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On deciding the existence of perfect entangled strategies for nonlocal games

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Abstract: First, we consider the problem of deciding whether a nonlocal game admits a perfect finite dimensional entangled strategy that uses projective measurements on a maximally entangled shared state. Via a polynomial-time Karp reduction, we show that independent set games are the hardest instances of this problem. Secondly, we show that if every independent set game whose entangled value is equal to one admits a perfect entangled strategy, then the same holds for all symmetric synchronous games. Finally, we identify combinatorial lower bounds on the classical and entangled values of synchronous games in terms of variants of the independence number of appropriate graphs. Our results suggest that independent set games might be representative of all nonlocal games when dealing with questions concerning perfect entangled strategies.

1 Introduction

Entanglement plays a central role in quantum information processing and is increasingly seen as a valuable resource for distributed tasks such as unconditionally secure cryptography [10], randomness certification [8, 22] and expansion [25, 9]. Given such a scenario it is interesting to understand how much and what kind of entanglement needs to be employed in an optimal entangled strategy. As is commonly done, we study these questions within the framework of nonlocal games. In the computer science community nonlocal games arise as one-round interactive proof systems, while in the physics community they are known as Bell inequalities [2].

A *nonlocal game* is specified by four finite sets A, B, Q, R, a probability distribution π on $Q \times R$ and a Boolean predicate $V : A \times B \times Q \times R \to \{0, 1\}$. The game proceeds as follows: Using π the verifier samples

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a pair $(q,r) \in Q \times R$ and sends q to Alice and r to Bob. Upon receiving their questions the players respond with $a \in A$ and $b \in B$, respectively. The players have knowledge of the distribution π and the predicate V and can agree on a common strategy before the start of the game, but they are not allowed to communicate after they receive their questions. We say the players *win* the game if $V(a,b|q,r) = 1$. A strategy is called *perfect* if it allows the players to win the game with probability one.

The goal of Alice and Bob is to maximize their probability of winning the game. The *classical value* of a game G, denoted $\omega(G)$, is the maximum expected winning probability when the players use deterministic strategies. An entangled strategy for a nonlocal game allows the players to determine their answers by performing joint measurements on a shared finite dimensional entangled state. The *entangled value* of a game G, denoted $\omega^*(G)$, is the supremum expected winning probability the players can achieve using entangled strategies. We stress that throughout this work we only consider finite dimensional quantum strategies without explicitly mentioning so.

Despite significant efforts, many fundamental questions concerning the properties of the entangled value have so far remained beyond reach:

(i) **The computability question:** Determine (or upper bound) the computational complexity of $\omega^*(G)$.

(ii) **The attainability question:** Determine if $\omega^*(G)$ can always be attained.

(iii) **The resources question:** How much and what kind of entanglement is needed to achieve $\omega^*(G)$.

The above questions are understood only for some very special classes of games. One notable example is the class of XOR games; for these games the answer sets, A and B, are binary and the verification predicate only depends on the XOR of the player's answers. For XOR games the entangled value can be formulated as a semidefinite program which can be approximated within arbitrary precision in polynomial time. Furthermore, the entangled value of an XOR game is always attained by a maximally entangled state [6].

At present, there has only been sporadic progress for other classes of nonlocal games. Some positive approximability results have been derived for the class of unique nonlocal games [15]. For general nonlocal games a hierarchy of semidefinite programming upper bounds for the entangled value was identified in [19]. Unfortunately, the quality of the approximation at each level of the hierarchy is not understood.

Given the lack of progress in addressing these questions there has been increasing interest in the study of restricted variants of the above problems. A decision problem that has gained some attention is the following:

`PERFECT`

Instance: A nonlocal game G.

Question: Does G admit a perfect finite dimensional entangled strategy?

It follows from recent work of Ji that `PERFECT` is NP-hard [13]. On the other hand, despite significant efforts `PERFECT` is currently not known to be decidable. Some partial progress has been documented concerning the decidability of `PERFECT` for special classes of nonlocal games. Specifically, Cleve and Mittal have shown that for BCS games, deciding the existence of a perfect entangled strategy can be reduced to deciding the existence of a self-adjoint operator solution to a polynomial system in non-commuting variables [7]. This reduction does not imply decidability since no algorithms are currently known for deciding the existence of operator solutions to non-commutative polynomial systems. In a follow up work Arkhipov studies parity BCS games with the additional requirement that each variable appears in exactly two clauses. To any such game he associates an undirected graph and shows that the game has a perfect entangled strategy if and only if the corresponding graph is non-planar [1]. Since non-planarity can be decided in linear time [12] this shows that `PERFECT` can also be decided in linear time for this special subclass of BCS games.

Motivation, results, and discussion. In view of the limited progress in understanding the computability and attainability questions and with the hope to gain new insights, in this work we study the decision problem PERFECT where we impose additional operational restrictions on the set of allowed strategies. Specifically, our goal is to decide whether a given nonlocal game G admits a perfect entangled strategy where the players are only allowed to apply *projective measurements* on a shared *maximally entangled* state (hereafter abbreviated as PME strategies). Formally, we focus on the following decision problem:

PERFECT-PME

Instance: A nonlocal game G.

Question: Does G admit a perfect PME strategy?

The study of PERFECT-PME is motivated by the following considerations. Firstly, given the impasse on the general question of deciding whether a nonlocal game admits a perfect strategy, PERFECT-PME can be viewed as an even further restricted variant of PERFECT that can hopefully provide useful insights into the general problem. Secondly, to the best of our knowledge, there are no known examples of nonlocal games that admit perfect strategies but cannot be won perfectly using PME strategies. Thus, PME strategies might even be sufficient to reach success probability one (whenever this can be done using some quantum strategy) which would imply that PERFECT is in fact equivalent to PERFECT-PME.

We note that the situation is quite different when one considers non-perfect strategies. Specifically, there are examples of nonlocal games whose entangled value is strictly smaller than one and for which maximally entangled states do not suffice to achieve the optimal success probability, e.g., [14, 26, 16, 23].

The decision problem PERFECT-PME has also been considered by Ji [13]. Similarly to [7], in this work Ji shows how one can associate to any nonlocal game G a polynomial system in non-commuting operator variables with the property that G admits a perfect PME strategy if and only if the corresponding system has a solution in self-adjoint operator variables. Using this reduction he proceeds to show that PERFECT-PME is NP-hard already when the input is restricted to be the BCS game corresponding to the 3-SAT problem.

Our main result in this work is given in Theorem 5.3 where we identify independent set games as being the *hardest instances* of PERFECT-PME. In the (X,t)-*independent set game* the players aim to convince a verifier that a graph X contains an independent set of size t (i. e., a set of t pairwise nonadjacent vertices). To play the game the verifier selects uniformly at random a pair of indices $(i,j) \in [t] \times [t]$ and sends i to Alice and j to Bob. The players respond with vertices $u,v \in V(X)$ respectively. In order to win, the players need to respond with the same vertex of X whenever they receive the same index. Furthermore, if they receive $i \neq j \in [t]$ they need to respond with nonadjacent (and distinct) vertices of X. The second decision problem relevant to this work is PERFECT where the input is restricted to be an independent set game.

Q-INDEPENDENCE

Instance: An (X,t)-independent set game.

Question: Does the game admit a perfect entangled strategy?

In our main result given in Theorem 5.3 we show that any instance G of PERFECT-PME can be transformed in polynomial-time to an instance G' of Q-INDEPENDENCE with the property that G admits a perfect PME strategy if and only if G' admits a perfect strategy. Formally:

Result 1: PERFECT-PME *is polynomial-time (Karp) reducible to* Q-INDEPENDENCE.

It is known that PME strategies suffice to win independent set games perfectly (whenever this can be done using some quantum strategy) [17]. As a result, all instances of Q-INDEPENDENCE can be identified with instances of PERFECT-PME and thus our first result can be understood as identifying Q-INDEPENDENCE to be among the most expressive subproblems of PERFECT-PME.

As an immediate consequence of our first result and the discussion in the previous paragraph it follows that PERFECT-PME is decidable if and only if Q-INDEPENDENCE is decidable. Currently, it is not known whether Q-INDEPENDENCE is decidable. Nevertheless, reducing the decidability question from arbitrary games to the special class of independent set games allows to narrow down our focus to this specific class of games for which it might be easier to make progress on the decidability question.

The proof of Result 1 consists of two steps which we now briefly describe. We first need to introduce some definitions. A nonlocal game is called *synchronous* if it satisfies the following three requirements: *(i)* Alice and Bob share the same question set Q and answer set A, *(ii)* $\pi(q,q) > 0$ for all $q \in Q$, and *(iii)* $V(a,b|q,q) = 0$ for all $q \in Q$ and all $a \neq b$. Notice that the (X,t)-independent set game defined above is an example of a synchronous nonlocal game. The second decision problem of interest in this paper is a variation of PERFECT where the input is restricted to be a synchronous game:

PERFECT-SYN
Instance: A synchronous nonlocal game G.
Question: Does G admit a perfect quantum strategy?

In Lemma 3.2 we show that any synchronous game that admits a perfect quantum strategy also has a perfect PME strategy. Notice that this implies that PERFECT-SYN is a subproblem of PERFECT-PME.

The first step in proving Result 1 is Lemma 3.5 where we show that PERFECT-PME is polynomial-time reducible to PERFECT-SYN. To achieve this we extend any nonlocal game G to a synchronous game $\tilde{G}$ where we can also ask Alice any of Bob's questions and vice versa (see Definition 3.4). The extended game $\tilde{G}$ has the property that G has a perfect PME strategy if and only if $\tilde{G}$ has a perfect strategy.

The second step in proving Result 1 is Lemma 5.2 where we show that PERFECT-SYN is polynomial-time reducible to Q-INDEPENDENCE. To achieve this, to any synchronous game G we associate an undirected graph $X(G)$ (see Definition 4.1) and show that G admits a perfect entangled strategy if and only if the $(X(G),|Q|)$-independent set game has a perfect strategy (where Q denotes the question set of G).

We note that following the completion of this work it was communicated to us by Ji that building on his recent results in [13] he has independently obtained Result 1. The proof of this fact has not been published but can be derived by appropriately combining the results in [13] together with two additional reductions (that are not stated in [13]). Furthermore, in contrast to [13] our approach is constructive and the final instance of Q-INDEPENDENCE is given explicitly in terms of the instance of PERFECT-PME.

As was already mentioned it is currently not known whether the entangled value of a nonlocal game (with finite question and answer sets) is always attained by some entangled strategy. In fact, there is evidence that this might not be true. The first example of a nonlocal game with answer sets of *infinite* cardinality for which the entangled value is only attained in the limit was identified recently in [18].

In this work we consider the attainability question restricted to perfect strategies for symmetric synchronous nonlocal games. A synchronous game is called *symmetric* if interchanging the roles of the players does not affect the value of the Boolean predicate (*cf.* Definition 5.4). We note that all games of relevance to this work (e.g. homomorphism) are in fact symmetric. In Theorem 5.5 we show that independent set games again capture the hardness of the attainability question for symmetric games.

Result 2: *Suppose that every independent set game G satisfying $\omega^*(G) = 1$ admits a perfect entangled strategy. Then the same holds for all symmetric synchronous nonlocal games.*

To obtain our second result we show that vanishing-error strategies for a symmetric synchronous game G give rise to vanishing-error strategies for an appropriate independent set game defined in terms of the game graph of G. Notice that since independent set games are synchronous, our second result can be understood as identifying a class of synchronous games which captures the hardness of the attainability question for perfect strategies for the entire class of symmetric synchronous nonlocal games. Nevertheless, we note that presently we do not know whether independent set games satisfy the assumption of Result 2.

A number of interesting results have been derived recently concerning the interplay between the theory of graphs and nonlocal games, e.g. [3, 17, 5, 1]. In Section 4 we take a similar approach and associate an undirected graph, called its *game graph*, to an arbitrary synchronous nonlocal game. As already mentioned this is an essential ingredient in showing that `PERFECT-SYN` is polynomial-time reducible to `Q-INDEPENDENCE`. In Theorem 4.6 we identify combinatorial lower bounds on the classical and entangled value of synchronous nonlocal games in terms of their corresponding game graphs.

Result 3: *Let G be a synchronous game with question set Q and uniform distribution of questions. If $X = X(G)$ is the game graph of G then $\omega(G) \geq \big(\alpha(X)/|Q|\big)^2$, and $\omega^*(G) \geq \big(\alpha_p(X)/|Q|\big)^2$, where $\alpha(X)$ denotes the independence number of X and $\alpha_p(X)$ the projective packing number of X (cf. Definition 4.2).*

2 Preliminaries

We denote the set of $d \times d$ Hermitian operators by $\mathcal{S}^d$. Throughout this work we equip $\mathcal{S}^d$ with the Hilbert-Schmidt inner product $\langle X, Y\rangle = \mathrm{Tr}(XY^*)$. An operator $X \in \mathcal{S}^d$ is called *positive*, denoted by $X \succeq 0$, if $\psi^* X \psi \geq 0$ for all $\psi \in \mathbb{C}^d$. The set of $d \times d$ positive operators is denoted by $\mathcal{S}^d_+$. We use the notation $X \succeq Y$ to indicate that $X - Y \succeq 0$. An operator X is called an (orthogonal) *projector* if it satisfies $X = X^* = X^2$. The *support* of an operator X, denoted $\mathrm{supp}(X)$, is defined as the projector on the range of X. The canonical orthonormal basis of $\mathbb{C}^d$ is denoted by $\{e_i : i \in [d]\}$, where $[d] := \{1, \ldots, d\}$.

Classical strategies and value. A deterministic strategy for a nonlocal game $G(\pi, V)$ consists of a pair of functions, $f_A : Q \to A$ and $f_B : R \to B$, which the players use in order to determine their answers. The *classical value* of the game G, denoted by $\omega(G)$, is equal to the maximum expected probability with which the players can win the game using deterministic strategies. Specifically,

$$\omega(G) := \max \sum_{q\in Q, r\in R} \pi(q,r) V\big(f_A(q), f_B(r)|q,r\big), \tag{2.1}$$

where the maximization ranges over all deterministic strategies.

Quantum strategies and value. In this section we briefly introduce those concepts from quantum information theory that are of relevance to this work. Readers without the required background are referred to [20] for a comprehensive introduction.

To any quantum system S we associate a complex inner product space $\mathbb{C}^d$, for some $d \geq 1$. The *state space* of the system S is defined as the set of unit vectors in $\mathbb{C}^d$. The most basic way one can extract classical information from a quantum system S is by measuring it. For the purposes of this paper, the most relevant mathematical formalism of the concept of a measurement is given by a Positive Operator-Valued Measure (POVM). A POVM is defined in terms of a family of positive operators $\mathcal{M} = (M_i \in \mathcal{S}^d_+ : i \in [m])$ that sum up to the identity operator, i. e., $\sum_{i\in[m]} M_i = I_d$. According to the axioms of quantum mechanics, if the measurement $\mathcal{M}$ is performed on a quantum system whose state is given by $\psi \in \mathbb{C}^d$ then the probability that the i-th outcome occurs is given by $\psi^* M_i \psi$. We say that a measurement $\mathcal{M}$ is *projective* if all the POVM elements M_i are orthogonal projectors.

Consider two quantum systems S_1 and S_2 with corresponding state spaces $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$ respectively. The state space of the joint system (S_1, S_2) is given by $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. Moreover, if S_1 is in state $\psi_1 \in \mathbb{C}^{d_1}$ and S_2 is in state $\psi_2 \in \mathbb{C}^{d_2}$ then the joint system is in state $\psi_1 \otimes \psi_2 \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. Lastly, if $(M_i \in \mathcal{S}^{d_1}_+ : i \in [m_1])$ and $(N_j \in \mathcal{S}^{d_2}_+ : j \in [m_2])$ define measurements on the individual systems S_1 and S_2 then the family of operators $(M_i \otimes N_j \in \mathcal{S}^{d_1 d_2}_+ : i \in [m_1], j \in [m_2])$ defines a product measurement on the joint system (S_1, S_2).

Given any bipartite quantum state $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$, it is possible to choose two orthonormal basis $\{\alpha_i : i \in [d]\}$ and $\{\beta_i : i \in [d]\}$ so that $\psi = \sum_{i=1}^d \lambda_i \alpha_i \otimes \beta_i$ and $\lambda_i \geq 0$ for all $i \in [d]$. This is known as the *Schmidt decomposition* of ψ and we refer to the λ_i as the *Schmidt coefficients* of ψ. We say that ψ has full Schmidt rank, if all its Schmidt coefficients are positive. We say that ψ is *maximally entangled* if all its Schmidt coefficients are the same. Throughout this paper we use ϕ to denote the canonical maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=1}^{d} e_i \otimes e_i$ and we make repeated use of the fact that $\phi^*(A\otimes B)\phi = \frac{1}{d}\operatorname{Tr}(AB^{\mathrm{T}})$ for any operators $A,B \in \mathbb{C}^{d\times d}$.

Consider a nonlocal game $G = (V,\pi)$ with question sets Q,R and answer sets A,B respectively. An *entangled strategy* for G consists of a bipartite state $\psi \in \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$, a POVM $\mathcal{M}_q = (M_{aq} \in \mathcal{S}_+^{d_1} : a \in A)$ for each of Alice's questions $q \in Q$ and a POVM $\mathcal{N}_r = (N_{br} \in \mathcal{S}_+^{d_2} : b \in B)$ for each of Bob's questions $r \in R$. Upon receiving questions $(q,r) \in Q\times R$, Alice performs measurement $\mathcal{M}_q$ on her part of ψ and Bob performs measurement $\mathcal{N}_r$ on his part of ψ. The probability that upon receiving questions $(q,r)\in Q\times R$ they answer $(a,b) \in A\times B$ is equal to $\psi^*(M_{aq}\otimes N_{br})\psi$. The *entangled value* of G, denoted by $\omega^*(G)$, is the supremum expected probability with which entangled players can win the game, i. e.,

$$\omega^*(G) := \sup \sum_{q\in Q, r\in R} \pi(q,r) \sum_{a\in A, b\in B} V(a,b|q,r)\psi^*(M_{aq}\otimes N_{br})\psi, \tag{2.2}$$

where the maximization ranges over all bipartite quantum states $\psi \in \mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$ and POVMs $(\mathcal{M}_q : q \in Q)$ and $(\mathcal{N}_r : r \in R)$. A strategy for G is called *projective* if all the measurements $\mathcal{M}_q$ and $\mathcal{N}_r$ are projective. We say that a nonlocal game G admits a perfect quantum strategy if $\omega^*(G) = 1$ and moreover, there exists a bipartite state $\psi \in \mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$ and POVMs $(\mathcal{M}_q : q \in Q)$ and $(\mathcal{N}_r : r \in R)$ that achieve this value.

Graph theory. A graph X is given by an ordered pair of sets $(V(X),E(X))$, where $E(X)$ is a collection of 2-element subsets of $V(X)$. The elements of $V(X)$ are called the *vertices* of the graph and the elements of $E(X)$ its *edges*. For every edge $e = \{u,v\} \in E(X)$ we say that u and v are *adjacent* and write $u \sim_X v$ or simply $u \sim v$ if the graph is clear from the context. A set of vertices $S \subseteq V(X)$ is called an *independent set* if no two vertices in S are adjacent. The cardinality of the largest independent set is denoted by $\alpha(X)$ and is called the *independence number* of X. The *complement* of a graph X, denoted by $\overline{X}$, has the same vertex set as X, but $u \sim v$ in $\overline{X}$ if and only if $u \neq v$ and $u \not\sim v$ in X. A set of vertices $C \subseteq V(X)$ is called a *clique* in X if S is an independent set in $\overline{X}$.

3 Synchronous games

In this section we introduce and study synchronous nonlocal games. We first show that synchronous games can always be won with perfect PME strategies (whenever a perfect strategy exists). Our main result in this section is Lemma 3.5 where we show that any instance of PERFECT-PME is polynomial-time reducible to an instance of PERFECT-SYN. The main ingredient in this proof is the notion of a synchronous extension of a nonlocal game.

3.1 Definition and basic properties

Throughout this section we focus on games where Alice and Bob share the same question and answer sets and furthermore, in order to win, they need to give the same answers upon receiving the same questions.

Definition 3.1. A nonlocal game $G = (V,\pi)$ is called *synchronous* if it satisfies the following properties:

(i) $A = B$ and $Q = R$;

(ii) $V(a,b|q,q) = 0$, if $a \neq b$;

(iii) for all $q \in Q$, we have $\pi(q,q) > 0$.

The notion of synchronous nonlocal games subsumes many classes of nonlocal games that have been recently studied [17, 4]. A related concept that has recently been considered is that of synchronous correlations, defined in [21]. These are correlations (joint conditional probability distributions) such that $\Pr(a,a'|q,q) = 0$ whenever $a \neq a'$.

We now study perfect entangled strategies for synchronous games and show that such strategies can, without loss of generality, be assumed to have a certain form.

Lemma 3.2. *Let G be a synchronous game which admits a perfect entangled strategy. Then there also exists a perfect* `PME` *strategy for G where Bob's projectors are the transpose of Alice's corresponding projectors.*

Proof. Let G be a synchronous game with answer set A and question set Q. Consider a perfect strategy for G given by a shared state $\psi \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, a POVM $\mathcal{M}_q = (M_{aq} : a \in A)$ for each of Alice's questions and a POVM $\mathcal{N}_q = (N_{aq} : a \in A)$ for each of Bob's questions. Without loss of generality, we can assume that the shared state is pure and has full Schmidt rank. Let $\rho_{aq} := \mathrm{Tr}_A\left((M_{aq} \otimes I)\psi\psi^*\right)$ denote Bob's residual states after Alice has responded $a \in A$ upon receiving question $q \in Q$. We first show that

$$\langle \rho_{aq}, \rho_{br} \rangle = 0, \text{ whenever } V(a,b|q,r) = 0 \text{ and } \pi(q,r) > 0. \tag{3.1}$$

For this consider a question/answer pair satisfying $V(a,b|q,r) = 0$ and assume that Bob has received question $r \in Q$. For the players to win, Bob needs to answer $b \in Q$ if he holds the state ρ_{br} since the game is synchronous. On the other hand, he cannot answer $b \in Q$ if he holds the state ρ_{aq}. Since the strategy is perfect, Bob never errs and we can use his answer to perfectly discriminate the states ρ_{aq} and ρ_{br}. Only orthogonal states can be perfectly discriminated and hence we must have that $\langle \rho_{aq}, \rho_{br} \rangle = 0$.

The last step is to use the support of Bob's residual states to construct a perfect `PME` strategy for G. For all $a \in A$ and $q \in Q$ define $P_{qa} := \mathrm{supp}(\rho_{aq})$. By definition of ρ_{aq} we have that $\sum_{a\in A} \rho_{aq} = \mathrm{Tr}_A(\psi\psi^*)$ and since ψ has full Schmidt rank it follows that $\mathrm{supp}(\mathrm{Tr}_A(\psi\psi^*)) = \mathrm{supp}(\sum_{i=1}^d \lambda_i e_i e_i^*) = I_d$. On the other hand, since G is a synchronous game we have that $\pi(q,q) > 0$ for all $q \in Q$. Thus, it follows from (3.1) that $\langle \rho_{aq}, \rho_{a'q} \rangle = 0$ for $a \neq a'$ and thus $\mathrm{supp}(\sum_{a\in A} \rho_{aq}) = \sum_{a\in A} \mathrm{supp}(\rho_{aq}) = \sum_{a\in A} P_{aq}$ for every $q \in Q$. Summarizing we have that $\sum_{a\in A} P_{aq} = I_d$ for all $q \in Q$ and thus we can define projective measurements $\mathcal{P}_q := (P_{aq} : a \in A)$ for Alice and $\mathcal{R}_q := (P_{aq}^{\mathrm{T}} : a \in A)$ for Bob.

Consider the strategy where the players share the state $\phi = \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d$, Alice uses the projective measurement $\mathcal{P}_q$ upon receiving question $q \in Q$ and Bob uses the projective measurement $\mathcal{R}_q$ upon receiving $q \in Q$. To see that this strategy never errs, note that the probability to answer $(a,b) \in A \times A$ upon receiving question pair $(q,r) \in Q \times Q$ is $\Pr(a,b|q,r) = \phi^*(P_{aq} \otimes P_{br}^{\mathrm{T}})\phi = \frac{1}{d}\mathrm{Tr}(P_{aq}P_{br})$. Since the supports of orthogonal states are orthogonal it follows from (3.1) that $\Pr(a,b|q,r) = 0$ whenever $V(a,b|q,r) = 0$ and $\pi(q,r) > 0$. ☐

This result was known for graph coloring [4] and graph homomorphism games [17]. Since both of these game classes are synchronous nonlocal games, Lemma 3.2 subsumes both of these results.

Remark 3.3. Notice that the perfect strategy guaranteed by Lemma 3.2 has the property that $\Pr(a,b|q,r) = \Pr(b,a|r,q)$ for all $a,b \in A$ and $q,r \in Q$. This observation is used in Lemma 5.2.

3.2 Synchronous extension

In this section we introduce the notion of the synchronous extension of a nonlocal game (*cf.* Definition 3.4). We also establish that `PERFECT-PME` is polynomial-time reducible to `PERFECT-SYN`.

In order to reduce instances of `PERFECT-PME` to those of `PERFECT-SYN`, to any game G we associate a synchronous game $\tilde{G}$ where we can also ask Alice any of Bob's questions and vice versa. The winning condition in $\tilde{G}$ is the same as in G if both players are asked their original questions or the other player's questions. When both players are given the same question, we require that their answers coincide, therefore ensuring that $\tilde{G}$ is synchronous. For simplicity we assume that the question sets and also the answer sets of the original game G are disjoint. Note however that this is not truly a restriction since any game can be converted into an equivalent game with disjoint question sets and disjoint answer sets, for instance by letting $Q' = \{(q,0) : q \in Q\}$ and $R' = \{(r,1) : r \in R\}$, and similarly for A and B.

Definition 3.4. Let G be a nonlocal game with disjoint question sets Q,R and disjoint answer sets A,B. The *synchronous extension* of G, denoted by $\tilde{G}$, is a new synchronous game with question and answer sets

$$\tilde{Q} = Q \cup R \quad \& \quad \tilde{A} = A \cup B.$$

The probability distribution $\tilde{\pi}$ on the question set $\tilde{Q} \times \tilde{Q}$ is any distribution of full support[1]. Lastly the verification predicate $\tilde{V}$ is given by:

$$\tilde{V}(a,b|q,r) = \tilde{V}(b,a|r,q) = V(a,b|q,r), \text{ for all } a \in A,\ b \in B,\ q \in Q,\ r \in R, \tag{3.2}$$

$$\tilde{V}(a,a'|q,q) = \delta_{aa'} \text{ and } \tilde{V}(b,b'|r,r) = \delta_{bb'}, \text{ for all } q \in Q,\ r \in R,\ a,a' \in A,\ b,b' \in B, \tag{3.3}$$

$$\tilde{V}(y,y'|x,x') = 0 \text{ if either } (x,y) \text{ or } (x',y') \text{ is an element of } (R \times A) \cup (Q \times B), \tag{3.4}$$

and it evaluates to one in all remaining cases. Notice that condition (3.2) ensures that players give correct answers upon receiving their original questions or when their roles are reversed. Furthermore, condition (3.3) ensures the game is synchronous and (3.4) ensures that only Alice's answers are accepted for Alice's questions and only Bob's answers are accepted for Bob's questions.

Generally the synchronous extension might be harder to win than the original game. However, as we will see in the next section, any perfect PME strategy for the game G, can be also be used to win $\tilde{G}$ perfectly.

3.3 Reducing PERFECT-PME to PERFECT-SYN

Using the notion of the synchronous extension we are now ready to prove the main result in this section.

Lemma 3.5. *A nonlocal game G has a perfect* PME *strategy if and only if its synchronous extension $\tilde{G}$ has a perfect entangled strategy. In particular,* PERFECT-PME *is polynomial-time reducible to* PERFECT-SYN.

Proof. First, assume that G has a perfect PME strategy using a maximally entangled state $\phi \in \mathbb{C}^d \otimes \mathbb{C}^d$ and projective measurements $\mathcal{P}_q = (P_{aq} : a \in A)$ and $\mathcal{R}_r = (R_{br} : b \in B)$ for Alice and Bob respectively. Also for all $q \in Q$ and $r \in R$ let $\mathcal{P}_q^{\mathrm{T}}$ and $\mathcal{R}_r^{\mathrm{T}}$ denote the projective measurements obtained by taking the transpose of all the projectors within the projective measurements $\mathcal{P}_q$ and $\mathcal{R}_r$ respectively. To play the game $\tilde{G}$ the players use the following strategy: Alice measures her part of ϕ using $\mathcal{P}_q$ upon receiving question $q \in Q$ and with $\mathcal{R}_r^{\mathrm{T}}$ upon receiving question $r \in R$. In the former case she responds with some $a \in A$, while in the latter she responds with some $b \in B$, where a and b are the respective measurement outcomes. Bob acts similarly, except that he uses his original measurements $\mathcal{R}_r$ for a question $r \in R$ and $\mathcal{P}_q^{\mathrm{T}}$ for a question $q \in Q$.

It remains to verify that this defines a perfect strategy for $\tilde{G}$. To do so we show that the players never return answers for which $\tilde{V}$ evaluates to zero. First, note that by construction both players only respond with Alice's answers when asked Alice's questions and similarly for Bob's questions and answers. Therefore they never lose due to condition (3.4). Next we will show that condition (3.2) never causes the players to lose $\tilde{G}$. If both players are given questions from their original question sets in G, then their strategies are exactly as

[1] We could also allow zero probabilities for questions that correspond to zero probability questions in the original game G.

they were in G, and since their strategy for G was perfect they will win in this case. If Alice is given $r \in R$ and Bob is given $q \in Q$, then they will respond with some $b \in B$ and $a \in A$ with probability equal to

$$\phi^* \left(R_{br}^{\mathrm{T}} \otimes P_{aq}^{\mathrm{T}}\right) \phi = \frac{1}{d} \operatorname{Tr}\left(R_{br}^{\mathrm{T}} P_{aq}\right) = \frac{1}{d} \operatorname{Tr}\left(P_{aq} R_{br}^{\mathrm{T}}\right) = \phi^* \left(P_{aq} \otimes R_{br}\right) \phi.$$

This is the probability of Alice and Bob outputting a and b respectively when receiving q and r in the original game G. If this probability is greater than 0, then $\tilde{V}(b,a|r,q) = V(a,b|q,r) = 1$ since they win G perfectly. Therefore condition (3.2) never causes Alice and Bob to lose $\tilde{G}$.

Lastly, for all $q \in Q$ and $a \neq a' \in A$ we have that

$$\Pr(a,a'|q,q) = \phi^* \left(P_{aq} \otimes P_{a'q}^{\mathrm{T}}\right) \phi = \frac{1}{d} \operatorname{Tr}(P_{aq} P_{a'q}) = 0,$$

and similarly for $b \neq b' \in B$ and $r \in R$. Therefore the players always give the same answer when asked the same question and thus they never lose $\tilde{G}$ due to condition (3.3). Since there are no other ways for the players to lose $\tilde{G}$, we have shown that they win this game perfectly.

To show the other direction let us assume $\tilde{G}$ has a perfect strategy. By construction $\tilde{G}$ is synchronous, hence Lemma 3.2 allows us to conclude that there exists a perfect `PME` strategy for $\tilde{G}$. Since $\tilde{G}$ contains the original game G, any perfect strategy for $\tilde{G}$ can also be used to win G perfectly. □

In fact, the proof of Lemma 3.5 shows that any (not necessarily perfect) `PME` strategy for $G = (V, \pi)$ can be used to win $\tilde{G} = (\tilde{V}, \tilde{\pi})$ with at least as high probability of success if $\tilde{\pi}|_G = \pi$. Here, we have used $\tilde{\pi}|_G$ to refer to the distribution obtained from $\tilde{\pi}$ by restricting to questions in G and re-normalizing.

4 Game graphs

In this section we introduce the notion of the game graph of a synchronous game (*cf.* Definition 4.1). Our main result in this section is Theorem 4.4 where we relate the existence of perfect entangled strategies for a synchronous game to the projective packing number of its game graph. This is used in Section 5.1 to reduce `PERFECT-SYN` to `Q-INDEPENDENCE`. Lastly, in Theorem 4.6 we identify combinatorial lower bounds on the classical and entangled values of synchronous games in terms of their game graphs.

4.1 Definition and some properties

A nonlocal game $G = (V, \pi)$ admits a perfect entangled strategy if there exist a quantum state $\psi \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and POVM measurements $(\mathcal{M}_{qa} : a \in A) \subseteq \mathcal{S}_+^{d_A}$ and $(\mathcal{N}_{rb} : b \in B) \subseteq \mathcal{S}_+^{d_B}$ such that

$$\psi^*(M_{qa} \otimes N_{rb})\psi = 0, \text{ when } V(a,b|q,r) = 0 \text{ and } \pi(q,r) > 0. \tag{4.1}$$

We have already seen in Lemma 3.2 that a synchronous game has a perfect entangled strategy if and only if it has a perfect `PME` strategy. This implies that for synchronous games Condition (4.1) reduces to a set of orthogonality relations between the measurement operators. Next, for every synchronous nonlocal game we associate an undirected graph which encodes these required orthogonalities as adjacencies.

Definition 4.1. Let G be a synchronous game with question set Q and answer set A. The *game graph* of G, denoted $X(G)$, is the undirected graph with vertex set $A \times Q$ where (a,q) is adjacent to (a',q') if $V(a,a'|q,q') = 0$ or $V(a',a|q',q) = 0$.

We note that the definition of the game graph is similar to the graphs that appear in the FGLSS-type reductions, introduced by Feige, Goldwasser, Lovász, Safra and Szegedy [11]. However, the scope of the FGLSS-reduction framework is much more general and thus the analogies with our notion of game graphs are quite limited.

An important feature of game graphs is that their vertex set admits a natural partition into cliques. Specifically, for a given question $q \in Q$ of a synchronous game G, the vertices of $V_q := \{(a,q) : a \in A\}$ are pairwise adjacent in $X(G)$. This observation will be important for the proofs in this section.

4.2 Synchronous games and the projective packing number

In this section we show that a synchronous game admits a perfect entangled strategy if and only if its game graph has a projective packing of value $|Q|$ (*cf.* Theorem 4.4).

We first recall the definition of the projective packing number of a graph [17, 24].

Definition 4.2. A *d-dimensional projective packing* of a graph $X = (V,E)$ consists of an assignment of projectors $P_u \in \mathcal{S}^d_+$ to every vertex $u \in V$ such that

$$\mathrm{Tr}(P_u P_v) = 0, \text{ whenever } u \sim_X v. \tag{4.2}$$

The *value* of a projective packing using projectors $P_u \in \mathcal{S}^d_+$ is defined as

$$\frac{1}{d} \sum_{u \in V} \mathrm{Tr}(P_u). \tag{4.3}$$

The *projective packing number* of a graph X, denoted $\alpha_p(X)$, is defined as the supremum of the values over all projective packings of the graph X.

Notice that the supremum in the definition of projective packing number is necessary because it is not clear that $\alpha_p(X)$ is always attained by some projective packing of the graph X. We now give an upper bound on the projective packing number of a game graph.

Lemma 4.3. *For any synchronous game G with question set Q we have that $\alpha_p\big(X(G)\big) \leq |Q|$.*

Proof. Let $(P_{aq} : a \in A, q \in Q)$ be a d-dimensional projective packing of $X(G)$. The vertices in $V_q = \{(a,q) : a \in A\}$ are pairwise adjacent and thus the projectors P_{aq} are pairwise orthogonal for every $q \in Q$. Therefore,

$$\sum_{a \in A} \mathrm{Tr}(P_{aq}) = \sum_{a \in A} \mathrm{rank}(P_{aq}) \leq d,$$

where $\mathrm{rank}(M)$ is the rank of matrix M. From the above inequality we further obtain that

$$\frac{1}{d} \sum_{(a,q) \in A \times Q} \mathrm{Tr}(P_{aq}) = \frac{1}{d} \sum_{q \in Q} \sum_{a \in A} \mathrm{Tr}(P_{aq}) \leq \frac{1}{d} |Q| \cdot d = |Q|,$$

and thus $\alpha_p\big(X(G)\big) \leq |Q|$. ☐

In view of Lemma 4.3 it is natural to ask when it is the case that $\alpha_p\big(X(G)\big) = |Q|$. As it turns out this happens exactly when there exists a perfect entangled strategy for G.

Theorem 4.4. *Let G be a synchronous game with question set Q. Then G has a perfect entangled strategy if and only if its game graph has a projective packing of value $|Q|$.*

Proof. Let G be a synchronous game with a perfect entangled strategy. By Lemma 3.2, there exists a perfect projective strategy for G that uses maximally entangled state $\phi \in \mathbb{C}^d \otimes \mathbb{C}^d$, where Alice's and Bob's projectors are transpose to each other. Let $P_{aq} \in \mathcal{S}_+^d$ be Alice's projector associated with question $q \in Q$ and answer $a \in A$. Since this strategy is perfect we have that

$$0 = \phi^*(P_{aq} \otimes P_{a'q'}^{\mathrm{T}})\phi = \frac{1}{d}\operatorname{Tr}(P_{aq}P_{a'q'}), \tag{4.4}$$

whenever $V(a,a'|q,q') = 0$ or $V(a',a|q',q) = 0$. It follows immediately from Equation (4.4) that the projectors P_{aq} form a d-dimensional projective packing of $X(G)$. Since $\sum_{a\in A} P_{aq} = I_d$ it follows that

$$\frac{1}{d}\sum_{(a,q)\in A\times Q}\operatorname{Tr}(P_{aq}) = \frac{1}{d}\sum_{q\in Q}\operatorname{Tr}(I_d) = |Q|, \tag{4.5}$$

and thus value of this packing is $|Q|$. Lastly, by Lemma 4.3 we get that $\alpha_p\big(X(G)\big) = |Q|$.

For the other direction, assume that $X(G)$ has a d-dimensional projective packing $(P_{aq} : a \in A, q \in Q)$ of value $|Q|$. Since G is a synchronous game we have that $(q,a) \sim (q,a')$ for $a \neq a' \in A$ and $q \in Q$. This implies that $\sum_{a\in A} P_{aq} \preceq I_d$, as the added projectors are mutually orthogonal. Furthermore, since the value of the projective packing is $|Q|$, we obtain

$$|Q| = \frac{1}{d}\sum_{(a,q)\in A\times Q}\operatorname{Tr}(P_{aq}) = \sum_{q\in Q}\Big(\frac{1}{d}\operatorname{Tr}\big(\sum_{a\in A} P_{aq}\big)\Big) \le \sum_{q\in Q}\frac{1}{d}\operatorname{Tr}(I_d) \le |Q|, \tag{4.6}$$

and thus Equation (4.6) holds throughout with equality. In particular, $\operatorname{Tr}\big(\sum_{a\in A} P_{aq}\big) = \operatorname{Tr}(I_d)$, and since $\sum_{a\in A} P_{aq} \preceq I_d$ we conclude that $\sum_{a\in A} P_{aq} = I_d$ and thus $\mathcal{P}_q = (P_{aq} : a \in A)$ forms a valid projective measurement. By the definition of the edge set of $X(G)$, we see that Alice and Bob can win with probability one, if they measure a maximally entangled state using projective measurements $\mathcal{P}_q$ and $\mathcal{P}_q^{\mathrm{T}}$ respectively. □

4.3 Lower bounding $\omega(G)$ and $\omega^*(G)$ for synchronous games

In this section we derive combinatorial lower bounds on the classical and entangled values of synchronous nonlocal games in terms of the independence number and the projective packing number of their game graphs respectively (*cf.* Theorem 4.6).

Our first result gives a necessary and sufficient condition for the existence of a perfect classical strategy.

Lemma 4.5. *Let G be a synchronous game with question set Q and let $X := X(G)$ be the its game graph. Then, G has a perfect classical strategy if and only if $\alpha(X) = |Q|$.*

Proof. Let $f_A, f_B : Q \to A$ be a perfect deterministic strategy for the game G. Since G is synchronous we have that $f_A = f_B =: f$. Set $V_q = \{(a,q) : a \in A\}$ and notice that $\{V_q : q \in Q\}$ forms a clique cover of X of cardinality $|Q|$. This shows that $\alpha(X) \le |Q|$. Lastly, we show that $S = \{(q, f(q)) : q \in Q\}$ is an independent set in X. Indeed, since f is a perfect strategy, for any $(q, f(q)), (r, f(r)) \in S$ we have that $V(f(q), f(r)|q,r) = V(f(r), f(q)|r,q) = 1$. This implies that $(q, f(q)) \not\sim (r, f(r))$.

Conversely, let S be an independent set in X of cardinality $|Q|$. Since $\{V_q : q \in Q\}$ is a clique cover of cardinality $|Q|$, for every $q \in Q$, the intersection $S \cap V_q$ contains exactly one vertex of X which we denote by (q, a_q). Define $f : Q \to A$ where $f(q) = a_q$ for every $q \in Q$ and consider the deterministic strategy for G where both players determine their answers using f. It remains to show that this is a perfect classical strategy. Assume for contradiction that there exist $q, r \in Q$ such that $V(f(q), f(r)|q,r) = 0$. By definition of X this implies that $(f(q), q) \sim (f(r), r)$, contradicting the fact that S is an independent set in X. □

As an immediate consequence of Lemma 4.5 we recover the well-known fact that there exist a graph homomorphism from a graph X to a graph Y if and only if $\alpha(X \ltimes Y) = |V(X)|$. Here $X \ltimes Y$ denotes the *homomorphic product* of X and Y whose vertex set is given by $V(X) \times V(Y)$ and $(x,y) \sim (x',y')$ if and only if $[(x = x')$ and $y \neq y']$ or $[x \sim x'$ and $y \not\sim y']$. To recover this result from Lemma 4.5 notice that the game graph for the (X,Y)-homomorphism game is given precisely by $X \ltimes Y$ (see also [17]).

We now proceed to lower bound the classical and entangled values of synchronous games.

Theorem 4.6. *Consider a synchronous game G with question set Q and uniform distribution of questions. If $X = X(G)$ is the game graph of G then,*

$$\omega(G) \geq \big(\alpha(X)/|Q|\big)^2 \textit{ and } \omega^*(G) \geq \big(\alpha_p(X)/|Q|\big)^2. \tag{4.7}$$

Proof. First, we consider the classical case. Our goal is to exhibit a deterministic strategy that wins on at least $\alpha(X)^2$ out of the $|Q|^2$ pairs of possible questions. Let S be an independent set in X of cardinality $\alpha(X)$. By definition of the edge set of X, for any pair $(a,q),(b,r) \in S$ we have that

$$V(a,b|q,r) = 1 \text{ and } V(b,a|r,q) = 1. \tag{4.8}$$

Set $Q' = \{q \in Q : \exists a \in A \text{ such that } (a,q) \in S\}$. Since G is synchronous and S is an independent set, for every $q \in Q'$ there exists a unique $a \in A$ such that $(a,q) \in S$, which we denote by $f(q)$. Furthermore, notice that $|Q'| = \alpha(X)$. Consider the following deterministic strategy: If a player receives as question an element $q \in Q'$ he responds with $f(q)$ and if $q \notin Q'$ his answer is arbitrary. It follows from (4.8) that for $q,r \in Q'$, the players win when asked (q,r) and (r,q). Since $|Q'| = \alpha(X)$ this strategy is correct on at least $\alpha(X)^2$ of the $|Q|^2$ possible questions.

Next we consider the entangled case. Let $(P_{aq} : a \in A, q \in Q)$ be a d-dimensional projective packing for X of value γ, i. e., $\gamma = \frac{1}{d}\sum_{a\in A, q\in Q} \mathrm{Tr}(P_{aq})$. We construct an entangled strategy whose value is at least $\gamma^2/|Q|^2$. Recall that for all $q \in Q$ the set $V_q = \{(a,q) : a \in A\}$ forms a clique in X. This implies that for fixed $q \in Q$ and $a \neq a' \in A$ the projectors P_{aq} and $P_{a'q}$ are pairwise orthogonal and thus $\sum_{a\in A} P_{aq} \preceq I_d$. Consider the following entangled strategy for G: The players share the maximally entangled state $\phi \in \mathbb{C}^d \otimes \mathbb{C}^d$ and for every $q \in Q$ Alice uses the projective measurement $(P_{aq} : a \in A) \cup (I - \sum_{a\in A} P_{aq})$ and Bob uses the measurement $(P_{aq}^{\mathrm{T}} : a \in A) \cup (I - \sum_{a\in A} P_{aq}^{\mathrm{T}})$. Using this strategy the players win with probability at least

$$\frac{1}{|Q|^2}\sum_{a,b,q,r} \phi^*(P_{aq} \otimes P_{br}^{\mathrm{T}})\phi\, V(a,b|q,r) = \frac{1}{d|Q|^2}\sum_{a,b,q,r} \mathrm{Tr}(P_{aq}P_{br})V(a,b|q,r). \tag{4.9}$$

If $V(a,b|q,r) = 0$ then $(a,q) \sim (b,r)$ in the game graph and by definition of the projective packing we have that $\mathrm{Tr}(P_{aq}P_{br}) = 0$. Thus (4.9) gives that

$$\omega^*(G) \geq \frac{1}{d|Q|^2}\sum_{a,b,q,r} \mathrm{Tr}(P_{aq}P_{br}) = \frac{1}{d|Q|^2}\mathrm{Tr}(P^2), \tag{4.10}$$

where $P = \sum_{a\in A, q\in Q} P_{aq}$. By the Cauchy-Schwartz inequality we get that $\mathrm{Tr}(P^2) \geq \mathrm{Tr}(P)^2/d$. Finally, since $\gamma = \mathrm{Tr}(P)/d$, it follows from (4.10) that $\omega^*(G) \geq \gamma^2/|Q|^2$ and the proof is completed. □

If a synchronous game G satisfies $\alpha_p(X(G)) = |Q|$, it follows from Theorem 4.6 that $\omega^*(G) = 1$. On the other hand we have seen in Theorem 4.4 that if there exists a projective packing for the game graph with value equal to $|Q|$ then G has a perfect quantum strategy. Notice that these two conditions are not equivalent since we could have $\alpha_p(X(G)) = |Q|$ without this value being attained.

In [5] the authors also propose a graph-theoretic lower bound on the entangled value of an arbitrary game. Although, at first sight this bound looks similar to our bound from Theorem 4.6, it is unclear if they are at all related. One main difference is that the graphs considered in [5] encode the verification function of the game in the vertices while our game graphs encode it into the edges.

5 Independent set games

In this section we show that `PERFECT-SYN` is polynomial-time reducible to `Q-INDEPENDENCE` (*cf.* Lemma 5.2). This fact combined with the reduction of `PERFECT-PME` to `PERFECT-SYN` derived in Lemma 3.5 implies that `PERFECT-PME` is polynomial-time reducible to `Q-INDEPENDENCE`, which is the main result in this paper. Additionally we consider the attainability problem for perfect strategies and synchronous games. In Theorem 5.5 we show that if any independent set game whose entangled value is one also admits a perfect strategy then the same is true for all symmetric synchronous games.

5.1 Reducing `PERFECT-PME` to `Q-INDEPENDENCE`

Recall that in the (X,t)-independent set game the players try to convince a verifier that the graph X contains an independent set of size t. The verifier selects uniformly at random $(i,j) \in [t] \times [t]$ and sends i to Alice and j to Bob. The players respond with vertices $u,v \in V(X)$ respectively. The verification predicate evaluates to zero in the following three cases: $[i = j$ and $u \neq v]$ or $[i \neq j$ and $u = v]$ or $[i \neq j$ and $u \sim_X v]$.

The independence number of a graph X can equivalently be defined as the largest integer $t \geq 1$ for which the (X,t)-independent set game admits a perfect classical strategy. Similarly, the *quantum independence number* of a graph X, denoted by $\alpha_q(X)$ is defined as the largest integer $t \geq 1$ for which the (X,t)-independent set game admits a perfect entangled strategy [17].

It is known that the projective packing number is an upper bound to the quantum independence number.

Lemma 5.1. *[24, 6.11.1] Let X be a graph and $k \in \mathbb{N}$. If $\alpha_q(X) \geq k$ then there exists a projective packing of X with value k. In particular, $\alpha_q(X) \leq \alpha_p(X)$.*

We are now ready to prove the main result in this section.

Lemma 5.2. *Let G be a synchronous game with question set Q. Then G has a perfect entangled strategy if and only if $\alpha_q(X(G)) = |Q|$. In particular,* `PERFECT-SYN` *is polynomial-time reducible to* `Q-INDEPENDENCE`.

Proof. Assume first there exists a perfect entangled strategy for G. By Lemma 3.2 there also exists a perfect `PME` strategy S for G where Bob's projectors are the transpose of Alice's corresponding projectors. For all $a,a' \in A$ and $q,q' \in Q$ let $\Pr(a,a'|q,q')$ be the probability that Alice and Bob answer a and b respectively upon receiving questions q and q' when employing strategy S. We now construct a perfect strategy for the $(X(G),|Q|)$-independent set game. Without loss of generality, we may assume that Q itself is used as the question set. Consider the following strategy: upon receiving q and q' respectively, Alice and Bob use strategy S to obtain answers a and a'. They then output vertices (a,q) and (a',q') respectively. If $\Pr(a,a'|q,q') > 0$ then by Remark 3.3 we have that also $\Pr(a',a|q',q) > 0$. From this we see that both $V(a,a'|q,q') = 1$ and $V(a',a|q',q) = 1$, since S was perfect. This implies that (a,q) and (a',q') are (possibly equal) nonadjacent vertices in $X(G)$. If $q \neq q'$, then these two vertices are not equal and are therefore distinct nonadjacent vertices of $X(G)$, as required by the independent set game. If $q = q'$, then since G is a synchronous game and S is perfect, we have that $a = a'$ and therefore the two outputted vertices are equal as required. This shows that using this strategy allows Alice and Bob to win the independent set game perfectly and thus $\alpha_q(X(G)) \geq |Q|$. On the other hand from Lemma 5.1 together with Lemma 4.3 it follows that $\alpha_q(X(G)) \leq |Q|$ and thus $\alpha_q(X(G)) = |Q|$.

Conversely, assume that $\alpha_q(X(G)) = |Q|$. By Lemma 5.1 and Lemma 4.3 there exists a projective packing of $X(G)$ of value $|Q|$, and therefore by Theorem 4.4 there exists a perfect entangled strategy for G. □

Lastly, combining Lemma 3.5 and Lemma 5.2 directly yields the main result of this paper.

Theorem 5.3. *A nonlocal game G with question sets Q and R admits a perfect* `PME` *strategy if and only if*

$$\alpha_q\big(X(\tilde{G})\big) = |Q| + |R|. \tag{5.1}$$

In particular, `PERFECT-PME` *is polynomial-time reducible to* `Q-INDEPENDENCE`.

5.2 Attainability problem for perfect strategies

In this section we focus on the the attainability problem for perfect strategies and show that the attainability question for symmetric synchronous games reduces to the attainability question for independent set games.

Definition 5.4. A synchronous game $G = (V, \pi)$ is called *symmetric* if

$$V(a,a'|q,q') = V(a',a|q',q), \text{ for all } a,a' \in A, q,q' \in Q.$$

Notice that all synchronous games we consider in this work (e. g., homomorphism and coloring games) are symmetric.

Theorem 5.5. *Suppose that any independent set game G satisfying $\omega^*(G) = 1$ admits a perfect entangled strategy. Then the same holds for all symmetric synchronous nonlocal games.*

Proof. Let $G = (V, \pi)$ be any symmetric synchronous game with question set Q and answer set A. Assume that $\omega^*(G) = 1$ and let $X := X(G)$ be its game graph. Define $G' = (V', \pi)$ to be the $(X, |Q|)$-independent set game with π as the distribution of questions. The crux of the proof is that from any strategy S that succeeds in G with probability at least $1-\varepsilon$, we can construct a strategy S' that wins G' with probability at least $1-\varepsilon$. Similarly to Lemma 5.2, using the strategy S for G we define the following strategy S' for G': Upon receiving $q \in Q$, Alice uses strategy S for G and obtains an answer $a \in A$. She then replies with vertex (q,a) of X. Similarly, Bob, upon receiving $q' \in Q$ he uses strategy S for G to obtain answer $a' \in A$. He then replies with vertex (q',a') of X. Let $\Pr_S(a,a'|q,q')$ denote the probability that using strategy S the players respond with $(a,a') \in A \times A$ upon receiving questions $q,q' \in Q$ respectively. By assumption we have that

$$\omega^*(G,S) := \sum_{q\in Q, a\in A} \pi(q,q)\Pr_S(a,a|q,q) + \sum_{q\neq q' \in Q} \pi(q,q') \sum_{a,a'\in A: V(a,a'|q,q')=1} \Pr_S(a,a'|q,q') \geq 1-\varepsilon. \tag{5.2}$$

Furthermore, by definition of the strategy S' we have that

$$\Pr_S(a,a'|q,q') = \Pr_{S'}\big((q,a),(q',a')|q,q'\big), \text{ for all } q,q' \in Q. \tag{5.3}$$

Since G' is an independent set game we have $V'\big((q,a),(q',a')|q,q'\big) = 1$ if and only if

$$[q = q' \text{ and } a = a'] \text{ or } [q \neq q' \text{ and } (a,q) \not\sim_X (a',q')]. \tag{5.4}$$

Since the game G is symmetric, Condition (5.4) is equivalent to

$$[q = q' \text{ and } a = a'] \text{ or } [q \neq q' \text{ and } V(a,a'|q,q') = 1]. \tag{5.5}$$

Combining (5.3) with (5.5), and the fact that $V(a,a'|q,q) = 1 \Rightarrow a = a'$, it follows that the probability of winning the game G' using strategy S' is at least $\omega^*(G,S) \geq 1-\varepsilon$. Since $\omega^*(G) = 1$ this argument can be repeated for any ε arbitrarily close to 0 which implies that the entangled value of G' is equal to one. By the assumption of the theorem, this implies that there is a perfect quantum strategy for G' and thus by Lemma 5.2 there exists a perfect quantum strategy for G. □

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