THE SPECIAL FUNCTION ش

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(Received March 12, 2005)

Abstract. The purpose of the work is to furnish a study of a discrete and special function, discovered by the author and named with the Arabian letter ش (Shin)[1], produced by a family of functions. A fundamental theorem is enunciated: it connects the rational values of this family with a natural number; to this aim two rational-value functions will be created, with the characteristic, in the field of real positive numbers, to be piecewise continuous. We state not only the existence of a separation element, but we prove that this element is just formed by one only integer constant function, the value of which is equal to 2. We point out a hypothetical, subtle connection among the special function ش , the Eulerian function Gamma and the second-order Eulerian numbers. It is finally proved that ش is completely monotonic: this characteristic is peculiar for the functions that have considerable applications in different fields of pure and applied Mathematics.

1. FROM THE CONTINUOUS TO THE DISCRETE

Let's consider the three *following* transcendental functions, determined by the letter ش :

$$\text{ش}\,[1] = \left(1 + \frac{1}{3k} \right)^{2k+1} \; ; \; \text{ش}\,[2] = \left(1 + \frac{1}{3k-1} \right)^{2k+1} \; ; \; \text{ش}\,[3] = \left(1 + \frac{1}{3k-2} \right)^{2k+1}$$

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[1] The letter ش is the thirteenth letter of the Arabian alphabet.

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and where the variable k will be restricted only in the real positive values, except some values, explained later.

The analytic study of the three functions permits to identify their progress, for the presence, in each of them, of two asymptotes: a horizontal asymptote, got by the limit for k going to infinity, represented by the horizontal straight line, of height:

$$e^{2/3} \approx 1,947734041$$

A vertical asymptote, specific for every function, got by the following limits:

$$\lim_{k \to 0^+} ش\,[1] = \infty \;;\; \lim_{k \to \frac{1}{3}^+} ش\,[2] = \infty \;;\; \lim_{k \to \frac{2}{3}^+} ش\,[3] = \infty$$

Besides by the calculus of the first derivative of each function and the study of its sign, it's possible to verify that the ش functions are decreasing in their whole field of existence, more precisely, if we consider for the variable k the whole positive real axis, we must exclude at least for the second and third function, respectively the intervals $\left(0, \frac{1}{3}\right]$ and $\left(0, \frac{2}{3}\right]$.

In fact, if we derive the ش functions we get respectively:

$$\frac{d}{dk}ش[1] = ش[1] \cdot \left[2 \cdot \log\left(1 + \frac{1}{3k}\right) - \frac{2k+1}{k(3k+1)}\right] \prec 0 \; \forall \; k \in \mathbf{R}^+$$

$$\frac{d}{dk}ش[2] = ش[2] \cdot \left[2 \cdot \log\left(1 + \frac{1}{3k-1}\right) - \frac{2k+1}{k(3k-1)}\right] \prec 0 \;\; \forall \; k \in \mathbf{R}^+ \quad \text{and} \quad k > \frac{1}{3}$$

$$\frac{d}{dk}ش[3] = ش[3] \cdot \left[2 \cdot \log\left(1 + \frac{1}{3k-2}\right) - \frac{3 \cdot (2k+1)}{(3k-1) \cdot (3k-2)}\right] \prec 0 \; \forall \; k \in \mathbf{R}^+ \quad \text{and} \quad k > \frac{2}{3}$$

In the Fig. 1 we have represented three functions where, among other things, it's evident that all of them have only one point of intersection with the horizontal straight line of height 2.
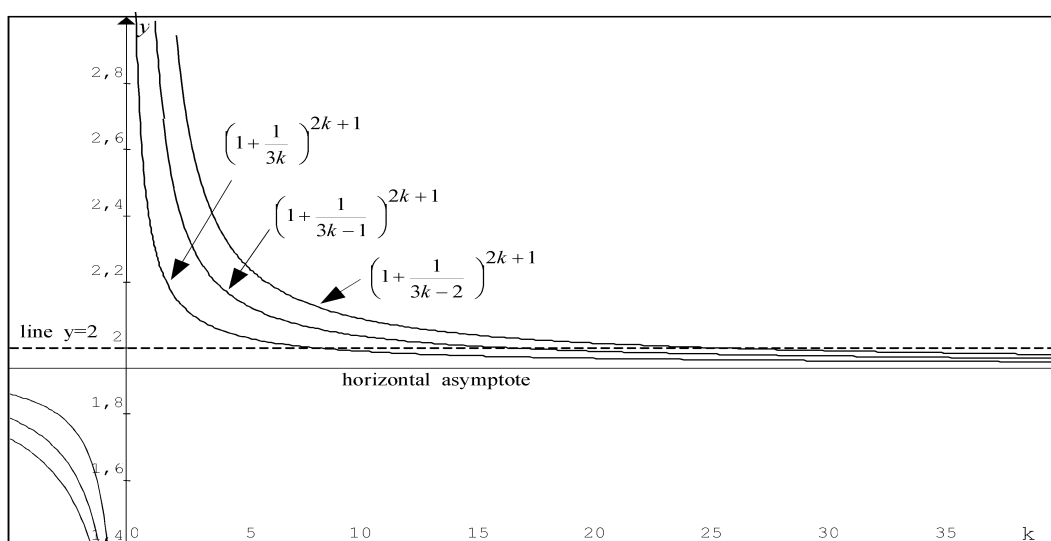
Fig. 1

That being stated, let's proceed in the passage from the continuous to the discrete, by considering for k, the only integer positive values; under these hypotheses it's possible to verify:

$$ش[1] \; \succ \; 2 \text{ for } 1 \leq k \leq 8 \qquad \text{and} \qquad ش[1] \; \prec \; 2 \text{ for } k \geq 9$$

$$ش[2] \; \succ \; 2 \text{ for } 9 \leq k \leq 16 \qquad \text{and} \qquad ش[2] \; \prec \; 2 \text{ for } k \geq 17$$

$$ش[3] \; \succ \; 2 \text{ for } 17 \leq k \leq 25 \qquad \text{and} \qquad ش[3] \; \prec \; 2 \text{ for } k \geq 26$$

and therefore, in the discrete it's possible to define some intervals[2] I_ℓ of integer values of the variable k, in order to characterize some limitations of the values that have the above stated functions; in fact it's possible to verify that for appropriate intervals $I_\ell; \ell = 2, 3$ the following are valid (see Fig. 2):

$$ش[1] \prec 2 \prec ش[2] \text{ for } 9 \leq k \leq 16 \Rightarrow I_2; \; ش[2] \prec 2 \prec ش[3] \text{ for } 17 \leq k \leq 25 \Rightarrow I_3$$

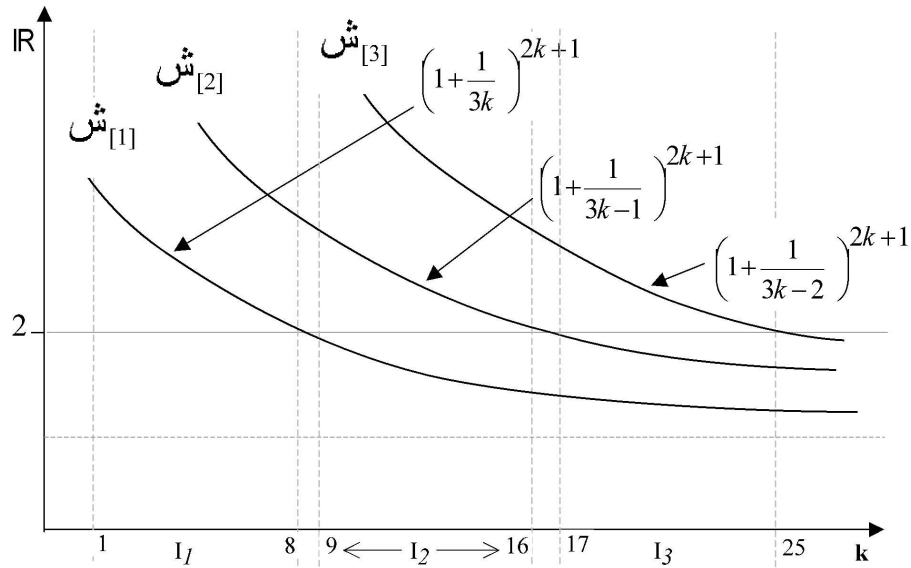[2] ℓ shows the "ordinal number" of the interval I_ℓ.

Fig. 2

In conclusion it's possible to introduce a family of ش functions, by the definition of appropriate arcs, whose separation element is the horizontal straight line of height 2.

The construction-algorithm of the above stated family is therefore describable in the following way: we begin from the first algebraic expression of the ش function, that is $\left(1+\frac{1}{3k}\right)^{2k+1}$, starting to calculate by growing values of the integer positive variable k, the corresponding rational values of ش[1] ; for the first 8 integer values of k, the function has rational values greater than 2 and it's therefore possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, each of them greater than 2.

After that we decrease of a unity the value of the denominator of the fraction inside the ش function, consequently we'll get the algebraic expression of a new function, that is $\left(1+\frac{1}{3k-1}\right)^{2k+1}$, that we have previously identified with ش[2] and that we can define as the *following* of ش[1] .

We will repeat, for it too, the same procedure and therefore we will calculate by growing values of the integer variable k, but greater than the previous 8, the corresponding rational values of ش[2] ; also in this case for 8 integer values of k , the function has rational values

greater than 2 and therefore it is possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, everyone greater than 2.

Now, for the same 8 integer values of k, utilized for ش$[2]$, it's besides possible to verify that the ش$[1]$ function has, on the contrary, rational values smaller than 2 and it's therefore possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, all smaller than 2.

If we repeat the procedure and then we decrease, as usual, the value of the denominator of the fraction inside the ش function, we can build the ش$[3]$ function, that so results the *following* of ش$[2]$.

In this case, differently from the first two, exactly for 9 integer values of k, greater than the previous 8, the function has rational values greater than 2 and therefore it's possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, all of them greater than 2.

Here, for the same 9 integer values of k, it's possible to verify that the ش$[2]$ function has, on the contrary, rational values smaller than 2, too, and it's so possible to associate to such values a bounded arc of the same function, represented, in the discrete field, by a sequence of rational numbers, each of them smaller than 2.

Consequently about what described, if we consider the second interval of 8 integer values of k, that is $k =$9,10,11,12,13,14,15,16, we can build two arcs, represented by two sequences of rational numbers: the first sequence formed by numbers greater than 2, because belonging to the arc of the ش$[2]$ function, the second sequence formed by numbers smaller than 2, because belonging to the arc of the ش$[1]$ function (see Fig. 2).

What shown is repeatable and it's possible to experiment, while the integer variable k grows, the determination of two appropriate arcs, belonging to two *following* ش functions.

To simplify the use and the control of the described algorithm a *vector* function has been defined by the software product *DERIVE*[3] *Version 6* for WINDOWS, which allows the display of 11 consecutive values of a generic ش function, by two only parameters: an

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[3] *DERIVE* is a powerful instrument of CAS (Computer Algebra System), spread by Texas Instruments.

integer value of k and a further integer value, corresponding to the value of the interval (decreased of a unity) that we wish to study.

In APPENDIX it's given, besides the macro function, which identifies the vector function, the result of a display got by its use.

2. THE FUNDAMENTAL THEOREM OF THE ش FUNCTION

Generalizing what described in the first paragraph and using the same method to build a couple of ش functions, identifiable by a precise interval I_ℓ, it's possible therefore to enunciate the following **Fundamental Theorem:**

Let's consider k, ℓ natural numbers different from zero, I_ℓ an interval of integer values of the variable k, $\Omega(I_\ell)$ an auxiliary integer function, then for each interval I_ℓ it's always possible the construction of a couple of functions with rational values, exclusively depending on k and that we denote with ش, so as the following boundary is always valid[4]:

$$\text{ش}[k, \Omega\,(I_\ell - 1)] \prec 2 \prec \text{ش}[k, \Omega\,(I_\ell)] \quad \text{with } \Omega\,(I_\ell - 1) = \Omega\,(I_\ell) - 1 \; ; \; \forall\, I_\ell \, , \; k, \ell \in \mathbf{N}$$

and for the generic ش function it's valid:

$$\text{ش}[k, \Omega\,(I_\ell)] \quad = \quad \left(1 + \frac{1}{3k - \Omega\,(I_\ell)}\right)^{2k+1} \tag{1}$$

The *auxiliary* integer function $\Omega(I_\ell)$, that really represents a growing "step function", is defined, for the intervals of 8 or 9 following values of k, in the following way:

- $\Omega\,(I_1) = 0$ for k=1,…,8

- $\Omega\,(I_2) = 1$ for k=9,…,16 ; $\Omega\,(I_3)$=2 for k=17,…,25 ; $\Omega\,(I_4)$=3 for k=26,…,34

- $\Omega\,(I_5) = 4$ for k=35,…,43 ; $\Omega\,(I_6)$ =5 for k=44,…,51 ; $\Omega\,(I_7)$=6 for k=52,…,60

- $\Omega\,(I_8) = 7$ for k=61,…,69; $\Omega\,(I_9)$ =8 for k=70,…,78 ; $\Omega\,(I_{10})$=9 for k=79,…,86

- $\Omega\,(I_{11}) = 10$ for k=87,…,95 ; $\Omega\,(I_{12})$=11 for k=96,…,104. ; etc.

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[4] The boundary can include the sign "=" if the integer variable k goes towards the infinity.

The sequence of growing values, defined for the *auxiliary* [5]function $\Omega(I_\ell)$, is therefore obtained to allow the individuation of two bounded arcs, belonging to two *following* ش functions, one above the line of height 2 and the other below it.

The set of the 11 specific intervals, by which the function $\Omega(I_\ell)$ is always positive and growing, represents a dominant characteristic of the family of ش functions.

For a better precision we write the *numerical series* that identifies the extent of such intervals, in terms of consecutive values of the variable k:

8, 9, 9, 9, 8, 9, 9, 9, 8, 9, 9 for a sum of 96 values of the variable k.

As we can see, the first interval of 8 values has been neglected: in it the function $\Omega(I_1)$ has value zero, but such interval has a peculiarity, as exactly at interval 31 the above mentioned *series* is interrupted and an interval of **8** values takes place [6] of one of **9,** giving origin again to the same sequence: the following 11 values of such function are determined by the extent of intervals, typical of the *series.*

Now this event is regularly repeated and precisely every **40** and **51** intervals: for instance the first substitutions happen at the intervals: 31, 71, 122, 162, 213, 253, 293, 344, 384, 435, 475, 526, 566, 617, 657, 697, 748, 788, 839, 879, 930, 970, 1021, 1061, 1112, 1152 and, all of them, in advantage of the 8 value intervals.

The first real effect of this phenomenon on a procedure that allows to calculate exactly the last value of k, present in a determined interval, foreseeing a constant and complete repetition of the *numerical series* of 11 intervals, is exactly quantifiable at the interval 122, where it is practically possible to verify that the last value of k inside it, results lower of a **unity**: this means that we could obtain the same result if at the interval 122, excluding the initial one, there were **120** integer intervals and one reduced of a unity, because of one only interruption.

Growing k, the substitutions immediately determinate a further but stable effect, for instance, at the interval 617 it's possible to verify that the last value of k, present in the interval is of **two** unities inferior than the computable one and, going further it's at the

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[5] The function $\Omega(I_\ell)$ is a depending function on the integer variable k.

[6] For precision it is the last 9, belonging to the second group of three consecutive 9 , that is of the eighth term of the *series.*

interval 1112, that it is possible to verify that the last value of k, present in the interval, is of **three** unities inferior than the computable one, and nothing short, for many following intervals it's possible to observe that such reduction of a unity is noticeable exactly every 495 intervals for indeed 10 times (**first case**) and every 484 intervals only once (**second case**).

In the **first case** we can get the same results by hypothesizing the constant presence of 494 integral intervals and one, reduced of a unity and this because **11** substitutions produce inside 495 intervals **38** complete *series* of 96 values of k and **11** *series,* reduced to only 61 values of k; under these conditions it's immediate to verify that we obtain 4319 values of k, that are inferior of one only unity in relation with the possible values (4320) inside 495 intervals, in case there were exclusively 45 numeric complete *series.*

In the **second case** we can get the same results by assuming the constant presence of 483 integral intervals and one, reduced of a unity and this because **11** substitutions produce inside 484 intervals **37** complete *series* of 96 values of k and **11** *series,* reduced to only 61 values of k; under these conditions it's immediate to verify that we obtain 4223 values of k, that are inferior of one only unity in relation with the possible values (4224) inside 484 intervals, in case there were exclusively 44 numeric complete *series.*

The peculiarity of the first interval of 8 values, for which the function $\Omega(I_\ell)$ is worthless, consists therefore in the fact that, as it was determined before the birth of the sequence, characteristic of the *series*, it is possible to think that at the origin, the number of integer intervals corresponds to **1.**

We specify that such characteristics, even if relevant, are minor, both in relation to the continuous and repetitive presence of the above described *series* and in relation with the largeness of the single intervals, which never descend under **8** values[7] for very great values of k.

The ratio of the number of integers inside the numeric *series* with the 11 intervals corresponds to $8,\overline{72}$ and this value diminishes in a little meaningful way while k tends to infinity, if we consider the ratio of the value of a very great k with its own belonging interval I_ℓ: a sufficiently precise value is obtainable with the following expression $\frac{96}{11} - \frac{1}{494}$, where

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[7] This means that every interval I_ℓ will never be empty.

we deduce by the denominator of the second fraction the importance of the number **494**.

Successively we give (Fig. 3) the graphs, related both to the family of ﺵ functions, or better, to the set of the arcs belonging to them, and to the auxiliary function $\Omega(I_\ell)$ and successively in particular the development of the graph of a couple of functions, characteristic of a precise interval I_ℓ (Fig. 4) and of the pointers than put into evidence the behaviour, growing k.

This last behaviour results particularly evident, by examining the various displays, produced by *DERIVE*, in relation with the first 11 intervals, typical of the *standard series*, while for the greater values of k it's necessary to outdistance in an appropriate way the intervals on which to do a comparison to have a further confirmation of such behaviour.

Now if k tends to infinity it's possible to compute the limit towards which the generic ﺵ function, that in this case represents the fusion, to infinity, of two arcs of *following* ﺵ functions, that refer themselves to a hypothetical and extreme interval I_ℓ.
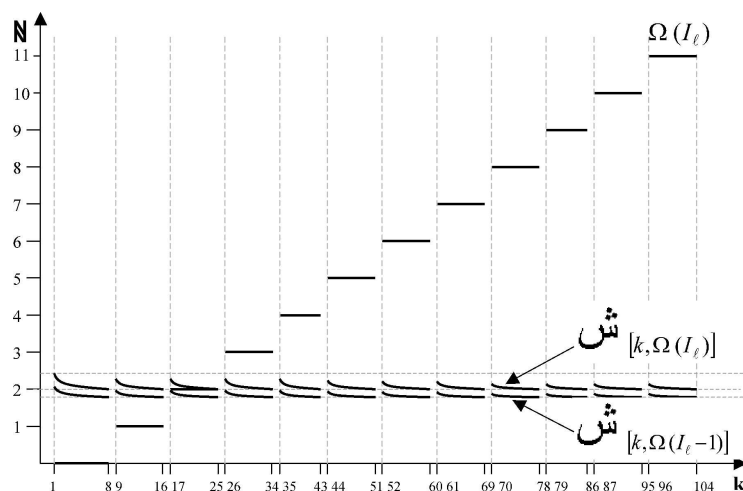
Fig. 3

Keeping in mind what noticed in the first and second case, previously described we can, first of all, calculate the two following values, by "excess" and "defect" of the ratio of a value of k, with the value of the integer function $\Omega(I_\ell)$, determined as a function of the I_ℓ interval, containing the same value of k:

$$1^{st}\text{case:}\, \frac{k}{\Omega(I_\ell)} \prec \frac{4319}{495} = 8,7\overline{25}$$

$$2^{nd}\text{case:}\frac{k}{\Omega(I_\ell)} \succ \frac{4223}{484} = 8,72520661157\overline{0247933884297}$$

These values, by considering the frequency of the two cases (10 times the first case and once the second case), permit to the approximate the ratio with:

$$\frac{k}{\Omega(I_\ell)} \cong \frac{10*8,7\overline{25} + 8,72520661157\overline{0247933884297}}{11} = 8,7252483512814091326488$$

$$= \frac{96}{11} - \frac{1}{493,9793814432989690721649} \cong \frac{96}{11} - \frac{1}{494}$$

In this way, according to the analysis of the progress of the various and *following* ش functions, we can furnish only an esteem of the ratio $\frac{k}{\Omega(I_\ell)}$.

If we want to determinate the effective value of such ratio, it's necessary to compare the values of the two quantities for sufficiently great values of k.

Keeping in mind the algorithm described at the end of paragraph 1, which puts in evidence the continuous oscillation of the rational values of the *following* ش functions, in proximity of the integer number 2, we will compare the values of the ratio $\frac{k}{\Omega(I_\ell)}$ as k grows with the value: $(3 - 2/\log 2)^{-1}$.
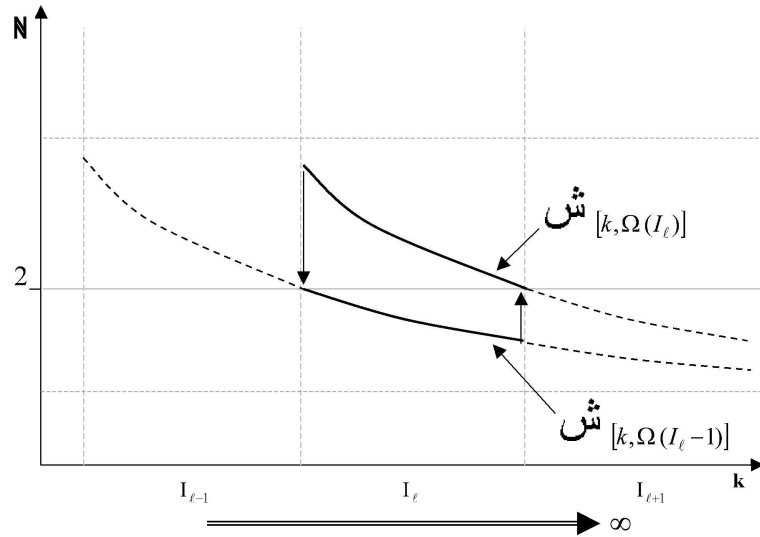

Fig. 4

More precisely, considering the values raised to the 18^{th}power ($k_1 = 10^{18}$, that is one quintillion), raised to the 33rd power ($k_2 = 10^{33}$, that is one decillion) and indeed raised to the 63rd power ($k_3 = 10^{63}$, that is one vigintillion) it's possible to observe that the real

value of the ratio $\Psi\left(k\right)=\frac{k}{\Omega\left(I_\ell\right)}$ tends to the previously shown value; in fact in the three various cases it's possible to determinate in order what follows:

$$\Rightarrow \Psi(k_1)=\frac{k_1}{\Omega(I_\ell)}-(3-2/\log 2)^{-1}\prec 5\cdot 10^{-19};\Rightarrow \Psi(k_2)\prec 5\cdot 10^{-34};\Rightarrow \Psi(k_3)\prec 5\cdot 10^{-64}$$

Definitively, going to the limit for k, which tends to infinity we can expect that:

$$\lim_{k\to\infty}\left(\frac{k}{\Omega\left(I_\ell\right)}\right)=(3-2/\log 2)^{-1}$$

This result has obviously an immediate consequence in the calculation of the following limit:

$$
\begin{aligned}
\lim_{k\to\infty}\text{ش}\left[k,\Omega\left(I_\ell\right)\right] &= \lim_{k\to\infty}\left(1+\frac{1}{3k-\Omega\left(I_\ell\right)}\right)^{2k+1}\\
&= \lim_{k\to\infty}e^{\frac{2k+1}{3k-\Omega\left(I_\ell\right)}} = e^{\lim_{k\to\infty}\frac{2k+1}{3k-k\left(3-\frac{2}{\log 2}\right)}}\\
&= e^{\log 2} = 2
\end{aligned}
$$

In conclusion, it happened that the natural number 2 (by the integer constant function Y=2) can represent the separation element of two arcs, belonging to two *following* ش functions, which therefore can be identified with two contiguous classes[8], which tend to approach indefinitely.

Even if we observe that the validity of the *fundamental* theorem is actually included in the construction algorithm[9] of the family of ش functions, keeping obviously in mind the characteristic of monotonicity (see previous paragraph and the following) of the generic ش function, by a more appropriate notation, due to Iverson, further we will give the elements to get a strict proof of the theorem in the modern sense of the term.

Let's extend the dependence of the integer step function $\Omega\left(I_l\right)$ to the real field and let's use the following definition:

$$\Omega\left(x\right)=\min\left\{k\in\mathbf{N}\colon \mathrm{S}_{k+1}\left(x\right)\geq 2\right\};\ x\in\mathbf{R}^+\ \text{and where}\ \mathrm{S}_k\left(x\right)=\left(1+\frac{1}{3x-k+1}\right)^{2x+1}$$

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[8] The contiguous classes are meant represented by two groups of rational non-integer numbers, greater and smaller than 2, separated by the rational integer number 2, which obviously doesn't belong to any of the two classes.

[9] Let's observe that the algorithm determinates for the generic ش function an inferior limit: it's established an inferior extreme which corresponds to the integer number 2.

By simple algebraic passages we have that:

$$\Omega\ (x) = \left\lceil 3x - \frac{1}{2^{\frac{1}{2x+1}} - 1} \right\rceil \tag{2}$$

where $\lceil x \rceil$ means the smallest integer, greater than x or equal to it.

So, from such a particular definition of $\Omega\ (x)$ the *fundamental* theorem immediately derives.

In fact the ش function possesses the following explicit formula:

$$S\left(x\right) = \left(1 + \frac{1}{3x - \Omega\left(x\right)}\right)^{2x+1} = \left(1 + \frac{1}{3x - \left\lceil 3x - \frac{1}{2^{\frac{1}{2x+1}} - 1} \right\rceil}\right)^{2x+1}$$

 and passing to the sequence we have:

$$S\left(n\right)\ =\ \left(1 + \frac{1}{\left\lfloor \frac{1}{2^{\frac{1}{2n+1}} - 1} \right\rfloor}\right)^{2n+1}$$

where $\lfloor x \rfloor$ shows the greatest integer smaller or equal to x.

From now on we will use the locution of " special function ش " for the discrete function ش$[k, \Omega\ (I_\ell)]$, defined by the formula (1).

Extending the field of definition of the variable k to the real positive numbers, it's possible to notice that such function, being represented by the union of continuous arcs (all above the straight line of height 2) is actually assimilable to a piecewise continuous function.

3. AN APPROACH OF THE SPECIAL FUNCTION ش WITH THE EULERIAN[10] GAMMA FUNCTION $\boldsymbol{\Gamma}$

One of the fundamental characteristics of the Eulerian *gamma* function is a certain condition of monotonicity, which is the fact that the function, which can be intended as the

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[10] Chapter II, [5].

most spontaneous extension of the factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$, out the field of the natural numbers, is logarithmically convex.

If we consider the expression (1) and we calculate its first derivative, we have:

$$\frac{d}{dk}\text{ش}[k, \Omega\left(I_\ell\right)] \;=\;$$

$$\text{ش}[k, \Omega\left(I_\ell\right)] \cdot \left[2\log\left(1 + \frac{1}{(3k - \Omega\left(I_\ell\right))}\right) - \frac{3 \cdot (2k+1)}{(3k - \Omega\left(I_\ell\right))\left(3k - \Omega\left(I_\ell\right) + 1\right)}\right] \quad (3)$$

$\forall \; k \in \mathbf{N}$ and $k > \frac{\Omega\left(I_\ell\right)}{3}$ it is always negative, therefore the function results monotonic and decreasing, and besides it's interesting to observe that, growing the k value, its absolute value diminishes.

That being stated we can also verify that the special function ش has the same characteristic of monotonicity of the Eulerian function Γ: it's sufficient in fact to verify that the second derivative of the logarithm of the same ش function is positive.

Such result is immediate, in fact for the (3), being:

$$\frac{d^2}{dk^2}\log\text{ش}[k, \Omega\left(I_\ell\right)] \;=\; \frac{d}{dk}\left\langle \;\; \frac{d}{dk}\text{ش}[k, \Omega\left(I_\ell\right)] \,/\, \text{ش}[k, \Omega\left(I_\ell\right)] \;\; \right\rangle$$

we have that:

$$\frac{d^2}{dk^2}\log\text{ش}[k, \Omega(I_\ell)] \;=\; \frac{d}{dk}\left[2\log\left(1 + \frac{1}{(3k - \Omega(I_\ell))}\right) - \frac{3 \cdot (2k+1)}{(3k - \Omega(I_\ell)) \cdot (3k - \Omega(I_\ell) + 1)}\right]$$

$$=\; \left[\frac{12 \cdot (3k \cdot \Omega\left(I_\ell\right) - \Omega\left(I_\ell\right)^2) + 6 \cdot (6k - \Omega\left(I_\ell\right)) + 9}{(3k - \Omega\left(I_\ell\right))^2 \cdot (3k - \Omega\left(I_\ell\right) + 1)^2}\right]$$

Since, for definition we have $k \succ \Omega\left(I_\ell\right), \forall\, k \in \mathbf{N}$, we have in conclusion:

$$\frac{d^2}{dk^2}\log\text{ش}[k, \Omega\left(I_\ell\right)] \;\; \succ \;\; 0$$

This demonstrates that also the special function ش is logarithmically convex.

The special function ش is actually an exponential general function; that being stated, considering its base, defined by the analysis of its behaviour to infinity, we consider the following two particular expressions:

$$\prod_{k=1}^{n}\left(1 + \frac{\log 2}{2k}\right) \quad \text{and} \quad \sum_{k=1}^{n}\left(\frac{\log 2}{2k + \log 2}\right)$$

In the previous paragraph we have practically put into evidence that for great values of k the rational non-integer term of the base of the special function ش tends ($\sim$) to:

$$\frac{1}{3k - \Omega\left(I_\ell\right)} \sim \frac{\log\ 2}{2k} \tag{4}$$

consequently, as there is the following relation $\Omega\left(I_\ell - 1\right) = \Omega\left(I_\ell\right) - 1$, we have:

$$\frac{1}{3k - \Omega\left(I_\ell - 1\right)} \sim \frac{\log\ 2}{2k + \log 2} \tag{5}$$

The value (4), if fixed for the base of the generic ش function, is very well fit to "interpolate" in the continuous field the union of the various arcs of the family of ش functions, got in the discrete field, above the straight line of height 2.

By this last value it's possible to calculate and verify[11] the following sizeable expression:

$$\prod_{k=1}^{n}\left(1 + \frac{\log 2}{2k}\right) \quad = \quad \frac{\Gamma\left(n + 1 + \frac{\log 2}{2}\right)}{\Gamma\left(n + 1\right) \cdot \Gamma\left(1 + \frac{\log 2}{2}\right)} \tag{6}$$

The value (5), on the contrary, if fixed for the base of the generic ش function, is very well fit to "interpolate" in the continuous field the union of the various arcs of the family of ش functions, got in the discrete field, below the straight line of height 2.

By it, it's possible to define the following partial sum:

$$\sum_{k=1}^{n}\frac{\log\ 2}{2k + \log 2} \quad = \quad \frac{\log\ 2}{2}\sum_{k=1}^{n}\frac{1}{k + \frac{\log 2}{2}} \tag{7}$$

From here, keeping in mind the formulae of recurrence of the logarithmic derivative of the *gamma,* function named *digamma* :

$$\psi(x + 1) = \psi(x) + \frac{1}{x}; \psi(x + n) = \psi(x) + \frac{1}{x} + \frac{1}{x + 1} + \frac{1}{x + 2} + ... + \frac{1}{x + n - 1} \text{ with } n \geq 1$$

we have, giving x the value $1 + \frac{\log 2}{2}$, the following identity:

$$\sum_{k=1}^{n}\frac{\log\ 2}{2k + \log 2} \quad = \quad \frac{\log\ 2}{2}\left[\psi\left(n + 1 + \frac{\log 2}{2}\right) - \psi\left(1 + \frac{\log 2}{2}\right)\right]$$

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[11] $\Gamma\left(x + n\right) = x \cdot \left(x + 1\right) \cdot \left(x + 2\right) \cdots \left(x + n - 1\right) \cdot \Gamma\left(x\right)$ with $n \in \mathbf{N}$ and $x \succ 0$

Now, going to the limit for $n \to \infty$, for the known properties of the *digamma* function, we have that the following series is divergent, that is:

$$\sum_{k=1}^{\infty} \frac{\log 2}{2k + \log 2} = \infty \tag{8}$$

That being stated, the importance of the result (6) must be evaluated above all according to the following identity:

$$\left(1 + \frac{\log 2}{2k}\right) = \left(1 - \frac{\log 2}{2k + \log 2}\right)^{-1}$$

From this last identity we can deduce:

$$\prod_{k=1}^{n} \left(1 - \frac{\log 2}{2k + \log 2}\right) = \frac{\Gamma(n+1) \cdot \Gamma\left(1 + \frac{\log 2}{2}\right)}{\Gamma\left(n + 1 + \frac{\log 2}{2}\right)} \tag{9}$$

and considering the infinite product:

$$\prod_{k=1}^{\infty} \left(1 - \frac{\log 2}{2k + \log 2}\right)$$

diverges [12] to zero.

Such conclusion can be immediately verified also by applying the result (8), modified in the sign and considering the following:

Theorem. *Supposed that* $-1 \prec a_n \leq 0$ *and the series* $\sum a_n$ *is divergent, then the infinite product* $\prod (1 + a_n)$ *diverges to zero.*

Proof. Assumed $b_n = -a_n \Rightarrow \quad 0 \leq \quad b_n \prec \quad 1$, as for $0 \leq x \prec 1$ results $1 - x \leq e^{-x}$, we can write:

$$0 \prec \quad P_n = \prod_{r=1}^{n} (1 + a_r) \quad \leq \quad e^{-(b_1 + b_2 + \dots + b_n)} \tag{10}$$

then if the series $\sum a_n$ is not convergent and necessarily diverges to $-\infty$, from (10) we have that $P_n \to 0$ that is the infinite product diverges to zero.

Besides, keeping in mind that, supposed a and b non negative, the following relation is valid:

$$\prod_{r=1}^{\infty} \frac{r \cdot (r + a + b)}{(r + a) \cdot (r + b)} = \frac{\Gamma(1 + a) \cdot \Gamma(1 + b)}{\Gamma(1 + a + b)}$$

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[12] Chapter II, pag. 33, [6].

verifiable, applying the known formula by Euler:

$$\Gamma\left(x\right) = \lim_{n\to\infty} \frac{n^x \cdot n!}{x \cdot (x+1) \cdot (x+2) \cdots (x+n)}$$

and considering the limit for $n \to \infty$ of the development of the following finished product:

$$\prod_{r=1}^{n+1} \frac{r \cdot (r+a+b)}{(r+a) \cdot (r+b)} = \frac{(1+a+b) \cdot (2+a+b) \cdots (1+a+b+n)}{(n^{1+a+b}) \cdot n!} \cdot$$

$$\cdot \frac{(n^{1+a}) \cdot n!}{(1+a) \cdot (2+a) \cdots (1+a+n)} \cdot \frac{(n^{1+b}) \cdot n!}{(1+b) \cdot (2+b) \cdots (1+b+n)} \cdot \frac{n+1}{n}$$

we can state that:

$$\frac{\Gamma\left(1+n\right) \cdot \Gamma\left(1 + \frac{\log 2}{2}\right)}{\Gamma\left(1 + n + \frac{\log 2}{2}\right)} = \prod_{r=1}^{\infty} \frac{r \cdot \left(r + n + \frac{\log 2}{2}\right)}{(r+n) \cdot \left(r + \frac{\log 2}{2}\right)}$$

By exploiting the properties of the *gamma* function, or resorting to the definition of the famous hypergeometric series, it's also possible to verify the following:

$$\frac{\Gamma\left(1+n\right) \cdot \Gamma\left(1 + \frac{\log 2}{2}\right)}{\Gamma\left(1 + n + \frac{\log 2}{2}\right)} = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\log 2}{2}\right) \cdot \left(\frac{\log 2}{2} - 1\right) \cdots \left(\frac{\log 2}{2} - r\right)}{(1+n+r) \cdot r!} \qquad (11)$$

In fact, adopting the classic symbolism for the hypergeometric function[13] :

$$\begin{aligned} F\left(a,b;c;x\right) &= 1 + \frac{a \cdot b}{c \cdot 1!} \cdot x + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{c \cdot (c+1) \cdot 2!} \cdot x^2 \\ &+ \frac{a \cdot (a+1) \cdot (a+2) \cdot b \cdot (b+1) \cdot (b+2)}{c \cdot (c+1) \cdot (c+2) \cdot 3!} \cdot x^3 + \ldots \end{aligned}$$

we can exploit the possibility to express it in terms of *gamma* functions, considering the hypergeometric integral, that is:

$$F\left(a,b;c;x\right) = \frac{\Gamma(c)}{\Gamma\left(b\right) \cdot \Gamma\left(c-b\right)} \cdot \int_0^1 t^{b-1} \cdot (1-t)^{c-b-1} \cdot (1 - t \cdot x)^{-a} \, dx \quad (\Re c \succ \Re b \succ 0)$$

From this, considering the limit for $x \to 1^-$ (Abel's theorem) and exploiting the properties of the Eulerian *Beta* function, we obtain the important relation of the hypergeometric Gauss's series:

$$F\left(a,b;c;\ 1\right) = \frac{\Gamma(c) \cdot \Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right) \cdot \Gamma\left(c-b\right)} \quad (c \neq 0, -1, -2, \ldots, \ \Re\left(c-a-b\right) \succ 0)$$

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[13] Chapter III, [5].

With the positions $a = -\frac{\log 2}{2}$, $b = n$ and $c = n + 1$, as all the required limitations are satisfied for the parameters, we easily reach the (11).

In conclusion, starting from the characteristic base of the generic ش function, defined by the analysis of its behaviour to infinity, we have also stated the following sizeable relation between an infinite product and a numeric series:

$$
\begin{aligned}
F\left(-\frac{\log 2}{2}, \ n; \ n+1; \ 1\right) &= \prod_{r=1}^{\infty} \frac{r \cdot \left(r + n + \frac{\log 2}{2}\right)}{(r+n)\left(r + \frac{\log 2}{2}\right)} \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\log 2}{2}\right) \cdot \left(\frac{\log 2}{2} - 1\right) \cdots \left(\frac{\log 2}{2} - r\right)}{(1 + n + r) \cdot r \, !}
\end{aligned}
$$

In the real field such result is graphically represented by Fig. 5, shown below, where it is evident that the x axis represents a horizontal asymptote for $x \to +\infty$.

Fig. 5

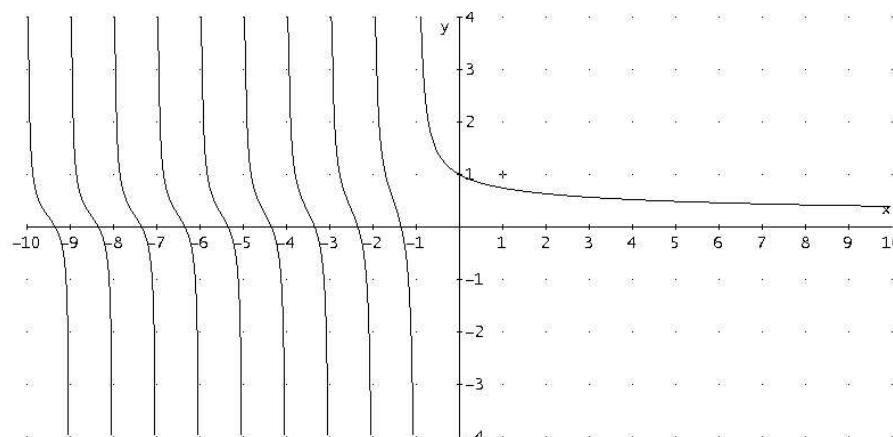Besides, in the field of the non-negative real numbers, both the infinite product and the series are absolutely and uniformly convergent.

4. THE SPECIAL FUNCTION ش AND THE SECOND-ORDER EULERIAN NUMBERS[14]

In the second paragraph we have shown that one of the most important characteristics in the construction of the family of ش functions is related with the repetitive presence of

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[14] pag. 247-251,[3].

11 specific intervals I_ℓ of the integer variable k: we have indeed seen that the extension of such 11 intervals is always characterized by the following numeric *series*:

8, 9, 9, 9, 8, 9, 9, 9, 8, 9, 9 for a sum of 96 values of the variable k.

But it also happened that such *series* undergoes inside a precise number of integer intervals some interruptions, as 8 value intervals substitute some 9.

More precisely this event is regularly repeated every **40** and **51** intervals and it's possible to observe that such intervals are aggregable so that to give origin to two groups: the first one formed by **484** intervals, produced by **7** interruptions at a distance of **40** intervals and **4** interruptions at a distance of **51** intervals, the second one formed by **495** intervals, produced by **6** interruptions at a distance of **40** intervals and **5** interruptions at a distance of **51** intervals.

For each group of **484** intervals we besides note always **10** groups of **495** intervals: this corresponds on average to a number of **494** intervals for all the groups.

In conclusion the characteristics of such phenomenon, keeping also in mind what explained in the initial part of the second paragraph, allow correctly some estimates for the integer values of k, referred to an interval I_ℓ, if we suppose the effective presence, even if virtual, of the following sequence of integer intervals:

1,120,494,494,. . . ,494, ≈494[15]**,494,..,494,. . . ,494,..,494,. . . , ≈494** (ad infinitum).

This interpretation is besides confirmed by the check carried out on the limit of the ratio $k/\Omega\ (I_\ell)$.

Now, as these numbers are closely connected to the nature of the complete and periodic sequence of the numerical series shown above, we have supposed that the numbers **1 , 120** and **494** can belong to a category of *special numbers;* particularly a research pointed to verify this hypothesis, has implied the following curious discovery even if it is always a conjecture on an almost light and remote connection: the numbers 1, 120 and 494 belong

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[15] **≈494:** with such notation we want to put into evidence that for great values of k the estimable number of integer intervals is, in some very near case, to such value, in fact it's as if every 495.000 intervals, two others of them, on average, undergo a reduction of one unity (from 9 to 8), but on the whole this further phenomenon is absolutely neglectable.

to the family of the so called "second-order Eulerian numbers", which result important for the tight connection that they have with Stirling's numbers.

They satisfy similar recurrence to the characteristic one of the "ordinary Eulerian numbers" , which are useful, above all because they give a connection among ordinary powers and consecutive binomial coefficients.

To be clearer we show below the recurrence, which characterizes the second-order Eulerian numbers, showing how it's possible, by it, to produce the three special numbers, typical of the special function ش.

In fact we have:

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (k+1) \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-1-k) \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle ;$$

$$\left\langle\!\!\left\langle {n \atop 0} \right\rangle\!\!\right\rangle = \mathbf{1} \quad \forall n \neq 0 \quad ; \quad \left\langle\!\!\left\langle {n \atop n} \right\rangle\!\!\right\rangle = 0 \text{ for } n \neq 0$$

$$\left\langle\!\!\left\langle {2 \atop 1} \right\rangle\!\!\right\rangle = (2) \left\langle\!\!\left\langle {1 \atop 1} \right\rangle\!\!\right\rangle + (2) \left\langle\!\!\left\langle {1 \atop 0} \right\rangle\!\!\right\rangle = 2$$

$$\left\langle\!\!\left\langle {3 \atop 2} \right\rangle\!\!\right\rangle = (3) \left\langle\!\!\left\langle {2 \atop 2} \right\rangle\!\!\right\rangle + (3) \left\langle\!\!\left\langle {2 \atop 1} \right\rangle\!\!\right\rangle = 6$$

$$\left\langle\!\!\left\langle {4 \atop 3} \right\rangle\!\!\right\rangle = (4) \left\langle\!\!\left\langle {3 \atop 3} \right\rangle\!\!\right\rangle + (4) \left\langle\!\!\left\langle {3 \atop 2} \right\rangle\!\!\right\rangle = 24$$

$$\left\langle\!\!\left\langle {5 \atop 4} \right\rangle\!\!\right\rangle = (5) \left\langle\!\!\left\langle {4 \atop 4} \right\rangle\!\!\right\rangle + (5) \left\langle\!\!\left\langle {4 \atop 3} \right\rangle\!\!\right\rangle = \mathbf{120}$$

$$\left\langle\!\!\left\langle {2 \atop 1} \right\rangle\!\!\right\rangle = (2) \left\langle\!\!\left\langle {1 \atop 1} \right\rangle\!\!\right\rangle + (2) \left\langle\!\!\left\langle {1 \atop 0} \right\rangle\!\!\right\rangle = 2$$

$$\left\langle\!\!\left\langle {3 \atop 1} \right\rangle\!\!\right\rangle = (2) \left\langle\!\!\left\langle {2 \atop 1} \right\rangle\!\!\right\rangle + (4) \left\langle\!\!\left\langle {2 \atop 0} \right\rangle\!\!\right\rangle = 8$$

$$\left\langle\!\!\left\langle {4 \atop 1} \right\rangle\!\!\right\rangle = (2) \left\langle\!\!\left\langle {3 \atop 1} \right\rangle\!\!\right\rangle + (6) \left\langle\!\!\left\langle {3 \atop 0} \right\rangle\!\!\right\rangle = 22$$

$$\left\langle\!\left\langle \begin{array}{c} 5 \\ 1 \end{array} \right\rangle\!\right\rangle = (2) \left\langle\!\left\langle \begin{array}{c} 4 \\ 1 \end{array} \right\rangle\!\right\rangle + (8) \left\langle\!\left\langle \begin{array}{c} 4 \\ 0 \end{array} \right\rangle\!\right\rangle = 52$$

$$\left\langle\!\left\langle \begin{array}{c} 6 \\ 1 \end{array} \right\rangle\!\right\rangle = (2) \left\langle\!\left\langle \begin{array}{c} 5 \\ 1 \end{array} \right\rangle\!\right\rangle + (10) \left\langle\!\left\langle \begin{array}{c} 5 \\ 0 \end{array} \right\rangle\!\right\rangle = 114$$

$$\left\langle\!\left\langle \begin{array}{c} 7 \\ 1 \end{array} \right\rangle\!\right\rangle = (2) \left\langle\!\left\langle \begin{array}{c} 6 \\ 1 \end{array} \right\rangle\!\right\rangle + (12) \left\langle\!\left\langle \begin{array}{c} 6 \\ 0 \end{array} \right\rangle\!\right\rangle = 240$$

$$\left\langle\!\left\langle \begin{array}{c} 8 \\ 1 \end{array} \right\rangle\!\right\rangle = (2) \left\langle\!\left\langle \begin{array}{c} 7 \\ 1 \end{array} \right\rangle\!\right\rangle + (14) \left\langle\!\left\langle \begin{array}{c} 7 \\ 0 \end{array} \right\rangle\!\right\rangle = \mathbf{494}$$

At the end, to be complete, we give besides the general formula of the second-order Eulerian numbers, which puts into evidence the connection with the binomial coefficients and with Stirling's numbers, a representation of the second order Eulerian triangle (Fig. 6):

$$\left\langle\!\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle\!\right\rangle = \sum_{k=0}^{m} \left(\begin{array}{c} 2n+1 \\ k \end{array} \right) \left\{ \begin{array}{c} n+m+1-k \\ m+1-k \end{array} \right\} \cdot (-1)^k \quad for \ n \succ m \geq 0 \qquad (12)$$

where for the binomial coefficients it is valid: $\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n\,!}{k\,!\,(n-k)\,!}$

and for the Stirling's numbers : $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \cdot \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$.

If we consider, for fundamental of the special function ش, the number $\left\langle\!\left\langle \begin{array}{c} 8 \\ 1 \end{array} \right\rangle\!\right\rangle$ from (12) we have:

$$\left\langle\!\left\langle \begin{array}{c} 8 \\ 1 \end{array} \right\rangle\!\right\rangle = \left(\begin{array}{c} 17 \\ 0 \end{array} \right) \left\{ \begin{array}{c} 10 \\ 2 \end{array} \right\} - \left(\begin{array}{c} 17 \\ 1 \end{array} \right) \left\{ \begin{array}{c} 9 \\ 1 \end{array} \right\}$$

but keeping in mind that for Stirling's numbers are valuable the following identities for $n \succ 0$:

$$\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 2^{n-1} - 1 \ \text{ and } \ \left\{ \begin{array}{c} n \\ 1 \end{array} \right\} = 1$$

we finally have:

$$\left\langle\!\left\langle \begin{array}{c} 8 \\ 1 \end{array} \right\rangle\!\right\rangle = 1 \cdot \left(2^9 - 1 \right) - 17 \cdot 1 = 511 - 17 = \mathbf{494}$$

N / K	0	1	2	3	4	5	6	7	8
0	1								
1	**1**	0	0	0	0	0	0	0	0
2	1	2	0	0	0	0	0	0	0
3	1	8	6	0	0	0	0	0	0
4	1	22	58	24	0	0	0	0	0
5	1	52	328	444	**120**	0	0	0	0
6	1	114	1452	4400	3708	720	0	0	0
7	1	240	5610	32120	58140	33984	5040	0	0
8	1	**494**	19950	195800	644020	785304	341136	40320	0
9	1	1004	67260	1062500	5765500	12440064	11026296	3733920	362880

Fig. 6

5. FROM DISCRETE TO COMPLEX FIELD

In the previous paragraphs we have discussed on arguments by infinitesimal, asymptotic, numerical and combinatorial analysis to characterize the special function ش .

It is known that in these fields the completely monotonic functions play a fundamental role.

We recall that a function $f : I \to \Re$ is said to be completely monotonic (c.m.) on a real interval I, if f has derivatives of all orders on I which alternate successively in sign, that is:

$$(-1)^n \cdot f^{(n)}(x) \geq 0 \quad \forall x \in I \ \text{and} \ \forall \, n \geq 0 \ \text{with} \ n = 0, 1, 2, 3, ...$$

In the recent past, various authors[16] showed that numerous functions, which are defined in terms of *gamma*, *polygamma* and other special functions, as the hypergeometrical ones, are completely monotonic and used this fact to derive many interesting new inequalities.

We shall confine ourselves only to prove, in very simple way, that the special function ش too, even if it is for definition a piecewise continuous function, in the real field, it possesses the same property (c.m.).

Lemma. *If* $f(x)$ *and* $g(x)$ *are c.m., then* $a \cdot f(x) + b \cdot g(x)$ *, where a and b are non-negative constants and* $f(x) \cdot g(x)$ *are also c.m..*

The proof of the first thesis is obvious, the second one is then easily seen from the Leibniz formula :

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[16]pag. 445-460, [1].

$$\frac{d^n}{dx^n} \cdot [f(x) \cdot g(x)] = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)$$

Theorem 1. *The special function* ش $[k, \Omega(I_\ell)]$ *is completely monotonic in each* I_ℓ .

Proof. The base function $f(k) = 1 + \frac{1}{3k - \Omega(I_\ell)}$ is c.m. in each I_ℓ ; in fact the n-th derivative of this function is:

$$\frac{d^n}{dk^n} f(k) = \frac{(-1)^n \cdot 3^n \cdot n!}{[3k - \Omega(I_\ell)]^{n+1}} \quad \Rightarrow \quad (-1)^n \cdot f^{(n)}(k) \geq 0$$

Thus by the Lemma, in case k is an integer, it's obvious that also the special function ش is c.m. in each I_ℓ ; therefore it remains to prove Theorem 1 for the case when k is a real and positive number[17].

To obtain this it is necessary to the use the following obvious Theorem 2, which is a consequence of the Lemma, for composed functions.

Theorem 2. *Let* $y = f(x)$ *c.m. and let the power series* $\varphi(y) = \sum_{j=0}^{\infty} a_j y^j$ *converge for all* y *in the range of the function* $y = f(x)$. *If* $a_j \geq 0$ *for all* $j = 0, 1, 2, 3, \ldots$ *then* $\varphi[f(x)]$ *is c.m..*

Corollary. *If* $f(x)$ *is c.m., then* $e^{f(x)}$ *is c.m..*

In particular, as the special function ش is equal:

$$ش[k, \Omega(I_\ell)] = e^{(2k+1)\cdot\log\left(1+\frac{1}{3k-\Omega(I_\ell)}\right)}$$

it is c.m. in each I_ℓ .

An interesting exposition of the main results on completely monotonic functions is given in Widder's[18] work.

That being stated, with some limitations due to the nature of the special function ش in the real field, we can describe a method for estimating the same function in the complex

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[17] We must also remember the (2) without giving up the characterization, determined for each interval I_ℓ .

[18] [7]

field with an important *improper integral.*

The ش function results to be a piecewise continuous function because of the presence of the step function $\Omega\,(I_\ell)$, that is discontinuous, and actually its complete monotonicity has been proved for all the closed intervals I_ℓ .

Passing to the interval $I\,[0,\infty\,)$ the continuity is not guaranteed and therefore the application of the following Hausdorff-Bernstein-Widder's[19] Theorem, must be done carefully, or it can be limited to characterize the behaviour of the special function ش at the origin ad infinity.

Theorem 3. *A necessary and sufficient condition for the function $f\,(s)$ in order to be completely monotonic in the interval $I\,[0,\infty\,)$ is that:*

$$f\,(s)\ = L_s\,[F\,(t)] = \int_0^\infty e^{-st}dF\,(t) \tag{13}$$

where $F\,(t)$ is non-decreasing and the integral converges in the interval $I\,[0,\infty\,)$.

The (13) represents the transformation of Laplace-Stieltjes of a locally and absolutely continuous function, with real values, in the interval $I\,[0,\infty\,)$.

Now, keeping in mind the relation existing in such case between Laplace-Stieltjes transform L_s and *ordinary*[20] Laplace transform L (observing that we can suppose $F\,(0) = 0$):

$$L_s\,[F\,(t)] = s\cdot \int_0^\infty e^{-st}F\,(t)\cdot dt = s\cdot L\,[F\,(t)] \tag{14}$$

we can determine the expression of the function $F\,(t)$ by the inverse Laplace transform.

From (13) and (14) we get $\Phi\,(s) = \frac{f(s)}{s} = L\,[F\,(t)]$ and successively we need to face the calculus of integral of the type[21]:

$$F\,(t)\ = \mathrm{L}^{-1}\,[\Phi\,(s)] = v.p.\,\frac{1}{2\pi i}\cdot \int_{x_0-i\cdot\infty}^{x_0+i\cdot\infty} e^{ts}\Phi\,(s)\cdot ds \tag{14.1}$$

with $t\succ 0$, $\Phi\,(s)$ holomorphic function in the half-plane $\Re e\,(s)\succ 0$ and x_0 arbitrary real positive number.

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[19] Chapter IV, pag. 160-161, [7].
[20] Chapter IV, [2].
[21] pag. 168, [2].

In general the calculus of Bromwich's integral (14.1), that is defined as a *Cauchy principal value*, we can only do it numerically, applying quadrature formulae, but in our case [$f(s) = $ ش (s)], being the $\Phi(s)$ function **piecewise analytic**, we could prove by the direct calculus of the considered integral (see the following paragraph), that $F(t) = 2 \cdot \Theta[t]$ and $\Theta[t]$ represents the unit step function or Heaviside's function ($\Theta[\,t < 0\,] = 0$; $\Theta[\,t > 0\,] = 1$) and in (13) we'll have $dF(t) = 2 \cdot d\Theta[t] = 2 \cdot \delta(t)$, with $\delta(t)$ that is the distribution of Dirac[22].

6. THE BEHAVIOUR OF THE SPECIAL FUNCTION ش AT THE ORIGIN AND INFINITY[23]

In the previous paragraph we have stated the following approximation "$\approx$", in terms of Laplace transform:

$$\Phi(s) = \quad \text{ش } (s) \, /_{\text{s}} \approx L\,[F(t)] = L\,[2 \cdot \Theta(t)] = \int_0^\infty e^{-st} 2 \cdot \Theta(t)\,dt = \frac{2}{s}$$

The approximation is essentially origined by neglecting the point of discontinuities of the first kind of the special function ش , between an interval I_ℓ and the following $I_{\ell+1}$ as far as the interval $I\,[0, \infty)$.

Said that to calculate the Bromwich's integral (14.1), we consider the path of integration rightly deformed, as we can see in the Fig. 7.
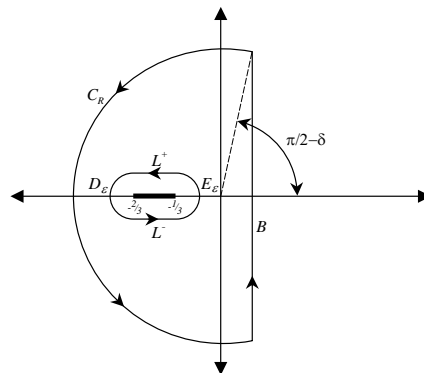

Fig. 7: The path of integration

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[22] pag. 30-35, [2].
[23] pag. 491-492, [4].

$\Phi\left(s\right)$ is a multiple-valued analytic function; in the interest of obtaining a single value mapping we consider the principal branch of the power in ش $\left(s\right)$, that allows us to treat a holomorphic branch of the same $\Phi\left(s\right)$ function.

In its field of single-values, the function $\Phi\left(s\right)$ possesses a simple pole at $s=0$ and two branch points at $s=-\frac{1}{3}$ and $s=-\frac{2}{3}$; let's observe, in fact, that in the interval $[-1,0]$ the special function ش is: ش $\left(s\right)=\left(1+\frac{1}{3s+1}\right)^{2s+1}=e^{(2s+1)\cdot\log\left(1+\frac{1}{3s+1}\right)}$

A cut, joining the two branch points, would prevent s to circulate around them, and the special function ش $\left(s\right)$ can be treated as a **piecewise holomorphic function**.

In conclusion, defined with C the boundary (shown in Fig. 7), with C_R the circular arc of radius R, with B the vertical line with $\Re\left(s\right)=x_0$, with L the boundary of the cut branch, formed by D_ε and E_ε, that is the semi-circles of radius ε capping the ends of the branch cut and by L^+ and L^-, lines above and below, we'll have:

$$\frac{1}{2\pi i}\cdot\int_{x_0-i\cdot\infty}^{x_0+i\cdot\infty}e^{ts}\Phi\left(s\right)\cdot ds=\frac{1}{2\pi i}\{\int_C\ldots-\int_{C_R}\ldots-\int_{L^+}\ldots-\int_{L^-}\ldots-\int_{D_\varepsilon}\ldots-\int_{E_\varepsilon}\ldots\} \qquad (15)$$

The first integral in the second member of (15), by the residue theorem, is $2\cdot\Theta\left(t\right)$, as the first order pole, at the origin, gives: $\lim_{s\to0}$ ش $\left(s\right)=2$.

The second integral with $s=R\cdot e^{i\vartheta}$ is:

$$\int_{C_R}e^{ts}\Phi\left(s\right)\cdot ds=\int_{\pi/2-\delta}^{\pi/2}\ldots d\vartheta+\int_{\pi/2}^{3\pi/2}\ldots d\vartheta+\int_{3\pi/2}^{3\pi/2+\delta}\ldots d\vartheta \qquad (16)$$

and therefore it vanishes: in fact the first and third integral in the second member of (16) vanish as $R\to\infty$ by the maximum modulus bound and the second integral vanished by Jordan's Lemma.

The third and fourth integral in the second member of (15) cancel each other along the paths L^+ and L^-: in fact their values, calculated along their opposite paths, eliminate each other.

In the end, the last two integrals of (15), by the maximum modulus bound, vanish as $\varepsilon\to0$.

Now, from the asymptotic behaviour of the function $F\left(t\right)$, so determined, we can therefore deduce asymptotic properties of the correspondent Laplace transform, that is to use the following Abelian theorems (initial and final value theorem):

Theorem. *Let F be a transformable function and let's suppose that the $\lim_{t\to\infty} F(t)$ exists, then the $\lim_{s\to 0} [\, s \cdot \Phi(s)\,]$ exists, too (let's suppose, for convenience $s \in \Re$) and is:*

$$\lim_{s\to 0} [\, s \cdot \Phi(s)\,] = \lim_{t\to\infty} F(t) \tag{17}$$

If the $\lim_{t\to 0^+} F(t)$ exists, then the $\lim_{s\to\infty} s \cdot \Phi(s)$ exists, too and is (with $s \in \Re$) :

$$\lim_{s\to\infty} [\, s \cdot \Phi(s)\,] = \lim_{t\to 0^+} F(t) \tag{18}$$

In our case we have seen that $F(t) = 2 \cdot \Theta(t)$, with $\Theta(t)$, that is Heaviside's function, and therefore we have that:

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 2 \cdot \Theta(t) = 2$$

and also:

$$\lim_{t\to 0^+} F(t) = \lim_{t\to 0^+} 2 \cdot \Theta(t) = 2$$

The existence and the calculus of such limits, in (17) and (18), give the following results (between them a further confirm of the *fundamental* theorem):

$$\lim_{s\to 0} \; ش(s) = 2 \; ; \; \lim_{s\to\infty} ش(s) = 2$$

APPENDIX

DERIVE Version 6 : **the vector function** : $S(k,\Omega) := \lim_{t \to k} \left(1 + \frac{1}{3t-\Omega}\right)^{2t+1}$

#1: $V(k, \Omega) := \text{VECTOR}([t, o, S(t,\Omega)], t, k, k + 10)$

a display result:

V(k,	Ω)	VECTOR	
09	1	2.0484148121729077984789748528464903812082646338028	
10	1	2.0379259208387064562838079920238964441117176933744	
11	1	2.0294161672236677191636908626945714916029532064944	
12	1	2.0223737073469397533461445484297949184415534016560	
13	1	2.0164491799135882361365114303236301480590762568587	
14	1	2.0113959747189663594458436566806886371655211482846	
15	1	2.0070350457364044054984268130457357906298862812402	
16	1	2.0032332566108411453651981972386971733090123101528	
17	1	1.9998895526624551656968593976078763119133058370586	
18	1	1.9969258468076576081148471529242828805292201418873	
19	1	1.9942808454379732420411582337304540085201659981375	

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