# Stabilising aggregation AMG

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**Abstract.** When applied to linear systems arising from scalar elliptic partial differential equations, algebraic multigrid (AMG) schemes based on aggregation exhibit a mesh size dependent convergence behaviour. As the number of iterations increases with the number of unknowns in the linear system, the computational complexity of such a scheme is non-optimal. This contribution presents a stabilisation of the aggregation AMG algorithm which adds a number of subspace projection steps at different stages of the algorithm and allows for variable cycling strategies. Numerical results illustrate the advantage of the stabilised algorithm over its original formulation.

**Keywords.** algebraic multigrid, aggregation, stabilisation

## 1  Introduction

Aggregation based algebraic multigrid schemes with piecewise constant restriction and prolongation operators result in an acceptable operator complexity [1] and are relatively easy to implement. The main drawback when applied to scalar, elliptic PDEs is their non-optimal computational complexity [2] as the number of iterations increases with the number of unknowns in the linear system.

In an effort to improve the convergence speed without increasing the operator complexity, Notay and Vassilevski [3] proposed the so-called *k-cycle*. The k-cycle uses the aggregation AMG as preconditioner for a flexible Krylov method to compute the coarse grid correction on each level.

The modification considered in this contribution relies on residual minimisation as the guiding principle for its subspace projection steps. By construction, the AMG solver is hence applicable to both, symmetric and mildly non-symmetric matrices arising from scalar elliptic PDEs. The remainder of this extended abstract contains the description of the stabilised algorithm in section 2, results of numerical experiments in section 3, and some short conclusions in section 4.

## 2  Stabilised aggregation AMG

To introduce the notation, let

$$A_f u_f = b_f \ \text{ with } A_f \in I\!R^{N_f \times N_f}, \ \ u_f, b_f \in I\!R^{N_f}$$

denote the linear system to be solved. We assume that the matrix $A_f$ is symmetric and positive definite. In the straightforward aggregation AMG, the prolongation operator $P : IR^{N_c} \to IR^{N_f}$ is piecewise constant, the restriction operator $R$ is chosen to be the transpose of the prolongation, and the coarse level system matrix $A_c \in IR^{N_c \times N_c}$ is given by the Galerkin product $A_c := RA_fP = P^TA_fP$. Algorithm 1 describes one iteration of the two-level cycle.

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**Algorithm 1** Basic two-level cycle

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Input: approximation $u^{(k)}$
Smooth $u^{(k)}$ in $A_f\,u^{(k)} = b_f$
$r_u = b_f - A_f\,u^{(k)}$
Coarse level correction:
    $\omega = PA_c^{-1}P^T\,r_u$
       Smooth $\omega$ in $A_f\,\omega = r_u$
$u^{(k+1)} = u^{(k)} + \omega$
Output: $u^{(k+1)}$

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In [4], Brandt mentions different strategies to improve the robustness of multilevel schemes in general. One of these options is the *recombination of iterants*, which aims to construct a better approximation $u^{(k)}$ at iteration $k$ through a linear combination of $m$ previous approximations $u^{(k-1)}, \ldots, u^{(k-m)}$. Our stabilisation approach includes a variant of his suggestion, in that it projects the residual after the pre-smoothing at iteration $k$ onto the vector $A_f\,\omega^{(k-1)}$, where $\omega^{(k-1)}$ is the correction at the previous iteration. In addition, two more projection steps and another smoothing stage are introduced after the coarse level correction and more than one coarse level correction per iteration is allowed. The numerical experiments in the next section show that already one additional correction step, which effectively replaces the V-cycle by a W-cycle, results in a marked improvement. Algorithm 2 shows the stabilised AMG iteration for a two-level scheme, the extension to several levels is done by recursion as usual.

## 3 Numerical experiments

We compare the basic aggregation AMG scheme to the stabilised one for the standard test case, the Laplace operator with homogeneous Dirichlet boundary conditions on the unit interval in 1D and the unit cube in 3D, respectively. The discretisation employs second order, centred finite differences on uniform grids. The right hand side vectors are chosen to yield the analytic solutions $u(x) = -x(x-1)$ in 1D and $u(x, y, z) = -x(x-1)y(y-1)z(z-1)$ in 3D. These analytic solutions, together with the starting vector $u^{(0)} = 0$, imply that the initial errors are smooth and dominated by low frequency components. As in [1], we employ double pairwise aggregation, which typically results in a reduction factor between 3 and 4 in the dimension of the matrix from one level to the

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**Algorithm 2** Stabilised two-level cycle

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Input: approximation $u^{(k)}$, previous correction $\omega^{(k-1)}$
Smooth $u^{(k)}$ in $A_f \, u^{(k)} = b_f$
$r_u = b_f - A_f \, u^{(k)}$
Find $\alpha \in I\!R$ such that $\|r_u - \alpha A_f \, \omega^{(k-1)}\| \le \|r_u - \beta A_f \, \omega^{(k-1)}\| \;\; \forall \beta \in I\!R$
$u^{(k)} \leftarrow u^{(k)} + \alpha \omega^{(k-1)}$
**for** $ci = 1$ to $2$ **do**
   Coarse level correction:
       $r_u = b_f - A_f \, u^{(k)}$
       $\omega = P A_c^{-1} P^T \, r_u$
       Smooth $\omega$ in $A_f \, \omega = r_u$
   Improving the correction
       $r_\omega = r_u - A_f \, \omega$
       $V_4 := span\{r_\omega, A_f \, r_\omega, A_f^2 \, r_\omega, A_f^3 \, r_\omega\}$
       Find $\nu \in V_4$ with $\|r_\omega - A_f \, \nu\|_{l_2} \le \|r_\omega - A_f \, z\|_{l_2} \forall z \in V_4$
       $\omega \leftarrow \omega + \nu$
       Smooth $\omega$ in $A_f \, \omega = r_u$
       Find $\alpha \in I\!R$ such that $\|r_u - \alpha A_f \, \omega\| \le \|r_u - \beta A_f \, \omega\| \;\; \forall \beta \in I\!R$
       $\omega \leftarrow \alpha \omega$
**end for**
$u^{(k+1)} = u^{(k)} + \omega$
Output: $u^{(k+1)}$

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next coarser one. Both schemes rely on two iterations of a symmetric Gauß-Seidel iteration as smoother and both schemes continue until the approximation satisfies the stopping criterion $\|b_f - A_f \, u^{(k)}\|_{l_2} < 10^{-7}$.

**Table 1.** Number of AMG iterations for the 1D problem. The first value in each field corresponds to the basic aggregation scheme (V-cycles), the value in brackets to the stabilised algorithm (W-cycles).

|            | 2 levels | 3 levels  | 4 levels  |
|------------|----------|-----------|-----------|
| h = 1/128  | 58 (17)  | 173 (18)  | 214 (18)  |
| h = 1/256  | 63 (22)  | 231 (23)  | 433 (23)  |
| h = 1/512  | 66 (22)  | 265 (26)  | 736 (26)  |

One observes from the results in table 1 that the basic aggregation scheme suffers from two types of problems. Not only does the number of iterations increase with the problem size, it also depends strongly on the number of levels in the multilevel scheme. Both features are clearly highly undesirable. The stabilised scheme with its computationally more expensive iterations is much more robust, even if a slight dependence on the problem size seems to remain.

Surprisingly perhaps, the number of iterations for the basic aggregation scheme increases more slowly in 3D than in 1D! However, the dependence on

**Table 2.** Number of AMG iterations for the 3D problem. The first value in each field corresponds to the basic aggregation scheme (V-cycles), the value in brackets to the stabilised algorithm (W-cycles).

|          | 2 levels | 3 levels | 4 levels | 5 levels |
|----------|----------|----------|----------|----------|
| h = 1/32 | 18 (7)   | 27 (7)   | 31 (7)   | 33 (7)   |
| h = 1/64 | 21 (10)  | 39 (11)  | 50 (11)  | 62 (11)  |

the number of levels is still evident. Again, the stabilised scheme manages to suppress the dependence on the number of levels. However, further tests are required to check whether the dependence on the mesh size remains.

## 4   Conclusions

The stabilisation of the aggregation AMG algorithm outlined in section 2 is flexible in the sense that the added stages do not rely on any property of the system matrix at a given level apart from being non-singular. The numerical experiments illustrate that the stabilisation strongly reduces the h-dependence present in the basic scheme and that it is robust with respect to the number of levels employed. However, the number of operations per cycle is significantly higher in the stabilised scheme than in the basic one, which obviously leaves some scope for improvement of the approach.

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