 3.1, we use the following result which characterizes the best payoff that any central planner can possibly hope to achieve. Fix a strategy G and its corresponding σ_G, define following *capped value*:

$$\kappa_G := \min\{x, \sigma_G\}, \quad x \sim G\ . \tag{3.4}$$

Given a strategy profile $(G_1, \ldots, G_n)$, the below lemma shows that the optimal agent's payoff is the highest capped value among senders.

[8]Let $W(y) := \int_0^y \left[H(x) - G(x)\right]dx$. We say H is an MPS of G over $[a, b]$ if and only if $W(a) = W(b) = 0$, and $W(y) \geq 0, \forall y \in [a, b]$.

Lemma 3.3. (Kleinberg et al. [27]) *The procedure defined in Theorem 2.1 can achieve the agent's optimal expected payoff $\mathbb{E}[\max_i \kappa_i]$, i.e., the highest expected capped value he obtains.*

Below we use $u^A(G_1, \ldots, G_n)$ to denote the agent's expected payoff when the agent is using the *optimal* inspection strategy, i.e., $u^A(G_1, \ldots, G_n) = \mathbb{E}_{G_1,\ldots,G_n}[\max_i \kappa_i]$. To simplify the notation, we interchangeably use κ_i and κ_{G_i}.

We are now ready prove Theorem 3.1. We first observe that for any strategy G such that $\mathbb{E}_{x\sim G}[x] = \lambda$, we have $\mathbb{E}_G[\kappa_G] = \lambda - c$. To see this, note that

$$\mathbb{E}_G[\kappa_G] = \int_0^{\sigma_G} xg(x)dx + \sigma_G \int_{\sigma_G}^1 g(x)dx = \lambda - \left(c + \sigma_G \int_{\sigma_G}^1 g(x)dx\right) + \sigma_G \int_{\sigma_G}^1 g(x)dx = \lambda - c \ .$$

Given a strategy profile $(G_1, \ldots, G_i, \ldots G_n)$, from Lemma 3.3, the agent's optimal expected payoff is the expectation of the maximum of n independent random variables $\{\kappa_i\}_{i\in[n]}$ with the mean $\lambda - c$. Let $\kappa_{-i} := \max_{G_{-i}}\{\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_n\}$, where $G_{-i} := (G_j)_{j\in[n], j\neq i}$. Now observe that,

$$\mathbb{E}_{G_1,\ldots,G_n}\left[\max_i \kappa_i\right] = \mathbb{E}_{G_{-i}}[\mathbb{E}_{G_i}[\max\{\kappa_i, \kappa_{-i}\}]] \ .$$

Below, we show that for all possible $\kappa_{-i} = b$, the following holds

$$\mathbb{E}_{G_i'}[\max\{\kappa_i, b\}] \geq \mathbb{E}_{G_i}[\max\{\kappa_i, b\}] \ . \tag{3.5}$$

Recall that from Proposition 3.1, we have $\sigma_{G_i'} \geq \sigma_{G_i}$. We now consider the following three cases:

- When $b \geq \sigma_{G_i'}$, (3.5) holds naturally as $\mathbb{E}_{G_i'}[\max\{\kappa_i, b\}] = b = \mathbb{E}_{G_i}[\max\{\kappa_i, b\}]$.
- When $\sigma_{G_i} \leq b < \sigma_{G_{i'}}$, we have $\mathbb{E}_{G_i}[\max\{\kappa_i, b\}] = b$, and

  $$\mathbb{E}_{G_i'}[\max\{\kappa_i, b\}] = \int_0^b \max\{x, b\}dG_i(x) + \int_b^1 \max\{\kappa_i, b\}dG_i(x) \geq b \ ,$$

  thus, (3.5) holds true.
- When $b < \sigma_{G_i}$, in this case, we have

  $$\begin{aligned}\mathbb{E}_{G_i}[\max\{\kappa_i, b\}] &= \int_0^b bdG(x) + \int_b^1 \max\{\kappa_i, b\}dG_i(x) \\ &= bG(b) + \int_b^1 \min\{x, \sigma_{G_i}\}dG_i(x) \\ &\overset{(a)}{=} bG(b) + \lambda - c - \int_0^b xdG_i(x) \overset{(b)}{=} \lambda - c + \int_0^b G_i(x)dx \ ,\end{aligned}$$

  where (a) uses the earlier observation $\mathbb{E}_G[\kappa_G] = \lambda - c$, and (b) uses integration by parts. Recall that G_i' is an MPS of G_i, we have $\int_0^b G_i(x)dx \leq \int_0^b G_i'(x)dx, \forall b$. As a result, we conclude that $\mathbb{E}_{G_i}[\max\{\kappa_i, b\}] \leq \mathbb{E}_{G_i'}[\max\{\kappa_i, b\}]$.

Putting all pieces together, (3.5) holds for any $b \in [0, 1]$, which completes the proof. ☐

The above proof essentially shows that the capped value of a more informative strategy is *second-order* stochastically dominated by the capped value of a less informative strategy. Then by the convexity of the maximum operator, one can also achieve the result in Theorem 3.1.

4 Equilibrium Analysis

In this section, we characterize the equilibrium for the senders' game of competitive information design for any prior H and any cost $c \geq 0$. In particular, we give sufficient and necessary conditions of the existence of pure symmetric equilibrium. We also characterize the unique equilibrium strategy if the pure symmetric equilibrium exists.

Before stating our results, we first define a special structure for senders' strategies.

Definition 4.1. (Alternating $(n-1)$–linear MPS – Hwang et al. [24]) *Given a prior H, G exhibits alternating $(n-1)$–linear MPS behavior in the interval $[a,b]$ if whenever G is not fully revealing information in a subinterval $[x_1,x_2] \subseteq [a,b]$, G^{n-1} is linear over $[x_1, \min\{x_2, \max_{x\in[a,b]}\{x : x \in \textbf{\textit{supp}}\,[G]\}]$ and $H \succeq_{[x_1,x_2]} G$.*

With the above structure, our main result in this section can be stated as follows.

Theorem 4.1. *For any prior H and any cost $c \geq 0$, given a strategy G and its $\bar{x}_G := \max\{x \in [0,\sigma_H] : x \in \textbf{\textit{supp}}\,[G]\}$, $(G,\dots,G)$ is an equilibrium if and only if*

(i) $\sigma_G = \sigma_H$;

(ii) G^{n-1} *is convex over* $[0,\bar{x}_G]$ *and* G *exhibits alternating* $(n-1)$*–linear MPS behavior over* $[0,\sigma_H]$;

(iii) *deviating to a strategy* F *where* $\sigma_F = \max\{\bar{x}_G, \lambda - c\}$ *is not profitable. More concretely,*

(a) *if* $\lambda - c \geq \bar{x}_G$*, then the optimal deviation value* $G(\lambda-c)^{n-1} \leq 1/n$;

(b) *if* $\lambda - c < \bar{x}_G$*, then the optimal deviation value* $\int_0^{x^\dagger} G(x)^{n-1} dH(x) + H(\sigma_H)^{n-1}(1-H(x^\dagger)) \leq 1/n$ *where* $x^\dagger$ *uniquely satisfies* $\int_{x^\dagger}^1 (x-\bar{x}_G) dH(x) = c$.

We interpret and examine each condition in the theorem below. Condition (i) indicates that the reservation value of the equilibrium strategy G must achieve its maximum. This aligns with the intuition that the sender prefers to be inspected earlier than later. Condition (ii) characterizes the structure of feasible equilibrium strategy. As we elaborate shortly, the first two conditions can uniquely[9] pin down a distribution G. Lastly, condition (iii) verifies whether G that satisfies the first two conditions is indeed an equilibrium strategy. Essentially, there are only two scenarios: (a) If $\lambda - c \geq \bar{x}_G$, deviating to no information strategy for a sender is the most profitable. (b) If $\lambda - c < \bar{x}_G$, it is the most profitable to deviate to a strategy F, with reservation value $\bar{x}_G$, which satisfies $F(x) = H(x), \forall x \leq x^\dagger$ and has no support between $x^\dagger$ and $\bar{x}_G$ (see the blue dotted line in Figure 1b). [10] In either case, the optimal deviation value can be computed in a closed form, so we can verify whether G is an equilibrium strategy.

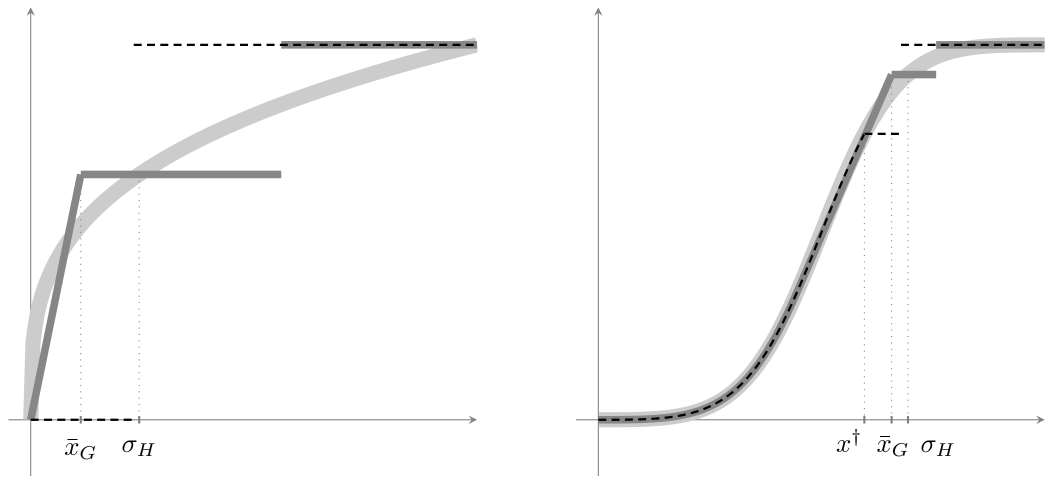
Note that in the special case where the inspection cost $c = 0$, our problem reduces to a simpler setting, in which the agent does not need to choose which senders to inspect and in what order as he can inspect all senders for free. In this setting, Hwang et al. [24][11] show that there always exists a unique equilibrium strategy that every sender takes G that satisfies the conditions that G^{n-1} is convex over the support of G, and G exhibits the above alternating behavior over $[0,1]$. Our result strictly generalizes their result. First, we can see that our conditions (i)–(iii) are always satisfied when $c = 0$: When there is no inspection cost, both σ_G and σ_H approach $+\infty$. For our condition (ii), G exhibiting alternating behavior over $[0,\sigma_H]$ is equivalent to exhibiting alternating behavior over $[0,1]$. For condition (iii), given a distribution G satisfying condition (ii) over $[0,1]$, we always have $\lambda - c = \lambda < \bar{x}_G$ as G has no support over $[\bar{x}_G, 1]$. When $c = 0$, we have $x^\dagger = 1$, and $\int_0^{x^\dagger} G(x)^{n-1} dH(x) + H(\sigma_H)^{n-1}(1-H(x^\dagger)) = \int_0^1 G(x)^{n-1} dH(x) \leq 1/n$ holds for sure. To see this, note that Hwang et al. [24] have showed that such G is the equilibrium strategy when $c = 0$. Thus, by definition, we have $\int_0^1 G(x)^{n-1} dH(x) \leq \int_0^1 G(x)^{n-1} dG(x) = 1/n$.

When inspection cost $c > 0$, a pure symmetric equilibrium might not exist. We present two examples (see Figure 1) where the pure symmetric equilibrium does not exist. Each of the examples violates one of the cases in condition (iii).

[9] The uniqueness here means the behavior of G over $[0,\sigma_H]$ is unique. Note that Theorem 4.1 only states the conditions for the support of G that is in $[0,\sigma_H]$. Indeed, one can show that if $(G,\dots,G)$ is an equilibrium, then $(G,\dots,G',\dots,G)$ is also an equilibrium as long as $G'(x) = G(x), \forall x \in [0,\sigma_H]$. The reason is that once we pin down the reservation value of all senders' strategies to be σ_H, each sender's expected payoff only depends on the behavior of his strategy in $[0,\sigma_H]$ (see Corollary 4.6 for detailed discussions).

[10] This specific structure of F is largely due to the convexity of G^{n-1} over $[0,\bar{x}_G]$, it will be proved in Lemma 4.4.

[11] In their model, the agent firstly observes all realized $\{x_i\}_{i\in[n]}$, and then selects the sender that has the maximum value. This is equivalent to our setting with $c = 0$. To see this, note when $c = 0$, the reservation value of any strategy goes to infinity. Thus, though the agent sequentially inspects senders, he would inspect all senders and select the best one.



(a) (b)

Figure 1: In both figures, the prior H is the gray solid line, the distribution G that satisfies conditions (i)–(ii) in Theorem 4.1 is the deep gray solid line. The profitable deviation F is then the black dashed line. See the detailed descriptions in Example 4.1 and Example 4.2. (a): Equilibrium does not exist as it violates the the case (a) in condition (iii). (b): Equilibrium does not exist as it violates the case (b) in condition (iii).

Example 4.1. (Pure symmetric equilibrium may not exist – violate case (a) in condition (iii)) *Consider prior $H(x) = x^{0.3}$ (the gray solid line in Figure 1a), $n = 2$, and a cost $c = 0.11$. With this prior, one can compute $\lambda = 0.2308$, $\sigma_H = 0.2431$, and $\sigma_{\mathsf{NI}} = \lambda - c = 0.1208$. Using the conditions (i)–(ii) in Theorem 4.1, one can compute a unique G (where $\bar{x}_G = 0.1122$). However, such G is not an equilibrium strategy as one can deviate to a No information disclosure strategy G_{NI} to achieve a higher payoff $G(\sigma_{\mathsf{NI}}) = 0.6542 > 0.5$.*

Example 4.2. (Pure symmetric equilibrium may not exist – violate case (b) in condition (iii)) *Consider prior $H(x) = \frac{1}{1+\left(\frac{x}{1-x}\right)^{-3}}$ (the gray solid line in Figure 1b), $n = 2, c = 0.005$, $\lambda = 0.5, \sigma_H = 0.6938$, $x^\dagger = 0.5961$, and $\bar{x}_G = 0.6571$. Such G is not an equilibrium as one can deviate to a strategy F to a higher payoff $0.5048 > 0.5$. F has reservation value $\sigma_F = \bar{x}_G$, and $F(x) = H(x), \forall x \in [0, x^\dagger]$, and F has no support over $[x^\dagger, \bar{x}_G]$.*

4.1 Applications and Implications of Theorem 4.1 Theorem 4.1 provides a general characterization of the equilibrium for competitive information design for Pandora's Box. Here we discuss the applications of the theorem in some interesting/important cases and their implications. All the proofs in this section are in Appendix B.

First of all, as discussed in Corollary 3.1, every sender deploying essentially full information strategy is a desired equilibrium as it leads to the highest agent payoff and the highest social welfare. Utilizing Theorem 4.1, we can characterize the sufficient and necessary condition for essentially full information strategy to be the equilibrium.

Corollary 4.1. *Essentially full information strategy is the equilibrium strategy if and only if H^{n-1} is convex over $[0, \sigma_H]$.*

We can observe a couple of interesting implications of Corollary 4.1. First, increasing competition makes it more likely to reach essential full information disclosure. This implication is from the the fact that when we fix inspection cost, the shape of the function H^{n-1} becomes more convex as n increases. Moreover, for an arbitrary prior H and any cost, one can show that there always exists a number of senders such that essentially full information is the equilibrium. We can also show that for any prior H, as long as the number of senders is high enough, essentially full information strategy can be the equilibrium strategy, as formalized below.

Corollary 4.2. *For any prior H and cost $c \geq 0$, there exists a $\underline{n} \in \mathbb{N}_+$, such that for any $n \geq \underline{n}$, essentially full information strategy is the equilibrium strategy.*

Another implication of Corollary 4.1 is that, increasing inspection cost makes it more likely to reach essential full information disclosure. This implication follows from when we fix the number of senders, if essentially full information is the equilibrium with a smaller inspection cost, it is also the equilibrium with a larger cost. This is because when increasing the cost, the corresponding reservation value σ_H is decreasing. Therefore, if H^{n-1} is already convex on a larger interval $[0, \sigma_H]$, it is also convex on a smaller interval. To illustrate this observation, for a general class of priors – the prior that has single-peaked density – we can characterize the lower bound cost for the essentially full information to be the equilibrium. In particular, when H^{n-1} has single-peaked density,[12] it is always first convex and then concave (see example in Figure 1b). Thus, as long as the reservation value σ_H falls below the inflection point of H^{n-1} (i.e., the point where the function H^{n-1} changes from being convex to concave), essentially full information is the equilibrium.

COROLLARY 4.3. *Fix n and H which H^{n-1} has single-peaked density over $[0, \sigma_H]$ and its inflection point $\underline{x}$, let $\underline{c}$ be an inspection cost where $\sigma_H = \underline{x}$, then for any cost $c \geq \underline{c}$, essentially full information is the equilibrium.*

In below, we exemplify the use of Corollary 4.3 to identify the condition of the inspection cost for common distributions that admit the existence of essentially full information equilibrium strategy when there are two senders.

EXAMPLE 4.3. (UNIFORM PRIOR) *Suppose H is the uniform prior over $[a, b]$ with $a \geq 0$, it can be shown that for any inspection cost $c \geq 0$, essentially full information strategy is an equilibrium strategy, namely, a strategy $G = H$ satisfies all conditions in Theorem 4.1.*

EXAMPLE 4.4. (GAUSSIAN PRIOR) *Suppose H is the Gaussian prior with mean $\lambda > 0$ and variance υ^2 where $\upsilon \geq 0$, it can be shown that essentially full information strategy is an equilibrium strategy if and only if the inspection cost c satisfies $c \geq {}^{\lambda}/_{2} + {}^{\upsilon}/_{\sqrt{2\pi}}$.*

In addition to characterizing the equilibirum conditions, we can also show that, under the condition that essentially full information is the equilibrium, the agent's payoff decreases as the inspection cost increases and increases as the number of senders increases.

COROLLARY 4.4. *Under essentially full information equilibrium, the agent's payoff is decreasing with respect to the inspection cost and increasing with respect to the number of senders.*

Below we provide one more example on how Theorem 4.1 can help us characterize the equilibrium in different cases. When H^{n-1} is concave over $[0, \sigma_H]$, using the conditions (i)–(ii), we can characterize a unique distribution G such that G^{n-1} will be firstly linear over $[0, \bar{x}_G]$ and then flat over $[\bar{x}_G, \sigma_H]$ (see the example in Figure 1a). Using the linearity of G^{n-1}, we can show that to verify whether such G is an equilibrium strategy, it only suffices to check whether $G(\lambda - c)^{n-1} \leq {}^{1}/_{n}$.

COROLLARY 4.5. *Given prior H in which H^{n-1} is concave over $[0, \sigma_H]$. Let G be a distribution satisfying the conditions (i)–(ii) in Theorem 4.1, then G is an equilibrium strategy if and only if $G(\lambda - c)^{n-1} \leq {}^{1}/_{n}$.*

We also exemplify below the use of corollary 4.5 to identify the condition of the inspection cost for common distribution that admit the existence of equilibrium strategy when there are two senders.

EXAMPLE 4.5. (EXPONENTIAL PRIOR) *Suppose H is the exponential prior over $[0, +\infty)$ with the parameter $\mu \geq 0$, namely, $H(x) = 1 - \exp(-\mu x)$. Since H is concave over the whole support $[0, +\infty)$, it can be shown that there exists an equilibrium strategy G (in particular, one can deduce the behavior of strategy G over $[0, \sigma_H]$ where $\sigma_H = {}^{-\ln(\mu c)}/_{\mu}$ as follows: $G(x) = \frac{H(\sigma_H)}{2\sigma_H - \frac{2(\sigma_H - (1/\mu - c))}{H(\sigma_H)}} x, \forall x \in [0, 2\sigma_H - \frac{2(\sigma_H - (1/\mu - c))}{H(\sigma_H)}]; G(x) = H(\sigma_H), \forall x \in [2\sigma_H - \frac{2(\sigma_H - (1/\mu - c))}{H(\sigma_H)}, \sigma_H]$) if and only if the inspection cost $c \geq 0$ and the parameter μ satisfy $(\mu c)^3 - 3(\mu c)^2 + 2\mu c + \mu c \ln(\mu c) \geq 0$. Note that when fixing any inspection cost $c > 0$, function $(xc)^3 - 3(xc)^2 + 2xc + xc\ln(xc)$ crosses x-axis over $(0, {}^{1}/_{c})$ once and it crosses from below. Intuitively, this suggests that for any fixed cost $c > 0$, it is more likely to admit the existence of a symmetric equilibrium if the parameter μ is larger, i.e., the prior has smaller variance.*

[12] As long as the density function h is log-concave over $[0, \sigma_H]$, H^{n-1} has single-peaked density for any n over $[0, \sigma_H]$.

4.2 Proof of Theorem 4.1

In this section, we present our proof for Theorem 4.1.

Technical challenges and proof overview. Determining whether a particular strategy profile $(G, \ldots, G)$ is an equilibrium can be challenging, as it depends on the full set $\mathcal{H}$ of feasible strategies, i.e., $\mathcal{H} := \{F : H \succeq F\}$, that each sender can deviate to. When the agent uses the optimal inspection strategy, however, using the observation we obtain in Proposition 3.1, one can first show that a strategy G can be an equilibrium strategy only if it satisfies $\sigma_G = \sigma_H$. This observation shrinks the set that contains any possible equilibrium strategy to the set $\mathcal{H}(\sigma_H) := \{F : H \succeq F \wedge \sigma_F = \sigma_H\}$. Next, using the conditions provided in Corollary 3.2, and examining the fixed point problem over the set $\mathcal{H}(\sigma_H)$, we can uniquely pin down the behavior of G over the interval $[0, \sigma_H]$ if G is the equilibrium strategy.

The above procedure helps us pin down the necessary conditions for G to be the equilibrium strategy. To verify whether the identified G is indeed the equilibrium strategy, we need to show that no sender has profitable deviation under the strategy profile $(G, \ldots, G)$. This step is challenging since we again need to examine all possible deviations that one sender can deviate to when all other senders use strategy G. Different deviation strategies have different reservation values, which impact the order that the agent inspects the senders, and change the deviation payoff. In a more detail, when a deviation strategy F has reservation value $\sigma_F = \sigma < \sigma_H$, let $U^S(x)$ be the sender's deviation payoff as a function of the realized value $x \sim F$, it can be shown that $U^S(x) = \min\{G(x)^{n-1}, G(\sigma)^{n-1}\}$, in which the shape of $U^S(\cdot)$ depends on the choice of σ. Thus, there is no single program that can encode sender's deviation problem. Instead, our solution is that, for every possible reservation value σ, we consider the corresponding linear program (note that the constraint $\sigma_F = \sigma$ can be formulated as a linear constraint), and then characterize its optimal deviation strategy. We then show that the optimal deviation value is single-peaked (with the peak at $\sigma^* := \max\{\sigma_{\mathsf{NI}}, \bar{x}_G\}$) w.r.t. $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$. To this end, to account for the additional constraint $\sigma_F = \sigma$, we extend the verification tool provided in Dworczak and Martini [17] to show what the optimal dual solution must look like, and then show there exists an optimal primal solution that satisfies complementary slackness.

To summarize, the analysis mainly consists of following steps:

- **Step 1**. In this step, we prove the condition (i) in Theorem 4.1, namely, for any H, if there exists an equilibrium $(G, \ldots, G)$, it must be that $\sigma_G = \sigma_H$ (see Lemma 4.1).
- **Step 2**. In this step, we show that no sender has profitable deviation to a strategy $F \in \mathcal{H}(\sigma_H)$ if all other senders use strategy satisfying conditions (i)–(ii) in Theorem 4.1 (see Lemma 4.2).
- **Step 3**. In this step, we show that when all other senders use strategy G satisfying conditions (i)–(ii) in Theorem 4.1, then no sender has profitable deviation if and only if condition (iii) holds (see Lemma 4.3).

Below, we first provide detailed analysis of the above steps. The proof of the main result Theorem 4.1 follows from combining the results of these steps.

Step 1 – Characterizing the reservation value of equilibrium strategy.

LEMMA 4.1. *For any H, if there exists an equilibrium $(G, \ldots, G)$, it must be that $\sigma_G = \sigma_H$. Each sender's expected payoff is $1/n$ at any equilibrium.*

Proof Sketch of Lemma 4.1. Given any symmetric strategy $(G, \ldots, G)$ where $\sigma_G < \sigma_H$, each sender i's expected payoff can be expressed as

$$u_i^S(G, \ldots, G) := \Pr[\mathbb{A}_i = 1 | \mathbb{I}_i = 1] \cdot \Pr[\mathbb{I}_i = 1] \ ,$$

where $\Pr[\mathbb{I}_i = 1]$ is the probability of sender i being inspected by the agent and $\Pr[\mathbb{A}_i = 1 | \mathbb{I}_i = 1]$ is the expected payoff conditional on being inspected. As there always exists probability such that sender i is never inspected by the agent, thus, we have

$$\Pr[\mathbb{I}_i = 1] \equiv 1 - \delta < 1 \ .$$

Now let $U_i^S(x)$ denote the sender i's expected payoff conditional on being inspected and the value x realizing. Then we have

$$\Pr[\mathbb{A}_i = 1 | \mathbb{I}_i = 1] = \int_0^1 U_i^S(x) dG(x) \ .$$

Now let $F : H \succeq F$ be a strategy satisfying $\sigma_F > \sigma_G$ and also

$$\int_0^1 U_i^S(x)dF(x) > \int_0^1 U_i^S(x)dG(x) - \varepsilon \ ,$$

for a small $\varepsilon > 0$. Note as $\sigma_G < \sigma_H$, such F must exist (we defer the detailed construction of such F to the Appendix B). Then by deviating to strategy F, from Proposition 3.1, we know sender i's probability of being inspected is increased to 1. Thus,

$$\begin{aligned} u_i^S(G, \ldots, F, \ldots, G) - u_i^S(G, \ldots, G) &> \int_0^1 U_i^S(x)dF(x) - \int_0^1 U_i^S(x)dG(x) \cdot (1-\delta) \\ &= \delta \cdot \int_0^1 U_i^S(x)dG(x) - \varepsilon > 0 \ , \end{aligned}$$

where the last inequality is by choosing a sufficiently small ε. As a result, such deviation is profitable.

Clearly, each sender's expected payoff is $1/n$ at any equilibrium. Suppose not, then the sender who has expected payoff smaller than $1/n$ can improve his expected payoff by simply mimicking another sender's strategy who has higher payoff than $1/n$. □

Step 2 – Characterizing the behavior of G over the interval $[0, \sigma_H]$. Now we use the result in Lemma 3.2 and the characterization in Corollary 3.2 to prove the condition (ii).

LEMMA 4.2. *Given prior H, under the strategy profile $(G, \ldots, G)$ where G satisfies the conditions (i)–(ii) in Theorem 4.1, then no sender has a profitable deviation to a strategy F where $\sigma_F = \sigma_H$. Meanwhile, if $(G, \ldots, G)$ is an equilibrium, then the behavior of G over the interval $[0, \sigma_H]$ must satisfy the condition (ii) in Theorem 4.1.*

The intuition for the proof is straightforward. Given all other senders using strategy G and sender i using strategy F where $\sigma_F = \sigma_H$, with the result in Lemma 3.2, it can be shown that sender i's expected payoff only depends on the behavior of F over the interval $[0, \sigma_H]$. Then using the characterization in Corollary 3.2, and the earlier results in Hwang et al. [24], we show sender i's best deviation in the set $\mathcal{H}(\sigma_H)$ is indeed G itself.

Proof of Lemma 4.2. We first prove the first part of the statement. Given a prior H, let G be the distribution satisfying conditions (i)–(ii) in Theorem 4.1. We now consider sender i's best response strategy F that is subject to $\sigma_F = \sigma_H$ given all other senders using strategy G. For notation simplicity, define following quantile value $p_F := 1 - F(\sigma_F), p_G := 1 - G(\sigma_G)$, and $p_H := 1 - H(\sigma_H)$. Observe that whenever sender i is inspected, there are two possible cases, either the realized $x_i \geq \sigma_H$ where the agent will stop the inspection and claim x_i from sender i; or the realized $x_i < \sigma_H$ where the agent claims x_i from sender i only if he inspects all senders and finds out $i = \operatorname{argmax}_j x_j$. With the above observation, we have following sender i's expected payoff on deviating to strategy F:

$$\begin{aligned} u_i^S(G, \ldots, F, \ldots, G) &= \sum_{j=1}^{n} \left(p_F \cdot (1-p_G)^{j-1} + \int_0^{\sigma_H} G(x)^{n-1} dF(x) \right) \cdot \frac{1}{n} \\ &\overset{\text{(a)}}{=} \frac{1}{n} \cdot \sum_{j=0}^{n-1} p_F \cdot (1-p_G)^j + \int_0^{\sigma_H} G(x)^{n-1} dF(x) \ , \end{aligned}$$

where in (a) we use $p_F = p_H = p_G$ due to Lemma 3.2. Now we consider following sender i's best response problem that is subject to deviating to strategies in $\mathcal{H}(\sigma_H)$:

$$\max_{F \in \mathcal{H}(\sigma_H)} \frac{1}{n} \cdot \sum_{j=0}^{n-1} p_H \cdot (1-p_H)^j + \int_0^{\sigma_H} G(x)^{n-1} dF(x) \ .$$

Given a prior H, p_H is a constant. The above program can be further reduced to

$$\max_{F \in \mathcal{H}(\sigma_H)} \int_0^{\sigma_H} G(x)^{n-1} dF(x) \ . \tag{4.6}$$

Recall that from Corollary 3.2, the constraint $\sigma_F = \sigma_H$ is equivalent to requiring $H \succeq_{[0,\sigma_H]} F$. To complete the proof, we note that Hwang et al. [24] have shown when $c = 0$, a strategy G that satisfies the properties in Definition 4.1 over the interval $[0, 1]$ is the best response strategy to itself, i.e., G is the solution to the program $\max_{F\in\mathcal{H}} \int_0^1 G(x)^{n-1} dF(x)$. Now given a strategy that satisfies the conditions (i)–(iii), it is easy to see that any strategy G^* that satisfies $G^*(x) = G(x), \forall x \in [0, \sigma_H]$ is the optimal solution to the program (4.6). The second part of the statement follows from the necessity the equilibrium strategy G when $c = 0$ in Hwang et al. [24]. $\square$

Step 3 – Verifying whether G is indeed an equilibrium strategy. Now to argue whether G, which satisfies the conditions (i)–(ii) in Theorem 4.1, is an equilibrium strategy, it remains to show that no sender has a profitable deviation to a strategy F that has $\sigma_F < \sigma_H$ if all other senders use the strategy G. In other words, we need to show that whenever we fix a $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$, the best payoff for a sender i to deviate to a strategy $F \in \mathcal{H}(\sigma) := \{F : H \succeq F \wedge \sigma_F = \sigma\}$ is no larger than $1/n$. Given sender i using F where $\sigma_F = \sigma < \sigma_H$, and other senders using G, we have

$$u_i^S(G, \ldots, F, \ldots, G) = G(\sigma)^{n-1} \cdot \int_\sigma^1 dF(x) + \int_0^\sigma G(x)^{n-1} dF(x) \ .$$

Using integral by parts and rearranging the terms, we can get

$$u_i^S(G, \ldots, F, \ldots, G) = \int_0^1 \min\left\{G(x)^{n-1}, G(\sigma)^{n-1}\right\} dF(x) \ . \tag{4.7}$$

The proof of Lemma 4.2 and the above deviation payoff have following implication that only the behavior over the interval $[0, \sigma_H]$ of the strategy G matters for the equilibrium.

Corollary 4.6. *Given a prior H, if $(G, \ldots, G)$ is an equilibrium, then the strategy profile $(G_1, \ldots, G_n)$ where $\forall i, G_i(x) = G(x), \forall x \in [0, \sigma_H]$ is also the equilibrium.*

Fix a $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$, we now consider following sender i's best response strategy that is subject to the constraint $\sigma_F = \sigma$

$$\max_{F\in\mathcal{H}(\sigma)} \int_0^1 \min\left\{G(x)^{n-1}, G(\sigma)^{n-1}\right\} dF(x) \ . \tag{4.8}$$

Given σ, let $\mathtt{OPT}_\sigma$ denote the optimal value of the above program. Essentially, G is equilibrium strategy must satisfy that

$$\max_{\sigma:\sigma\in[\sigma_{\mathsf{NI}},\sigma_H)} \mathtt{OPT}_\sigma \le \frac{1}{n} \ . \tag{4.9}$$

In below analysis, we characterize the most profitable deviation given all other senders using strategy G. In particular, to guarantee (4.9), we show that, depending on the relative value σ_{NI} and $\bar{x}_G$, it suffices to only consider one deviation: either deviating to no information disclosure strategy (if $\sigma_{\mathsf{NI}} > \bar{x}_G$) or deviating to a strategy whose reservation value equals to $\bar{x}_G$ (if $\sigma_{\mathsf{NI}} \le \bar{x}_G$).

Lemma 4.3. *Fix a prior H and the cost $c > 0$, given all other senders using G that meets the conditions (i)–(ii) in Theorem 4.1, then*

(a) if $\sigma_{\mathsf{NI}} = \lambda - c > \bar{x}_G$, the most profitable deviation is no information strategy;

(b) if $\sigma_{\mathsf{NI}} = \lambda - c \le \bar{x}_G$, the most profitable deviation is a strategy F where $\sigma_F = \bar{x}_G$.

The condition (iii) in Theorem 4.1 simply follows by ensuring that the value of most profitable deviation is no larger than $1/n$. To prove Lemma 4.3, for the case $\sigma_{\mathsf{NI}} \le \bar{x}_G$, we separate our discussions in two regimes: for $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G)$ we show the optimal value $\mathtt{OPT}_\sigma$ is increasing w.r.t. σ; for $\sigma \in [\bar{x}_G, \sigma_H)$, we show the optimal value $\mathtt{OPT}_\sigma$ is decreasing w.r.t. σ. The analysis of other case where $\sigma_{\mathsf{NI}} > \bar{x}_G$ follows similarly. To show the monotoncity of $\mathtt{OPT}_\sigma$, we first characterize optimal solution F_σ for any $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$, and then examine the optimal deviation value $\mathtt{OPT}_\sigma$ under the deviation F_σ. In the remaining of the paper, due to the space limit, we mainly present the proof for first regime of the case $\sigma_{\mathsf{NI}} \le \bar{x}_G$.

LEMMA 4.4. *Given a prior H, and distribution G satisfying the conditions (i)–(ii) in Theorem 4.1, when $\sigma_{\mathsf{NI}} \le \bar{x}_G$, then for any $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$, a distribution F_σ that satisfies following structure is an optimal solution to the program (4.8)*

$$F_\sigma(x) = \begin{cases} H(x), & \forall x \in [0, x^\dagger) \\ H(x^\dagger), & \forall x \in [x^\dagger, x^\ddagger) \\ 1, & \forall x \in [x^\ddagger, 1] \end{cases} \tag{4.10}$$

where $x^\dagger$ satisfies that $\int_0^\sigma F_\sigma(x)dx = \sigma - (\lambda - c)$. Furthermore, the optimal value $\texttt{OPT}_\sigma$ is increasing w.r.t. $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$.

The structure of the optimal solution F_σ admits the following interpretations. Let $u(x) := \min\{G(x)^{n-1}, G(\sigma)^{n-1}\}$. As we can see, for any $\sigma \le \bar{x}_G$, u is convex over $[0, \sigma]$ (recall the convexity G^{n-1} in $[0, \bar{x}_G]$) and is constant over $[\sigma, 1]$. Then if a solution F has support below σ, ideally, by Jensen's inequality, F should allocate its support as much dispersed as possible in this interval. In other words, the MPS constraint should bind for the support of F that is in $[0, \sigma]$. At the same time, u attains maximum for any values above σ, F thus should put as much mass as possible above σ. Due to the equal-mean constraint (i.e., $\int x dF(x) = \lambda$), F should put their support that is in $[0, \sigma]$ as close to 0 as possible (and simultaneously as much dispersed as possible) so that F can allocate more mass above σ. Note that the constraint $\sigma_F = \sigma$ is a linear constraint, and it thus determines the cutoff $x^\dagger$ of the portion where F satisfies the property in Lemma 3.1.

Proof of Lemma 4.4. We first prove the optimal structure of F_σ for $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$. We begin with analyzing following general problem for any $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$,

$$\max_{F \in \mathcal{H}} \int_0^1 u(x) dF(x) \quad \text{s.t.} \int_0^\sigma F(x)dx = \sigma - (\lambda - c) \ . \tag{4.11}$$

The above program has two major constraints, one is $F \in \mathcal{H}$ to account for the feasibility of strategy F, and the other one accounts for $\sigma_F = \sigma$ (recall Lemma 3.1). The above optimization problem is non-trivial as sender i can deviate to any possible strategy $F \in \mathcal{H}(\sigma)$, and this is an infinite-dimensional linear program. Nevertheless, some recent technical developments in the information design literature are useful to our problem. In particular, we use the following result obtained by Dworczak and Martini [17], which provides a duality theory for optimization problems with MPS constraints. To be more precise, they consider the problem $\max_{F: H \succeq F} \int_0^1 u(x)dF(x)$, and show that if F is the solution to this program, then there must exist a convex function $p(x) : [0, 1] \to \mathbb{R}$ such

$$\int_0^1 p(x)dF(x) = \int_0^1 p(x)dH(x) \ , \tag{4.12}$$

and F is also the optimal solution to the program $\max_{\widetilde{F} \in \Delta([0,1])} \int_0^1 (u(x) - p(x))d\widetilde{F}(x)$. In our problem, additional to the MPS constraint, we also have a linear constraint that the strategy F has $\sigma_F = \sigma$. Follow the similar analysis, one can deduce that if F_σ is the optimal solution to the program (4.11), it must also exist a convex function $p(\cdot)$ where (4.12) holds for F_σ, and there exists $\alpha \in \mathbb{R}$ such that

$$F_\sigma \in \text{argmax}_{\widetilde{F} \in \Delta([0,1])} \left\{ \int_0^1 (u(x) - p(x))d\widetilde{F}(x) - \alpha \cdot \left(\sigma \int_0^\sigma d\widetilde{F}(x) - \int_0^\sigma x d\widetilde{F}(x) - \sigma + (\lambda - c) \right) \right\} \ ,$$

where we have used integration by parts in the reservation value constraint. Observe that we can always add a constant to $p(\cdot)$ without changing any of its properties. Thus, by complementary slackness, one must have

$$\begin{aligned} &\text{if } x \in [0, \sigma) \wedge x \in \mathsf{supp}\,[F_\sigma] : \quad u(x) = p(x) + \alpha \cdot (\sigma - x) \\ &\text{if } x \in [0, \sigma) \wedge x \notin \mathsf{supp}\,[F_\sigma] : \quad u(x) \le p(x) + \alpha \cdot (\sigma - x) \\ &\text{if } x \in [\sigma, 1] \wedge x \in \mathsf{supp}\,[F_\sigma] : \quad u(x) = p(x) \\ &\text{if } x \in [\sigma, 1] \wedge x \notin \mathsf{supp}\,[F_\sigma] : \quad u(x) \le p(x) \ . \end{aligned} \tag{4.13}$$

Now to prove the optimal solution defined as in (4.10), it suffices to show that there exists a convex function $p(\cdot)$ and a value $\alpha \in \mathbb{R}$ that satisfies the conditions in (4.12) and (4.13) with $u(x) = \min\left\{G(x)^{n-1}, G(\sigma)^{n-1}\right\}$. We consider

$$\alpha = -\frac{G(\sigma)^{n-1} - G(x^\dagger)^{n-1}}{\sigma - x^\dagger} \ ; \qquad p(x) = \begin{cases} G(x)^{n-1} - \alpha \cdot (\sigma - x), & \forall x \in [0, x^\dagger) \\ G(\sigma)^{n-1}, & \forall x \in [x^\dagger, 1] \end{cases}$$

To check the convexity of p, note that $\frac{\partial p(x)}{\partial x} = \frac{\partial G(x)^{n-1}}{\partial x} + \alpha$ is increasing over $[0, x^\dagger]$ since G^{n-1} is convex over $[0, x^\dagger]$. Moreover, $\frac{\partial p(x^\dagger)}{\partial x^\dagger} = (G(x^\dagger)^{n-1})' + \alpha \le 0$ as G^{n-1} is convex over $[0, \sigma]$, and $\lim_{x \to (x^\dagger)^-} p(x) = G(\sigma)^{n-1}$. Thus, $p(\cdot)$ is global convex over $[0, 1]$.

To satisfy the condition (4.13), note for $x \in [x^\dagger, \sigma]$, we have

$$\begin{aligned} p(x) + \alpha \cdot (\sigma - x) - G(x)^{n-1} &= G(\sigma)^{n-1} - \frac{G(\sigma)^{n-1} - G(x^\dagger)^{n-1}}{\sigma - x^\dagger} \cdot (\sigma - x) - G(x)^{n-1} \\ &= (\sigma - x) \cdot \left(\frac{G(\sigma)^{n-1} - G(x)^{n-1}}{\sigma - x} - \frac{G(\sigma)^{n-1} - G(x^\dagger)^{n-1}}{\sigma - x^\dagger} \right) \ge 0 \\ \Rightarrow \quad p(x) + \alpha \cdot (\sigma - x) \ge G(x)^{n-1}, & \quad \forall x \in [x^\dagger, \sigma] \ . \end{aligned}$$

Together with $p(x) = G(\sigma)^{n-1}, \forall x \in [\sigma, 1]$, we know that $p(\cdot)$ satisfies the condition (4.13).

Lastly, to satisfy the condition (4.12), as $F_\sigma(x) = H(x), \forall x \in [0, x^\dagger]$, it suffices to ensure

$$\int_{x^\dagger}^1 p(x) dF_\sigma(x) = \int_{x^\dagger}^1 p(x) dH(x) \ ,$$

where the above holds true as they both equal to $G(\sigma)^{n-1} \cdot (1 - H(x^\dagger))$. Thus the constructed p and α satisfy the conditions in (4.12)–(4.13), implying the solution in (4.10) is an optimal solution.

With the above characterized F_σ, we now prove the second part of the above result, i.e., $\texttt{OPT}_\sigma$ is monotone increasing w.r.t. $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$. By definition, we have

$$\texttt{OPT}_\sigma = \int_0^{x^\dagger} G(x)^{n-1} dH(x) + G(\sigma)^{n-1} \cdot (1 - H(x^\dagger)) \ . \tag{4.14}$$

Recall that $x^\dagger$ satisfies $\int_0^{x^\dagger} H(x)dx + (\sigma - x^\dagger) \cdot H(x^\dagger) = \sigma - (\lambda - c)$, thus, $\sigma = \frac{\int_0^{x^\dagger} H(x)dx - x^\dagger H(x^\dagger) + (\lambda - c)}{1 - H(x^\dagger)}$. Define a function $\sigma(x) := \frac{\int_0^x H(t)dt - xH(x) + (\lambda - c)}{1 - H(x)}$. Now back to (4.14), we have

$$\texttt{OPT}_\sigma = \int_0^{x^\dagger} G(x)^{n-1} dH(x) + G(\sigma(x^\dagger))^{n-1} \cdot (1 - H(x^\dagger)) \ .$$

Consider a function $f(x) := \int_0^x G(t)^{n-1} dH(t) + G(\sigma(x))^{n-1} \cdot (1 - H(x))$. Let $g(\cdot)$ denote the density function of distribution G. Now observe that

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= G(x)^{n-1} h(x) + (n-1) G(\sigma(x))^{n-2} g(\sigma(x)) \sigma(x)' (1 - H(x)) - G\left(\sigma(x)\right)^{n-1} h(x) \\ &= h(x) \cdot \left(\left(G(x)^{n-1} - G\left(\sigma(x)\right)^{n-1} \right) + (n-1) G(\sigma(x))^{n-2} g(\sigma(x)) \cdot (\sigma(x) - x) \right) \\ &= h(x) \cdot \left(\left(G(x)^{n-1} - G\left(\sigma(x)\right)^{n-1} \right) + \frac{\partial G(\sigma(x))^{n-1}}{\partial \sigma(x)} \cdot (\sigma(x) - x) \right) \overset{\text{(a)}}{\ge} 0 \ , \end{aligned}$$

where in (a), we use the convexity of G^{n-1} over its support in $[0, \bar{x}_G]$, and $\sigma(x) \ge x, \forall x \in [0, \bar{x}_G]$, and $h(x) \ge 0, \forall x$. This implies that the optimal deviation payoff is increasing w.r.t. $x^\dagger$, and thus increasing w.r.t. $\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$. ☐

For the value $\texttt{OPT}_\sigma$ for $\sigma \in [\bar{x}_G, \sigma_H)$, we show that it is monotone decreasing w.r.t. $\sigma \in [\bar{x}_G, \sigma_H)$.

LEMMA 4.5. *For any prior H, given a strategy G that satisfies conditions (i)–(ii) in Theorem 4.1, the value $\texttt{OPT}_\sigma$ is monotone decreasing w.r.t. $\sigma \in [\bar{x}_G, \sigma_H)$.*

To prove this result, for each $\sigma \in [\bar{x}_G, \sigma_H)$, we first characterize the optimal solution F_σ to the program (4.8) using a much more involved duality argument (see Lemma B.1 and its proof in Appendix B). Then with the obtained F_σ, we prove the monotonicity of $\texttt{OPT}_\sigma$. The proof uses the convexity of G^{n-1} over $[0, \bar{x}_G]$, and is in Appendix B. Combine Lemma 4.4 and Lemma 4.5 will prove Lemma 4.3. Putting all pieces together can prove Theorem 4.1 (see the end of Appendix B).

5 Conclusion

In this paper, we study the competitive information design for the Pandora's Box problem. We characterize the informational properties of Pandora's Box by analyzing how a box's partial information disclosure affects the agent's optimal decisions. We fully characterize the pure symmetric equilibrium for the boxes' competitive information disclosure with providing necessary and sufficient conditions that guarantee the existence and uniqueness of competition equilibrium, and reveal various insights regarding information competition and the resultant agent payoff at equilibrium.

Acknowledgement. Haifeng Xu is supported by NSF grant CCF-2132506.

References

[1] Marek Adamczyk, Maxim Sviridenko, and Justin Ward. Submodular stochastic probing on matroids. *Mathematics of Operations Research*, 41(3):1022–1038, 2016.

[2] Simon P Anderson and Régis Renault. Advertising content. *American Economic Review*, 96(1):93–113, 2006.

[3] Pak Hung Au. Competition in designing pandora's boxes. *Available at SSRN 3141387*, 2018.

[4] Pak Hung Au and Keiichi Kawai. Competitive disclosure of correlated information. *Economic Theory*, pages 1–33, 2019.

[5] Pak Hung Au and Keiichi Kawai. Competitive information disclosure by multiple senders. *Games and Economic Behavior*, 119:56–78, 2020.

[6] Robert J Aumann, Michael Maschler, and Richard E Stearns. *Repeated games with incomplete information*. MIT press, 1995.

[7] Dirk Bergemann and Stephen Morris. Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95, 2019.

[8] Hedyeh Beyhaghi and Robert Kleinberg. Pandora's problem with nonobligatory inspection. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 131–132, 2019.

[9] David Blackwell. Equivalent comparisons of experiments. *The annals of mathematical statistics*, pages 265–272, 1953.

[10] David A Blackwell and Meyer A Girshick. *Theory of games and statistical decisions*. Courier Corporation, 1979.

[11] Simon Board and Jay Lu. Competitive information disclosure in search markets. *Journal of Political Economy*, 126(5):1965–2010, 2018.

[12] Raphael Boleslavsky and Christopher Cotton. Limited capacity in project selection: Competition through evidence production. *Economic Theory*, 65(2):385–421, 2018.

[13] Shant Boodaghians, Federico Fusco, Philip Lazos, and Stefano Leonardi. Pandora's box problem with order constraints. *Mathematics of Operations Research*, 2022. URL https://doi.org/10.1287/moor.2022.1271.

[14] Shuchi Chawla, Evangelia Gergatsouli, Yifeng Teng, Christos Tzamos, and Ruimin Zhang. Pandora's box with correlations: Learning and approximation. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1214–1225. IEEE, 2020.

[15] Laura Doval. Whether or not to open pandora's box. *Journal of Economic Theory*, 175:127–158, 2018.

[16] Shaddin Dughmi. Algorithmic information structure design: a survey. *ACM SIGecom Exchanges*, 15(2):2–24, 2017.

[17] Piotr Dworczak and Giorgio Martini. The simple economics of optimal persuasion. *Journal of Political Economy*, 127(5):1993–2048, 2019.

[18] Matthew Gentzkow and Emir Kamenica. Competition in persuasion. *The Review of Economic Studies*, 84(1): 300–322, 2016.

[19] Matthew Gentzkow and Emir Kamenica. Bayesian persuasion with multiple senders and rich signal spaces. *Games and Economic Behavior*, 104:411–429, 2017.

[20] Negin Golrezaei, Vahideh Manshadi, and Vahab Mirrokni. Two-stage pandora's box for product ranking. *Available at SSRN 3130378*, 2018.

[21] Ronen Gradwohl, Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. Reaping the informational surplus in bayesian persuasion. *American Economic Journal: Microeconomics*, 2022, forthcoming.

[22] Anupam Gupta, Viswanath Nagarajan, and R Ravi. Approximation algorithms for optimal decision trees and adaptive tsp problems. *Mathematics of Operations Research*, 42(3):876–896, 2017.

[23] Wei He and Jiangtao Li. Competitive information disclosure in random search markets. 2021.

[24] Ilwoo Hwang, Kyungmin Kim, and Raphael Boleslavsky. Competitive advertising and pricing. *Emory University and University of Miami*, 2019.

[25] Emir Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11:249–272, 2019.

[26] Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

[27] Robert Kleinberg, Bo Waggoner, and E Glen Weyl. Descending price optimally coordinates search. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 23–24, 2016.

[28] Anton Kolotilin. Optimal information disclosure: A linear programming approach. *Theoretical Economics*, 13 (2):607–635, 2018.

[29] Jeffrey Mensch. Screening inattentive buyers. *American Economic Review*, 2021.

[30] Wojciech Olszewski and Richard Weber. A more general pandora rule? *Journal of Economic Theory*, 160: 429–437, 2015.

[31] Martin L Weitzman. Optimal search for the best alternative. *Econometrica: Journal of the Econometric Society*, pages 641–654, 1979.

[32] Mark Whitmeyer. Persuasion produces the (diamond) paradox. *arXiv preprint arXiv:2011.13900*, 2020.

A Missing proofs of Section 3

COROLLARY 3.1. *Let G be an essentially full information strategy. The agent obtains the highest expected payoff $\sigma_H - \int_0^{\sigma_H} H(x)^n dx$ under $(G, \ldots, G)$ among all possible (symmetric or asymmetric) strategy profiles.*

Proof of Corollary 3.1. Recall that from Theorem 3.1, we know

$$u^A(H, \ldots, H) = \max_{G_i : H \succeq G_i, \forall i} u^A(G_1, \ldots, G_n) \ . \tag{A.1}$$

From Lemma 3.3, we know

$$\begin{aligned} u^A(H, \ldots, H) &= \mathbb{E}\Big[\max_i \kappa_i\Big] = \mathbb{E}_{x_i \sim H, \forall i}\big[\max\big\{ \min\{x_1, \sigma_H\}, \ldots, \min\{x_n, \sigma_H\}\big\}\big] \\ &= \sigma_H \cdot (1 - H(\sigma_H)^n) + \mathbb{E}_{x_i \sim H, \forall i}[\max\{x_1, \ldots, x_n\} | x_i < \sigma_H, \forall i] \\ &= \sigma_H \cdot (1 - H(\sigma_H)^n) + \int_0^{\sigma_H} x dH(x)^n \\ &= \sigma_H - \int_0^{\sigma_H} H(x)^n dx \ . \end{aligned}$$

For an essentially full information disclosure strategy G, we have $G(x) = H(x), \forall x \in [0, \sigma_H]$. Thus,

$$u^A(G, \ldots, G) = \sigma_H - \int_0^{\sigma_H} G(x)^n dx \overset{\text{(a)}}{=} \sigma_H - \int_0^{\sigma_H} H(x)^n dx = u^A(H, \ldots, H) \ , \tag{A.2}$$

where (a) is from the definition of strategy G. □

COROLLARY 3.2. *Given the prior H and the cost $c \geq 0$, for any strategy G that satisfies $H \succeq G$, we have $\lambda - c \leq \sigma_G \leq \sigma_H$. Moreover,*

- *$\sigma_G = \lambda - c$ if and only if G has no support over $[0, \lambda - c]$;*
- *$\sigma_G = \sigma_H$ if and only if H is an MPS of G over the interval*[13] *$[0, \sigma_H]$, denoted by $H \succeq_{[0,\sigma_H]} G$.*

Proof of Corollary 3.2. The condition for $\sigma_G = \lambda - c$ is straightforward from Lemma 3.1. We next prove the condition for $\sigma_G = \sigma_H$. For the "if" direction, note that from the definition of $H \succeq_{[0,\sigma_H]} G$, we know $W(\sigma_H) = 0$, i.e., $\int_0^{\sigma_H} H(x) = \int_0^{\sigma_H} G(x)$, thus $\int_0^{\sigma_H} G(x) = \sigma_H - (\lambda - c)$. From Lemma 3.1, we then know $\sigma_G = \sigma_H$. For the "only if" direction, from $\sigma_G = \sigma_H$, we know $\int_0^{\sigma_H} G(x) = \sigma_H - (\lambda - c)$, thus $\int_0^{\sigma_H} G(x)dx = \int_0^{\sigma_H} H(x)dx$, implying $W(\sigma_H) = 0$. As $H \succeq G$, we know $W(y) \geq 0, \forall y \in [0, \sigma_H]$. Thus, $H \succeq_{[0,\sigma_H]} G$. □

LEMMA 3.1. *For any G with mean λ and for any $c \geq 0$, $\sigma_G = \sigma$ if and only if $\int_0^{\sigma} G(x)dx = \sigma - (\lambda - c)$.*

Proof of Lemma 3.1. By definition, we have

$$\begin{aligned} c = \mathbb{E}_{x \sim G}[\max\{x - \sigma_G, 0\}] = \int_{\sigma_G}^1 (x - \sigma_G) dG(x) &= \lambda - \int_0^{\sigma_G} x dG(x) - \sigma_G(1 - G(\sigma_G)) \\ &= \lambda + \int_0^{\sigma_G} G(x)dx - \sigma_G \ , \end{aligned}$$

where we have used the fact $\int x dG(x) = \lambda$ and integral by parts. Rearranging the terms gives us the result. □

LEMMA 3.2. *For any H, a strategy $G : H \succeq G$ satisfying $\sigma_G = \sigma_H$ must have $G(\sigma_H) = H(\sigma_H)$.*

Proof of Lemma 3.2. Recall that if G satisfies $\sigma_G = \sigma_H$, from Lemma 3.1, we have $\int_0^{\sigma_H} G(x)dx = \sigma_H - (\lambda - c) = \int_0^{\sigma_H} H(x)dx$. We now consider following two possible cases:

[13] Let $W(y) := \int_0^y \big[H(x) - G(x)\big]dx$. We say H is an MPS of G over $[a, b]$ if and only if $W(a) = W(b) = 0$, and $W(y) \geq 0, \forall y \in [a, b]$.

- Suppose that $G(\sigma_H) > H(\sigma_H)$, as H is continuous over $[0,1]$, and G is nondecreasing, then there exists $x' > \sigma_H$ such that $G(x) > H(x), \forall x \in (\sigma_H, x')$, then we have

$$\int_0^{x'} G(x)dx = \int_0^{\sigma_H} G(x)dx + \int_{x'}^{\sigma_H} G(x)dx > \int_0^{\sigma_H} H(x)dx + \int_{x'}^{\sigma_H} H(x)dx = \int_0^{x'} H(x)dx \ ,$$

which violates the definition of $H \succeq G$.

- Suppose that $G(\sigma_H) < H(\sigma_H)$, as H is continuous over $[0,1]$, and G is nondecreasing, then there exists $x' < \sigma_H$ such that $H(x) > G(x), \forall x \in (x', \sigma_H)$, then consider

$$\int_0^{\sigma_H} H(x)dx = \int_0^{x'} H(x)dx + \int_{x'}^{\sigma_H} H(x)dx > \int_0^{x'} G(x)dx + \int_{x'}^{\sigma_H} G(x)dx = \int_0^{\sigma_H} G(x)dx \ ,$$

which violates the condition that $\sigma_G = \sigma_H$.

□

B Missing proofs of Section 4

Corollary 4.1. *Essentially full information strategy is the equilibrium strategy if and only if H^{n-1} is convex over $[0, \sigma_H]$.*

Proof of Corollary 4.1. When H^{n-1} is convex over $[0, \sigma_H]$, it is easy to see that the unique distribution G that meets conditions (i)–(ii) in Theorem 4.1 must satisfy that $G(x) = H(x), \forall x \in [0, \sigma_H]$. We now show how the condition (iii) always holds when H^{n-1} is convex over $[0, \sigma_H]$. In this case, we know $\bar{x}_G = \sigma_H$, and $\sigma_{\mathsf{NI}} = \lambda - c < \sigma_H = \bar{x}_G$, thus, it suffices to show the case (b) in condition (iii) holds. Clearly, when $\bar{x}_G = \sigma_H$, we have $x^\dagger = \bar{x}_G$, and

$$\begin{aligned}\int_0^{x^\dagger} G(x)^{n-1}dH(x) + H(\sigma_H)^{n-1}(1 - H(x^\dagger)) &= \int_0^{\sigma_H} H(x)^{n-1}dH(x) + H(\sigma_H)^{n-1}(1 - H(\sigma_H) \\ &= \frac{1}{n} \cdot H(\sigma_H)^n + H(\sigma_H)^{n-1}(1 - H(\sigma_H)) \le \frac{1}{n} \ ,\end{aligned}$$

where the last inequality always holds by algebra for any $n \ge 2$. Thus, G, i.e., the essentially full information disclosure, is the equilibrium strategy. □

Corollary 4.4. *Under essentially full information equilibrium, the agent's payoff is decreasing with respect to the inspection cost and increasing with respect to the number of senders.*

Proof of Corollary 4.4. Recall that from Corollary 3.1, we know under essentially full information equilibrium, we have $u^A(G, \ldots, G) = \sigma_H - \int_0^{\sigma_H} H(x)^n dx$. Consider function $f(x, n) := x - \int_0^x H(t)^n dt$. Clearly, we have $\frac{\partial f(x,n)}{\partial x} = 1 - H(x)^n > 0$. Thus, agent's payoff under essentially full information equilibrium is strictly increasing w.r.t. σ_H. This implies that agent's payoff is decreasing w.r.t. the cost. On the other hand, when n increases, we have H^n is more convex and the integral $\int_0^x H(t)^n dt$ is smaller, implying that agent's payoff is increasing. □

Corollary 4.5. *Given prior H in which H^{n-1} is concave over $[0, \sigma_H]$. Let G be a distribution satisfying the conditions (i)–(ii) in Theorem 4.1, then G is an equilibrium strategy if and only if $G(\lambda - c)^{n-1} \le 1/n$.*

Proof of Corollary 4.5. When H^{n-1} is concave over $[0, \sigma_H]$, it is easy to see that the unique distribution G that meets condition (i)–(ii), must be that G^{n-1} is linear over $[0, \bar{x}_G]$, and G has no support over $[\bar{x}_G, \sigma_H]$. If $\lambda - c \ge \bar{x}_G$, then G is equilibrium strategy if and only if $G(\lambda - c)^{n-1} \le 1/n$. If $\lambda - c < \bar{x}_G$, we now show that the case (b) in condition (iii) is equivalent to ensure $G(\lambda - c)^{n-1} \le 1/n$. To see this, let $k := \frac{H(\sigma_H)^{n-1}}{\bar{x}_G}$ denote the slope of the linear portion of G^{n-1}. Then, for $x^\dagger$ satisfying $\int_{x^\dagger}^1 (x - \bar{x}_G)dH(x) = c$, i.e., $\int_0^{x^\dagger} H(x)dx + (\bar{x}_G - x^\dagger)H(x^\dagger) = \bar{x}_G - (\lambda - c)$, note that

$$
\begin{aligned}
&\int_0^{x^\dagger} G(x)^{n-1} dH(x) + H(\sigma_H)^{n-1}(1 - H(x^\dagger)) \\
=\ & G(x^\dagger)^{n-1} H(x^\dagger) - k \int_0^{x^\dagger} H(x)dx + H(\sigma_H)^{n-1}(1 - H(x^\dagger)) \\
=\ & G(x^\dagger)^{n-1} H(x^\dagger) - k \cdot \left(\bar{x}_G - (\lambda - c) - (\bar{x}_G - x^\dagger)H(x^\dagger)\right) + H(\sigma_H)^{n-1}(1 - H(x^\dagger)) \\
=\ & k \cdot (\lambda - c) = G(\lambda - c)^{n-1} \ ,
\end{aligned}
$$

where we have used the linearity of G^{n-1} over $[0, \bar{x}_G]$. Thus, combining above two cases, to guarantee G is the equilibrium strategy, it suffices to ensure $G(\lambda - c)^{n-1} \leq {}^1\!/n$. □

LEMMA 4.1. *For any H, if there exists an equilibrium $(G, \ldots, G)$, it must be that $\sigma_G = \sigma_H$. Each sender's expected payoff is ${}^1\!/n$ at any equilibrium.*

Proof of Lemma 4.1. We prove the lemma using two senders case. The analysis for multiple senders can be easily carried over. Given a symmetric strategy (G, G) where $\sigma_G \neq \sigma_H$, let $\bar{x}_G = \max\{x : x \in \mathsf{supp}\,[G] \wedge x \leq \sigma_G\}$, we now consider following possible scenarios:

- $G(x) = H(x), \forall x \in [0, \bar{x}_G]$. In this case, we must have $\bar{x}_G < \sigma_G$, otherwise we have $\sigma_G = \sigma_H$. Consider (sufficiently small) ε and ε', and let $x^\dagger := \min\{x : G(x) \geq H(\bar{x}_G + \varepsilon)\}$. Consider sender 1 deviating to a new strategy F where

$$
F(x) = \begin{cases} G(x), & \forall x \in [0, \bar{x}_G) \\ H(x), & \forall x \in [\bar{x}_G, \bar{x}_G + \varepsilon) \\ H(\bar{x}_G + \varepsilon), & \forall x \in [\bar{x}_G + \varepsilon, x^\dagger + \varepsilon') \\ G(x), & \forall x \in [x^\dagger + \varepsilon', 1], \end{cases} \tag{B.3}
$$

  where ε' further satisfies that

$$
\int_{\bar{x}_G}^{x^\dagger} (F(x) - G(x))dx = \int_{x^\dagger}^{x^\dagger + \varepsilon'} (G(x) - F(x))dx. \tag{B.4}
$$

  By construction, we have $F \succeq G$ as $\int_0^\sigma (F(x) - G(x))dx \geq 0, \forall \sigma$, and $H \succeq F$ as $\int_0^\sigma (H(x) - F(x))dx \geq 0, \forall \sigma$. Let $\Delta_\varepsilon := H(\bar{x}_G + \varepsilon) - H(\bar{x}_G)$. Now consider

$$
\begin{aligned}
\int_{\sigma_G}^1 (x - \sigma_G)dF(x) - \int_{\sigma_G}^1 (x - \sigma_G)dG(x) &= \int_{\sigma_G}^1 xdF(x) - \int_{\sigma_G}^1 xdG(x) - \sigma_G \cdot \left(\int_{\sigma_G}^1 dF(x) - \int_{\sigma_G}^1 dG(x) \right) \\
&= \int_0^{\sigma_G} xdG(x) - \int_0^{\sigma_G} xdF(x) + \sigma_G \cdot \Delta_\varepsilon \\
&= \sigma_G G(\sigma_G) - \int_0^{\sigma_G} G(x)dx - \sigma_G F(\sigma_G) + \int_0^{\sigma_G} F(x)dx + \sigma_G \cdot \Delta_\varepsilon \\
&= \int_0^{\sigma_G} F(x)dx - \int_0^{\sigma_G} G(x)dx > 0, \\
\Rightarrow \quad \int_{\sigma_G}^1 (x - \sigma_G)dF(x) > c \ .
\end{aligned}
$$

  As $\int_\sigma^1 (x - \sigma)dF(x)$ is strictly decreasing w.r.t σ, we thus have $\sigma_F > \sigma_G$. Now let $u_a^S := p_G + \int_0^{\sigma_G} G(x)dG(x)$ and consider

$$
\begin{aligned}
u_1^S(F, G) - u_a^S &= \int_{\sigma_G}^1 dF(x) + \int_0^{\sigma_G} G(x)dF(x) - u_a^S \\
&= \int_{\bar{x}_G}^{\bar{x}_G + \varepsilon} G(x)dF(x) - \Delta_\varepsilon = (1 - p_G) - \Delta_\varepsilon = -p_G \Delta_\varepsilon
\end{aligned}
$$

Choose ε such that $p_G\Delta_\varepsilon < u_a^S - \frac{1}{2}$, we then have

$$u_1^S(F,G) = u_a^S - p_G\Delta_\varepsilon > \frac{1}{2} = u_1^S(G,G).$$

- $\exists x \in [0,\bar{x}_G], G(x) \neq H(x)$. In this case, we consider two possible scenarios:
  1. When $G(\sigma_G) > H(\sigma_G)$. In this case, as we have $\int_0^{\sigma_G} G(x)dx < \int_0^{\sigma_G} H(x)dx$, there must exist a point $x^\dagger := \max\{x \in [0,\bar{x}_G] : G(x) \geq H(x) \wedge G(x) < H(x)\}$. Now consider following new strategy F:

$$F(x) = \begin{cases} G(x), & \forall x \in [0, x^\dagger - \varepsilon) \\ H(x), & \forall x \in [x^\dagger - \varepsilon, x^\dagger) \\ G(x), & \forall x \in [x^\dagger, x^\ddagger) \\ G(x^\ddagger), & \forall x \in [x^\ddagger, x^\ddagger + \varepsilon') \\ G(x), & \forall x \in [x^\ddagger + \varepsilon', 1], \end{cases}$$

  where $x^\ddagger \geq \sigma_G$ and $\varepsilon, \varepsilon'$ are sufficiently small such that they satisfy the following

$$\int_{x^\dagger-\varepsilon}^{x^\dagger} (F(x) - G(x))dx = \int_{x^\ddagger}^{x^\ddagger+\varepsilon'} (G(x) - F(x))dx.$$

  By construction, $F \succeq G$ as $\int_0^\sigma (F(x) - G(x))dx \geq 0, \forall\sigma$, and $H \succeq F$ as $\int_0^\sigma (H(x) - F(x))dx \geq 0, \forall\sigma$. Now consider

$$\begin{aligned} \int_{\sigma_G}^1 (x-\sigma_G)dF(x) - \int_{\sigma_G}^1 (x-\sigma_G)dG(x) &= \int_{\sigma_G}^1 xdF(x) - \int_{\sigma_G}^1 xdG(x) - \sigma_G \cdot \left(\int_{\sigma_G}^1 dF(x) - \int_{\sigma_G}^1 dG(x)\right) \\ &= \int_0^{\sigma_G} xdG(x) - \int_0^{\sigma_G} xdF(x) \\ &= \sigma_G G(\sigma_G) - \int_0^{\sigma_G} G(x)dx - \sigma_G F(\sigma_G) + \int_0^{\sigma_G} F(x)dx \\ &= \int_0^{\sigma_G} F(x)dx - \int_0^{\sigma_G} G(x)dx > 0. \end{aligned}$$

  Thus, we have $\sigma_F > \sigma_G$. As a result, let $u_a^S := p_G + \int_0^{\sigma_G} G(x)dG(x)$ and

$$\begin{aligned} u_1^S(F,G) - u_a^S = \int_{\sigma_G}^1 dF(x) + \int_0^{\sigma_G} G(x)dF(x) - u_a^S &= \int_{x^\dagger-\varepsilon}^{x^\dagger} G(x)dF(x) - \int_{x^\dagger-\varepsilon}^{x^\dagger} G(x)dG(x) \\ &= \int_{x^\dagger-\varepsilon}^{x^\dagger} G(x)\cdot(h(x) - f(x))dx \end{aligned}$$

  Choose ε such that $\int_{x^\dagger-\varepsilon}^{x^\dagger} G(x)\cdot(h(x) - f(x))dx < u_a^S - \frac{1}{2}$, we then have

$$u_1^S(F,G) = u_a^S - \int_{x^\dagger-\varepsilon}^{x^\dagger} G(x)\cdot(h(x) - f(x))dx > \frac{1}{2} = u_1^S(G,G).$$

  2. When $G(\sigma_G) \leq H(\sigma_G)$. In this case, consider the point $x^\dagger := \max\{x \in [0,\bar{x}_G] : G(x) \leq H(x) \wedge G(x) > H(x)\}$. Now consider following new strategy F:

$$F(x) = \begin{cases} G(x), & \forall x \in [0, x^\dagger) \\ H(x), & \forall x \in [x^\dagger, x^\dagger + \varepsilon) \\ H(x^\dagger + \varepsilon), & \forall x \in [x^\dagger + \varepsilon, \bar{x}) \\ G(x), & \forall x \in [\bar{x}, x^\ddagger) \\ G(x^\ddagger), & \forall x \in [x^\ddagger, x^\ddagger + \varepsilon') \\ G(x), & \forall x \in [x^\ddagger + \varepsilon', 1], \end{cases}$$

where $x^\ddagger \geq \sigma_G$, and $\bar{x}$ satisfies $G(\bar{x}) = H(x^\dagger + \varepsilon)$. Moreover, $\varepsilon, \varepsilon'$ are sufficiently small such that they satisfy the following

$$\int_{x^\dagger}^{\bar{x}} (F(x) - G(x))dx = \int_{x^\ddagger}^{x^\ddagger+\varepsilon'} (G(x) - F(x))dx.$$

Follow the earlier analysis, we have $\sigma_F > \sigma_G$, and with sufficiently small $\varepsilon, \varepsilon'$, we have $u_1^S(F, G) > u_1^S(G, G)$. Putting pieces together, the proof then completes. ☐

COROLLARY 4.6. *Given a prior H, if $(G, \ldots, G)$ is an equilibrium, then the strategy profile $(G_1, \ldots, G_n)$ where $\forall i, G_i(x) = G(x), \forall x \in [0, \sigma_H]$ is also the equilibrium.*

Proof of Corollary 4.6. It suffices to show that given $(G_1, \ldots, G_n)$, no sender has profitable deviation. Consider following two kinds of deviation: one is deviating to a strategy that has reservation value σ_H, then from Lemma 4.2, we know there exists no such profitable deviation; for any $\sigma < \sigma_H$, the other is deviating to a strategy that has reservation value σ, then from (4.7) and the definition of $(G, \ldots, G)$, we know there exists no such profitable deviation. ☐

LEMMA B.1. *Given a prior H, and a unique distribution G satisfying the conditions (i)–(ii) in Theorem 4.1, for any $\sigma \in [\max\{\sigma_{\mathsf{NI}}, \bar{x}_G\}, \sigma_H)$, let Δ satisfy $\sigma - (\lambda - c) + H(\sigma + \Delta) \cdot \Delta = \int_0^{\sigma+\Delta} H(x)dx$, and let $x^* := x_m$, i.e., the last point where G^{n-1} is strictly convex, a distribution F_σ satisfying following structure is an optimal solution to the program (4.8)*

1. *if $\int_0^{x^*} H(x)dx + (\bar{x}_G - x^*) \cdot H(x^*) + (\sigma - \bar{x}_G) \cdot H(\sigma + \Delta) > \sigma - (\lambda - c)$, then*

$$F_\sigma(x) = \begin{cases} H(x), & \forall x \in [0, x^\dagger) \\ H(x^\dagger), & \forall x \in [x^\dagger, \bar{x}_G) \\ H(\sigma + \Delta), & \forall x \in [\bar{x}_G, x^\ddagger) \\ 1, & \forall x \in [x^\ddagger, 1] \end{cases} \tag{B.5}$$

   where $x^\dagger \in [0, x^*)$ satisfies $\int_0^\sigma F_\sigma(x)dx = \sigma - (\lambda - c)$.

2. *if $\int_0^{x^*} H(x)dx + (\bar{x}_G - x^*) \cdot H(x^*) + (\sigma - \bar{x}_G) \cdot H(\sigma + \Delta) \leq \sigma - (\lambda - c)$, then*

$$F_\sigma(x) = \begin{cases} H(x), & \forall x \in [0, x^*) \\ H(x^*), & \forall x \in [x^*, x') \\ H(\sigma + \Delta), & \forall x \in [x', x^\ddagger) \\ 1, & \forall x \in [x^\ddagger, 1] \end{cases} \tag{B.6}$$

   where $x' \in [x^*, \bar{x}_G]$ satisfies $\int_0^\sigma F_\sigma(x)dx = \sigma - (\lambda - c)$.

Proof of Lemma B.1. We first show the unique existence of Δ such that $\sigma - (\lambda - c) + H(\sigma + \Delta) \cdot \Delta = \int_0^{\sigma+\Delta} H(x)dx$. Fix $\sigma \in [\bar{x}_G, \sigma_H)$, consider a function $f(x) := \sigma - (\lambda - c) + H(\sigma + x) \cdot x - \int_0^{\sigma+x} H(t)dt$. Clearly, $f(\cdot)$ is continuously differentiable and increasing over $[0, 1 - \sigma]$. Note that

$$f(\sigma_H - \sigma) = \sigma - (\lambda - c) + H(\sigma_H) \cdot (\sigma_H - \sigma) - \int_0^{\sigma_H} H(t)dt = (\sigma_H - \sigma) \cdot (H(\sigma_H) - 1) \leq 0$$
$$f(1 - \sigma) = \sigma - (\lambda - c) + H(1) \cdot (1 - \sigma) - \int_0^1 H(t)dt = c > 0 \ .$$

Thus, there must exist a unique $\Delta \in (\sigma_H - \sigma, 1 - \sigma)$ such that $f(\Delta) = 0$. In below, we show the optimality of solution (B.5) and (B.6) via constructing a dual solution that satisfies the complementary slackness conditions in Equations (4.12) and (4.13). Fix a $\sigma \in [\bar{x}_G, \sigma_H)$, and its corresponding Δ. For notation simiplicity, we define $p^\dagger := H(x^\dagger)^{n-1}$ in first case and $p^* := H(x^*)^{n-1}$ in second case, and $p_H := H(\sigma_H)^{n-1}$.

- When $\int_0^{x^*} H(x)dx + (\bar{x}_G - x^*) \cdot H(x^*) + (\sigma - \bar{x}_G) \cdot H(\sigma + \Delta) > \sigma - (\lambda - c)$, in this case, let $\alpha_G := \frac{p_H - p^\dagger}{\bar{x}_G - x^\dagger}$, and consider following dual solution

$$\alpha = -\frac{\alpha_G \cdot (\sigma + \Delta - x^\dagger) + p^\dagger - p_H}{\Delta} \ ;$$
$$p(x) = \begin{cases} G(x)^{n-1} - \alpha \cdot (\sigma - x), & \forall x \in [0, x^\dagger) \\ \alpha_p \cdot (x - x^\dagger) + p^\dagger - \alpha \cdot (\sigma - x^\dagger), & \forall x \in [x^\dagger, \sigma + \Delta) \\ p_H, & \forall x \in [\sigma + \Delta, 1] \end{cases}$$

where $\alpha_p := \alpha + \alpha_G$. We now show that the above constructed $p(\cdot)$ is global convex over $[0, 1]$, and $p(\cdot), \alpha$ satisfy the complementary slackness conditions in Equations (4.12) and (4.13).

To see the convexity of p, note that for any $x \in [0, x^\dagger]$, $\frac{\partial p(x)}{\partial x} = (G(x)^{n-1})' + \alpha$ is increasing due to the convexity G^{n-1} over $[0, x^\dagger]$. Moreover,

$$\lim_{x \to (x^\dagger)^-} \frac{\partial p(x)}{\partial x} = (G(x^\dagger)^{n-1})' + \alpha \le \alpha_p = \alpha + \alpha_G \ ;$$
$$\begin{aligned} \lim_{x \to (\sigma+\Delta)^-} \frac{\partial p(x)}{\partial x} &= \alpha_p = \alpha + \alpha_G = -\frac{\alpha_G \cdot (\sigma + \Delta - x^\dagger) + p^\dagger - p_H - \alpha_G \cdot \Delta}{\Delta} \\ &= -\frac{\alpha_G \cdot (\sigma - x^\dagger) - (p_H - p^\dagger)}{\Delta} = -(p_H - p^\dagger) \cdot \frac{\frac{\sigma - x^\dagger}{\bar{x}_G - x^\dagger} - 1}{\Delta} \le 0 \ . \end{aligned}$$

To check the continuity of p, note that

$$\lim_{x \to (x^\dagger)^-} p(x) = G(x^\dagger)^{n-1} - \alpha \cdot (\alpha - x^\dagger) = p^\dagger - \alpha \cdot (\alpha - x^\dagger) = p(x^\dagger) \ ;$$
$$\begin{aligned} \lim_{x \to (\sigma+\Delta)^-} p(x) &= \alpha_p \cdot (\sigma + \Delta - x^\dagger) + p^\dagger - \alpha \cdot (\sigma - x^\dagger) \\ &= \alpha \cdot \Delta + \alpha_G \cdot (\sigma + \Delta - x^\dagger) + p^\dagger = p_H \ . \end{aligned}$$

Thus, $p(\cdot)$ is convex over $[0, 1]$.

To satisfy the condition (4.13), note that for $x \in [x^\dagger, \bar{x}_G)$, we have

$$\begin{aligned} p(x) + \alpha \cdot (\sigma - x) - G(x)^{n-1} &= \alpha_p \cdot (x - x^\dagger) + p^\dagger - \alpha \cdot (\sigma - x^\dagger) + \alpha \cdot (\sigma - x) - G(x)^{n-1} \\ &= (x - x^\dagger)(\alpha_p - \alpha) - (G(x)^{n-1} - p^\dagger) \\ &= \alpha_G \cdot (x - x^\dagger) - (G(x)^{n-1} - p^\dagger) \overset{\text{(a)}}{\ge} 0 \ , \\ \Rightarrow \quad p(x) + \alpha \cdot (\sigma - x) &\ge G(x)^{n-1}, \quad \forall x \in [x^\dagger, \bar{x}_G) \ . \end{aligned}$$

where (a) is from the convexity of G^{n-1} over $[0, \bar{x}_G)$. Note F_σ has non-zero support on $\bar{x}_G$. For $x \in [\bar{x}_G, \sigma)$, we know

$$\begin{aligned} p(\bar{x}_G) = \alpha_p \cdot (\bar{x}_G - x^\dagger) + p^\dagger - \alpha \cdot (\sigma - x^\dagger) &= (\alpha + \alpha_G) \cdot (\bar{x}_G - x^\dagger) + p^\dagger - \alpha \cdot (\sigma - x^\dagger) \\ &= p_H - \alpha(\sigma - \bar{x}_G) \ ; \\ p(x) + \alpha \cdot (\sigma - x) - G(x)^{n-1} &= p(x) + \alpha \cdot (\sigma - x) - p_H \\ &= \alpha_G \cdot (x - x^\dagger) - (p_H - p^\dagger) \ge 0 \\ \Rightarrow \quad p(x) + \alpha \cdot (\sigma - x) &\ge p_H, \quad \forall x \in [\bar{x}_G, \sigma) \ . \end{aligned}$$

For $x \in [\sigma, \sigma + \Delta]$, we already know $\alpha_p \le 0$ and $p(\sigma + \Delta) = p_H$, thus we have $p(x) \ge p_H, \forall x \in [\sigma, \sigma + \Delta]$.

Lastly, to satisfy condition (4.12), as $F_\sigma(x) = H(x), \forall x \in [0, x^\dagger]$, it suffices to ensure

$$\int_{x^\dagger}^1 p(x) dF_\sigma(x) = \int_{x^\dagger}^1 p(x) dH(x) \ .$$

Now note that

$$\int_{x^\dagger}^{1} p(x)dF_\sigma(x) = (H(\sigma+\Delta) - H(x^\dagger))\cdot p(\bar{x}_G) + (1 - H(\sigma+\Delta))\cdot p_H \ . \tag{B.7}$$

$$\begin{aligned}\int_{x^\dagger}^{1} p(x)dH(x) &= p_H - p(x^\dagger)\cdot H(x^\dagger) - \int_{x^\dagger}^{1} H(x)dp(x)\\ &= p_H - p(x^\dagger)\cdot H(x^\dagger) - \alpha_p \cdot \int_{x^\dagger}^{\sigma+\Delta} H(x)dx \ .\end{aligned} \tag{B.8}$$

Consider

$$\begin{aligned}(\text{B.7}) - (\text{B.8}) = {} & H(x^\dagger)\cdot(p(x^\dagger) - p(\bar{x}_G)) + H(\sigma+\Delta)\cdot(p(\bar{x}_G) - p_H) + \alpha_p \cdot \int_{x^\dagger}^{\sigma+\Delta} H(x)dx\\ = {} & -\alpha_p \cdot \left(H(x^\dagger)\cdot \frac{p(x^\dagger) - p(\bar{x}_G)}{-\alpha_p} + H(\sigma+\Delta)\cdot\frac{p(\bar{x}_G) - p_H}{-\alpha_p} - \int_{x^\dagger}^{\sigma+\Delta} H(x)dx\right)\\ \overset{\text{(a)}}{=} {} & -\alpha_p \cdot \left(H(x^\dagger)\cdot(\bar{x}_G - x^\dagger) + H(\sigma+\Delta)\cdot(\sigma+\Delta-\bar{x}_G) - \int_{x^\dagger}^{\sigma+\Delta} H(x)dx\right)\\ \overset{\text{(b)}}{=} {} & -\alpha_p \cdot \left(\sigma - (\lambda - c) + H(\sigma+\Delta)\Delta - \int_{0}^{\sigma+\Delta} H(x)dx\right) \overset{\text{(c)}}{=} 0 \ ,\end{aligned}$$

where (a) uses the definition of $p(\cdot)$ over $[x^\dagger, \sigma+\Delta]$, (b) uses the definition of $x^\dagger$, namely, $\int_0^{x^\dagger} H(x)dx + (\bar{x}_G - x^\dagger)H(x^\dagger) + (\sigma - \bar{x}_G)H(\sigma+\Delta) = \sigma - (\lambda - c)$, and (c) is from the definition of Δ.
Putting all pieces together, we know the above α, and p is a dual solution that satisfies the complementary slackness, leading the optimality of F_σ in (B.5).

- When $\int_0^{x^*} H(x)dx + (\bar{x}_G - x^*)\cdot H(x^*) + (\sigma - \bar{x}_G)\cdot H(\sigma+\Delta) \le \sigma - (\lambda - c)$, in this case, let $\alpha_G := \frac{p_H - p^*}{\bar{x}_G - x^*}$, i.e., the slope of the last linear portion of G, and consider following dual solution

$$\begin{aligned}\alpha &= -\frac{\alpha_G\cdot(\sigma+\Delta-x^*) + p^* - p_H}{\Delta} \ ;\\ p(x) &= \begin{cases} G(x)^{n-1} - \alpha\cdot(\sigma - x), & \forall x\in[0,x^*)\\ \alpha_p\cdot(x - x^*) + p^* - \alpha\cdot(\sigma - x^*), & \forall x \in [x^*, \sigma+\Delta)\\ p_H, & \forall x\in[\sigma+\Delta, 1]\end{cases}\end{aligned} \tag{B.9}$$

where $\alpha_p := \alpha_G + \alpha$. Follow the analysis in earlier case, one can show that the above constructed p is convex over $[0,1]$, and α, p satisfy the complementary slackness conditions in (4.12) and (4.13), showing that the solution in (B.6) is an optimal solution.

The proof then completes. □

Lemma 4.5. *For any prior H, given a strategy G that satisfies conditions (i)–(ii) in Theorem 4.1, the value $\texttt{OPT}_\sigma$ is monotone decreasing w.r.t. $\sigma \in [\bar{x}_G, \sigma_H)$.*

We first show following monotonicity result.

Claim B.1. *Fix a $\sigma \in (\bar{x}_G, \sigma_H)$ and its corresponding Δ such that $\sigma - (\lambda - c) + H(\sigma+\Delta)\cdot\Delta = \int_0^{\sigma+\Delta} H(x)dx$. When σ increases, the value $\sigma + \Delta$ will decrease.*

Proof of Claim B.1. To prove the above result, consider a function $\nu(\sigma, y) := \sigma - (\lambda - c) + H(y)\cdot(y - \sigma) - \int_0^y H(t)dt$. Clearly $\frac{\partial \nu(\sigma,y)}{\partial\sigma} = 1 - H(y) \ge 0$ and $\frac{\partial\nu(\sigma,y)}{\partial y} = H(y) + h(y)(y-\sigma) - H(y) \ge 0$ for $y \ge \sigma$. Consider σ_1, σ_2 where $\sigma_1 < \sigma_2$, and their corresponding Δ_1, Δ_2 such that $\nu(\sigma_1, \sigma_1+\Delta_1) = 0$ and $\nu(\sigma_2, \sigma_2+\Delta_2) = 0$. Then by monotonicity of $\tau(\sigma, \cdot)$ and $\tau(\cdot, y)$, we have

$$\tau(\sigma_2, \sigma_2+\Delta_2) = 0 = \tau(\sigma_1, \sigma_1+\Delta_1) \le \tau(\sigma_2, \sigma_1+\Delta_1) \ \Rightarrow\ \sigma_2 + \Delta_2 \le \sigma_1 + \Delta_1 \ . \tag{B.10}$$

□

We are now ready to present our proof for Lemma 4.5.

Proof of Lemma 4.5. We consider following possible cases based on the value of $\sigma_{\mathsf{NI}} = \lambda - c$ and $\bar{x}_G$.

- When $\lambda - c \geq \bar{x}_G$, we know that $\sigma > \bar{x}_G, \forall \sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$. Thus, for any $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$, the optimal deviation F_σ follows the characterizations in Lemma B.1. Fix a σ and its corresponding Δ where $\sigma - (\lambda - c) + H(\sigma + \Delta) \cdot \Delta = \int_0^{\sigma+\Delta} H(t)dt$.

  In first case of Lemma B.1, with structure of F_σ, we can write the payoff of deviating to F_σ as follows:

$$\mathtt{OPT}_\sigma = \int_0^{x^\dagger} G(x)^{n-1} dH(x) + H(\sigma_H)^{n-1} \cdot (1 - H(x^\dagger)) \ . \tag{B.11}$$

  We will now show that $\mathtt{OPT}_\sigma$ is decreasing w.r.t $\sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)$. Recall that $x' = x^\dagger$ satisfies

$$\int_0^{x^\dagger} H(x)dx + (\bar{x}_G - x^\dagger)H(x^\dagger) + (\sigma - \bar{x}_G)H(\sigma + \Delta) = \sigma - (\lambda - c) \ .$$

  Thus, with the definition of Δ, we have

$$H(\sigma + \Delta) \cdot (\sigma + \Delta - \bar{x}_G) + H(x^\dagger) \cdot (\bar{x}_G - x^\dagger) = \int_{x^\dagger}^{\sigma+\Delta} H(t)dt \ .$$

  Now consider following function $\tau : [\sigma, 1] \times [0, \bar{x}_G] \to \mathbb{R}$

$$\tau(y, x) := H(y) \cdot (y - \bar{x}_G) + H(x) \cdot (\bar{x}_G - x) - \int_x^y H(t)dt \ .$$

  Clearly, we have

$$\frac{\partial \tau(y, x)}{\partial y} = h(y) \cdot (y - \bar{x}_G) \geq 0; \quad \frac{\partial \tau(y, x)}{\partial x} = h(x)(\bar{x}_G - x) \geq 0 \ .$$

  Consider σ_1, σ_2 where $\sigma_1 < \sigma_2$, and their corresponding Δ_1, Δ_2, $x_1^\dagger, x_2^\dagger$, such that $\tau(\sigma_1 + \Delta_1, x_1^\dagger) = 0$ and $\tau(\sigma_2 + \Delta_2, x_2^\dagger) = 0$ Then by monotonicity of $\tau(y, \cdot)$ and $\tau(\cdot, x)$, we have

$$\tau(\sigma_2 + \Delta_2, x_2^\dagger) = 0 = \tau(\sigma_1 + \Delta_1, x_1^\dagger) \geq \tau(\sigma_2 + \Delta_2, x_1^\dagger) \ \Rightarrow \ x_2^\dagger \geq x_1^\dagger \ ,$$

  where we have used the result in Claim B.1. Thus, we have showed that when σ increases, the value $x^\dagger$ will also increase.

  Now back to (B.11), consider a function $f(x) := \int_0^x G(t)^{n-1} dH(t) + H(\sigma_H)^{n-1} \cdot (1 - H(x))$, then $\forall x \in [0, \bar{x}_G]$,

$$\frac{\partial f(x)}{\partial x} = G(x)^{n-1} h(x) - H(\sigma_H)^{n-1} h(x) = h(x) \cdot (G(x)^{n-1} - H(\sigma_H)^{n-1}) \leq 0 \ ,$$

  implying that $f(x)$ is strictly decreasing w.r.t $x \in [0, \bar{x}_G]$. Consequently, we have showed that the value $\mathtt{OPT}_\sigma$ is decreasing w.r.t σ.

  In second case of Lemma B.1, we have

$$\mathtt{OPT}_\sigma = \int_0^{x^*} H(x)dx + (H(\sigma + \Delta) - H(x^*)) \cdot G(x_1)^{n-1} + H(\sigma_H)^{n-1} \cdot (1 - H(\sigma + \Delta)) \ , \tag{B.12}$$

  where x_1 satisfies that

$$\begin{aligned} &\int_0^{x^*} H(x)dx + (x_1 - x^*)H(x^*) + (\sigma - x_1)H(\sigma + \Delta) = \sigma - (\lambda - c) \\ \Rightarrow \quad x_1 &= \frac{\int_0^{x^*} H(x)dx - x^* H(x^*) + \sigma H(\sigma + \Delta) - (\sigma - (\lambda - c))}{H(\sigma + \Delta) - H(x^*)} \\ &= \frac{\int_0^{x^*} H(x)dx - x^* H(x^*) + (\sigma + \Delta) H(\sigma + \Delta) - \int_0^{\sigma+\Delta} H(t)dt}{H(\sigma + \Delta) - H(x^*)} \ , \end{aligned}$$

where we have used the definition of Δ in last equation. Recall that $\Delta \in (\sigma_H - \sigma, 1-\sigma)$, and $x_1 \in [x^*, \bar{x}_G]$. Define a function $\kappa(x) : [\sigma_H, 1] \to [x^*, \bar{x}_G]$

$$\kappa(x) := \frac{\int_0^{x^*} H(x)dx - x^* H(x^*) + xH(x) - \int_0^x H(t)dt}{H(x) - H(x^*)} \ .$$

Now back to (B.12) and consider following function $f : [\sigma_H, 1] \to \mathbb{R}$:

$$f(x) := \int_0^{x^*} H(t)dt + (H(x) - H(x^*)) \cdot G(\kappa(x))^{n-1} + H(\sigma_H)^{n-1}(1 - H(x)) \ .$$

Observe that

$$\begin{aligned}
\frac{\partial f(x)}{\partial x} &= h(x)G(\kappa(x))^{n-1} + (n-1)(H(x) - H(x^*))G(\kappa(x))^{n-2}g(\kappa(x))(\kappa(x))' - H(\sigma_H)^{n-1}h(x) \\
&= h(x) \cdot (G(\kappa(x))^{n-1} - H(\sigma_H)^{n-1}) + (n-1)(H(x) - H(x^*))G(\kappa(x))^{n-2}g(\kappa(x)) \cdot \\
&\quad \frac{xh(x) \cdot (H(x) - H(x^*)) - \left(\int_0^{x^*} H(x)dx - x^* H(x^*) + xH(x) - \int_0^x H(t)dt\right) h(x)}{(H(x) - H(x^*))^2} \\
&= h(x) \cdot (G(\kappa(x))^{n-1} - H(\sigma_H)^{n-1}) + (n-1)G(\kappa(x))^{n-2}g(\kappa(x)) \cdot (xh(x) - h(x)\kappa(x)) \\
&= h(x) \cdot \left(G(\kappa(x))^{n-1} - H(\sigma_H)^{n-1}) + \frac{\partial G(\kappa(x))^{n-1}}{\partial \kappa(x)} \cdot (x - \kappa(x))\right) \geq 0 \ . \qquad \text{(B.13)}
\end{aligned}$$

Recall that in Claim B.1, we have showed larger σ will induce smaller $\sigma + \Delta$. Together with (B.13), we can conclude that the value $\mathtt{OPT}_\sigma$ is decreasing w.r.t σ.

Combined with the earlier analysis for the first case of Lemma B.1, we can conclude that

$$\max_{\sigma : \sigma \in [\sigma_{\mathsf{NI}}, \sigma_H)} \mathtt{OPT}_\sigma = \mathtt{OPT}_{\sigma_{\mathsf{NI}}} = G(\sigma_{\mathsf{NI}})^{n-1} = G(\lambda - c)^{n-1} \ .$$

Thus, to ensure $\mathtt{OPT}_\sigma \leq 1/n$, it suffices to ensure $G(\lambda - c)^{n-1} \leq 1/n$.

- When $\lambda - c < \bar{x}_G$. Follow the analysis in case (i), for any $\sigma \in [\bar{x}_G, \sigma_H)$, we know

$$\mathtt{OPT}_\sigma \leq \mathtt{OPT}_{\bar{x}_G} \ .$$

Now consider the deviation F which satisfies $\sigma_F \in [\sigma_{\mathsf{NI}}, \bar{x}_G]$, from the proof for Lemma 4.4, we know

$$\max_{\sigma \in [\sigma_{\mathsf{NI}}, \bar{x}_G]} \mathtt{OPT}_\sigma = \mathtt{OPT}_{\bar{x}_G} = \int_0^{x^\dagger} G(x)^{n-1} dH(x) + H(\sigma_H)^{n-1}(1 - H(x^\dagger)) \ , \qquad \text{(B.14)}$$

where $x^\dagger$ satisfies $\int_0^{x^\dagger} H(x)dx + (\bar{x}_G - x^\dagger) \cdot H(x^\dagger) = \bar{x}_G - (\lambda - c)$, i.e., $\int_{x^\dagger}^1 (x - \bar{x}_G)dH(x) = c$. As a result, to ensure $\mathtt{OPT}_\sigma \leq 1/n$, it suffices to ensure (B.14) $\leq 1/n$.

□

Combine the above results, we now prove our main theorem.

Proof of Theorem 4.1. For the "if" direction, it suffices to show that no sender has profitable deviation under the strategy profile $(G, \ldots, G)$ where G satisfies conditions (i)–(iii) in Theorem 4.1. Consider following two kinds of deviations: one is deviating to a strategy F where $\sigma_F = \sigma_H$, i.e., $F \in \mathcal{H}(\sigma_H)$, and the other is deviating to a strategy F where $\sigma_F = \sigma < \sigma_H$, i.e., $F \in \mathcal{H}(\sigma)$. From the first part of Lemma 4.2, we know there is no such profitable deviation to a strategy $F \in \mathcal{H}(\sigma_H)$. From Lemma 4.3 and Lemma 4.4, we know there is no such profitable deviation to a strategy $F \in \mathcal{H}(\sigma), \forall \sigma < \sigma_H$. Thus, $(G, \ldots, G)$ must be an equilibrium. For the "only if" direction, Lemma 4.1 proves the condition (i). The condition (ii) follows from the second part of Lemma 4.2. The conditions (iii) follows from the definition of equilibrium. Namely, it is not profitable to deviate to a strategy that has the reservation value $\max\{\sigma_{\mathsf{NI}}, \bar{x}_G\}$, thus the optimal deviation value is no larger than $1/n$, with Lemma 4.4, this is exactly the statement of the condition (iii). □