p-Colorings of Weaving Knots

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Abstract

We explore the p-colorability of a family of knots known as weaving knots. First, we show some general results pertaining to p-coloring any (m, n) weaving knot. We then determine the p-colorability of any $(m, 3)$ weaving knot, and designate these p-colorings into two separate types, whose characteristics we begin to explore. We also partially classify some of the p-colorings for any (m, n) weaving knot when n is even. Finally, we discuss different ways to count the number of p-colorings of a knot and then use these counting methods to calculate the number of one type of p-coloring of $(m, 3)$ weaving knots.

1 Introduction

The knot invariant known as p-colorability, accredited to Ralph Fox, has been studied in context with certain families of knots. Previous research involving p-colorability includes several methods for completely classifying the p-colorablity of the family of torus knots [3] [4]. Torus knots are an example of a family of knots where every member is the closure of a braid having a braid word of the form w^m, where w is a base word. Hence, each torus knot is defined by two parameters: the number of strands in the braid and the number of times the base word is repeated. We say that such a family of knots is generated by a base word. This paper focuses on classifying the p-colorability of weaving knots, another family of knots generated by a base word.

This paper is an exploration into the p-colorablity of weaving knots and presents several notable results. Our main results include a complete classification of the p-colorablity of all $(m, 3)$ weaving knots, and a partial classification of the p-colorablity of (m, n) weaving knots when n is even. We also exhibit several other relationships between p-colorablity and (m, n) weaving knots.

When investigating p-colorablity, a natural extension is to count the number of p-colorings, another knot invariant. A result concerning the number of p-colorings of torus knots by finite Alexander quandles appears in a recent paper by Asami and Kuga [2]. A special case of this result, using only elementary techniques, appears in [4]. The number of fundamentally different p-colorings or p-coloring classes has also been considered in relationship to pretzel knots [5]. We use two different techniques to count the number of one type of p-coloring of $(m, 3)$ weaving knots.

The following is a brief outline of the organization of this paper.

In Section 2 we present basic definitions of knots, invariants, p-colorability, determinants, crossing matrices, p-nullity, and braids. Even though p-colorability can be defined for any integer $p \geq 2$, in this paper we will only address cases where p is an odd prime.

In Section 3 we begin our exploration of weaving knots. We present and prove some basic properties of all weaving knots in relationship to p-colorability.

Section 4 contains a complete classification of all $(m, 3)$ weaving knots, our first main result. We accomplish this by calculating the determinant and hence, the p-colorability of any such knot. We can partition these p-colorings into two types, each with distinct properties.

In Section 5 we present a partial classification of (m, n) weaving knots by showing that an (m, n) weaving knot is p-colorable for all $p|m$ when n is even. This is our second main result.

In Section 6 we discuss the different ways to count the number of p-colorings of a knot. We then use these techniques to count one type of p-coloring of $(m, 3)$ weaving knots.

2 Definitions and Background Information

We begin by presenting some basic definitions and background pertaining to knots and p-colorability. See also [1] and [7].

2.1 Knots and Invariants

A mathematical knot is a fairly intuitive concept. Imagine taking a piece of string, tying a knot in it and then attaching the ends of the string together. Assume that such an object has no thickness and you have a *mathematical knot* [1]. In more precise terminology, a *mathematical knot* is a simple, closed polygonal curve in $\mathbb{R}^3$ [7]. In order to study these objects, mathematicians often use a 2-dimensional representation of a knot called a *projection*. In this paper all knots will be represented using such 2-dimensional projections. A projection of a knot is not unique, as seen in Figure 1.

The field of knot theory is precisely what its name describes, the study of mathematical knots. One of the main goals of this field is to be able to discriminate between any two distinct knots. Given any two knot projections, we would like to be able to tell whether or not they represent the same knot. If a person is given two pieces of knotted string, a logical way to try and distinguish if they were the same knot would be to try and deform one piece of knotted string to look like the other one. Such an approach also makes sense in order to

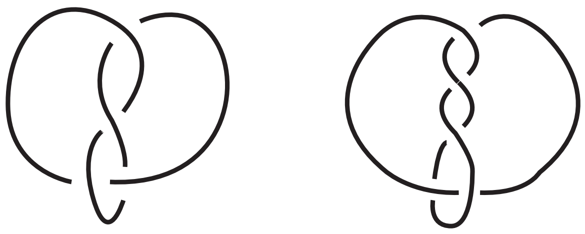

Figure 1: Two projections of the figure-eight knot.

distinguish between mathematical knots. However, there are some restrictions on how you are allowed to deform a projection of a mathematical knot.

There are two basic maneuvers we can use to deform a knot projection. The first involves deformations that do not affect the crossings in the projection. This type of deformation is called *planar isotopy*, and involves imagining the plane as being made of rubber with the projection drawn on it. You can then deform the rubber and hence the projection without affecting the crossings (see Figure 2) [1].

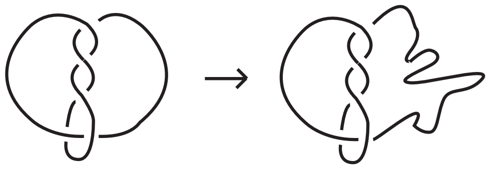

Figure 2: An example of planar isotopy.

The second way to change a projection of a knot involves three allowable alterations of the crossings in the projection called the *Reidemeister moves*. The *first Reidemeister move* allows us to add or remove a kink in the knot (see Figure 3). The *second Reidemeister move* allows us to add or remove two crossings by sliding one strand across another (see Figure 4). The *third Reidemeister move* allows us to slide one strand over or under a crossing (see Figure 5) [1].

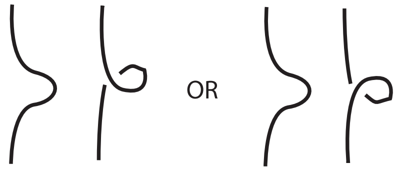

Figure 3: Type I Reidemeister move.

In 1926 the German mathematician Kurt Reidemeister showed that given two projections of the same knot, there exists some sequence of planar isotopies and Reidemeister moves that deform one projection into the other. We say that

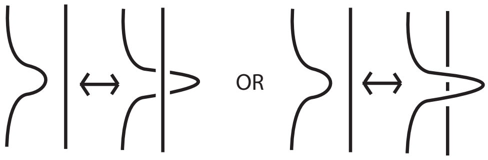

Figure 4: Type II Reidemeister move.

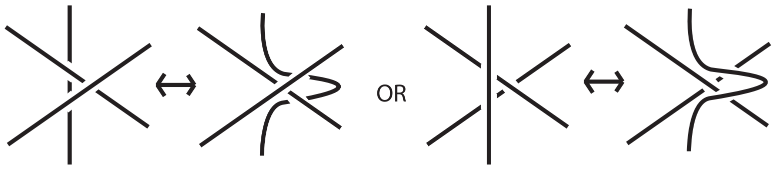

Figure 5: Type III Reidemeister move.

a property of a knot is an *invariant* if it always gives equivalent knots the same value. Similarly, we also say that a property of a knot is *invariant under the Reidemeister moves* if it does not depend upon the projection [1]. Some simple examples of knot or link invariants are linking number, bridge number, and the Jones polynomial.

2.2 p-colorablity

Definition 1. *[7] Given an odd prime number p we say that a projection of a knot K is* p-colorable *if every strand in the projection can be labeled using the numbers 0 to $p-1$, with at least 2 of the labels distinct, so that at each crossing we have*

$$2x - y - z = 0 \bmod p, \tag{1}$$

where x is the value assigned to the overstrand and y and z are the values assigned to the understrands of the crossing.

This definition requires a brief explanation on why we choose p to be a odd prime. We exclude 2 from consideration since the only knot or link that could possibly be 2-colorable is a link with at least two components. In fact, every such link is 2-colorable [8]. While we could consider all integers > 2, we choose to exclude non-primes since their p-colorability is determined by their prime divisors.

Proposition 1. *If a projection of a knot K is p-colorable then every projection of K is p-colorable.*

Proposition 1 is equivalent to saying that p-colorablity is invariant under the Reidemeister moves, and hence is a knot invariant. In fact, the well known knot invariant of tricolorability is actually a special case of p-colorability with $p = 3$.

While p-colorability is far from a complete knot invariant, it is indeed an invariant and does distinguish between some knots. It also provides a platform for defining other knot invariants, like the number of p-colorings.

2.3 Matrices, determinants, and p-nullity

A *circulant matrix* is a square matrix where the entries of each row are identical to the entries of the previous row, except that they have been shifted to the right one position and wrapped around [6]. Equation 2 is an example of such a matrix.

$$C = \mathrm{Circ}(c_1, c_2, \cdots, c_n) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}. \tag{2}$$

A p-coloring of a projection P of a knot K determines a linear equation at each crossing as indicated in Equation 1. The matrix defined by the set of linear equations for all crossings in P is called the *(major) crossing matrix* of P. This matrix can be constructed by labeling all crossings and strands in P with distinct labels. Since the number of crossings and strands is always equal we can then construct a square matrix with a row for each crossing and column for each strand. For each row j place -2 in the (i, j) position where the i^{th} column represents the overstrand of the j^{th} crossing. Also place a 1 in the (k, j) and (l, j) positions where the k^{th} and l^{th} columns represent the understrands of the j^{th} crossing. All other entries in the matrix are 0 [7].

A *minor crossing matrix* for P is the matrix obtained by deleting any row and column from the crossing matrix (or major crossing matrix) of P. The absolute value of the determinant of any minor crossing matrix for any projection of K is called the *determinant of* K [7].

Proposition 2. *Suppose p is a prime number. A knot K is p-colorable if and only if p divides* $\det(K)$*.*

Proof. A system of linear equations has a solution if and only if the determinant of the corresponding matrix is 0. So, when working $\bmod p$, a solution exists if the determinant is equal to $0 \bmod p$. Hence, a knot K is p-colorable if and only if p-divides $\det(K)$. □

Definition 2. *The* p-nullity *of a knot K is the dimension of the null space of any associated crossing matrix mod p.*

The p-nullity of a knot K is well defined since it is independent of projection and choice of labeling scheme [7].

2.4 Braids

A *braid* is a set of n strings which can be interpreted as attached to a horizontal bar at the top and the bottom. Each string in a braid must always head downwards; in other words, each string will intersect a horizontal plane exactly once. By connecting each of the strands on the top bar with the corresponding strands on the bottom bar we obtain a knot or link. The knot or link obtained in this manner is referred to as the *closure* of the braid. In 1923 J.W. Alexander showed that every knot is the closure of some braid [1]. When a braid represents the knot K created by its closure, it is called the *braid representation* of K [1].

Definition 3. *Intersect a braid B with a horizontal line l such that no strand in the braid intersects l at the same place as any other strand. At the height of l we say that the i^{th}* strand of B *is the i^{th} strand from the left to intersect l.*

Using Definition 3 any crossing in a braid can be described by which strands cross over and under each other at the height of the specified crossing. A *braid word* is a description of a projection of a braid that is arranged so that no two crossings occur at the same height. Every crossing where strand i crosses over strand $i+1$ is denoted as σ_1, and every crossing where strand $i+1$ crosses over strand i is denoted as σ_i^{-1}. The concatenation of these symbols as we travel downward from the top crossing of a braid B is the braid word for B. For example, the braid representation of the knot 6_1 in Figure 6 has braid word $\sigma_3\sigma_3\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}$.

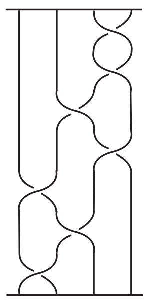

Figure 6: The braid representation of the 6_1 knot.

We say that a knot is *generated by a base word* if the braid word for any member can be written as w^m where w is called a *base word*. Each time the base word is repeated in the braid representation is called a *cycle* through the base word. For example, the (m,n) torus knot is the closure of the n-strand braid word

$$(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1})^m.$$

So, in this case we see that every (m,n) torus knots is generated by a base word, with that base word being $(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1})$. Figure 7 provides an example of what the braid representation of a torus knot looks like.

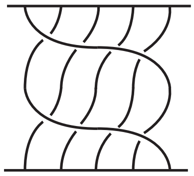

Figure 7: The braid representation of the $(5, 2)$ torus knot.

3 Some Observations about Weaving Knots

In this section we describe the family of weaving knots and some basic p-colorability properties it exhibits.

3.1 Weaving knots

Weaving knots can be described using two parameters, so we will henceforth refer to any weaving knot as an (m, n) weaving knot. We define an (m, n) *weaving knot* to be the knot K that has a braid representation of the form

$$(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\dots\sigma_{n-1}^{\pm 1})^m$$

where the final term of the base word is

$$\begin{cases} \sigma_{n-1}, & \text{if } n \text{ is even} \\ \sigma_{n-1}^{-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, weaving knots are generated by a base word. Figure 8 provides some examples of what braid representations of weaving knots look like. The two parameters for an (m, n) weaving knot translate directly to its braid representation where m is the number of cycles through the base word, and n is the number of strands in the braid.

For purposes of notation, in this paper when we refer to a knot as $W_{m,n}$ we are indicating that we are discussing the (m, n) weaving knot.

3.2 p-coloring properties of $W_{m,n}$

In this subsection we will explore some general properties of p-colorings of all (m, n) weaving knots.

Theorem 1. *For any odd prime p and any $n \in \mathbb{Z}^+$, there exists some $m \in \mathbb{Z}^+$ such that $W_{m,n}$ is p-colorable.*

Before we begin our proof, we need to first add to our framework of definitions and properties.

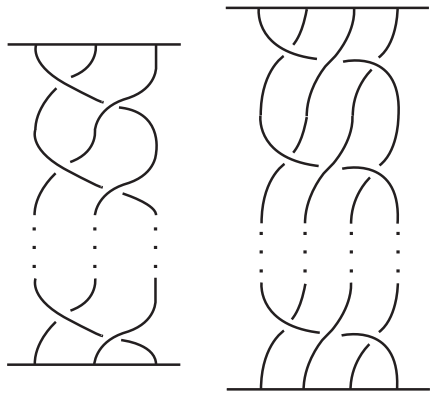

Figure 8: Braid representations of the $(m,3)$ and $(m,4)$ weaving knots.

Definition 4. *Let K be a knot that is generated by a base word. Given a p-colored braid representation of K, the j^{th}* color array *of K is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color of the i^{th} strand (see Definition 3) of the braid representation of K after j cycles of the base word have been completed.*

In order to capture the idea that the top and bottom strands in a braid representation are continuous we require the following stipulation. In addition to the standard rules of p-colorability, a coloring of a braid representation of a knot K is a valid p-coloring if and only if its initial color array is exactly the same as its final color array.

Observation 1. *A p-coloring of any knot K is entirely determined by the initial color array of a braid representation for K.*

This observation can be justified by examining the crossings in a braid representation of a knot where the crossings are arranged to occur at different heights. Once the braid is given an initial coloring array the overstrand and the incoming understrand of the top crossing must be colored. Using the p-colorablity equation, Equation 1, this forces the coloring of the outgoing understrand. Now, for the next crossing down in the braid the overstrand and the incoming understrand will be colored and hence the coloring of the outgoing understrand is forced by Equation 1. This process continues all the way down the braid. After the last crossing is passed, all strands will be colored. If the induced coloring is a valid p-coloring, then it is clear to see that it was entirely determined by the initial color array.

We now proceed with the proof of Theorem 1.

Proof. Let p be any odd prime and n any positive integer. There exist a finite number of permutations of n of these p elements. So, there are a finite number, p^n to be exact, of possible distinct coloring arrays (each having n elements) that can arise when attempting to p-color a knot with our given p. So, let's take a knot $W_{p^n,n}$ and give it an initial color array of $(c_1, c_2, \ldots, c_n)$ where each $c_i \in \mathbb{Z}_p$ and at least two labels are distinct. This coloring will induce p^n more coloring arrays, one after each cycle. Since we have a total of $p^n + 1$ coloring arrays and p^n possible distinct coloring arrays, for some integers r and s, with $0 \geq r, s \geq p^n$ and $r < s$, the r^{th} color array equal to $(c'_1, c'_2, \ldots, c'_n)$ must equal the s^{th} color array.

We now want to show that $W_{s-r,n}$ is p-colorable with initial color array equal to $(c'_1, c'_2, \ldots, c'_n)$. To do so, we need only to show that 2 distinct labels are used in the p-coloring induced by this initial coloring array. Seeking a contradiction, let's assume not, so $c'_1 = c'_2 = \cdots = c'_n$. By Equation 1, this implies that the $(r-1)^{st}$ color array of $W_{p^n,n}$ must equal $(c'_1, c'_2, \ldots, c'_n)$. By the same argument, the $(r-2)^{nd}$ color array of $W_{p^n,n}$ must also equal $(c'_1, c'_2, \ldots, c'_n)$. Continuing with this line of reasoning, we get that the initial color array for $W_{p^n,n}$ is $(c'_1, c'_2, \ldots, c'_n)$. However, this is a contradiction to our original decision that the initial color array have at least two distinct labels. Hence, the knot $W_{s-r,n}$ is p-colorable with initial color array $(c'_1, c'_2, \ldots, c'_n)$.

□

Theorem 2. *Given a knot $W_{m,n}$, if q divides m and $W_{q,n}$ is p-colorable for a given prime p, then $W_{m,n}$ is p-colorable.*

Proof. Since q divides m, there exists $r \in \mathbb{Z}^+$ such that $m = qr$. Recall that the braid word for $W_{m,n}$ is

$$\begin{aligned}(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\ldots\sigma_{n-1}^{\pm1})^m &= (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\ldots\sigma_{n-1}^{\pm1})^{qr} \\ &= ((\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\ldots\sigma_{n-1}^{\pm1})^q)^r\end{aligned}$$

which is the braid word for $W_{q,n}$ to the r^{th} power. Since $W_{q,n}$ is p-colorable there exists some initial coloring array $(c_1, c_2, \ldots, c_n)$ that induces a valid p-coloring of $W_{q,n}$. So, give $W_{m,n}$ the initial coloring array $(c_1, c_2, \ldots, c_n)$ and induce the coloring of all strands. After q cycles $W_{m,n}$ will have q^{th} color array equal to the initial color array since $W_{q,n}$ is p-colorable. In fact this color array will reappear every q cycles due to the same reasoning. Hence, for all $s \in \mathbb{Z}^+$ the qs^{th} color array of $W_{m,n}$ will be $(c_1, c_2, \ldots, c_n)$. Hence, the qr^{th} color array of $W_{m,n}$ will be $(c_1, c_2, \ldots, c_n)$. So, the initial and final color arrays of $W_{m,n}$ are equal, and two distinct labels must be used since $(c_1, c_2, \ldots, c_n)$ induces a valid p-coloring of $W_{q,n}$. Therefore, $W_{m,n}$ is p-colorable. □

4 p-Colorings of $W_{m,3}$

In this section we will classify p-colorability of all knots $W_{m,3}$ and discuss some properties of these p-colorings.

4.1 Calculating p-colorability of $W_{m,3}$ for small m

We would like to be able to calculate the p-colorability of any knot $W_{m,n}$, so we begin with the case where $n = 3$. We started this process by directly calculating the p-colorablity of knots $W_{m,3}$ for small values of m shown in Table 2.

number of cycles (m)	p-colorability
2	5
3	-
4	3,5
5	11
6	5
7	29
8	3,5,7
9	19
10	5, 11
11	199
12	3,5
13	521
14	5,13,29
15	11,31
16	3,5,7,47
17	3571
18	5,17,19
19	9349
20	3,5,11,41

Table 1: The p-colorability of $(m, 3)$ weaving knots for small values of m.

When this table is first analyzed it might appear that for every prime $m > 3$ there is exactly one p for which $W_{m,3}$ is p-colorable. However, this apparent pattern is not true. When $m = 23$, $W_{23,3}$ is 461-colorable as well as 139-colorable.

In order to get a better grasp on the data and look for patterns, we wrote a computer program to calculate the p-colorability $W_{m,3}$. After running the program for about a month we were only able to calculate the p-colorability of the $W_{m,3}$ for m up to 397. However, this did provide the basis for discovering a pattern within the p-colorablity of $W_{m,3}$.

4.2 p-colorability of $W_{m,3}$ using determinants

In this subsection we calculate the p-colorability of any knot $W_{m,3}$. We do this by calculating the determinant of $W_{m,3}$ and applying Proposition 2.

Main Theorem 1. *An knot $W_{m,3}$ is p-colorable only for all odd primes p dividing*

$$-(C_{2m-2}+1)(C_{2m})+(C_{2m-1})^2$$

where $C_j=\sum_{i=1}^{j}(-1)^{i+1}f_i$ *and* f_i *is the* i^{th} *fibonacci number* ($f_1=1, f_2=1$).

In order to prove Main Theorem 1 we want to first calculate the determinant of $W_{m,3}$. To do so we will first construct a crossing matrix for $W_{m,3}$. The choice of labeling crossings and strands in Figures 9 and 10 induce the $2m\times 2m$ crossing matrix M for $W_{m,3}$ in Equation 3.

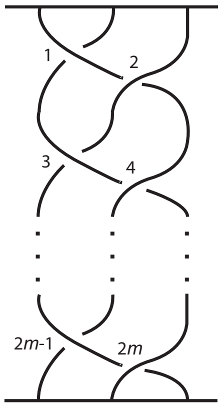

Figure 9: Labeling of crossings.

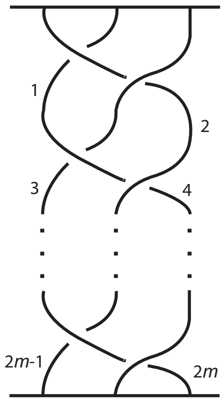

Figure 10: Labeling of strands.

$$M = Circ(1,0,\ldots,0,1,-2,0) \tag{3}$$

Write M as the block matrix

$$M=\begin{pmatrix} I & A \\ Z & R \end{pmatrix},$$

where I is $(2m-3)\times(2m-3)$ and R is 3×3.

In order to calculate the determinant of $W_{m,3}$ we will begin by applying the row operations in Equation 4 that reduce M to the matrix M' whose upper left square block I' is $(2m-3)\times(2m-3)$ identity matrix.

$$\begin{array}{rcll} R_3 & \longrightarrow & R_3+2R_1, & \\ R_k & \longrightarrow & R_k+2R_{k-2}-R_{k-3}, & \text{for } 4\le k\le 2m-1, \\ R_{2m} & \longrightarrow & R_{2m}-R_{2m-3}. & \end{array} \tag{4}$$

After these row operations M is now of the form:

$$M'=\begin{pmatrix} I' & A' \\ Z' & R' \end{pmatrix}$$

where I' is the $(2m-3) \times (2m-3)$ identity matrix and Z' is the $(2m-3) \times 3$ zero matrix. So, M' is now a block upper triangular matrix. We now want to look more closely at the specific entries in A' and R'. Let's begin by studying the entries in A'.

Prior to applying to row operations in Equation 4 to M, the matrix A was of the form:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$A_1 = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}, \text{ and } A_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

So, after the reductions in Equation 4 every row of A' will be some linear combination of A_1, A_2, and A_3. Since the first two rows of M are left unchanged by the reductions steps in Equation 4, $A'_1 = A_1$ and $A'_2 = A_2$.

Proposition 3. *For j such that $j \geq 3$, row A'_j in the matrix A' will have the following form after the reduction steps in Equation 4:*

$$A'_j = C_j A'_1 + C_{j-1} A'_2 + C_{j-2} A'_3$$

where $C_j = \sum_{i=1}^{j} (-1)^{i+1} f_i$ and f_i is the i^{th} fibonacci number ($f_1 = 1, f_2 = 1$).

In order to prove Proposition 3 we need the following identity.

$$2C_{n-2} - C_{n-3} = C_n \qquad \text{for all } n > 3. \tag{5}$$

To derive this identity we split our work into two cases.

Case 1. Assume n is even.

$$\begin{aligned} 2C_{n-2} - C_{n-3} &= 2\sum_{i=1}^{n-2} (-1)^{i+1} f_i - \sum_{i=1}^{n-3} (-1)^{i+1} f_i \\ &= \sum_{i=1}^{n-2} (-1)^{i+1} f_i - f_{n-2} \\ &= C_{n-2} - f_{n-2} \\ &= C_{n-2} + f_{n-1} - (f_{n-1} + f_{n-2}) \\ &= C_{n-2} + f_{n-1} - f_n \\ &= C_n \end{aligned}$$

Case 2. Assume n is odd.

$$\begin{aligned}
2C_{n-2} - C_{n-3} &= 2\sum_{i=1}^{n-2}(-1)^{i+1}f_i - \sum_{i=1}^{n-3}(-1)^{i+1}f_i \\
&= \sum_{i=1}^{n-2}(-1)^{i+1}f_i + f_{n-2} \\
&= C_{n-2} + f_{n-2} \\
&= C_{n-2} - f_{n-1} + (f_{n-1} + f_{n-2}) \\
&= C_{n-2} - f_{n-1} + f_n \\
&= C_n
\end{aligned}$$

We can now prove Proposition 3.

Proof. We will proceed by induction on j. The row reductions in Equation 4 prescribe that $A_3' = 2A_1 + 0A_2 + A_3$. Notice that $C_3 = 1 - 1 + 2 = 2, C_2 = 1 + 1 = 2, C_1 = 1$. Hence, $A_3' = 2A_1 + 0A_2 + A_3 = C_3A_1 + C_2A_2 + C_1A_3$, and the proposition holds.

Similarly, $A_4' = -A_1 + 2A_2$ after the reduction steps. Since, $C_4 = -1, C_3 = 2, C_2 = 2$, we see that $A_4' = C_4A_1' + C_3A_2' + C_2A_3'$ and the proposition holds.

Now, we assume that the proposition holds for rows up to $n-1$. We want to show that the proposition holds for row n. By the reduction steps in Equation 4 we see that

$$\begin{aligned}
A_n' &= 2A_{n-2}' - A_{n-3}' \\
&= 2(C_{n-2}A_1 + C_{n-3}A_2 + C_{n-4}A_3) - (C_{n-3}A_1 + C_{n-4}A_2 + C_{n-5}A_3) \\
&\text{(by inductive hypothesis)} \\
&= (2C_{n-2} - C_{n-3})A_1 + (2C_{n-3} - C_{n-4})A_2 + (2C_{n-4} - C_{n-5})A_3.
\end{aligned}$$

Using Identity 5 we see that $2C_{n-2} - C_{n-3} = C_n$, $2C_{n-3} - C_{n-4} = C_{n-1}$ and $2C_{n-4} - C_{n-5} = C_{n-2}$. Thus, we see that $A_n' = C_nA_1 + C_{n-1}A_2 + C_{n-2}A_3$. So, the proposition holds for $j = n$. Hence, by induction the proposition holds for all $j \geq 3$. ☐

Now that we know the makeup of the entries in A' we can determine the entries in R'.

Prior to the row operations in Equation 4 we had

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

Let

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ \text{and } R_3 = \begin{pmatrix} -2 & 0 & 1 \end{pmatrix}.$$

Notice that we can rewrite these rows in terms of A_1, A_2, and A_3 where

$$\begin{aligned} R_1 &= A_1 + 2A_2 + 4A_3 \\ R_2 &= A_2 + 2A_3 \\ R_3 &= -2A_1 - 4A_2 - 7A_3. \end{aligned}$$

Using these calculations, the reduction steps in Equation 4, and Proposition 3, we see that the rows of R' will have the following form:

$$\begin{aligned} R_1' &= C_{2m-2}A_1 + C_{2m-3}A_2 + C_{2m-4}A_3 + R_1 \\ &= C_{2m-2}A_1 + C_{2m-3}A_2 + C_{2m-4}A_3 + (A_1 + 2A_2 + 4A_3) \\ &= (C_{2m-2} + 1)A_1 + (C_{2m-3} + 2)A_2 + (C_{2m-4} + 4)A_3 \end{aligned}$$

$$\begin{aligned} R_2' &= C_{2m-2}A_1 + C_{2m-3}A_2 + C_{2m-4}A_3 + R_2 \\ &= C_{2m-2}A_1 + C_{2m-3}A_2 + C_{2m-4}A_3 + (1A_2 + 2A_3) \\ &= C_{2m-2}A_1 + (C_{2m-3} + 1)A_2 + (C_{2m-4} + 2)A_3 \end{aligned}$$

$$\begin{aligned} R_3' &= -(C_{2m-3}A_1 + C_{2m-4}A_2 + C_{2m-5}A_3) + R_3 \\ &= -(C_{2m-3}A_1 + C_{2m-4}A_2 + C_{2m-5}A_3) + (-2A_1 - 4A_2 - 7A_3) \\ &= (-C_{2m-3} - 2)A_1 + (-C_{2m-4} - 4)A_2 + (-C_{2m-5} - 7)A_3. \end{aligned}$$

We have now been able to write all rows in A' and R' in terms of A_1, A_2, and A_3. Using this knowledge, we want to look at the specific elements in R' in order to help us calculate the determinant of the knot $W_{m,3}$. Since row reductions do not alter the determinant of a matrix $det(W_{m,3})$ is equal to the absolute value of the determinant of any minor crossing matrix of M'.

Since we know A_1, A_2 and A_3, and all entries in R' are just linear combinations of these rows it is a simple exercise to determine the exact elements in R' as shown below.

$$R' = \begin{pmatrix} C_{2m-2}+1 & (C_{2m-2}+1)(-2)+(C_{2m-3}+2) & (C_{2m-3}+2)(-2)+(C_{2m-4}+4) \\ C_{2m-1} & (C_{2m-1})(-2)+(C_{2m-2}+1) & (C_{2m-2}+1)(-2)+(C_{2m-3}+2) \\ (-C_{2m-3}-2) & (-C_{2m-3}-2)+(-C_{2m-4}-4) & (-C_{2m-4}-4)(-2)+(-C_{m-5}-7) \end{pmatrix}$$

Using some basic algebra and Identity 5 we can simplify many entries in R' to yield the following matrix:

$$R' = \begin{pmatrix} C_{2m-2}+1 & -C_{2m} & -C_{2m-1} \\ C_{2m-1} & -2C_{2m-1}+C_{2m-2}+1 & -C_{2m} \\ -C_{2m-3}-2 & C_{2m-1} & C_{2m-2}+1 \end{pmatrix}. \qquad (6)$$

Now that we know all the entries in M' we can begin our proof of Main Theorem 1.

Proof. In order to find the determinant of $W_{m,3}$ let's form a minor matrix M'' of M' by deleting the bottom or $2m^{th}$ row of M' and the second to last or $(2m-1)^{th}$ column of M'. This leaves M'' in the following form:

$$M'' = \begin{pmatrix} I' & A'' \\ Z'' & R'' \end{pmatrix}.$$

So A'' is the $2 \times (2m-3)$ matrix equals to the matrix A' with the 2^{nd} column removed, Z'' is the $(2m-3) \times 2$ zero matrix, and R'' is of the following form:

$$R'' = \begin{pmatrix} C_{2m-2}+1 & -C_{2m-1} \\ C_{2m-1} & -C_{2m} \end{pmatrix}.$$

Since M'' is in upper block triangular form $det(W_{m,3}) = |det(M'')| = |det(I')det(R'')|$. But, since I' is the identity matrix we can explicitly determine $det(W_{m,3})$ as shown in Equation 7.

$$\begin{aligned} det(W_{m,3}) &= |det(M'')| \\ &= |det(R'')| \\ &= |(C_{2m-2}+1)(-C_{2m}) - (-C_{2m-1})(C_{2m-1})| \end{aligned} \tag{7}$$

By Proposition 2, $W_{m,3}$ is p-colorable only for all odd primes p dividing

$$\begin{aligned} &(C_{2m-2}+1)(-C_{2m}) - (-C_{2m-1})(C_{2m-1}) \\ &= -(C_{2m-2}+1)(C_{2m}) + (C_{2m-1})(C_{2m-1}) \\ &= -(C_{2m-2}+1)(C_{2m}) + (C_{2m-1})^2. \end{aligned}$$

□

4.3 Types of p-colorings of $W_{m,3}$

By Main Theorem 1 we know that any arbitrary knot $W_{m,3}$ is p-colorable if and only if p divides $-(C_{2m-2}+1)(C_{2m}) + (C_{2m-1})^2$. This only occurs when

$$(C_{2m-2}+1)(C_{2m}) \bmod p = (C_{2m-1})(C_{2m-1}) \bmod p.$$

We will split this up into two cases in order to distinguish two different types of p-colorings that arise.

Case 1. $(C_{2m-2}+1)(C_{2m}) \bmod p = (C_{2m-1})(C_{2m-1}) \bmod p = 0 \bmod p$

We say that p-colorings that arise under Case 1 are *Type I* p-colorings. The 3-coloring of $W_{4,3}$, the 29-coloring of $W_{7,3}$ and the 5-coloring of $W_{10,3}$ are examples of Type I colorings.

Case 2. $(C_{2m-2}+1)(C_{2m}) \bmod p = (C_{2m-1})(C_{2m-1}) \bmod p \neq 0 \bmod p$

We say that p-colorings that arise under Case 2 are *Type II* p-colorings. The 5-colorings of $W_{2,3}, W_{4,3}$, and $W_{8,3}$ are all examples of Type II colorings.

Let's look at some properties pertaining to these two different types of p-colorings.

Lemma 1. *If a knot $W_{m,3}$ has a Type I p-coloring then p^2 must divide $det(W_{m,3})$.*

Proof. Suppose $W_{m,3}$ has a Type I p-coloring. Since $(C_{2m-2}+1)(C_{2m}) \bmod p = (C_{2m-1})(C_{2m-1}) \bmod p = 0 \bmod p$, p must divide C_{2m-1} and either $(C_{2m-2}+1)$ or (C_{2m}). In our original crossing matrix M (see Equation 3) for $W_{m,3}$ the sum of any row or column is always 0. The row operations we performed on M will not change this fact. Recall that the only non zero elements in row $(2m-2)$ in the reduced matrix M' are $(C_{2m-2}+1), (-C_{2m})$, and $(-C_{2m-1})$, as seen in Equation 6. Hence, we know that the following must be true:

$$(C_{2m-2}+1) + (-C_{2m}) + (-C_{2m-1}) = 0.$$

But since p must divide C_{2m-1} and either $(C_{2m-2}+1)$ or (C_{2m}), then p must divide both $(C_{2m-2}+1)$ and (C_{2m}). Hence, p^2 must divide $-(C_{2m-2}+1)(C_{2m})+(C_{2m-1})^2$. Therefore, p divides $|-(C_{2m-2}+1)(C_{2m})+(C_{2m-1})^2| = det(W_{m,3})$. □

Theorem 3. *A knot $W_{m,3}$ has a Type I p-coloring if and only if all entries of the matrix R' are divisible by p, where R' is the bottom right 3×3 matrix of the crossing matrix M' for $W_{m,3}$ after the reduction steps in Equation 4.*

Proof. Recall from the proof of Lemma 1 that p divides C_{2m}, C_{2m-1}, and $C_{2m-2}+1$. So clearly every entry in the first two rows of R' seen in Equation 6 are divisible by p. Also recall that the sum of the entries in any row of R' must equal 0 since all non zero entries in these rows in the crossing matrix M' occur in R'. Hence, $(-C_{2m-3}-2) + C_{2m-1} + (C_{2m-2}+1) = 0$, which implies that p must divide $-C_{2m-3}-2$. Hence p divides all entries in the third row of R'. Therefore, p divides all entries in R'. □

Corollary 4. *A Type I p-coloring of a knot $W_{m,3}$ has p-nullity 3.*

Proof. Let p be an odd prime such that a knot $W_{m,3}$ is p-colorable with a Type I p-coloring. Let M be the associated crossing matrix as seen in Equation 3, and M' the matrix after the reduction steps in Equation 4. In order to calculate the p-nullity of $W_{m,3}$ we take all entries in M' mod p. Since we know that M' is upper block triangular with a $(2m-3)\times(2m-3)$ identity block on the diagonal, then we know the first $(2m-3)$ rows of M' are linearly independent. Now using the fact that M' is upper block triangular and Theorem 3, we know that the bottom 3 rows of M' taken mod p are all rows of zeros. Hence, $W_{m,3}$ has p-nullity 3. □

Type I p-colorings appear to be much more prevalent than Type II p-colorings. In fact, in our calculations of the p-colorablity of $W_{m,3}$ for $2 \leq m \leq 397$ we have seen very few and only very specific Type II p-colorings. However, we have yet been unable to prove the following conjecture that captures the essence of this observation.

Conjecture 1. *The only Type II p-colorings for a knot $W_{m,3}$ are the p-colorings when $p = 5$ and m is not divisible by* 10.

5 p-Colorings of (m, n) Weaving Knots

In this section we will begin to classify p-colorability of the (m, n) weaving knots when n is even.

5.1 Calculating p-colorability of $W_{m,n}$ when n is even

After calculating the p-colorablity for all knots $W_{m,3}$ it makes sense to try and expand on this to calculate the p-colorablity of all knots $W_{m,n}$. However, this task is more involved than simply being a generalization of the cases we have already calculated. Hence, we have so far only been able to calculate the p-colorablity for some limited cases.

Main Theorem 2. *Suppose $W_{m,n}$ is a weaving knot and p is an odd prime. If n is even and $p|m$, then $W_{m,n}$ is p-colorable.*

Before beginning the proof of this theorem, we need to build up a little more notation. Let K be a knot generated by a base word have m cycles through that base word. Let $(c_1, c_2, \cdots, c_n)$ be the j^{th} color array for the braid representation of K, and consider the map $\phi : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ where

$$\phi(j^{th} \text{ color array of } K) = (j+1)^{\text{st}} \text{ color array of } K$$

and the $(j+1)^{\text{st}}$ color array of K is the color array induced according to the rules of p-colorablity (see Equation 1).

We also define the map ϕ^j to be the composition of j copies of ϕ. Notice that $\phi^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)^n$ is then the map defined by

$$\phi^j(\text{ initial color array of } K) = j^{\text{th}} \text{ color array of } K.$$

For K to be p-colorable there must exist some initial color array $(c_1, c_2, \ldots, c_n)$ such that $\phi^m(c_1, c_2, \ldots, c_n) = (c_1, c_2, \ldots, c_n)$. Lastly, we define the map $\phi_i^j : (\mathbb{Z}_p)^n \to (\mathbb{Z}_p)$ to be the i^{th} coordinate map of ϕ^j. We now can proceed with our proof of Main Theorem 2.

Proof. Let $W_{m,n}$ be a weaving knot where n is even, and let p be an odd prime such that $p|m$. We are going to show that giving $W_{m,n}$ an initial color array of the form $(c, c+1, c, c+1, \ldots, c, c+1)$ where $c \in \mathbb{Z}_p$ will induce a valid p-coloring.

In this proof we want to view the strands of our braid like pieces of string that cross over and under each other. So, for now we will consider the i^{th} strand to be the entire curve that starts in the i^{th} position from the left at the top of the braid representation. At any given height in the braid representation we will refer to the horizontal placement of a strand in the braid as its strand position.

We want to start by directly calculating $\phi(c, c+1, c, c+1, \ldots, c, c+1)$. To do this we will individually calculate $\phi_i(c, c+1, c, c+1, \ldots, c, c+1)$ for $1 \leq i \leq n$ and then compile this information to determine the desired result.

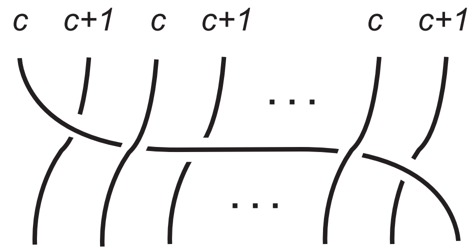

Figure 11: The j^{th} cycle of a knot $W_{m,n}$ with n even given a j^{th} color array of $(c, c+1, c, c+1, \ldots, c, c+1)$.

First, notice that $\phi_i(c, c+1, c, c+1, \ldots, c, c+1) = c$ for even i such that $1 \leq i < n$. This is clear to see by looking at Figure 11. After one cycle through the braid word the strands in the even strand positions, except for the n^{th} position are precisely the strands that are the overstrands of the crossings in the cycle through the base word. Hence, they do not change color anywhere in the cycle.

Now, we want to look at $\phi_i(c, c+1, c, c+1, \ldots, c, c+1)$ for $i = n$. Let's start by looking at the first strand in our colored cycle through the base word. This is the strand that starts out in the first strand position and ends up in the n^{th} strand position after weaving over and under the intermediate strands. Notice that in Figure 11 this strand passes under every other strand, and only those strands. Since we have an initial color array of $(c, c+1, c, c+1, \ldots, c, c+1)$ this means that this strand, initially colored with c, passes only under strands of the color c. Using Equation 1, our p-colorablity equation, we see that this strand remains colored c throughout the entire cycle since $2c - c = c$. Hence, we see that $\phi_n(c, c+1, c, c+1, \ldots, c, c+1) = c$.

Now, we can calculate $\phi_i(c, c+1, c, c+1, \ldots, c, c+1)$ for odd i such that $1 \leq i < n$. Notice by looking at Figure 11 that each of these strands is the outgoing understrand of an intersection with the overstrand being colored c (since the first strand remains colored c as seen above) and the incoming understrand being colored $c+1$. Using, the p-colorablity equation we see that for odd i

$$\begin{aligned}\phi_i(c, c+1, c, c+1, \ldots, c, c+1) &= 2(c) - (c+1) \\ &= c - 1.\end{aligned}$$

So, by compiling all of this information we see that

$$\phi(c, c+1, c, c+1, \ldots, c, c+1) = (c-1, c, c-1, c, \ldots, c-1, c). \tag{8}$$

Using Equation 8 we can now calculate the m^{th} color array of $W_{m,n}$.

$$\begin{aligned}\phi^m(c, c+1, c, c+1, \ldots, c, c+1) &= (c-m, c+1-m, c-m, c+1-m, \ldots, c-m, c+1-m) \\ &= (c, c+1, c, c+1, \ldots, c, c+1).\end{aligned}$$

We get this final equation by using the fact that $p|m$ and all elements of the color array are elements of $\mathbb{Z}_p$ and hence are taken mod p. Therefore, our initial and final color arrays are identical. Note that since our initial color array uses both c and $c+1$ two distinct labels are used. Hence, $W_{m,n}$ is p-colorable.

□

It is important to note that this is only a partial classification of the p-colorings of Wm, n where n is even. Other p-colorings do exist that are not detected in this fashion.

6 Counting the Number of p-Colorings

While p-colorability is a knot invariant, it would be nice to have a more sensitive knot invariant that has the ability to distinguish between more knots. One way to modify p-colorability to make it a better knot invariant is by counting the number of different ways to p-color a knot. However, there are several ways of managing this task. What exactly should qualify as a different coloring is not totally obvious. We will now describe two processes for counting the number of p-colorings.

6.1 Distinct p-colorings

Definition 5. *Given any projection P of a knot K we define* significant strands *of P to be a set of strands that once colored with any color, force the p-coloring of all other strands of P to form a valid p-coloring.*

Notice that the significant strands of a projection P of a knot K correspond directly to the number of free variables in a solution to the crossing matrix M associated with P. Hence, the number of significant strands for P is the p-nullity of K, since, as mentioned earlier, p-nullity is independent of labeling and projection. So, the number of significant strands for any knot K is invariant under the Reidemeister moves.

Two p-colorings of a knot K are *distinct* only when the p-colorings are not exactly identical. Hence, two p-colorings of a knot K are considered to be the *same* if one projection can be lain on top of the other without any rotation or reflection and have all colors of all strands match up. For example, in Figure 12 all of the six 3-colorings of the trefoil knot are distinct since no coloring is exactly identical to another. The p-nullity or number of significant strands of K directly determines the number of ways to count distinct p-colorings of K. This is the first of two ways to ways to count p-colorings.

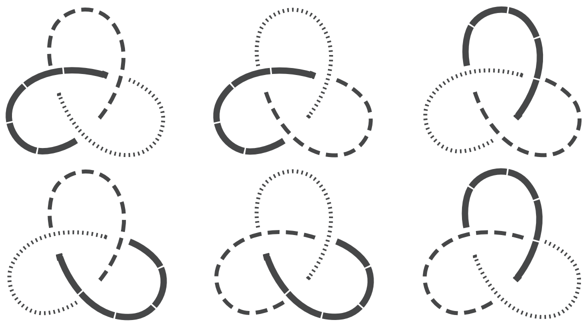

Figure 12: The six distinct 3-colorings of the trefoil knot.

Theorem 5. *A p-colorable knot K with p-nullity m has $p^m - p$ distinct p-colorings.*

Proof. Since, K has p-nullity m, then K has m significant strands. So, we can choose the color for each of the m strands. Since we have p colors this gives us p^m possible combinations of colorings, all of which induce a p-coloring. However, note that p of these choices will color all significant strands the same color, and hence induce the trivial coloring (the coloring where only one label is used). So, we subtract p from p^m to get the total number of actual p-colorings equal to $p^m - p$. □

6.2 Fundamentally different p-colorings and p-coloring classes

An alternate way of calculating the number of p-colorings is presented in [5]. In that paper two p-colorings of a knot K are defined to be *fundamentally different* if given a fixed projection of K, one p-coloring cannot be obtained from the other by permuting the colors assigned to the strands (see Figure 13). A formula for counting the number of fundamentally different p-colorings of any knot K also appears in [5]. This formula takes in the p-nullity of a knot as a parameter. Tables 2 and 3 present the number of fundamentally different p-colorings for small p-nullity according to [5].

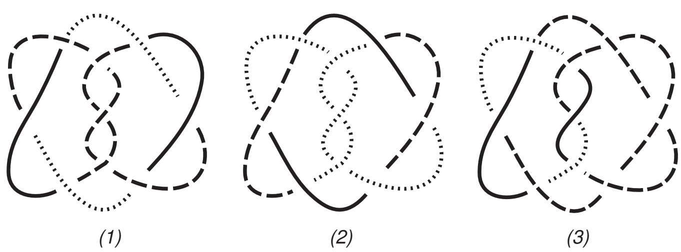

Figure 13: Colorings (1) and (2) of the 9_{35} knot are equivalent, while coloring (3) is fundamentally different from colorings (1) and (2).

p-nullity	fundamental p-colorings
1	0
2	1
3	4
4	14
5	51
6	202

Table 2: The number of fundamentally different p-colorings for a given p-nullity n (the table holds for any p provided $p \geq n$).

3-nullity	fundamental 3-colorings
1	0
2	1
3	4
4	13
5	40
6	121

Table 3: The number of fundamentally different 3-colorings for a 3-nullity n knot (note the tables differ only when $p < n$).

A redefinition of the concept of fundamentally different colorings as an equivalence relation whose equivalence classes partition the fundamentally different p-colorings of a knot K into p-coloring classes appears in [4]. This is accomplished by defining $G_p(K)$ be the set of all p-colorings for a knot K. Note that $G_p(K)$ is empty only if K is not p-colorable. In order to count the number of p-colorings in $G_p(K)$ that differ by more than just a permutation of the colors assigned to the strands we need to reformulate our original definition of p-colorability. The following definition is clearly equivalent to Definition 1.

Definition 6. *Suppose $\mathcal{S}_K$ is the set of all strands of K. A* p-coloring *of a knot K is a map*

$$\gamma : \mathcal{S}_K \to \mathbb{Z}_p$$

satisfying the condition that

$$2\gamma(s_j) - \gamma(s_i) - \gamma(s_k) = 0 \bmod p$$

for all $s_i, s_j, s_k \in \mathcal{S}_k$ at a crossing of K, where s_j is the overcrossing strand and s_i, s_k are the undercrossing strands.

For all $\gamma, \delta \in G_p(K)$, consider the relation $\sim$ defined by

$$\gamma \sim \delta \iff \gamma = \rho \circ \delta \text{ for some permutation } \rho\colon \mathbb{Z}_p \to \mathbb{Z}_p. \qquad (9)$$

Theorem 6. *[4] The relation $\sim$ defined in Equation (9) is an equivalence relation on $G_p(K)$.*

The equivalence classes defined by $\sim$ for a knot K are the *p-coloring classes of K*.

6.3 Counting the number of p-colorings for $W_{m,3}$

We would like to be able to apply these different means for counting the number of p-colorings to all p-colorable $W_{m,n}$. However, we need to know the p-nullity of all p-colorable $W_{m,n}$ in order to accomplish this goal. Therefore, we will only look at Type I p-colorings of $W_{m,3}$ since these are the only p-colorings of weaving knots whose p-nullity we know.

Observation 2. *A knot $W_{m,3}$ with a Type I p-coloring has $p^3 - p$ distinct p-colorings.*

We can validate this observation by noticing that Corollary 4 tells us that $W_{m,3}$ with a Type I p-coloring has p-nullity 3. Then, a direct application of Theorem 5 tells us that $W_{m,3}$ has $p^3 - p$ distinct p-colorings.

We can also calculate the number of p-coloring classes for Type I p-colorings of $W_{m,3}$.

Observation 3. *A knot $W_{m,3}$ with a Type I p-coloring has 4 fundamentally different p-colorings or p-coloring classes.*

The justification for this observation also uses Corollary 4 which tells us that $W_{m,3}$ with a Type I p-coloring has p-nullity 3. Tables 2 and 3 then show us that $W_{m,3}$ has precisely 4 fundamentally different p-colorings or p-coloring classes.

7 Further Questions

(1) So far the only Type II p-colorings of $W_{m,3}$ that we have seen are those where $p = 5$ and m is not divisible by 10. Can it be shown that these are the only Type II p-colorings are can another Type II p-coloring be found?

(2) In Section 5 we were only able to provide a partial classification for p-colorablity of all knots $W_{m,n}$ when n is even. It would be nice to be able to completely classify the p-colorablity of all (m, n) weaving knots.

(3) While the different methods of counting the number of p-colorings of knots do define a finer knot invariant than p-colorability alone, they appear to still leave room for improvement.

None of the mentioned methods for counting the number of p-colorings are totally invariant under planar isotopy. Notice that in Figure 12 the top left 3-coloring can be rotated 120 degree clockwise to obtain the top right 3-coloring. But since the definition of distinct p-colorings does not allow such rotation, these p-colorings are considered to be distinct. On some level it feels like these p-colorings should be considered to be equivalent.

The invariant of p-coloring classes also has a problematic nature. In Figure 13 it is unclear whether or not there is some series of planar isotopies that deform an element of one p-coloring class into an element of another p-coloring class.

An interesting and potentially useful endeavor would be to take these faults in the current ways of counting p-colorings and form a new way of counting p-colorings that accounts for them. So, overall the goal would be to form a way in which to count the number of p-colorings that is invariant under planar isotopy.

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References

[1] C. Adams, *The Knot Book: An Elementary Introduction To The Mathematical Theory of Knots*, W.H. Freeman and Company, New York (2001).

[2] S. Asami and K. Kuga, *Colorings of Torus Knots and their Twist-Spins by Alexander Quandles over Finite Fields*, Technical Reports of Mathematical Science, Chiba University, Vol. 20 (2004).

[3] S. Asami and S. Satoh, *An Infinite Family of Non-invertible Surfaces in 4-Space*, to appear in Bull. London Math. Soc., pre-print at: `http://www.math.s.chiba-u.ac.jp/~ satoh/PDF%20file/invertible.pdf`

[4] A. Breiland, L. Oesper, and L. Taalman, *p-Coloring Classes of Torus Knots*, REU Thesis, James Madison University (2004).

[5] K. Brownell, K. O'Neil, and L. Taalman, *Pretzel Knots and Colorablility*, REU Thesis, James Madison University (2003), pre-print at: `http://www.math.jmu.edu/~ taal/reu2003/brown_oneil_taal.pdf`

[6] P. Davis, *Circulant Matrices*, John Wiley & Sons, Inc., USA (1979).

[7] C. Livingston, *Knot Theory*, The Mathematical Association of America, Washington DC (1993).

[8] K. Murasugi, *Knot Theory and its Applications*, Birkchäuser, Boston (1996).