Optimum Codebook Design and Average SEP Loss Analysis of Spatially Independent and Correlated Feedback Based MISO Systems with Rectangular QAM Constellation

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***Abstract*—In this paper we present an optimum codebook design algorithm that minimizes the loss in average symbol error probability (SEP) of a spatially correlated multiple input single output (MISO) system with finite-rate feedback under both perfect and imperfect channel estimate assumptions. Towards the goal of designing an optimum codebook that minimizes average SEP (ASEP) loss due to finite-rate channel quantization, we derive the distortion function as a first order approximation of the instantaneous SEP loss. Utilizing high resolution quantization theory and assuming perfect channel estimation at the receiver, we analyze the loss in ASEP for spatially independent and correlated finite-rate feedback transmit beamforming MISO systems with $M_1 \times M_2$-QAM constellation. We then consider the high-SNR regime and show that the loss associated with quantizing the spatially independent channels is related to the loss associated with quantizing the spatially correlated channels by a scaling constant given by the determinant of the correlation matrix. We also present simulation results that illustrate the effectiveness of the new codebook design and validate the derived analytical expressions for ASEP loss.**

Index Terms: MISO systems, transmit beamforming, codebook design, spatial correlation, finite-rate feedback, rectangular QAM.

I. INTRODUCTION

In this paper we focus on multiple input single output (MISO) systems where channel state information (CSI) is conveyed from the receiver to the transmitter through a finite-rate feedback link, a topic of much recent interest [1]-[7]. These works develop and analyze finite-rate feedback schemes under a variety of conditions. [8] discussed the importance of using right metric for codebook design. However, [8] did not provide codebook construction based on the average bit error probability (metric considered in [8]) criteria. The importance of the choice of performance metric and the effect of mismatch in the channel statistics assumptions are the main focus of this paper.

The effect of limited feedback on the ergodic capacity is a well studied concept. Average Symbol Error Probability (ASEP), another important communication system performance metric, has received much less attention. For a limited set of constellations and for independent and identically distributed (i.i.d.) fading channels it has been analyzed utilizing an approximation to the statistical distribution of the key random variable that characterizes the system performance. Specifically for spatially i.i.d. channels both [1] and [2] characterized the absolute amplitude square of the inner product between the channel direction and its quantized version as a truncated beta distribution and used it to study effect of quantization on ASEP. Similar to the capacity analysis, ASEP analysis for correlated channels using such statistical methods have not met with much success. In this paper we first design codebooks that are optimum for minimizing the average symbol error probability loss assuming perfect channel knowledge at the receiver. For this scenario, we then make use of the source coding based framework developed in [4] to analyze the ASEP loss in correlated Rayleigh fading channels with rectangular QAM constellation. The application of the theory in [4] to this problem is quite involved because of the complicated dependency of the objective function on the random variables involved as well as the nature of the constellation ($M_1 \times M_2$-QAM). The impact of the performance metric on the performance of the quantizer is highlighted by comparing the performance with past quantizer designs which utilize capacity loss as a metric. The quantizer design problem in the presence of channel estimation errors is also addressed and compared to the designs that assume perfect channel knowledge at the receiver.

The rest of this paper is organized as follows. In Section II,

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we introduce the system model. Optimum transmit beamforming vector under both perfect and imperfect channel estimate scenarios and optimum codebook design specific to minimizing the loss in average symbol error probability are developed in Section III. The average SEP loss expressions for spatially independent and correlated channels are derived for rectangular $M_1 \times M_2$-QAM modulation in Section IV. Numerical and simulation results are presented in Section V. We conclude this paper in Section VI.

Notation: Small and upper case bold letters indicate vector and matrix respectively. $E(.)$, $(.)^T$, $(.)^H$, $|.|$, and $\|.\|$ denote expectation, transpose, Hermitian, absolute value, and 2-norm respectively. $x \sim p(x)$ indicates that the random variable x is distributed as $p(x)$. $\mathbf{x} \sim \mathcal{NC}\left(\mathbf{0}, \mathbf{\Sigma}\right)$ indicates a circularly symmetric complex Gaussian random variable $\mathbf{x}$ with mean $\mathbf{0}$ and covariance $\mathbf{\Sigma}$.

II. System Model

We consider a multiple input single output system with t antennas at the transmitter and one antenna at the receiver. The wireless channel $\mathbf{h} \in \mathbb{C}^{t\times 1}$ between the transmitter and the receiver is modeled as a correlated frequency-flat Rayleigh fading channel with spatial distribution given by $\mathbf{h} \sim \mathcal{NC}\big(\mathbf{0}, \mathbf{\Sigma}_{\mathrm{h}}\big)$[1]. Let $\mathbf{w} \in \mathbb{C}^{t\times 1}$ be the unit norm beamforming vector (BV) at the transmitter. Then, the received signal is given by

$$y = \mathbf{h}^H \mathbf{w}\, s_m + \eta\,, \tag{1}$$

where $\eta \sim \mathcal{NC}\big(0, 1\big)$. For simplicity the time indices are ignored in the above equation. The transmitted two dimensional modulation symbol is denoted by s_m with $E[|s_m|^2] = \rho$. Note that ρ represents the SNR. The channel is estimated at the receiver and is partially available at the transmitter through a finite-rate feedback link of B bits per CSI update. More specifically, a quantization codebook $\mathcal{W} = \big\{\widehat{\mathbf{v}}_1, \cdots, \widehat{\mathbf{v}}_N\big\}$, composed of $N = 2^B$ unit-norm transmit BV's is assumed to be known to both the receiver and the transmitter. Based on the channel estimate, the receiver selects the best code point $\widehat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter through an error free link [1]-[7].

III. Optimum Transmit Beamforming Vector and Codebook Design

In this section, under the assumption of perfect and imperfect estimates of the channel at the receiver, we design the optimum codebook matched to the distortion function, the average SEP loss.

[1]We normalize the channel covariance matrix such that the mean of the eigenvalues equals to one, i.e. $Trace(\mathbf{\Sigma}_{\mathrm{h}}) = t$.

A. Perfect Channel Estimate at the Receiver

Assuming perfect knowledge of the channel vector $\mathbf{h}$, the optimum transmit BV, a well-known result, is given by the channel direction vector

$$\mathbf{v} = \frac{\mathbf{h}}{\|\mathbf{h}\|}\,.$$

In a low rate feedback link based system, the receiver selects the code point $\widehat{\mathbf{v}}$ that is closest to $\mathbf{v}$. Assuming no errors in the feedback link, the unit-norm vector $\widehat{\mathbf{v}}$ is employed as the BV at the transmitter. The received signal can now be written as

$$\begin{aligned} y &= \langle \mathbf{h}, \widehat{\mathbf{v}} \rangle\, s_m + \eta \\ &= \sqrt{\alpha}\, \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle\, s_m + \eta \end{aligned} \tag{2}$$

where

$$\alpha = \|\mathbf{h}\|^2$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \mathbf{y}.$$

1) Distortion Function - Average SEP of Rectangular QAM: In this subsection, we derive the *non-mean-squared distortion* function, the average SEP loss for a rectangular $M_1 \times M_2$-QAM constellation of size $M = M_1 M_2$. The transmitting symbol $s_m = s_x + j s_y$, $m = 0, 1, \ldots, M-1$, $x = 0, 1, \ldots, M_1 - 1$, $y = 0, 1, \ldots, M_2 - 1$. Here $s_x = a_x d$, and $s_y = a_y d$, where $a_x = -(M_1 - 1) + 2x$ (i.e., $a_x d$ is the in-phase M_1-PAM constellation symbol) and $a_y = -(M_2 - 1) + 2y$ (i.e., $a_y d$ is the quadrature-phase M_2-PAM constellation symbol). Average symbol error probability without channel quantization for the in-phase M_1-PAM is given by [9]

$$P_{_{PM_1}} = 2\left(1 - \frac{1}{M_1}\right)\, E\left[Q\left(\sqrt{\lambda\,\alpha}\right)\right] \tag{3}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-u^2/2)\, du\,,$$

$$\lambda = \rho\, \phi\,,$$

and

$$\phi = \frac{6}{M_1^2 + M_2^2 - 2}\,.$$

The ASEP for Quadrature M_2-PAM is given by (3), with M_1 replaced by M_2. The ASEP of $M_1 \times M_2$-QAM with perfect feedback is given by

$$P_{_{P-QAM}} = P_{_{PM_1}} + P_{_{PM_2}} - P_{_{PM_1}}\, P_{_{PM_2}}.$$

The ASEP with finite-rate channel quantization for M_1-PAM is given by,

$$P_{_{QM_1}} = 2\left(1 - \frac{1}{M_1}\right)\, E\left[Q\left(\sqrt{\lambda\,\alpha\, |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2}\right)\right]. \tag{4}$$

The ASEP for M_2-PAM with channel quantization is given by (4), with M_1 replaced by M_2. The ASEP of $M_1 \times M_2$-QAM with finite-rate quantization is given by

$$P_{Q-QAM} = P_{QM_1} + P_{QM_2} - P_{QM_1} P_{QM_2}.$$

The finite-rate quantization results in an increase in the average symbol error probability, which is given by

$$P_{Loss} = P_{Q-QAM} - P_{P-QAM}.$$

The instantaneous SEP loss due to finite-rate CSI quantization is taken to be the system distortion function $D_{\mathrm{Q}}(\mathbf{v}, \hat{\mathbf{v}}; \alpha)$, and is given by

$$\begin{aligned} D_{\mathrm{Q}}(\mathbf{v}, \widehat{\mathbf{v}}; \alpha) \quad &\triangleq \quad \left[Q\left(\sqrt{\lambda\alpha|\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle|^2}\right) - Q\left(\sqrt{\lambda\alpha}\right)\right] \cdot \\ &\left[A + C\left(Q\left(\sqrt{\lambda\alpha}\right) + Q\left(\sqrt{\lambda\alpha|\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle|^2}\right)\right)\right] \quad (5) \end{aligned}$$

where

$$A = 2\left(2 - \frac{1}{M_1} - \frac{1}{M_2}\right), \qquad (6)$$

and

$$C = -4\left(1 - \frac{1}{M_1} - \frac{1}{M_2} + \frac{1}{M_1 M_2}\right). \qquad (7)$$

Under high resolution assumption, the quantized beamforming vector $\widehat{\mathbf{v}}$ is close to $\mathbf{v}$, and the inner product $|\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle|$ is close to one. In this case, the distortion function $D_{\mathrm{Q}}(\mathbf{v}, \widehat{\mathbf{v}}; \alpha)$ can be approximated using a first order Taylor series expansion w.r.t. the random variable $|\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle|^2$. After some manipulations, the distortion function can be written as

$$\begin{aligned} D_{\mathrm{Q}}(\mathbf{v}, \widehat{\mathbf{v}}; \alpha) \quad &\approx \quad \exp\left(-\frac{\lambda\,\alpha}{2}\right) \cdot \sqrt{\frac{\lambda\,\alpha}{8\pi}} \cdot \\ &\left[A + 2C\,Q\left(\sqrt{\lambda\alpha}\right)\right]\left(1 - |\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle|^2\right). \quad (8) \end{aligned}$$

In this paper, we only consider the case with same distance d in both in-phase and quadrature-phase. The analysis can be easily extended to the case where the distances are not the same.

2) Optimum Codebook Design for Rectangular QAM with Perfect Channel Estimate : The codebook is designed to minimize the average SEP loss. The cost function for SEP loss given in (8) is different compared to the ergodic capacity loss function employed in [1]. However, the general vector quantization (VQ) framework can still be used with appropriate modification. The criteria in this case is given by

$$\begin{aligned} &\max_{\mathcal{Q}(.)} E|\langle\tilde{\alpha}\mathbf{v}, \mathcal{Q}(\mathbf{h})\rangle|^2, \\ &\mathcal{Q}(\mathbf{h}) = \hat{\mathbf{v}} \end{aligned} \qquad (9)$$

where

$$\tilde{\alpha}^2 = \exp\left(-\frac{\lambda\,\alpha}{2}\right)\sqrt{\frac{\lambda\,\alpha}{8\pi}}\left[A + 2C\,Q\left(\sqrt{\lambda\alpha}\right)\right].$$

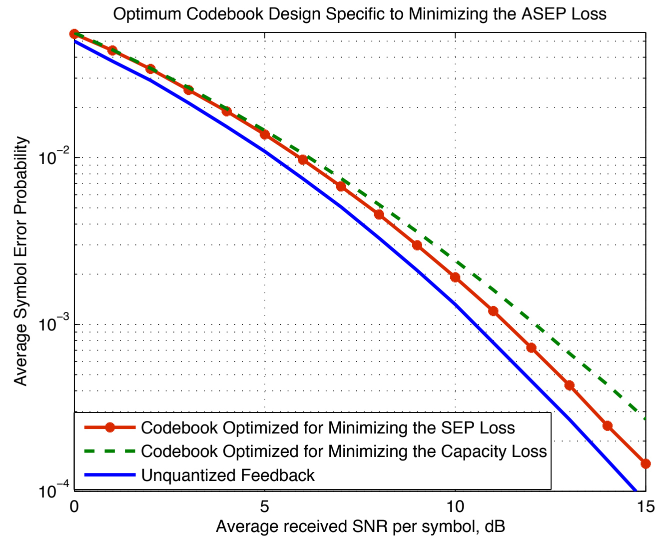

Fig. 1. Comparison between the codebook optimized to minimize the average capacity loss and ASEP loss with BPSK constellation, number of transmit antennas t=3, and the number of feedback bits B=4.

With this new design criterion, the codebook is designed by iterating the two conditions of Lloyd algorithm, the nearest neighbor-hood condition and centroid condition, until convergence. More details on the algorithm design can be found in [1]. It should be noted that similar to the case of capacity loss, because of the form of the SEP loss function, the codebook designed for spatially i.i.d. channel for the SEP distortion is also optimum for the capacity loss function. A drawback with the new codebook is that the codebook has to be designed for each operating SNR, constellation and correlation matrix.

Fig. 1 shows the ASEP resulting from using codebooks optimized for ergodic capacity loss and average SEP loss when evaluated using the ASEP metric. Note that optimum codebook implies that there is a separate codebook for each SNR point. The gains with the optimum codebook designed for SEP loss are evident in Fig. 1. As seen in Fig. 1 there is a gain of 1.2dB at an SNR of 15dB, with number of transmit antennas $t = 3$ and the feedback bits $B = 4$, between using the codebook optimized for ASEP loss and the codebook optimized for average capacity loss. The gain is increasing with SNR indicating that at medium-high SNR it is important to use the optimum codebook designed specifically for minimizing ASEP loss. The spatial correlation matrix in Fig. 1 is assumed to have a Toeplitz structure with the first row being $[1,\ 0.9,\ 0.81]$. In section.IV we quantify the loss due to quantization under i.i.d. and correlated scenarios assuming perfect channel estimation at the receiver and the optimum codebook designed in this section.

B. Erroneous Channel Estimate at the Receiver

With channel estimation errors, the optimum transmit beamforming vector is no longer given by the channel direction vector. We consider design of a codebook that takes into account the statistics of the channel estimate. The effect of estimation errors are abstracted using the modeling approach as in [12]. This modeling results in the channel estimate, $\widetilde{\mathbf{h}}$ and the actual channel, $\mathbf{h}$, being related in the following manner:

$$\mathbf{h} = \widetilde{\mathbf{h}} + \mathbf{n} \tag{10}$$

where

$$\widetilde{\mathbf{h}} \sim \mathcal{NC}\left(\mathbf{0}, \mathbf{\Sigma}_{\text{im}}\right),$$

$$\mathbf{\Sigma}_{\text{im}} = \mathbf{\Sigma}_{\text{ce}}\mathbf{\Sigma}_{\text{ee}}^{-1}\mathbf{\Sigma}_{\text{ec}}$$

and the uncorrelated error term $\mathbf{n} \sim \mathcal{NC}\left(\mathbf{0}, \mathbf{\Sigma}_{\text{n}}\right)$ where

$$\mathbf{\Sigma}_{\text{n}} = \mathbf{\Sigma}_{\text{h}} - \mathbf{\Sigma}_{\text{ce}}\mathbf{\Sigma}_{\text{ee}}^{-1}\mathbf{\Sigma}_{\text{ec}}.$$

$\mathbf{\Sigma}_{\text{h}}$ and $\mathbf{\Sigma}_{\text{ee}}$ are the autocorrelation matrices of $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ respectively, $\mathbf{\Sigma}_{\text{ce}}$ and $\mathbf{\Sigma}_{\text{ec}}$ are the cross-correlation matrices. The correlation between the two processes indicates the quality of the channel estimate. This modeling can be justified for pilot based channel estimation schemes.

With estimation errors, the received signal with an arbitrary unit norm beamforming vector $\mathbf{w}$ is given by

$$\begin{aligned} y &= \left(\widetilde{\mathbf{h}} + \mathbf{n}\right)^H \mathbf{w}\, s_m + \eta \\ &= \widetilde{\mathbf{h}}^H \mathbf{w}\, s_m + \zeta \end{aligned} \tag{11}$$

where conditioned on $\mathbf{w}$ and assuming that s_m belongs to PSK constellation,

$$\zeta \sim \mathcal{NC}\left(0, 1 + \rho\, \mathbf{w}^H \mathbf{\Sigma}_{\text{n}} \mathbf{w}\right).$$

The appearance of signal term in the noise is due to the fact that only the channel estimate $\widetilde{\mathbf{h}}$ is available at the receiver instead of actual channel $\mathbf{h}$. Selection of $\mathbf{w}_{opt}$, the optimum BV, is based on maximizing the following received SNR

$$\begin{aligned} \mathbf{w}_{opt} &= \arg \max_{\|\mathbf{w}\|=1} \left(\frac{\rho\, \mathbf{w}^H \widetilde{\mathbf{h}} \widetilde{\mathbf{h}}^H \mathbf{w}}{1 + \rho\, \mathbf{w}^H \mathbf{\Sigma}_{\text{n}} \mathbf{w}} \right) \\ &= \arg \max_{\|\mathbf{w}\|=1} \left(\frac{\mathbf{w}^H \widetilde{\mathbf{h}} \widetilde{\mathbf{h}}^H \mathbf{w}}{\mathbf{w}^H \mathbf{\Sigma}_{\text{d}} \mathbf{w}} \right) \end{aligned}$$

where

$$\mathbf{\Sigma}_{\text{d}} = \rho \mathbf{\Sigma}_{\text{n}} + \mathbf{I}.$$

The solution to the above maximization problem is given by

$$\mathbf{w}_{opt} = \frac{\mathbf{\Sigma}_{\text{d}}^{-1} \widetilde{\mathbf{h}}}{\|\mathbf{\Sigma}_{\text{d}}^{-1} \widetilde{\mathbf{h}}\|}. \tag{12}$$

With this selection of the beamforming vector, the received SNR ρ_e is given by

$$\begin{aligned} \rho_e &= \rho\, \omega, \\ \omega &= \widetilde{\mathbf{h}}^H \mathbf{\Sigma}_{\text{d}}^{-1} \widetilde{\mathbf{h}}. \end{aligned} \tag{13}$$

1) Distortion Function - Average SEP of BPSK Constellation: In this section, to illustrate how the codebook design changes because of estimation errors, we focus on BPSK constellation. The extension of codebook design for rectangular QAM constellation is relatively straightforward. The average symbol error probability with un-quantized version of optimum beamforming vector given in (12) is

$$P_{_P} = E\left[Q\left(\sqrt{2\rho\, \omega}\right)\right], \tag{14}$$

The ASEP with quantized feedback (i.e., $\mathbf{w} = \widehat{\mathbf{v}}$) is given by

$$P_{_Q} = E\left[Q\left(\sqrt{2\rho\, \tau}\right)\right] \tag{15}$$

where

$$\tau = \frac{\widehat{\mathbf{v}}^H \widetilde{\mathbf{h}} \widetilde{\mathbf{h}}^H \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^H \mathbf{\Sigma}_{\text{d}} \widehat{\mathbf{v}}}.$$

Under high resolution assumption, the instantaneous loss in SEP due to quantization can be approximated by taking the first order Taylor series expansion w.r.t. the variable τ around ω as

$$\begin{aligned} D_{\text{Q-BPSK}}(\mathbf{v}, \widehat{\mathbf{v}}; \omega) &= \left[Q\left(\sqrt{2\rho\, \tau}\right) - Q\left(\sqrt{2\rho\, \omega}\right)\right] \\ &\approx \exp\left(-\rho\omega\right) \sqrt{\frac{\rho}{\omega 4\pi}}\ \left(\omega - \tau\right). \end{aligned} \tag{16}$$

2) Optimum Codebook Design for BPSK with Estimation Errors: The design criteria is to minimize the average symbol error probability loss

$$\min_{\mathcal{Q}(.)} E\left(\frac{\widehat{\mathbf{v}}^H \mathbf{\Sigma}_{\text{v}} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^H \mathbf{\Sigma}_{\text{d}} \widehat{\mathbf{v}}} \right) \tag{17}$$

where

$$\mathbf{\Sigma}_{\text{v}} = \exp\left(-\rho\, \omega\right) / \sqrt{\omega} \left\{ \omega\, \mathbf{\Sigma}_{\text{d}} - \widetilde{\mathbf{h}} \widetilde{\mathbf{h}}^H \right\}.$$

We now briefly discuss the two conditions of Lloyd algorithm.

Nearest Neighborhood Condition: Beginning with an arbitrary set of unit vectors $\widehat{\mathbf{v}}_i,\ i = 1, \cdots, N$ forming the codebook $\mathcal{W}$, the optimum Voronoi Regions $\mathcal{R}_i,\ i = 1, \cdots, N$ are found from the following condition

$$\mathcal{R}_i = \left\{ \mathbf{v} \in \mathbb{C}^t : \frac{\widehat{\mathbf{v}}_i^H \mathbf{\Sigma}_{\text{v}} \widehat{\mathbf{v}}_i}{\widehat{\mathbf{v}}_i^H \mathbf{\Sigma}_{\text{d}} \widehat{\mathbf{v}}_i} \leq \frac{\widehat{\mathbf{v}}_j^H \mathbf{\Sigma}_{\text{v}} \widehat{\mathbf{v}}_j}{\widehat{\mathbf{v}}_j^H \mathbf{\Sigma}_{\text{d}} \widehat{\mathbf{v}}_j}, \forall j \neq i \right\}.$$

Centroid Condition: The codebook $\mathcal{W}$ is updated in this step. For a given partition $\mathcal{R}_i$ obtained from the previous step, the new set of beamforming vectors satisfy

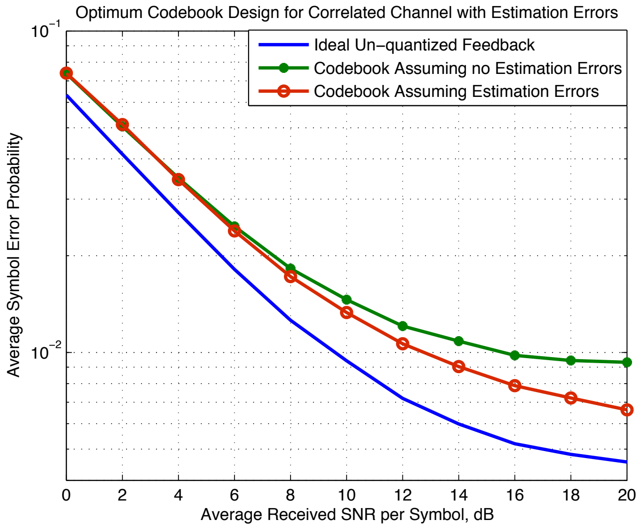

Fig. 2. Effectiveness of codebook design that takes channel estimation errors into account as compared to codebook designed specific to ASEP loss but ignoring estimation errors - BPSK constellation, number of transmit antennas t=3, and the number of feedback bits B=4.

$$\widehat{\mathbf{v}}_i = \arg \min_{\|\widehat{\mathbf{v}}\|=1, \widehat{\mathbf{v}}\in\mathcal{R}_i} E\left\{\frac{\widehat{\mathbf{v}}^H \boldsymbol{\Sigma}_{\mathrm{v}} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^H \boldsymbol{\Sigma}_{\mathrm{d}} \widehat{\mathbf{v}}}\right\}$$

$$= \arg \min_{\|\widehat{\mathbf{v}}\|=1, \widehat{\mathbf{v}}\in\mathcal{R}_i} \left\{\frac{\widehat{\mathbf{v}}^H \boldsymbol{\Sigma}_{\mathrm{m}} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^H \boldsymbol{\Sigma}_{\mathrm{d}} \widehat{\mathbf{v}}}\middle| \mathbf{v}\in\mathcal{R}_i\right\}, \; i = 1, \cdots, N$$

where

$$\boldsymbol{\Sigma}_{\mathrm{m}} = E(\boldsymbol{\Sigma}_{\mathrm{v}}).$$

In the implementation of the algorithm $\boldsymbol{\Sigma}_{\mathrm{m}}$ has to be estimated from the training unit norm vectors belonging to $\mathcal{R}_i$. The generalized eigenvalue equation for $\boldsymbol{\Sigma}_{\mathrm{d}}$ and $\boldsymbol{\Sigma}_{\mathrm{m}}$ is

$$\boldsymbol{\Sigma}_{\mathrm{m}}\mathbf{F} = \Lambda \boldsymbol{\Sigma}_{\mathrm{d}}\mathbf{F}$$

where

$$\Lambda = \mathrm{diag}\left(\lambda_1, \cdots, \lambda_t\right),$$

and

$$\mathbf{F} = \left(\mathbf{f}_1, \cdots, \mathbf{f}_t\right).$$

Assuming that $\lambda_1 > \lambda_2 \cdots > \lambda_t$, the solution to minimization function is given by

$$\widehat{\mathbf{v}}_i = \frac{\mathbf{f}_t}{\|\mathbf{f}_t\|}.$$

The above two conditions are iterated until convergence. Note that compared to the perfect channel estimation scenario, the encoding process is also different. Fig. 2 shows the effectiveness of designing the codebook (for each SNR point) taking estimation errors into account. For the results shown in Fig. 2, $\boldsymbol{\Sigma}_{\mathrm{h}}$ is simulated following the correlation model in [16]: A linear antenna array with antenna spacing of half wavelength, angle of arrival $\phi = 0^\circ$ and an uniform angular spread of $[-\pi/5, \pi/5]$. $\boldsymbol{\Sigma}_{\mathrm{im}}$ is simulated in a similar fashion with an uniform angular spread of $[-\pi/5.5, \pi/5.5]$ and the resulting correlation matrix is scaled by 0.7582. Note that the various auto and cross correlation matrices are included in $\boldsymbol{\Sigma}_{\mathrm{im}}$, so they are not specified separately. The noise correlation matrix is given by

$$\boldsymbol{\Sigma}_{\mathrm{n}} = \boldsymbol{\Sigma}_{\mathrm{h}} - \boldsymbol{\Sigma}_{\mathrm{im}}.$$

IV. Average SEP Loss Analysis

To obtain insights into the performance of quantized feedback schemes developed, we make use of the analytical results based on high resolution theory developed in [4]. Though the optimum codebook under both perfect and imperfect channel estimates were developed above, due to analytical tractability reasons, the ASEP loss analysis is carried out only under the assumption that perfect channel estimate is available at the receiver. In the last subsection, we consider the high-SNR regime for an insight into the effect of quantization on a correlated channel. We only present the end results and the details are relegated to the Appendices. For the purpose of completeness, in Appendix-I we briefly summarize the asymptotic distortion analysis of the generalized vector quantizer results that are relevant for the analysis of average SEP loss of $M_1 \times M_2$-QAM constellation. The distortion analysis results presented are really lower bounds that become more accurate as the number of feedback bits increase.

A. Distortion Analysis for spatially i.i.d. Channels

The final expression for the loss in ASEP of a spatially i.i.d. MISO system with rectangular QAM is given by

$$D_{\text{Q-iid}} = \left(\frac{\sqrt{\lambda}(t-1)A\,2^{t-1}\Gamma\left(t+\frac{1}{2}\right)}{\sqrt{\pi}\,t!\,\left(\lambda+2\right)^{\left(t+\frac{1}{2}\right)}}\right)\cdot 2^{-\frac{B}{t-1}} + \left[\frac{\lambda(t-1)\,C\,\Gamma\left(t+\frac{1}{2}\right)}{4\pi\left(1+\lambda\right)^{t+1}\,\Gamma\left(t+\frac{3}{2}\right)}\right]\cdot {}_2F_1\left(1, t+1; t+\frac{3}{2}; \frac{1}{1+\kappa}\right)\cdot 2^{-\frac{B}{t-1}}, \quad (18)$$

A and C are defined in (6) and (7), ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function [15], and

$$\kappa = \frac{\lambda}{\lambda+2}.$$

Important steps in the derivation of above equation are given in Appendix-II.

B. Distortion Analysis for Spatially Correlated Channels

Assuming that the optimum codebook is used, the ASEP loss of a spatially correlated MISO system is given by

$$D_{\text{Q-cor}} = \left[\beta_1\big(t, \lambda, \boldsymbol{\Sigma}_{\mathrm{h}}\big)\,T_{\mathcal{D}} + \beta_2\big(t, \lambda, \boldsymbol{\Sigma}_{\mathrm{h}}\big)\sqrt{\frac{\lambda}{2}}\,T_{\mathcal{E}}\right]\cdot \gamma_t^{-1}\cdot\left|\boldsymbol{\Sigma}_{\mathrm{h}}\right|^{-1}\cdot 2^{-\frac{B}{t-1}} \quad (19)$$

where

$$\beta_1\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big) = \left(\int_{\mathbf{v}:\,\mathbf{g}(\mathbf{v})=0} \left(\frac{\lambda}{2} + \mathbf{v}^{\mathsf{H}} \mathbf{\Sigma}_{\mathsf{h}}^{-1} \mathbf{v} \right)^{\frac{(1-t)(t+\frac{1}{2})}{t}} \cdot \, d\,\mathbf{v} \right)^{\frac{t}{t-1}},$$

and

$$\beta_2\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big) = \left(\int_{\mathbf{v}:\,\mathbf{g}(\mathbf{v})=0} \left(\lambda + \mathbf{v}^{\mathsf{H}} \mathbf{\Sigma}_{\mathsf{h}}^{-1} \mathbf{v} \right)^{\frac{(1-t)(t+1)}{t}} \cdot \, {}_2F_1\Big(1, t+1; t+\frac{3}{2}; \frac{1}{1+\nu}\Big)^{\frac{t-1}{t}} \, d\,\mathbf{v} \right)^{\frac{t}{t-1}}.$$

$T_{\mathcal{D}}$, $T_{\mathcal{E}}$, and ν are defined in (37), (38), and (39) respectively. The derivation details of the above equation can be found in Appendix-III.

C. Mismatched Distortion Analysis for Correlated Channels

As pointed out in section III-A2, if the codebook designed for capacity is used for average symbol error probability analysis there will be a loss due to the mismatch in codebook design. From the results in [10], the loss due to mismatch can be calculated as

$$D_{\text{Low-mm}} = 2^{-\frac{B}{t-1}} \int_{\mathbb{Q}} I^{\text{w,mm}}_{\text{c,opt}}(\mathbf{v}) \, [\lambda_{cap}(\mathbf{v})]^{\frac{1}{t-1}} \; p\big(\mathbf{v}\big) \, d\,\mathbf{v}, \tag{20}$$

$I^{\text{w,mm}}_{\text{c,opt}}$ is the constrained, weighted, and mismatched inertial profile. Due to the nature of mismatch[2], using the results in [10] and [11], it is easy to show that $I^{\text{w,mm}}_{\text{c,opt}}$ is same as $I^{\text{w}}_{\text{c, opt}}\big(\mathbf{v}\big)$ given by (36). λ_{cap}, the point density function of the capacity loss metric (clearly sub-optimal for ASEP metric), is given by (eq.(30) in [11]) and $p\big(\mathbf{v}\big)$ is given by (34). Even if the codebook is designed specific to minimizing the average SEP loss, a mismatch (e.g. by using codebook designed for a different SNR) is still possible. By selecting different system parameters (SNR, correlation matrix, constellation type) for point density function and the constrained and weighted inertial profile, (20) can also be used to analytically characterize the loss due to the usage of a wrong codebook.

D. Distortion Analysis in High-SNR Regime

The analytical expressions for SEP loss of $M_1 \times M_2$-ary QAM constellation for transmit beamforming of a MISO system are given by (18) and by (19) for spatially i.i.d. and correlated cases. The equations are lengthy and complex providing limited insight into the system behavior. In high-SNR regime it is easy to see that $\kappa \approx 1$. For spatially i.i.d. MISO fading channels, the average distortion, $D_{\text{Q-H-SNR-iid}}$, under high-SNR assumption can be simplified as

$$\begin{aligned} D_{\text{Q-H-SNR-iid}} \;=\; & \left(\frac{2^{t-1}\,(t-1)\,A\,\Gamma\left(t+\frac{1}{2}\right)}{\sqrt{\pi}\,t!\,\phi^t} \right) \cdot 2^{-\frac{B}{t-1}} \rho^{-t} + \\ & \left[\frac{(t-1)\,C\,\Gamma\left(t+\frac{1}{2}\right)}{4\pi\,\Gamma\left(t+\frac{3}{2}\right)\,\phi^t} \right] \cdot \\ & {}_2F_1\left(1, t+1; t+\frac{3}{2}; \frac{1}{2}\right) 2^{-\frac{B}{t-1}} \rho^{-t}. \end{aligned} \tag{21}$$

From the above equation it is clear that the diversity order is 't' and increasing the number of feedback bits has an exponential impact on the system distortion function, notice that this fact is true even without the high-SNR assumption. The rest of the terms in (21) depend on the number of transmitting antennas and the size of the rectangular QAM constellation. For spatially correlated channel, the functions $\beta_1\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big)$ and $\beta_2\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big)$ are difficult to evaluate. However, we can evaluate them in closed form under high SNR assumption as follows:

$$\beta_{1-H-SNR}\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big) = \lambda^{-\left(t+\frac{1}{2}\right)}\, 2^{\left(t+\frac{1}{2}\right)}\, \gamma_t^{\frac{t}{t-1}},$$

and

$$\beta_{2-H-SNR}\big(t,\lambda,\mathbf{\Sigma}_{\mathsf{h}}\big) = 2\,\lambda^{-(t+1)}\, \gamma_t^{\frac{t}{t-1}}$$

where $\beta_{1-H-SNR}$ and $\beta_{2-H-SNR}$ are high-SNR versions of β_1 and β_2. After some manipulations we arrive at an interesting simple relation between the ASEP loss associated with spatially correlated and i.i.d. channel scenarios as

$$D_{\text{Q-H-SNR-iid}} = |\mathbf{\Sigma}_{\mathsf{h}}|\, D_{\text{Q-H-SNR-cor}}. \tag{22}$$

In the correlated case the loss is a simple scaling of the loss associated with i.i.d. case, the scaling factor being the determinant of the correlation matrix. Note that this analysis is quite general in the sense that we can have an arbitrary correlation structure across the antennas. The quantization parameter B, and number of antennas, t, both appear in the exponent for the correlated scenario under general and high-SNR regimes. In the correlated scenario, the additional loss in ASEP due to quantization is independent of the constellation size. The diversity order is also not effected as a result of quantization. For both i.i.d. and correlated channels, in high-SNR regime, the ASEP without quantization can be written in terms of ASEP with quantization and ASEP loss due to quantization as follows:

$$\begin{aligned} P_{P-QAM} \;&\approx\; c_1\,\rho^{-t}\left(1 + c_2\,2^{-\frac{B}{t-1}}\right) \\ &=\; c_1\,(\rho - \Delta\rho)^{-t} \end{aligned}$$

where

$$\Delta\rho = \left[1 - \left(1 + c_2\,2^{-\frac{B}{t-1}} \right)^{-\frac{1}{t}} \right] \rho\,.$$

$\Delta\rho$ can be viewed as the SNR penalty caused by the finite-rate quantization of the CSI, c_1 and c_2 are constants. Note that exact values of c_1 and c_2 can be calculated but are

[2]The mismatched distortion function is a scaled version of the true optimal distortion function, hence the optimal Voronoi shapes remains same.

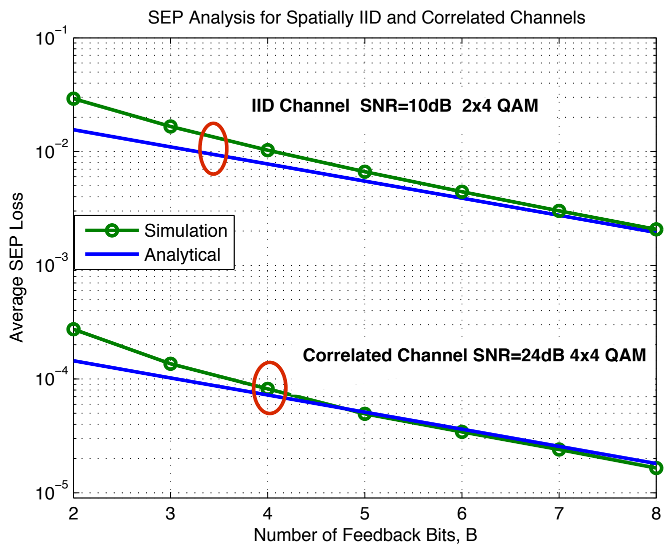

Fig. 3. ASEP loss due to finite-rate quantization with $M_1 \times M_2$-QAM constellation for both spatially i.i.d. and spatially correlated channels, number of transmit antennas t=3.

not relevant for the present discussion. The insights from the above equation are as follows: The system performance in terms of SEP is more sensitive to the finite-rate channel quantization in the high-SNR regime and in order to maintain the same SNR penalty due to finite-rate feedback, the quantization resolution B has to increase as the system average SNR increases. Since $\Delta\rho/\rho \ll 1$, after some manipulation, we can obtain a relation for number of feedback bits B as

$$\begin{aligned}\frac{B}{t-1} &= -\log_2\left[\left((1-\Delta\rho/\rho)^{-t}-1\right)/c_2\right]\\ &\approx -\log_2(\Delta\rho)+\log_2(\rho)-\log_2(t)+\log_2(c_2),\end{aligned}$$

which means for a fixed number of antennas t, in order to maintain a fixed SNR loss $\Delta\rho$,

$$B \approx (t-1)\log_2(\rho)+c\,.$$

V. Numerical and Simulation Results

A sample simulation in Fig. 3 plots the average SEP loss due to the finite-rate quantization of the channel direction versus feedback rate B, for a 3×1 MISO system over perfectly estimated spatially i.i.d. and correlated Rayleigh fading channels with different rectangular $M_1 \times M_2$-QAM constellations at system SNRs $\rho = 10$dB, and 24dB, respectively. Codebooks are designed by using optimal criterion, suitable for minimizing ASEP loss, as explained in section III-A2. The spatially correlated channel is simulated by the correlation model in [16]: A linear antenna array with antenna spacing of half wavelength, angle of arrival $\phi = 0°$ and uniform angular-spread in $[-30°, 30°]$.

Fig. 3 shows the analytical and simulation plots for both spatially i.i.d. and correlated channels. The analytical expression for i.i.d. is closed form, and for correlated channel the expression is closed form under high SNR assumption. The simulation and analytical results match well as the number of feedback bits increase. The distortion function we have is a first order approximation and this approximation becomes accurate as the number of feedback bits increase. Also note that the analytical expression for distortion is not optimum but a lower bound on the optimum, which becomes more tight as the number of feedback bits increases.

VI. Conclusion

We considered the problem of designing an optimum codebook that minimizes the loss in average SEP and analyzing the effect of finite-rate feedback on the ASEP of a transmit beamforming MISO system with rectangular QAM utilizing a high-resolution source coding perspective. We derived the distortion function as a first order approximation of the instantaneous SEP loss and used it to design optimum codebook under both perfect and imperfect channel estimate assumptions. Assuming perfect channel estimation at the receiver, no feedback delay and under high resolution assumptions, we provided analytical expressions for loss in ASEP due to finite-rate channel quantization for spatially independent and correlated channels. We then considered the high-SNR regime and showed that the loss associated with the spatially i.i.d. case is the loss associated with the spatially correlated case scaled by the determinant of the correlation matrix. The simulation results are in agreement with the analytical expressions. The presented framework of analysis can be extended to analyze the loss in SEP or BEP of other two dimensional linear modulation schemes.

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APPENDIX-I: HIGH RESOLUTION THEORY

It is assumed that the source variable $\mathbf{h}$ is a two-vector tuple, $(\mathbf{v}, \alpha)$, where vector $\mathbf{v} \in \mathbb{Q}$ represents the actual quantization variable of dimension $2t$ and $\alpha \in \mathbb{Z}$ is the additional side information of dimension 1. The *side information* α is available at the receiver but not at the transmitter. The encoding or the quantization process is denoted as $\widehat{\mathbf{v}} = \mathcal{Q}(\mathbf{v}, \alpha)$. The distortion introduced by a finite-rate quantizer is defined as

$$D = E\Big[D_{\mathrm{Q}}\big(\mathbf{v}, \widehat{\mathbf{v}}\,;\,\alpha\big)\Big]$$

where $D_{\mathrm{Q}}\big(\mathbf{v}, \widehat{\mathbf{v}}\,;\,\alpha\big)$ is a general *non-mean-squared distortion* function between $\mathbf{v}$ and $\widehat{\mathbf{v}}$ that is parameterized by α. It is further assumed that function D_{Q} has a continuous second order derivative $\mathbf{W}_\alpha(\mathbf{v})$, the sensitivity matrix, with the $(i,j)^{\text{th}}$ element given by

$$w_{i,j} = \frac{1}{2}\,\frac{\partial^2}{\partial\, v_i \partial\, v_j} D_{\mathrm{Q}}\big(\,\mathbf{v}\,, \widehat{\mathbf{v}}\,;\,\alpha\big)\bigg|_{\mathbf{v}=\widehat{\mathbf{v}}}. \qquad (23)$$

$\mathbf{W}_\alpha(\mathbf{v})$ represents sensitivity matrix of an unconstrained source. However, the beamforming vector has a norm constraint $\|\mathbf{v}\| = 1$, and a phase constraint $\measuredangle\langle\mathbf{v}, \widehat{\mathbf{v}}\rangle = 0$. We denote the constrained space as $\mathbf{g}(\mathbf{v}) = 0$. Since we are operating in the constrained space, the degrees of freedom in $\mathbf{v}$ reduce from $2t$ to $2t-2$. The sensitivity matrix is replaced by its constrained version $\mathbf{W}_{\mathrm{c},\,\alpha}(\mathbf{v})$ given by

$$\mathbf{W}_{\mathrm{c},\,\alpha}\big(\mathbf{v}\big) = \mathbf{V}_n^T\,\mathbf{W}_\alpha\big(\mathbf{v}\big)\,\mathbf{V}_n\,, \qquad (24)$$

where $\mathbf{V}_n \in \mathbb{R}^{2t\times 2t-2}$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space $\mathcal{N}\big(\frac{\partial}{\partial\,\mathbf{v}}\,\mathbf{g}(\mathbf{v})\big)$. Under high resolution assumption, the asymptotic distortion of the generalized finite-rate quantization system can be lower bounded by the following form

$$D_{\text{Low}} = 2^{-\frac{B}{t-1}}\left(\int_{\mathbb{Q}}\left(I^{\mathrm{w}}_{\mathrm{c,opt}}(\mathbf{v})\,p\big(\mathbf{v}\big)\right)^{\frac{t-1}{t}}\,d\,\mathbf{v}\right)^{\frac{t}{t-1}}, \quad (25)$$

where $I^{\mathrm{w}}_{\mathrm{c,opt}}\big(\mathbf{v}\big)$ is the constrained average optimal inertial profile defined as [4]

$$I^{\mathrm{w}}_{\mathrm{c,opt}}\big(\mathbf{v}\big) = \int_{\mathbb{Z}} I_{\mathrm{c,opt}}\big(\mathbf{v}\,;\,\alpha\big)\,p\big(\alpha\big|\,\mathbf{v}\big)\;d\,\alpha\,. \qquad (26)$$

The normalized inertial profile of an optimal quantizer is defined as the minimum inertia of all admissible Voronoi regions. The inertial profile of any Voronoi shape, including the constrained optimal inertial profile, $I_{\mathrm{c,opt}}\big(\mathbf{v}\,;\,\alpha\big)$, can be tightly lower bounded by that of an M-shaped hyper-ellipsoid

$$I_{\mathrm{c,opt}}\big(\mathbf{v}\,;\,\alpha\big) \gtrapprox \frac{t-1}{t}\left(\frac{\big|\mathbf{W}_{\mathrm{c},\alpha}(\mathbf{v})\big|}{\kappa^2_{2t-2}}\right)^{\frac{1}{2t-2}} \qquad (27)$$

where $|\cdot|$ represents determinant and κ_n is the volume of an n-dimensional unit sphere.

APPENDIX-II: AVERAGE SEP LOSS ANALYSIS FOR SPATIALLY I.I.D. CHANNEL

In this appendix, we make use of the asymptotic distortion bounds presented in Appendix-I and show the main steps in arriving at the loss in average symbol error probability for the $M_1 \times M_2$-QAM constellation. The relevant distortion function $D_{\mathrm{Q}}\big(\mathbf{v}, \widehat{\mathbf{v}}\,;\,\alpha\big)$ is given in (8). Due to space limitations, we only outline the important steps and present the final results.

The lower bound on asymptotic distortion given by (25), requires the computation of constrained sensitivity matrix (24), lower bound on constrained normalized inertial profile of an optimal quantizer (27) and the weighted constrained inertial profile (26). After some simplification the constrained sensitivity matrix for the distortion function of SEP loss (instantaneous) can be shown to be given by

$$\mathbf{W}_{\mathrm{c},\,\alpha}\big(\mathbf{v}\big) = \exp\left(-\frac{\lambda\,\alpha}{2}\right)\sqrt{\frac{\lambda\,\alpha}{8\pi}}\,\Big[A + 2C\,Q\left(\sqrt{\lambda\alpha}\right)\Big]\cdot \mathbf{I}_{2t-2}. \qquad (28)$$

For spatially independent and correlated channels, the optimal inertial profile is obtained by substituting (28), the constrained sensitivity matrix, into the hyper-ellipsoidal approximation given by (27). The optimal constrained inertial profile is given by

$$I_{\mathrm{c,\,opt}}\,(\mathbf{v}\,;\,\alpha) = \left(\frac{(t-1)}{t}\,\exp\left(-\frac{\lambda\,\alpha}{2}\right)\gamma_t^{-\frac{1}{t-1}}\sqrt{\frac{\lambda\,\alpha}{8\pi}}\right)\cdot \Big[A + 2C\,Q\left(\sqrt{\lambda\alpha}\right)\Big] \qquad (29)$$

where

$$\gamma_t = \frac{\pi^{t-1}}{(t-1)!}\,.$$

For spatially i.i.d. channel, $\mathbf{h} \sim \mathcal{NC}(\mathbf{0},\,\mathbf{I}_t)$, the random variable α $(\alpha = \|\mathbf{h}\|^2)$ has a pdf

$$p_\alpha(x) = p_{\alpha|\mathbf{v}}(x) = \frac{\exp(-x)\, x^{t-1}}{(t-1)!}, \qquad x \geq 0\,. \quad (30)$$

Since the channel is spatially independent α does not depend on the channel direction $\mathbf{v}$. Using (30) and (29) in (26), $I_{\text{c,opt}}\big(\mathbf{v}\,;\,\alpha\big)$, the weighted constrained inertial profile coefficient can be obtained.

After some simplification an intermediate step in the derivation, with a change in variable using

$$y = x\left(\frac{\lambda}{2}+1\right),$$

is given by

$$I^{\text{w}}_{\text{opt}}(\mathbf{v}) = D_{\mathcal{A}}\, A\, \Gamma\left(t+\frac{1}{2}\right) + D_{\mathcal{A}}\, 2C \int_0^\infty Q\left(\sqrt{\mu y}\right)\, \exp\left(-y\right)\, y^{t-\frac{1}{2}}\, dy \quad (31)$$

where

$$D_{\mathcal{A}} = \frac{\sqrt{\lambda}(t-1)\,\gamma_t^{-\frac{1}{t-1}}}{\sqrt{8\pi}\, t!\left(\frac{\lambda}{2}+1\right)^{\left(t+\frac{1}{2}\right)}},$$

and

$$\mu = \frac{2\lambda}{\lambda+2},$$

and $\Gamma(n)$ is the standard Gamma function [15]. We use

$$Q(x) = \frac{1}{\pi}\int_{\theta=0}^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \quad x \geq 0,$$

an alternative definition of Q function [9], to simplify the second term with integral in (31) and arrive at

$$\int_0^\infty Q\left(\sqrt{\mu y}\right)\, \exp\left(-y\right)\, y^{t-\frac{1}{2}}\, dy = \frac{\Gamma\left(t+\frac{1}{2}\right)}{\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\sin^2\theta}{\kappa+\sin^2\theta}\right)^{t+\frac{1}{2}} d\theta \quad (32)$$

where

$$\kappa = \frac{\mu}{2}.$$

We make use of [9, Eqn. (5.17)] to arrive at a closed form expression

$$\int_{\theta=0}^{\pi/2}\left(\frac{\sin^2\theta}{\kappa+\sin^2\theta}\right)^{t+\frac{1}{2}} d\theta = \frac{\sqrt{\kappa\pi}\;\Gamma\left(t+1\right)}{2\left(1+\kappa\right)^{t+1}\Gamma\left(t+\frac{3}{2}\right)}\, {}_2F_1\left(1, t+1; t+\frac{3}{2}; \frac{1}{1+\kappa}\right) \quad (33)$$

for the finite integral in (32). By substituting the weighted constrained inertial profile coefficient (31) and

$$p(\mathbf{v}) = \frac{1}{\gamma_t},$$

into the distortion integral (25), the ASEP loss of an i.i.d. MISO system can be shown to be given by (18).

Appendix-III: Average SEP Loss Analysis for Spatially Correlated Channel

In this appendix, loss in average symbol error probability is evaluated for the spatially correlated scenario. All the steps until the derivation of constrained normalized inertial profile (29) are same for both spatially independent and correlated channels. For correlated MISO fading channels $\mathbf{h} \sim \mathcal{NC}\big(\mathbf{0},\, \mathbf{\Sigma}_{\text{h}}\big)$ with channel correlation matrix $\mathbf{\Sigma}_{\text{h}}$ having distinct eigen-values, i.e. [3] $\lambda_{\text{h},1} > \cdots > \lambda_{\text{h},t} > 0$. The marginal pdf of $\mathbf{v}$ and conditional distribution of $\alpha|\mathbf{v}$ can be shown to be [14]:

$$p_{\,\mathbf{v}}\left(\mathbf{x}\right) = \gamma_t^{-1}\,\left|\mathbf{\Sigma}_{\text{h}}\right|^{-1}\left(\mathbf{x}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{x}\right)^{-t}, \quad (34)$$

$$p_{\alpha|\mathbf{v}}\left(x\right) = \frac{x^{t-1}\left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{v}\right)^{t}\exp\left(-x\,\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{v}\right)}{(t-1)!}. \quad (35)$$

By substituting the conditional pdf $p_{\alpha|\mathbf{v}}\left(x\right)$ given by (35) and the constrained normalized inertial profile (29) into equation (26), the average inertial profile can be obtained as

$$I^{\text{w}}_{\text{c, opt}}(\mathbf{v}) = \left(\frac{\left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{v}\right)^{t}}{\left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{v}+\frac{\lambda}{2}\right)^{t+\frac{1}{2}}}\right)\cdot \left[T_{\mathcal{D}} + \frac{T_{\mathcal{E}}\sqrt{\nu}}{\left(1+\nu\right)^{t+1}}\, {}_2F_1\left(1, t+1; t+\frac{3}{2}; \frac{1}{1+\nu}\right)\right] \quad (36)$$

where

$$T_{\mathcal{D}} = \frac{\sqrt{\lambda}(t-1)\gamma_t^{-\frac{1}{t-1}} A\,\Gamma\left(t+\frac{1}{2}\right)}{\sqrt{8\pi}\, t!}, \quad (37)$$

$$T_{\mathcal{E}} = \frac{\sqrt{\lambda}(t-1)\,\gamma_t^{-\frac{1}{t-1}}\, C\,\Gamma\left(t+\frac{1}{2}\right)}{\Gamma\left(t+\frac{3}{2}\right)\sqrt{8\pi}}, \quad (38)$$

and

$$\nu = \frac{\lambda}{\left(2\,\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{v}+\lambda\right)}. \quad (39)$$

Using (34) and $I^{\text{w}}_{\text{c, opt}}\big(\mathbf{v}\big)$ in (36), with the help of the alternative representation of Q function, the average symbol error probability loss of a spatially correlated transmit beamforing multiple input single output system is given by (19).

[3] In this paper, we assume that the channel covariance matrix $\mathbf{\Sigma}_{\text{h}}$ has distinct positive eigen-values. The result can be extended to any covariance matrix that is positive definite. If the channel covariance matrix is singular, the quantization should be carried out in a space with reduced dimension.