PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 31—38

THE MAIN PART OF THE SPECTRUM, DIVISORS AND SWITCHING OF GRAPHS

Dragoš M. Cvetković

(Communicated September 20, 1977)

Abstract. The relationship between three apparently unrelated notions from the theory of graph spectra, mentioned in the title, is discussed.

1. Introduction

1° Let $\mu_1, \ldots, \mu_m$ be the distinct eigenvalues of an (undirected) graph G. It was noticed in [2] that for the number N_k of walks of length k in G the formula

$$N_k = C_1 \mu_1^k + \cdots + C_m \mu_m^k \tag{1}$$

holds, where $C_1, \ldots, C_m$ are certain numbers (not depending on k). In the same paper the main part of the spectrum was defined as the set of those eigenvalues μ_i for which $C_i \neq 0$.

The way of finding the C_i's is given in [7]. (Similar calculations appear also in the theory of Markov chains [8]). The adjacency matrix A of G is symmetric and so there exists a system of mutually orthogonal eigenvectors $u_1, \ldots, u_n$ belonging to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. If these vectors are normalized (so that their moduli are equal to 1), the matrix $U = \| u_1, \ldots, u_n \|$, whose columns are the mentioned eigenvectors, satisfies the relation $A = U \Lambda U^{-1}$, where Λ is a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Since U is orthogonal, we have $A^k = U \Lambda^k U^T$. The number of walks N_k is, of course, equal to the sum of all elements of A^k and a straightforward computation gives

$$N_k = \sum_{i=1}^{n} (u_{1i} + u_{2i} + \cdots + u_{ni})^2 \lambda_i^k, \tag{2}$$

where u_{si} are coordinates of the vector u_i.

Formula (2) is derived in [5] and applied to some chemical problems. The formula was known to the author much earlier and was included in the manuscript of the monograph [4]. It is independently discovered also in [11], where it was used to find the number of dissimilar walks of length n.

By comparing (1) and (2) we see that an eigenvalue μ_i is a main eigenvalue if and only if it has an eigenvector in which the sum of coordinates is not zero, i. e. which is not orthogonal to the vector j, whose all components are equal to 1.

Following this observation (see also [11]) we introduce a suitable terminology.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Let $\lambda_j = \lambda_{j+1} = \cdots \lambda_{j+p} = \lambda$ for some j, p, λ and let no other eigenvalue be equal to λ. If the eigenspace of λ contains an eigenvector not orthogonal to j, then we shall call λ_j a main eigenvalue and $\lambda_{j+1}, \ldots, \lambda_{j+p}$ will be called non-main eigenvalues. If all the eigenvectors of λ are orthogonal to j then all the eigenvalues $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+p}$ are non-main. If $\lambda_j = \lambda$ is a main eigenvalue we can find an orthogonal basis $u^{(0)}, u^{(1)}, \ldots, u^{(p)}$ of its eigenspace such that $u^{(0)} j \neq 0$ and $u^{(i)} j = 0$ for $i = 1, \ldots, p$. Again, $u^{(0)}$ is a main eigenvector and $u^{(1)}, \ldots, u^{(p)}$ are non-main eigenvectors. If λ is not a main eigenvalue, all its eigenvectors are called non-main.

2° Given a square matrix $B = \| b_{ij} \|_1^s$, let the vertex set X of a graph G be partitioned into (non-empty) subsets $X_1, X_2, \ldots, X_s$ so that for any $i, j = 1, \ldots, s$ each vertex from X_i is adjacent to exactly b_{ij} vertices of X_j. Then the multidiagraph H with the adjacency matrix B is called a front divisor of G [14], or briefly, a divisor of G.

The existence of a divisor means that the graph has a certain structure. (The divisor can be interpreted as a homomorphic image of the graph). On the other hand the characteristic polynomial of a divisor divides the characteristic polynomial of the graph (i. e. the spectrum of divisor is contained in the spectrum of the graph) [13]. In this way the notion of a divisor can be understood as a link between spectral and structural properties of a graph [9]. The concept of divisor was also used in coding theory [6].

3° A partition of the vertex set of a graph G into two (disjoint) subsets (one of which may be empty) will be represented as a colouring c of vertices by two colours (say black and white) such that the vertices from the same subset are coloured by the same colour. The graph G together with its colouring c will be denoted by G_c.

Switching a graph G with respect to a colouring c (or partition c) means deleting all edges between black and white vertices in G_c and introducing a new edge between a black and a white vertex whenever they were non-adjacent in G_c [12]. The graph obtained after switching will be denoted by $\mathcal{S}(G_c)$. Graphs G and H are switching equivalent if $H = \mathcal{S}(G_c)$ for some colouring c. Switching relation is an equivalence relation in the set of graphs and we can speak about switching classes of graphs.

A remarkable fact about switching is that switching equivalent graphs have the same Seidel spectrum (i. e. the spectrum of the Seidel adjacency matrix $S = \| s_{ij} \|_1^n$ with $s_{ii} = 0$ and (for $i \neq j$) $s_{ij} = -1$ if vertices i and j are adjacent and $s_{ij} = 1$ otherwise).

2. The main part of the spectrum and the switching

Some relations between the spectrum of a graph and its complement were found in [3]. Since the connection of the adjacency matrix of a graph and the adjacency matrix of its complement goes via the matrix J whose all entries are unity, the same type of relations must exist between the spectrum and the Seidel spectrum of a graph. (If A is the adjacency matrix of a graph and if S denotes its Seidel matrix, we have $S = J - 2A - I$).

For the Seidel matrix (more generally, for any symmetric matrix) we can again define main and non-main eigenvalues and eigenvectors by orthogonality conditions with respect to the vector j. The matrix J has a main eigenvector j belonging to the (main) eigenvalue n (dimension of J) and there are $n-1$ non-main independent vectors belonging to the eigenvalue 0. Therefore each non-main eigenvector of A is a non-main eigenvector of S and the converse also holds. If λ is a non-main eigenvalue of A, then $-2\lambda-1$ is a non-main eigenvalue of S and vice versa. Hence, the number of non-main (and, of course, also the number of the main) eigenvalues is the same for A and S. The multiplicity of λ in A and of $-2\lambda-1$ in S need not be the same, since a main eigenvector can exist for λ in A and not for $-2\lambda-1$ in S, or vice versa. Hence, we have proved the following lemma.

Lemma. *Let G be a graph with the adjacency matrix A and the Seidel matrix S. For any λ, the multiplicity of λ in A and of $-2\lambda-1$ in S differ by at most* 1.

Remark. In regular graphs all eigenvalues except for the greatest one are non-main and therefore they are transformed by $\lambda \to -2\lambda-1$ when going from A to S. To the greatest eigenvalue r of A there corresponds the eigenvalue $n-1-2r$ of S and it may coincide with some other eigenvalues.

Theorem 1. *Let graphs G_1 and G_2 be switching equivalent. For any λ, the multiplicity of λ in G_1 (as the eigenvalue of the adjacency matrix) and the multiplicity of λ in G_2 differ by at most* 2.

Proof. Let A_1, A_2 and S_1, S_2 be the adjacency and the Seidel matrices of G_1, G_2 respectively. Then S_1 and S_2 have the same spectrum. The multiplicities of the corresponding eigenvalues of A_1, S_1 and of S_2, A_2 are related by lemma and the theorem readily follows.

Remark. If we switch G_1 into G_2 then the Seidel spectrum is not changed but the main part of the Seidel spectrum is changed in general. For example, the graph $3K_1$ has the spectrum 0, 0, 0 and the main part is $\{0\}$. The Seidel spectrum is $2, -1, -1$, the main part being $\{2\}$. If we switch $3K_1$ into $K_{1,2}$ the main part of the Seidel spectrum becomes $\{2, -1\}$ and when going to the ordinary spectrum we get $\sqrt{2}, 0, -\sqrt{2}$, the main part being $\{\sqrt{2}, -\sqrt{2}\}$. Hence, the multiplicity of 0 in $3K_1$ has decreased by 2 after switching. This example suggests that the change of multiplicity by 2 is connected with a conversion of a non-main eigenvalue to a main eigenvalue or vice versa.

There is another aspect of the relation between the main part of the spectrum and switching. We shall derive a formula connecting the characteristic polynomial $P_G(\lambda)$ and the Seidel polynomial $S_G(\lambda)$.

Let N_k be the number of walks of length k in a graph G and let $H_G(t)=\sum_{k=0}^{+\infty} N_k t^k$ be the generating function for the numbers of walks (N_0 being equal to the number of vertices n of G). As shown in [2] we have

$$(3) \qquad H_G(t)=\frac{1}{t}\left[(-1)^n\frac{P_{\bar G}\left(-\frac{t+1}{t}\right)}{P_G\left(\frac{1}{t}\right)}-1\right].$$

This formula can be useful in some combinatorial enumeration problems, but it is important to notice [11] that this formula, when rewritten in the form

$$(4) \qquad P_{\bar G}(\lambda)=(-1)^n P_G(-\lambda-1)\left(1-\frac{1}{1+\lambda}H_G\left(-\frac{1}{1+\lambda}\right)\right),$$

gives the connection between the characteristic polynomials of the graph G and of its complement $\bar G$.

Formula (4) can be easily derived using the following simple relation

$$(5) \qquad \det(Y+tJ)=\det Y+t\operatorname{sum}\operatorname{adj} Y,$$

where Y is any square matrix, t is a number, sum X means the sum of all entries of the matrix X and adj Y is the matrix of cofactors of elements of Y.

Applying (5) with $Y=I-tA$ and $t=-1$ we get

$$(6) \qquad \operatorname{sum}\operatorname{adj}(I-tA)=\frac{1}{t}(\det((t+1)I+t\bar A)-\det(I-tA)),$$

where $\bar A=J-I-A$ is the adjacency matrix of $\bar G$. On the other hand

$$(7) \qquad H_G(t)=\sum_{k=0}^{+\infty} t^k \operatorname{sum} A^k=\operatorname{sum}\sum_{k=0}^{+\infty}A^k t^k=\operatorname{sum}(I-tA)^{-1}=$$
$$=\operatorname{sum}\operatorname{adj}(I-tA)/\det(I-tA).$$

Combining (6) and (7) we get (3).

Since $S=J-2A-I$, putting $Y=\lambda I+2A+I$ and $t=-1$ in (5) we get in a similar way:

Theorem 2.

$$(8) \qquad P_G(\lambda)=\frac{(-1)^n}{2^n}S_G(-2\lambda-1)\frac{1}{1+\frac{1}{2\lambda}H_G\left(\frac{1}{\lambda}\right)}.$$

In this way again the function H_G plays the role of the link between the characteristic polynomial $P_G(\lambda)$ and the Seidel polynomial $S_G(\lambda)$. It should be noticed that $P_G(\lambda)$ depends on two functions $S_G(\lambda)$ and $H_G(\lambda)$. The first one is the same for all graphs in a switching class and the second one is changed under switching. Hence, it will be of some interest to investigate the behaviour of $H_G(\lambda)$ under switching.

3. The main part of the spectrum and divisors

The function $\frac{1}{u} H_G\left(\frac{1}{u}\right)$ has only simple poles and they represent the main part of the spectrum of G, since

$$\text{(9)} \qquad \frac{1}{u} H_G\left(\frac{1}{u}\right) = \frac{1}{u} \sum_{k=0}^{+\infty} N_k \left(\frac{1}{u}\right)^k = \frac{1}{u} \sum_{k=0}^{+\infty} \frac{1}{u^k} \sum_{i=1}^{m} C_i \mu_i^k =$$

$$= \frac{1}{u} \sum_{i=1}^{m} C_i \sum_{k=0}^{+\infty} \left(\frac{\mu_i}{u}\right)^k = \frac{1}{u} \sum_{i=1}^{m} C_i \frac{1}{1-\frac{\mu_i}{u}} = \sum_{i=1}^{m} \frac{C_i}{u-\mu_i}.$$

Let $n_k^{(j)}$ be the number of walks of length k in a multidigraph H, which starts from the vertex j. Then $n_k^{(i)}$ is the sum $\mathrm{sum}^{(i)} B^k$ of all entries of the i-th row in B^k, where B is the adjacency matrix of H. Consider the function $\frac{1}{u} H_H^{(i)}\left(\frac{1}{u}\right)$, where $H_H^{(i)}(t) = \sum_{k=0}^{+\infty} n_k^{(i)} t^k$. We have

$$\text{(10)} \qquad \frac{1}{u} H_H^{(i)}\left(\frac{1}{u}\right) = \frac{1}{u} \sum_{k=0}^{+\infty} \frac{n_k^{(i)}}{u^k} = \frac{1}{u} \sum_{k=0}^{+\infty} \frac{1}{u^k} \mathrm{sum}^{(i)} B^k = \frac{1}{u} \mathrm{sum}^{(i)} \sum_{k=0}^{+\infty} \frac{B^k}{u^k} =$$

$$= \frac{1}{u} \mathrm{sum}^{(i)} \left(I - \frac{1}{u} B\right)^{-1} = \mathrm{sum}^{(i)} (uI - B)^{-1}.$$

Theorem 3. *The spectrum of any divisor H of a graph G includes the main part of the spectrum of G.*

Proof. Let H be formed on the basis of the partition $X = X_1 \cup X_2 \cup \cdots \cup X_s$ of the vertex set X of G. Let $|X_i| = n_i$ $(i = 1, \ldots, s)$. Let i be the vertex of H corresponding to X_i. H is a homomorphic image of G and the image of any walk in G is a walk in H, where, of course, different walks in G can have the same image in H. If a walk in H starts at vertex i then the corresponding walks start in X_i. If we fix a vertex in X_i as the starting point of the walk, then the walk in G is uniquely determined by the walk in H (which starts at vertex i), since there exists a $1-1$ correspondence between the edges starting at i and the edges starting at any fixed vertex from X_i. In this way we get $N_k = n_1 n_k^{(1)} + \cdots + n_s n_k^{(s)}$ and by (10) we have

$$\frac{1}{u} H_G\left(\frac{1}{u}\right) = \sum_{i=1}^{s} n_i \,\mathrm{sum}^{(i)} (uI - B)^{-1}.$$

We see that the function at the right hand side is rational with the characteristic polynomial $P_H(u)$ of H in denominator. (Some factors of $P_H(u)$ may be cancelled). Since by (9) the (simple) poles of $\frac{1}{u} H_G\left(\frac{1}{u}\right)$ form the main part of the spectrum of G, all these main eigenvalues must be zeros of $P_H(u)$.

This completes the proof.

Corollary. *The greatest eigenvalue r of a graph always belongs to the main part of the spectrum. Hence, any divisor of a graph G contains the eigenvalue r.*

It was conjectured in [11] that the spectrum of a divisor H with the smallest number of vertices is just the main part of the spectrum of G. Theorem 3 confirms one part of this conjecture. The remaining part is not true because of the following counterexample.

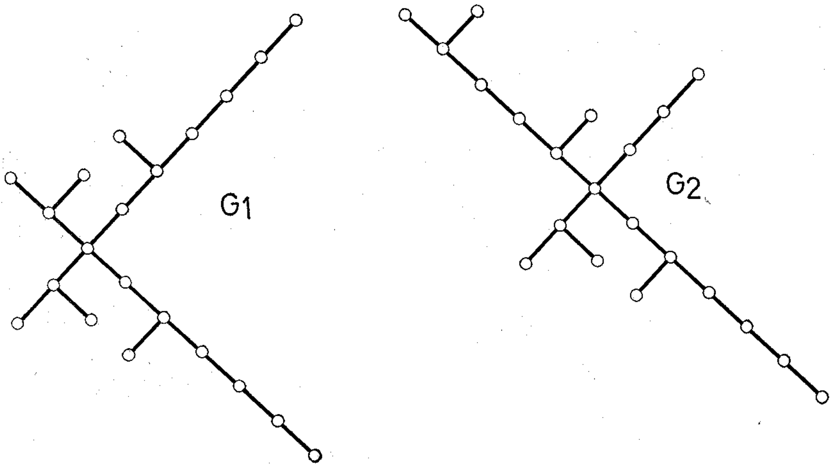

Fig. 1

According to Theorem 9.3 of [10], graphs G_1 and G_2 of Fig. 1 are cospectral and have cospectral complements. According to (3), G_1 and G_2 have the same main part of the spectrum. (Notice that cospectral graphs need not have the same main part of the spectrum). Because of the symmetry G_1 has a divisor on 10 vertices. By Theorem 3 the main part of the spectrum of G_1 and G_2 has at most 10 eigenvalues. But the divisor of G_2 with a minimal number of veitices has 19 vertices. So G_2 is a counterexample to the above conjecture.

Remark. It can easily be seen that the graphs with only one eigenvalue in the main part of the spectrum are just the regular graphs. It will be interesting to know which graphs have exactly k eigenvalues in the main part of the spectrum for any k. In particular, semiregular bipartite graphs have exactly two eigenvalues in the main part of the spectrum. Are there other graphs with this property? Note that the eigenvalue -2 in line graphs is never a main eigenvalue.

4. Divisors and switching of graphs

Theorem 4. *If a regular graph G of degree r with n vertices can be switched into a regular graph of degree r^*, then* $r^*-\dfrac{n}{2}$ *is an eigenvalue of G.*

Proof. If G has the mentioned property then the switching sets form a divisor with the adjacency matrix

$$\begin{Vmatrix} r-\frac{1}{2}(n-x-r^*+r) & \frac{1}{2}(n-x-r^*+r) \\ \frac{1}{2}(x-r^*+r) & r-\frac{1}{2}(x-r^*+r) \end{Vmatrix},$$

where x is the size of a switching set $(1\leqslant x<n)$. This matrix has the eigenvalues r and $r^*-\frac{n}{2}$, which proves the theorem.

Corollary. 1. *If n is odd then G cannot be switched into a regular graph since an eigenvalue of a graph cannot be a non-integer rational number.*

Corollary 2. *If G can be switched into a regular graph of the same degree and if q is the least eigenvalue of G, then $r-\frac{n}{2}\geqslant q$, i.e. $n\leqslant 2r-2q$. Since $q\geqslant -r$, we get $r-\frac{n}{2}\geqslant -r$, i.e. $r\geqslant n/4$.*

Example 1. There is no cospectral pair of non-isomorphic cubic graphs with less than 14 vertices [1]. Therefore the existence of such pairs cannot be explained by switching.

Example 2. Under which condition the graph $L(K_s)$ cannot be switched into another regular graph of the same degree? We have $n=\binom{s}{2}$, $r=2s-4$ and $q=-2$. Hence $r-\frac{n}{2}<-2$, $2s-4-\frac{s(s-1)}{2}<-2$, i.e. $s>8$. As known, for $s=8$ there are three graphs cospectral, switching equivalent and not isomorphic to $L(K_8)$.

Acknowledgement. The author is thankful to F. Harary, who read the paper in manuscript and gave some useful coments.

REFERENCES

[1] F. C. Bussemaker, S. Čobeljić, D. M. Cvetković, J. J. Seidel, *Computer investigation of cubic graphs*, Technological University Eindhoven, T. H. — Report 76 — WSK-01; *Cubic graphs on $\leqslant$14 vertices*, J. Comb. Theory (B), 23 (1977), 234—235.

[2] D. M. Cvetković, *The generating function for variations with restrictions and paths of the graph and self-complementary graphs*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 320—No. 328 (1970), 27—34.

[3] D. M. Cvetković, *Graphs and their spectra*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 354—No. 356 (1971), 1—50.

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs*, A monograph, in print.

[5] D. M. Cvetković, I. Gutman, *Note on branching*. Croat. Chem. Acta 49 (1977), 115—121.

[6] D. M. Cvetković, J. H. van Lint, *An elementary proof of Lloyd's theorem*, Proc. Kon. Ned. Akad. v. Wet. A 80 (1) (1977), 6—10.

[7] D. M. Cvetković, S. K. Simić, *On enumeration of certain types of sequences*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412—No. 460. (1973), 159—164.

[8] W. Feller, *An introduction to probability theory and its applications*, Vol. I, New York 1964.

[9] H. J. Finck, H. Sachs, *Über Beziehungen zwischen Struktur und Spektrum regulärer Graphen*, Wiss. Z. Techn. Hochsch. Ilmenau 19 (1973), 83—99.

[10] C. Godsil, B. McKay, *Some computational results on the spectra of graphs*, Combinatorial Mathematics, IV, Proc. 4-th Australian Conf. Berlin—Heidelberg—New York 1976, 73—92.

[11] F. Harary, A. J. Schwenk, *The spectral approach to determining the number of walks in a graph*, Pacific J. Math. to appear.

[12] J. H. van Lint, J. J. Seidel, *Equilateral point sets in elliptic geometry*, Nederl. Akad. Wetensch. Proc. Ser. A 69 (1966), 335—348.

[13] M. Petersdorf, H. Sachs, *Über Spektrum, Automorphismengruppe und Teiler eines Graphen*, Wiss. Z. Techn. Hochsch. Ilmenau 15 (1969), 123—128.

[14] H. Sachs, *Über Teiler, Faktoren und charakteristische Polynome von Graphen I, II*, Wiss. Z. Techn. Hochsch. Ilmenau, 12 (1966), 7—12; 13 (1967), 405—412.