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On divisors of a quadratic form

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Abstract. An approximation is given for the number of divisors of the quadratic form $n^2 + m^2 + t^2$.

Keywords: Approximation, divisor function, quadratic form.

1. Introduction.

Let $\tau(n)$ denote the number of divisors of n. Hooley [2] studied the behavior of the sum

$$S(x) = \sum_{n \leq x} \tau(n^2 + a)$$

where a is a fixed non zero integer such that $-a$ is not a perfect square. Gafurov [3,4] proved an asymptotic formula for the sum

$$S(x) = \sum_{1 \leq n,m \leq x} \tau(n^2 + m^2).$$

The main purpose of this paper is to obtain an approximation for the sum

$$S(x) = \sum_{1 \leq n,m,t \leq x} \tau(n^2 + m^2 + t^2). \tag{1.1}$$

From definition of $\tau(n)$, we can write (1.1) as

$$\begin{aligned} S(x) =& 2 \sum_{d \leq x\sqrt{3}} \sum_{\substack{1 \leq n,m,t \leq x \\ n^2+m^2+t^2 \equiv 0 (\mathrm{mod}\, d)}} 1 - \sum_{d \leq x\sqrt{3}} \sum_{\substack{1 \leq n,m,t \leq x \\ n^2+m^2+t^2 \equiv 0 (\mathrm{mod}\, d) \\ n^2+m^2+t^2 \leq dx\sqrt{3}}} 1 \\ =& 2S_1 - S_2. \end{aligned} \tag{1.2}$$

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Let $e(s)$ denote the exponential function, $e(s) = e^{2\pi i s}$, and $\zeta(s)$ the Riemann Zeta Function. Let γ be Euler's constant. Let $\rho(n)$ denote the number of solutions of the congruence

$$x^2 + y^2 + z^2 \equiv 0 (\mathrm{mod}\, n), \quad 1 \leq x, y, z \leq n. \tag{1.3}$$

Our first result is an asymptotic formula for the function $\rho(n)$.

Theorem 1. *The following formula holds*

$$\sum_{n \leq x} \rho(n) = \frac{4}{15} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \tag{1.4}$$

Using Theorem 1 and (1.2) we shall get the following approximation for the sum (1.1).

Theorem 2. *The divisor function of a quadratic form verifies the following formulas*

$$\begin{aligned} &\left(2C - \frac{222\sqrt{3}}{25} \frac{\zeta(3)}{\zeta(4)}\right) x^3 + O(x^2 \log x) \\ &\quad < \sum_{1 \leq n,m,t \leq x} \tau(n^2 + m^2 + t^2) - \frac{8}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x \\ &\quad \leq \left(2B - \frac{2}{225\sqrt{3}} \frac{\zeta(3)}{\zeta(4)}\right) x^3 + O(x^2 \log x) \end{aligned} \tag{1.5}$$

where

$$B = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 + 8\sqrt{3}), \quad C = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 - 8\sqrt{3})$$

for

$$A = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} \gamma + 2 \left(\frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} - 6 \left(\frac{1}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4}. \tag{1.6}$$

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2. Preliminary results

Expressing the number of solutions of the congruence (1.3) by Gauss Sums we obtain the following lemma.

Lemma 1. *Let $\rho(p^\alpha)$ the number of solutions of the congruence $x^2+y^2+z^2 \equiv 0(\bmod p^\alpha)$, $1 \le x, y, z \le p^\alpha$. Then we have*

(a) when p is an odd prime

$$\rho(p^\alpha) = \begin{cases} p^{2\alpha} + p^{2\alpha-1} - p^{3\alpha/2-1}, & \text{if } \alpha \text{ is even} \\ p^{2\alpha} + p^{2\alpha-1} - p^{(3\alpha-1)/2}, & \text{if } \alpha \text{ is odd,} \end{cases}$$

(b) when $p = 2$

$$\rho(2^\alpha) = \begin{cases} 2^{3\alpha/2}, & \text{if } \alpha \text{ is even} \\ 2^{(3\alpha+1)/2}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

Proof. (a) We can express the number of solutions of the congruence in the form

$$\rho(p^\alpha) = \frac{1}{p^\alpha} \sum_{a=1}^{p^\alpha} \left(\sum_{u=1}^{p^\alpha} e\left(\frac{au^2}{p^\alpha}\right)\right)^3 \tag{2.1}$$

Moreover, (see [6], Chapter 1, §3) if p is an odd prime number and $p \nmid a$,

$$S(a, p^\gamma) = \sum_{u=1}^{p^\gamma} e\left(\frac{au^2}{p^\gamma}\right) = \begin{cases} \left(\frac{a}{p}\right) i^{((p-1)/2)^2} p^{\gamma/2}, & \text{if } \gamma \equiv 1(\bmod 2) \\ p^{\gamma/2}, & \text{if } \gamma \equiv 0(\bmod 2) \end{cases} \tag{2.2}$$

where $\left(\frac{a}{p}\right)$ is Legendre's Symbol. Therefore, we can write

$$\rho(p^\alpha) = \frac{1}{p^\alpha} \sum_{a=1, p\nmid a}^{p^\alpha} (S(a, p^\alpha))^3 + \frac{1}{p^\alpha} \sum_{\beta=1}^{\alpha} \sum_{a=1, p\nmid a}^{p^{\alpha-\beta}} p^{3\beta} \left(S(a, p^{\alpha-\beta})\right)^3 .$$

First, we suppose $\alpha \equiv 0(\bmod 2)$. We have

$$\rho(p^\alpha) = \frac{1}{p^\alpha} \sum_{a=1, p\nmid a}^{p^\alpha} p^{3\alpha/2} + \frac{1}{p^\alpha} \sum_{\gamma=1}^{\alpha/2} \sum_{a=1, p\nmid a}^{p^{\alpha-2\gamma}} p^{6\gamma} p^{3(\alpha-2\gamma)/2} +$$

$$+ \frac{1}{p^\alpha} \sum_{\gamma=1}^{\alpha/2} \sum_{a=1, p\nmid a}^{p^{\alpha-2\gamma+1}} p^{6\gamma-3} p^{3(\alpha-2\gamma+1)/2} \left(\frac{a}{p}\right)^3 i^{3((p-1)/2)^2} .$$

As $\sum_{a=1, p\nmid a}^{p} (\frac{a}{p}) = 0$, we obtain

$$\rho(p^\alpha) = p^{2\alpha} + p^{2\alpha-1} - p^{3\alpha/2-1}. \tag{2.3}$$

Analogously, for $\alpha \equiv 1(\bmod 2)$, we obtain

$$\rho(p^\alpha) = p^{2\alpha} + p^{2\alpha-1} - p^{(3\alpha-1)/2}. \tag{2.3'}$$

(b) In the case $p = 2$, when $2 \nmid a$, the Gauss sum holds

$$S(a, 2^\gamma) = \sum_{x=1}^{2^\gamma} e\left(\frac{ax^2}{2^\gamma}\right) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma = 1 \\ 2^{\gamma/2}(1 + i^a), & \text{if } \gamma > 0 \text{ and even} \\ 2^{\gamma/2}(1 + i)e^{(i\pi/4)(a-1)}, & \text{if } \gamma > 1 \text{ and odd} \end{cases}$$

(see [5] Thm. 3, Chapter 11). Then

$$\rho(2^\alpha) = \frac{1}{2^\alpha} \sum_{a=1,2\nmid a}^{2^\alpha} (S(a, 2^\alpha))^3 + \frac{1}{2^\alpha} \sum_{\beta=1}^{\alpha} \sum_{a=1,2\nmid a}^{2^{\alpha-\beta}} 2^{3\beta} \left(S(a, 2^{\alpha-\beta})\right)^3.$$

When $\alpha \equiv 0(\bmod 2)$, $\alpha > 0$, we have

$$\rho(2^\alpha) = \frac{1}{2^\alpha} \sum_{a=1,2\nmid a}^{2^\alpha} 2^{\frac{3\alpha}{2}} (1 + i^a)^3 + \frac{1}{2^\alpha} \sum_{\gamma=1}^{\frac{\alpha}{2}-1} \sum_{a=1,2\nmid a}^{2^{\alpha-2\gamma}} 2^{6\gamma} 2^{\frac{3}{2}(\alpha-2\gamma)} (1 + i^a)^3$$
$$+ \frac{(1+i)^3}{2^\alpha} \sum_{\gamma=1}^{\alpha/2-1} \sum_{a=1,2\nmid a}^{2^{\alpha-2\gamma+1}} 2^{6\gamma-3} 2^{\frac{3}{2}(\alpha-2\gamma+1)} e^{\frac{3i\pi}{4}(a-1)} + 2^{2\alpha}.$$

It says

$$\rho(2^\alpha) = 2^{3\alpha/2}. \tag{2.4}$$

In same form, when $\alpha \equiv 1(\bmod 2)$, we deduce

$$\rho(2^\alpha) = 2^{(3\alpha+1)/2}. \tag{2.4'}$$

□

Let $F(s)$ be the Dirichlet function of $\rho(n)$,

$$F(s) = \sum_{n=1}^{\infty} \frac{\rho(n)}{n^s}.$$

By Lemma 1 and the properties of Dirichlet series we have

$$F(s)=\zeta(s-2)\zeta(2s-3)\sum_{n=1}^{\infty}\frac{\mu(n)f(n)}{n^{2s-2}}$$

where $\mu(n)$ is the Möbius function and $f(n)$ is the multiplicative function

$$f(n)=\begin{cases}1, & \text{if } n \text{ is odd}\\ 4, & \text{if } n \text{ is even.}\end{cases}$$

Hence, we can express

$$\rho(n)=n^2\sum_{d^2|n}\frac{g(d)}{d^2} \tag{2.5}$$

with

$$g(d)=\begin{cases}\varphi(d), & \text{if } d \text{ is odd}\\ -2\varphi(d), & \text{if } d \text{ is even}\end{cases} \tag{2.6}$$

where $\varphi(d)$ is Euler's function,

$$\varphi(d)=\#\{n\in\mathbb{N}\mid 1\le n\le d,\ (n,d)=1\}.$$

It is known that

$$\sum_{\substack{n=1\\(2,n)=1}}^{\infty}\frac{\varphi(n)}{n^s}=\prod_{p\neq 2}\left\{\sum_{\alpha=0}^{\infty}\frac{\varphi(p^\alpha)}{p^{\alpha s}}\right\}=\frac{2^s-2}{2^s-1}\frac{\zeta(s-1)}{\zeta(s)},\qquad s>2$$

and (see Thm. 11.12 of [1])

$$\sum_{\substack{n=1\\(2,n)=1}}^{\infty}\frac{\varphi(n)\log n}{n^4}=-\left(\frac{\zeta(s-1)}{\zeta(s)}\right)'_{s=4}+\left(\frac{1}{2^s-1}\frac{\zeta(s-1)}{\zeta(s)}\right)'_{s=4},\qquad s>2.$$

Lemma 2. *It holds*

$$\sum_{n\le x}\frac{g(n)}{n^4}=\frac{4}{5}\frac{\zeta(3)}{\zeta(4)}+O\Big(\frac{\log x}{x^3}\Big). \tag{2.7}$$

Proof. From (2.6) this sum is

$$\sum_{n\le x}\frac{g(n)}{n^4}=3\sum_{\substack{n\le x\\(2,n)=1}}\frac{\varphi(n)}{n^4}-2\sum_{n\le x}\frac{\varphi(n)}{n^4}. \tag{2.8}$$

Notice that, from Abel summation Formula

$$\sum_{n\leq x}\frac{\varphi(n)}{n^4}=\frac{\zeta(3)}{\zeta(4)}-\frac{1}{2\zeta(2)x^2}+O\left(\frac{\log x}{x^3}\right). \tag{2.9}$$

As $\varphi(n)=n\sum_{d|n}\frac{\mu(d)}{d}$, we can write

$$\sum_{\substack{n\leq x\\(2,n)=1}}\frac{\varphi(n)}{n^4}=\sum_{\substack{n\leq x\\(2,n)=1}}\frac{\mu(n)}{n^4}\sum_{\substack{m\leq x/n\\(2,m)=1}}\frac{1}{m^3}. \tag{2.10}$$

We need the following relations

$$\sum_{\substack{n\leq x\\(2,n)=1}}\frac{1}{n^3}=\frac{7}{8}\zeta(3)-\frac{1}{4}x^{-2}+O(x^{-3}) \tag{2.11}$$

and

$$\sum_{\substack{n\leq x\\(q,n)=1}}\frac{\mu(n)}{n^\alpha}=\frac{q^\alpha}{\zeta(\alpha)J_\alpha(q)}+O\left(\frac{\sigma^*_{-1+\epsilon}(q)\delta(x)}{x^{\alpha-1}}\right) \tag{2.12}$$

where $J_\alpha(n)=\sum_{dq=n}\mu(d)q^\alpha$, where $\sigma^*_s(q)$ is the sum of s-th powers of the square-free divisors of q, and where $\delta(x)=\exp\{-A\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\}$, for A a positive constant (see [7], Lemma 3.6). From (2.10),(2.11) and (2.12) we obtain

$$\sum_{\substack{n\leq x\\(2,n)=1}}\frac{\varphi(n)}{n^4}=\frac{14}{15}\frac{\zeta(3)}{\zeta(4)}-\frac{1}{3\zeta(2)x^2}+O\left(\frac{\log x}{x^3}\right). \tag{2.13}$$

Then, replacing (2.9) and (2.13) in (2.8) we obtain (2.7). □

3. Proofs of Theorems

Proof of Theorem 1. By (2.5), one has

$$\begin{aligned}\sum_{n\leq x}\rho(n)&=\sum_{n\leq\sqrt{x}}n^2g(n)\sum_{m\leq\frac{x}{n^2}}m^2\\&=\frac{x^3}{3}\sum_{n\leq\sqrt{x}}\frac{g(n)}{n^4}+O\left(x^2\sum_{n\leq\sqrt{x}}\frac{|g(n)|}{n^2}\right).\end{aligned}$$

Notice that $\sum_{n\le x}\frac{g(n)}{n^2}=O(\log x)$. From this bound and Lemma 2 we have

$$\sum_{n\le x}\rho(n)=\frac{x^3}{3}\left\{\frac{4}{5}\frac{\zeta(3)}{\zeta(4)}+O(\frac{\log x}{x^{3/2}})\right\}+O(x^2\log x)$$

and (1.4) is proved. □

Proof of Theorem 2. From (1.2) we can write

$$\sum_{1\le n,m,t\le x}\tau(n^2+m^2+t^2)=2S_1-S_2, \tag{3.1}$$

where

$$S_2=\sum_{d\le x\sqrt{3}}\ \sum_{\substack{1\le n,m,t\le x\\ n^2+m^2+t^2\equiv 0(\operatorname{mod} d)\\ n^2+m^2+t^2\le dx\sqrt{3}}} 1=\sum_{d\le x\sqrt{3}}S(d).$$

Now, the inner sum $S(d)$ is bounded by the number of integer points (n,m,t) such that $1\le n,m,t\le\sqrt{dx\sqrt{3}-2}$ and $n^2+m^2+t^2\equiv 0(\operatorname{mod} d)$.

For x any real number, let $[x]$ denote the largest integer $\le x$. Making the following change of variables, $n=n_1d+v_1,\ m=m_1d+v_2,\ t=t_1d+v_3$, $1\le v_1,v_2,v_3\le d$, we have

$$S_2\le\sum_{d\le x\sqrt{3}}\ \sum_{1\le v_1,v_2,v_3\le d}\theta(v_1)\theta(v_2)\theta(v_3)$$

where we denote

$$\theta(v_i)=\left[\left(\sqrt{dx\sqrt{3}-2}-v_i\right)/d\right]+1$$

for every $i=1,2,3$.

On the other hand, $S(d)$ is bigger than the number of integer points (n,m,t) such that $1\le n,m,t\le\sqrt{dx/\sqrt{3}-2}$ and $n^2+m^2+t^2\equiv 0(\operatorname{mod} d)$. So

$$S_2\ge\sum_{d\le x/\sqrt{3}}\ \sum_{1\le v_1,v_2,v_3\le d}\Phi(v_1)\Phi(v_2)\Phi(v_3)$$

where

$$\Phi(v_i)=\left[\left(\sqrt{(dx/\sqrt{3})-2}-v_i\right)/d\right]+1$$

for every $i = 1, 2, 3$.

Using the properties of $[x]$, we obtain

$$\sum_{d \le x/\sqrt{3}} \rho(d) \left[\frac{y_1}{d}\right]^3 < S_2 \le \sum_{d \le x\sqrt{3}} \rho(d) \left(\left[\frac{y}{d}\right] + 1\right)^3 \tag{3.2}$$

where $y = \sqrt{dx\sqrt{3} - 2}$ and $y_1 = \sqrt{\frac{dx}{\sqrt{3}} - 2}$. Then we can write

$$\begin{aligned} S_2 \le & 3^{3/4} x^{3/2} \sum_{d \le x\sqrt{3}} \frac{\rho(d)}{d^{3/2}} + 3^{3/2} x \sum_{d \le x\sqrt{3}} \frac{\rho(d)}{d} \\ & + 3^{5/4} x^{1/2} \sum_{d \le x\sqrt{3}} \frac{\rho(d)}{d^{1/2}} + \sum_{d \le x\sqrt{3}} \rho(d). \end{aligned} \tag{3.3}$$

Using Abel summation formula and Theorem 1, we get

$$\sum_{n \le x} \frac{\rho(n)}{n} = \frac{1}{x} A(x) + \int_1^x \frac{A(t)}{t^2} dt + O(x) = \frac{2}{5} \frac{\zeta(3)}{\zeta(4)} x^2 + O(x \log x), \tag{3.4}$$

where $A(t) = \sum_{n \le t} \rho(n)$. In the same form we deduce

$$\sum_{n \le x} \frac{\rho(n)}{\sqrt{n}} = \frac{8}{25} \frac{\zeta(3)}{\zeta(4)} x^{5/2} + O(x^{3/2} \log x) \tag{3.5}$$

and

$$\sum_{n \le x} \frac{\rho(n)}{n^{3/2}} = \frac{8}{15} \frac{\zeta(3)}{\zeta(4)} x^{3/2} + O(x^{1/2} \log x). \tag{3.6}$$

Therefore, from (3.3), (3.4)-(3.6) and Theorem 1 we deduce

$$S_2 \le \frac{222\sqrt{3}}{25} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \tag{3.7}$$

Analogously, from (3.2) we have

$$\begin{aligned} S_2 > & 3^{-3/4} x^{3/2} \sum_{d \le x/\sqrt{3}} \frac{\rho(d)}{d^{3/2}} - 3^{1/2} x \sum_{d \le x/\sqrt{3}} \frac{\rho(d)}{d} \\ & + 3^{3/4} x^{1/2} \sum_{d \le x/\sqrt{3}} \frac{\rho(d)}{d^{1/2}} - \sum_{d \le x/\sqrt{3}} \rho(d) + O(x^2). \end{aligned}$$

Then, from (3.4),(3.5), (3.6) and Theorem 1, we have

$$S_2 > \frac{2}{225\sqrt{3}} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \tag{3.8}$$

For the first sum, S_1, it has

$$\sum_{d \le x\sqrt{3}} \rho(d) \left[\frac{x}{d}\right]^3 < S_1 \le \sum_{d \le x\sqrt{3}} \rho(d) \left(\left[\frac{x}{d}\right] + 1\right)^3$$

As before, by Theorem 1 and partial summation, we obtain

$$\sum_{n \le x} \frac{\rho(n)}{n^2} = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} x + O(\log x). \tag{3.9}$$

In a similar manner to the proof of Theorem 1, one has from (2.5)

$$\sum_{n \le x} \frac{\rho(n)}{n^3} = \sum_{n \le \sqrt{x}} \frac{g(n)}{n^4} \sum_{m \le x/n^2} \frac{1}{m}$$

and from Thm. 3.2 of [1]

$$\begin{aligned} \sum_{n \le x} \frac{\rho(n)}{n^3} &= \sum_{n \le \sqrt{x}} \frac{g(n)}{n^4} \left\{ \log(\frac{x}{n^2}) + \gamma + O(\frac{n^2}{x}) \right\} \\ &= (\log x + \gamma) \sum_{n \le \sqrt{x}} \frac{g(n)}{n^4} - 2 \sum_{n \le \sqrt{x}} \frac{g(n) \log n}{n^4} + O(\frac{\log x}{x}). \end{aligned}$$

As in (2.8) and (2.9)

$$\sum_{n=1}^{\infty} \frac{g(n) \log n}{n^s} = 3 \sum_{\substack{n=1 \\ (n,2)=1}}^{\infty} \frac{\varphi(n) \log n}{n^s} - 2 \sum_{n=1}^{\infty} \frac{\varphi(n) \log n}{n^s}, \quad s > 2$$

$$\sum_{n \le \sqrt{x}} \frac{g(n) \log n}{n^4} = 3 \left(\frac{1}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} - \left(\frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} + O(\frac{\log x}{x}).$$

So

$$\sum_{n \le x} \frac{\rho(n)}{n^3} = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} \log x + A + O\left(\frac{\log x}{x}\right), \tag{3.10}$$

where A is the constant (1.6).

From (1.4), (3.4), (3.9) and (3.10), in analogy to (3.7) and (3.8), we obtain that

$$S_1 \leq \frac{4}{5}\frac{\zeta(3)}{\zeta(4)}x^3\log x + Bx^3 + O(x^2\log x) \tag{3.11}$$

where

$$B = A + \frac{\zeta(3)}{\zeta(4)}\frac{2}{5}(\log 3 + 9 + 8\sqrt{3}),$$

and

$$S_1 > \frac{4}{5}\frac{\zeta(3)}{\zeta(4)}x^3\log x + Cx^3 + O(x^2\log x) \tag{3.12}$$

where

$$C = A + \frac{\zeta(3)}{\zeta(4)}\frac{2}{5}(\log 3 + 9 - 8\sqrt{3}).$$

From (3.1), (3.7) and (3.12) we get the following inequality

$$\sum_{1\leq n,m,t\leq x} \tau(n^2+m^2+t^2) > \\ > \frac{8}{5}\frac{\zeta(3)}{\zeta(4)}x^3\log x + \left(2C - \frac{222\sqrt{3}}{25}\frac{\zeta(3)}{\zeta(4)}\right)x^3 + O(x^2\log x). \tag{3.13}$$

And from (3.1), (3.8) and (3.11) we get

$$\sum_{1\leq n,m,t\leq x} \tau(n^2+m^2+t^2) \leq \\ \leq \frac{8}{5}\frac{\zeta(3)}{\zeta(4)}x^3\log x + \left(2B - \frac{2}{225\sqrt{3}}\frac{\zeta(3)}{\zeta(4)}\right)x^3 + O(x^2\log x). \tag{3.14}$$

The Theorem 2 follows from inequalities (3.13) and (3.14). □

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