Removing Even Crossings, Continued

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Abstract

In this paper we investigate how certain results related to the Hanani-Tutte theorem can be lifted to orientable surfaces of higher genus. We give a new simple, geometric proof that the weak Hanani-Tutte theorem is true for higher-genus surfaces. We extend the proof to prove that bipartite generalized thrackles in a surface S can be embedded in S.

We also show that a result of Pach and Tóth that allows the redrawing of a graph removing intersections on even edges remains true on higher-genus surfaces. As a consequence, we can conclude that $\mathrm{cr}_S(G)$, the *crossing number* of the graph G on surface S, is bounded by $2\,\mathrm{ocr}_S(G)^2$, where $\mathrm{ocr}(G)_S$ is the *odd crossing number* of G on surface S.

Finally, we begin an investigation of optimal crossing configurations for which $\mathrm{ocr} \cong \mathrm{cr}$.

1 Introduction

We continue the investigation of the Hanani-Tutte theorem and its close relatives begun in "Removing Even Crossings" [10, 11], aiming for analogues on orientable surfaces of higher genus. The theorem of Hanani and Tutte states that every drawing in the plane of a non-planar graph contains two non-adjacent edges which intersect an odd number of times.[1] There are several proofs of the theorem starting with the original papers by Hanani and Tutte [2, 13, 3, 4, 12, 6]. Kleitman's proof [4] is particularly short and elegant. All of these proofs invoke Kuratowski's theorem and then verify the result for subdivisions of $K_{3,3}$ and K_5. This approach seems hopeless for surfaces of higher genus (the list of excluded

[1] We make the usual assumptions on drawings of graphs, see [7, page 230].

minors is not yet known even for the torus). In [10, 11] we used geometric methods to give a new proof of the Hanani-Tutte theorem in the plane that does not rely on Kuratowski's theorem. As a first step, we can extend those methods to prove a weak version of the Hanani-Tutte theorem for higher-genus surfaces: a graph of genus g embedded on a surface of genus less than g must contain two edges that cross an odd number of times. While this result has previously been shown by Cairns and Nikolayevsky using homology theory [1], our proof is entirely geometric. We also show how to use our methods to obtain another result of [1], namely that a bipartite generalized thrackle on a surface is embeddable on that surface.

As our goal is the strong version of the Hanani-Tutte theorem for higher-genus surfaces, it is instructive to see where our proof for the plane breaks down. At the core of that proof is the following result from [10, 11]. An *even* edge in a drawing is an edge that intersects every other edge an even number of times (including the possibility that it does not intersect it at all).

Theorem 1.1 (Pelsmajer, Schaefer, Štefankovič) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

This theorem is a strengthening of the following, earlier result of Pach and Tóth [8, Theorem 1].

Theorem 1.2 (Pach, Tóth) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection.*

In Section 3 we will see that the result of Pach and Tóth can be lifted to higher-genus surfaces, whereas Theorem 1.1 cannot even be extended to the torus. That thwarts our original approach to the Hanani-Tutte theorem. However, we can still conclude that

$$\mathrm{cr}_S(G) \leq 2\,\mathrm{ocr}_S(G)^2$$

is true on arbitrary surface S, where $\mathrm{cr}_S(G)$ is the crossing number of G on S and $\mathrm{ocr}_S(G)$ is the odd crossing number of G on S. This generalizes a result by Pach and Töth who proved $\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$ [8].

In Section 4 we explore another route to understand the relationship between the odd crossing number of a graph and its crossing number. The goal is to establish a linear bound of the form $\mathrm{cr}(G) \leq c\,\mathrm{ocr}(G)$. Our approach is to study possible minimal crossing configurations that can occur. A first such case is investigated here.

2 The weak Hanani-Tutte theorem for higher-genus surfaces

In this section we want to show that the weak Hanani-Tutte theorem is true for arbitrary orientable surfaces. This result is known, with a short and elegant proof using homology theory [1, Lemma 3]. In the spirit of our earlier paper we present an entirely geometric proof of that result.

We do require some background on surfaces. Let S be a compact, connected, orientable surface without boundary—unless we explicitly say otherwise, this is what we mean by *surface* from now on. By the classification theorem for surfaces, S is homeomorphic to a sphere with some number g of handles attached, and g is called the *genus* of S. A closed curve on S is *contractible* if it can be continuously deformed on S to a point. Assume that C is a non-contractible closed curve on S. Locally near C, since S is orientable, $S - C$ is partitioned into exactly two sides. So $S - C$ is a surface with two boundary components, and since each side is in a single component of $S - C$, $S - C$ has at most two components. We say that C is *S-separating* or simply *separating* if $S - C$ has two components. Otherwise, $S - C$ is connected, and we call C *nonseparating* (remember that we assume that C is non-contractible).

Given a nonseparating curve C, we can lower the genus of S by cutting along C and then attaching a disk to each boundary component of $S-C$. This creates a new surface S', which we call the *C-reduced surface*. Euler's formula can be applied to show that the genus of S' is one less than the genus of S.

The following lemma will help us find nonseparating curves.

Lemma 2.1 *Suppose that C and C' are closed curves on a surface S. If C and C' intersect an odd number of times, then neither is S-separating; indeed, both are nonseparating curves.*

Proof Suppose that C is separating. Then $S - C$ has two components S_1, S_2, each with C as its boundary. If we trace the curve C', it must switch between S_1 and S_2 each time it crosses C, and never otherwise. Hence there must be an even number of switches, contradicting the assumption that C and C' intersect oddly. Hence, neither C nor C' can be separating. Since they cannot be contractible either (a closed contractible curve intersects another closed curve an even number of times), they are both nonseparating. □

Of course, we wish to study graphs drawn on surfaces; the next proposition tells us how to find nonseparating curves in a graph.

Proposition 2.2 *Let G be a graph with a single vertex v, drawn on a surface of genus $g > 0$, so that all edges are even. Then either G contains an edge e that is a nonseparating curve, or else we can draw a new nonseparating curve through v that intersects each edge of G an even number of times.*

In the latter case, we can add a new edge e to G and let it be drawn as the new curve.

Proof Since $g > 0$ the surface must contain a nonseparating curve C, and we may assume that $v \notin C$. If C crosses no edge of G an odd number of times then we can use the curve C itself: deform a small segment of it (without crossing any vertices) to approach v between two consecutive edges in the rotation, so the curve eventually contains v. Otherwise there is a loop e in G that intersects C an odd number of times. By Lemma 2.1, e is nonseparating. Since every edge in G is even, we can use the curve e. □

We are now ready to state and prove the weak Hanani-Tutte theorem for arbitrary (orientable) surfaces. The inductive part of the argument is similar to the analogous result for the plane; the interesting new aspect in the proof is the base case.

Theorem 2.3 (Cairns, Nikolayevsky) *If G can be drawn on a surface of genus g so that all its edges are even, then G can be embedded on that surface, i.e. drawn intersection-free, without changing the rotation system.*

Proof We can assume that G is connected. Fix a drawing D of G on a surface of genus g. The proof will be a double induction over the genus g of the surface and the number of vertices of G. We need to keep track of the cyclic ordering of the edges leaving each vertex in D. This ordering is called the *rotation* at that vertex. The *rotation system* of the drawing is the collection of the rotations at all vertices.

To make the inductive step work, we prove the following slightly stronger statement:

> If D is a drawing of a multigraph G on a surface of genus g so that any pair of edges intersects an even number of times in D, then G can be drawn without intersections on a surface of genus g and without changing the rotation system.

If G has more than a single vertex, D must contain an even non-loop edge uv. We proceed as in a previous paper [10, Theorem 1.1]: contract the edge uv by pulling v towards u as shown in Figure 1.

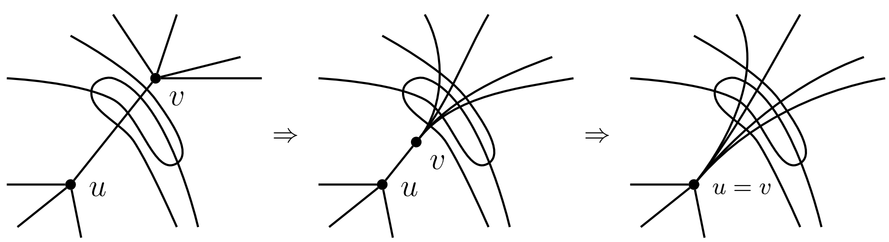

Figure 1: Pulling an endpoint (left to middle) and contracting the edge (middle to right); illustration taken from [10].

In the new drawing the edges incident to v remain even, since uv was an even edge. We might have introduced self-intersections by contracting uv, but these self-intersections are easily eliminated as shown in Figure 2.

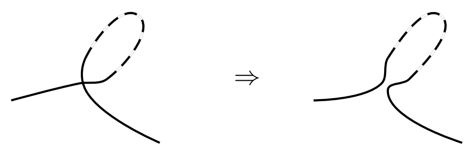

Figure 2: Removing a self-intersection; illustration taken from [10].

Finally, we contract uv to a single vertex u and join the rotations at the two vertices appropriately (see the right part of Figure 1). Call the new graph G'. By inductive assumption, there is a drawing of G' on a surface of genus g without intersections that respects the rotation system. In such a drawing we can split the vertex u into vertices u and v and reintroduce the edge between the two vertices (without any intersections).

This leaves us with the case that G consists of a single vertex u with loops. If $g = 0$ we can redraw the loops without any intersections. We can therefore assume that $g > 0$. By Proposition 2.2 there is an edge e in G—or one can be added to G so as to not create any odd intersections—which is drawn as a nonseparating curve. We will create the e-reduced surface, but first we remove intersections with e. Since e is even, the intersections with e can be partitioned into pairs such that each pair involves the same edge (other than e). Now erase all intersections with e, and for each pair, on each side of e, draw a curve alongside e to connect the severed ends. (See Figure 3 for an example.)

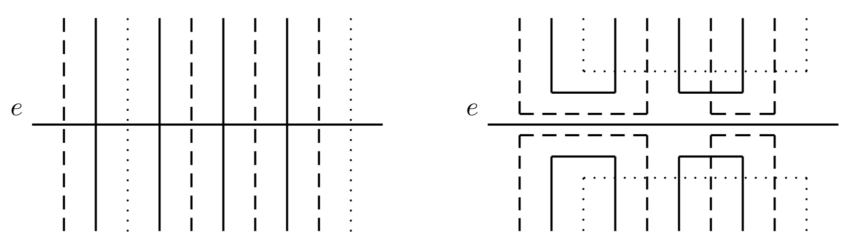

Figure 3: Eliminating intersections with e.

Note that this procedure does not change the parity of the number of intersections between any pair of edges, although it does lead to "curves" with multiple components, only one of which contains v; we will deal with these shortly.

Since we have removed all intersections from e, we can now contract e to a single vertex, and split the surface at that vertex. Since e is intersection-free, it naturally splits the rotation of u into two rotations; we split the vertex u into two vertices u and v with those rotations. Call the resulting graph G'. Our current drawing of G' on the e-reduced surface is not quite proper: some of the edges are represented by "curves" that have more than one component. However, we can deform a small segment of a component close to another component of the same curve, without crossing any vertices; then the local redrawing move shown in Figure 4 combines the two components. Repeatedly doing this for all curve components will yield a single curve. (At this point we made use of the fact that e is nonseparating.)

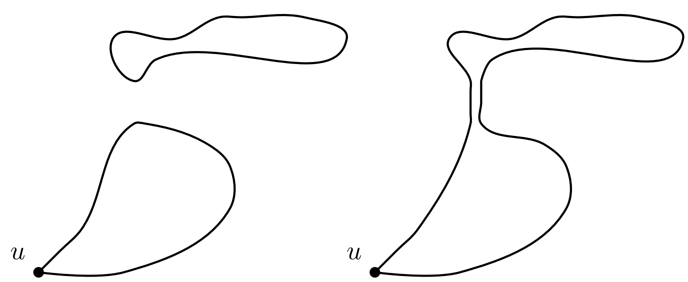

Figure 4: Reconnecting a closed component to the main curve.

Repeating this for each curve gives us a drawing of G' on a surface of genus $g - 1$ such that every pair of edges intersect an even number of times. By the inductive assumption G' has a drawing on a surface of genus $g - 1$ without intersections with the rotation system unchanged. We can now attach a handle to that surface with one end near u, where edges incident to v belong in the rotation, and the other end near v, where the edges incident to u belong in the rotation. Then we can move u and v close together along the handle and merge them, recovering the original rotation of u. Finally we can add back the loop e around the handle (or not, if it we artificially added the loop to G), without introducing any intersections. This finishes the proof. □

Cairns and Nikolayevsky used the weak Hanani-Tutte theorem to prove an interesting result on generalized thrackles. A graph is a *thrackle* if it can be drawn such that any pair of edges intersects exactly once, where a common endpoint of two edges counts as an intersection of these two edges (a convention we will use only in the remainder of this section while talking about thrackles). A *generalized thrackle* is a graph that can be drawn such that any pair of edges intersects an odd number of times (again counting endpoints).

Remark Two edges *cross* (rather than intersect) if they intersect at a point which is not an endpoint of either edge (we also assume they do not touch). We can easily see that a graph is a generalized thrackle if and only if it can be drawn such that any pair of edges crosses (rather than intersects) an odd number of times: Simply redraw the graph locally at each vertex so every pair of edges incident to the vertex cross an odd number of times as illustrated in Figure 5. This suffices since for each intersection of two edges at an endpoint there is one new crossing, and all other intersections are crossings. The desired redrawing can always be done: We can assume that, locally, the edges leaving the vertex look as shown on the left of Figure 5; that is, the edges are straight lines leaving upwards. We now reverse the rotation at the vertex and reconnect. (Cairns and Nikolayevsky use an equivalent idea.)

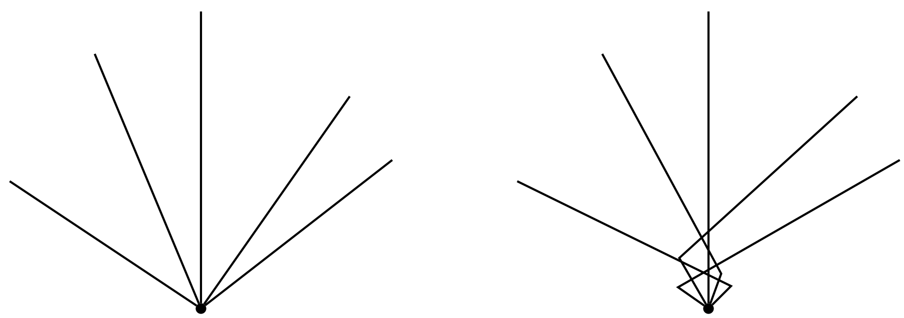

Figure 5: Flipping the parity of crossings at a vertex.

Now suppose G is a generalized thrackle on some surface of genus g. Let us also assume that G is bipartite, that is, all edges are between some vertex sets V_0 and V_1. Draw G so that any pair of edges intersects evenly (counting endpoints). Apply the parity-flipping operation described in Remark 2 to each vertex in V_1 only. Then two edges cross an even number of times if and only if they share an endpoint in V_0.

Pick a set of edges such that every vertex of V_0 is contained in exactly one edge. One by one, select one of these edges $e = uv$, with $v \in V_0, u \in V_1$, and contract v to u along e, merging the rotations appropriately. This can create edges with both endpoints in V_1. For convenience, call this set of edges E_1; then $E \backslash E_1$ are the edges that still go from V_0 to V_1.

Claim: After each step, every edge in E_1 is even and the parity of crossings between every other pair of edges in $E \backslash E_1$ is unchanged. This can be shown inductively: At each step, each edge f incident to v other than e is extended along the old drawing of e, adding an odd number of crossings with the edges that cross e oddly—which are precisely the edges in $E \backslash E_1$ that are not incident to v. Since f already had an even number of crossings with each other edge incident to v and with each edge in E_1, the contraction makes f even. Since the edges incident to v other than e are the edges that get moved from $E \backslash E_1$ to E_1, the claim follows.

Thus we obtain a drawing of a graph on a surface of genus g in which all edges are even (We might have to remove self-intersections as we did earlier). By Theorem 2.3 the graph can be embedded on the surface without any intersections, without changing the rotation. Since the rotation did not change, we can uncontract edges and obtain a planar drawing of the original graph. This gives an easy geometric proof of the following theorem.

Theorem 2.4 (Cairns, Nikolayevsky) *If G is a bipartite, generalized thrackle on a surface of genus g, then G can be embedded on that surface.*

The special case $g = 0$ of the theorem was first proved by Lovász, Pach, and Szegedy[5]: if a bipartite graph is a generalized thrackle, then it is planar.

3 Removing even crossings in higher-genus surfaces

In Section 2 we gave a new proof of the weak Hanani-Tutte theorem for arbitrary surfaces (as first proved by Cairns and Nikolayevsky). In the plane we know that a stronger result is true: Pach and Tóth first showed the following result [8]:

Theorem 3.1 (Pach, Tóth) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection.*

Pach and Tóth applied their result to show a relationship between two different notions of crossing numbers. The *crossing number*, $\text{cr}(G)$, of a graph G is the smallest number of intersections in a drawing of G.[2] The *odd crossing number*, $\text{ocr}(G)$, is the smallest number of pairs of edges that intersect oddly in a drawing of G. By definition $\text{ocr}(G) \leq \text{cr}(G)$; however there are graphs for which the two numbers differ [9]. Pach and Tóth showed that on the other hand that $\text{cr}(G) \leq 2\,\text{ocr}(G)^2$.

The redrawing procedure used in the proof of Theorem 3.1 can lead to an increase of the odd crossing number, and will, therefore, probably not lead to better bounds of $\text{cr}(G)$ in terms of $\text{ocr}(G)$ (a linear bound is suspected). In a previous paper we showed that Theorem 3.1 can be strengthened to avoid an increase in the odd crossing number:

Theorem 3.2 (Pelsmajer, Schaefer, Štefankovič) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

[2] We make all the standard assumptions on a drawing, such as requiring that not more than two edges intersect in an intersection point [7, page 230].

As a consequence, we were able to show that crossing number and odd crossing number are the same when they are at most 3.

As it turns out the stronger Theorem 3.2 fails for higher-genus surfaces, indeed, it already fails for the torus, as the following example shows:

Example 3.3 Consider the graph G shown in Figure 6. The graph consists of

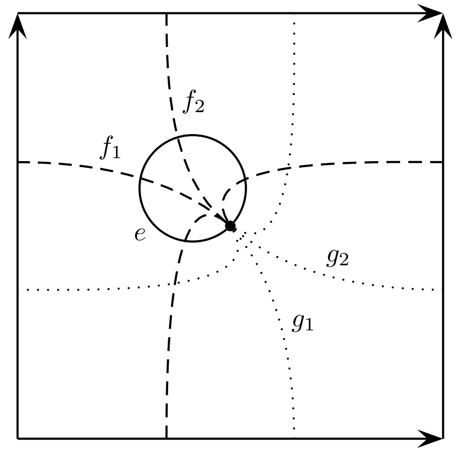

Figure 6: An example showing that Theorem 3.2 is not true for the torus.

an even loop e and two pairs of loops, one pair with its ends inside of e (f_1, f_2 in Figure 6) and the other with its ends outside of e (g_1 and g_2). The two loops in each pair alternate ends at the vertex, but do not intersect each other. Each loop with ends inside of e intersects exactly one of the loops with ends outside of e oddly (and the other loop not at all). We claim that G cannot be embedded in the torus in such a way that e is free of intersections and the pairs of edges that intersect oddly do not change, if the rotation at the vertex is fixed. For a contradiction, assume that G can be embedded thus. Then e is either contractible or nonseparating (there are no separating curves on the torus). If e is contractible, then one of the two pairs has to lie entirely within e, i.e. be embedded in the plane. Consequently, the two loops in that pair must intersect oddly, which they did not do before. If e is nonseparating, both pairs of loops are embedded in a region homeomorphic to an annulus (a plane with two holes), which forces the loops in each pair to intersect each other. Again this changes which pairs of edges intersect oddly.

This example can be modified to create a simple graph without a fixed rotation system, for which Theorem 3.2 on the torus would fail. Erase the drawing in a small ball containing the vertex of G, and draw the wheel W_{10} there without crossings, such that its interior has ten 3-faces and one vertex of degree 10. The five edges of G have ten ends, which we can extend to meet each of the other vertices of W_{10}, without creating any crossings. Let G' be the resulting graph drawn on the torus.

Suppose that there is an ocr-optimal drawing in which the even edges of G' (namely, e and the edges of W_{10}) are crossing-free. We can apply the previous argument if W_{10} is drawn the same way. Otherwise some triangle of W_{10} does not bound a face, either because it is nonseparating or because its interior intersects W_{10}. In both cases, the rest of G must be drawn in a subsurface of genus 0, and it is easy to see that there must be new odd pairs.

In these particular examples, it is easy to redraw the graphs so that e is intersection-free and the odd crossing number does not increase, while keeping the rotation system the same. We do not know whether this is true in general.

However, the original result of Pach and Tóth *is* true for higher-genus surfaces.

Theorem 3.4 *If D is a drawing of a connected graph G in some surface S, and E_0 is the set of even edges in D, then G can be drawn in S so that no edge in E_0 is involved in an intersection.*

Proof Fix a drawing D of G in S. The proof will be a double induction over the genus g of the surface S and the number of vertices of G. As usual we keep track of the rotation system.

For the induction, we prove the following slightly stronger statement:

> If D is a drawing of a multigraph G on a surface of genus g with even edges E_0, then G can be drawn in S so that no edge in E_0 is involved in an intersection and the rotation system of the drawing is the same as that of D.

As we have done before we contract even edges while this is possible, maintaining the rotation system. In this way we obtain a graph in which all non-loop edges are odd, i.e. they intersect at least one other edge an odd number of times. We continue by contracting the odd edges as well. The important observation is that since all even edges are loops at this point, even edges remain even.

We obtain a graph with a single vertex u and a bouquet of loops, some odd and some even. If the drawing contains a nonseparating even loop e, we proceed as in the proof of Theorem 2.3: we remove all intersections with e, contract e to a single vertex, and split the surface at that vertex, and also split the vertex into two vertices u and v with the rotations dictated by the surface split. Call the resulting graph G'. As we did earlier, we reconnect the components of each curve.

By the inductive assumption we know that G' can be drawn in a surface of genus $g - 1$ so that no edge in E_0 is involved in an intersection, while keeping the rotation system the same. We can then add a handle close to u and v and identify these vertices, reconstructing the rotation system of G.

If the drawing does not contain a nonseparating even loop, all even loops are either contractible or surface-separating. Consider the rotation system at u and an even loop e at u. Since the surface is split by e into two pieces, any other loop f at u—which must cross e an even number of times—must begin and end on the same side of e. Hence the ends of e and f cannot alternate at u. This means that we can redraw all the even loops in a very small neighborhood of u, without changing the rotation at u, so that no two of them intersect each other. The even loops might still intersect odd loops, indeed might intersect odd loops that they did not intersect before, but by the argument we made, they will intersect them an even number of times; hence, after the redrawing none of the edges in E_0 intersect each other, and they remain even, so we are justified in still calling them even. Note that for this process to work, it was essential that none of the even loops is nonseparating.

At this point even loops only intersect odd loops. Consider an even loop e at u. If an odd loop f crosses e, we simply remove the segment between the first and last crossing with e and directly connect the two ends alongside the boundary of e (we argued earlier that the two ends are not separated by e). Repeating this finishes the proof. □

Observe that the proof potentially increases the odd crossing number of the drawing, both when contracting odd edges, as well as in the last step when reconnecting the ends of odd loops. Nevertheless, it is good enough to extend the result by Pach and Tóth that $\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$ [8] in the plane to any (orientable) surface.

Corollary 3.5 *For any surface S we have*

$$\mathrm{cr}_S(G) \leq 2\,\mathrm{ocr}_S(G)^2.$$

Proof Let D be an ocr-optimal drawing of $G = (V, E)$ in surface S, i.e. a drawing realizing $\mathrm{ocr}_S(G)$. Let E_0 be the set of even edges in D. Using Theorem 3.4 we can obtain a drawing of G in which all edges of E_0 are free of intersections. In other words, only the edges not in E_0 are involved in intersections, and there are at most $|E - E_0| \leq 2\,\mathrm{ocr}_S(G)$ of them. Erase all of the edges in $E - E_0$ and redraw them so as to minimize the number of crossings between them. Obviously no pair of these edges needs to intersect more than once, so the new drawing has crossing number at most

$$\binom{2\,\mathrm{ocr}_S(G)}{2} \leq 2\,\mathrm{ocr}_S(G)^2.$$

□

If the surface S is the plane, then the above proof can be simplified even further by removing the induction on the genus, giving a really simple proof of the fact that

$$\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$$

in the plane.

4 Odd-crossing Minimal Configurations in Planar Drawings

Theorem 4.1 *Suppose that $G = (V, E)$ has an* ocr*-optimal D in the plane with an edge e^* that intersects each of $e_1, \ldots, e_k \in E$ an odd number of times, and all edges in $E - \{e^*, e_1, \ldots, e_k\}$ are even. Then* $\text{ocr}(G) = \text{cr}(G)$.

Proof We can assume that G is connected (otherwise we can deal with each component separately). Let $E' = \{e^*, e_1, \ldots, e_k\}$. Then we can redraw G using the even crossing lemma [10] so that every even edge in D is free of intersections in the new drawing D', and no new odd pairs are created. Since $\text{ocr}(D) = \text{ocr}(G)$, no odd pair of D can become even in D'. In particular e^* crosses each e_i in D', and hence all of E' lies in a single face of the plane graph $D' - E'$.

Without loss of generality, this face is the outer face. Contract each component of $D' - E'$ within the plane to a vertex, preserving rotations. Any remaining even edges are loops whose interiors only contain other even loops; we remove them all. We may now redraw while maintaining rotation, after which the loops can be added back and edges can be then uncontracted, without adding any crossings. Note that the set of remaining edges is E'.

Case 0 If there is a single vertex, then the parity of crossings between each pair of edges is determined by whether their ends alternate or not in the rotation, so we can easily redraw (as in [10]) to achieve cr = ocr. Hence we may assume that more than one vertex remains.

Case 1 Assume e^* is a loop at vertex v, say. By redrawing e^* near v, we can ensure that e^* intersects only edges with one end at v and that lie "between" the two ends of v in the rotation (if we look at it the right way, clockwise or counter-clockwise, as appropriate). See Figure 7. The intersections of the redrawn e^* must include every edge, since otherwise the redrawing lowers the number of odd intersections involving e^*. By redrawing e^* to be drawn on the other side near v (see Figure 7, we come to the same conclusion, implying that every edge e_i must be a loop at v with ends that alternate with the ends of e^* in the rotation. But this is Case 0.

Figure 7: Two ways of drawing e^* near v.

Case 2 e^* has two different endpoints $u \neq v$.

Suppose that x is another vertex; x is adjacent to some e_i. Consider the portion of e_i from x to the first crossing with e^*. (See Figure 8 on the left.) We can redraw all edges that cross this segment as shown in the middle of Fig. 8;

note that this doesn't affect the parity of any pair's crossings. Then we can pull x along the segment until it crosses e^*, as shown on the right of Fig 8. This last move changes the parity of crossings of e^* with an edge e_j if and only if e_j has exactly one endpoint at x. Since D is ocr-optimal, no odd intersection with e^* can be eliminated; thus every edge incident to x is a loop. But then x and e^* are in different components, so G was not connected, contradicting our assumption.

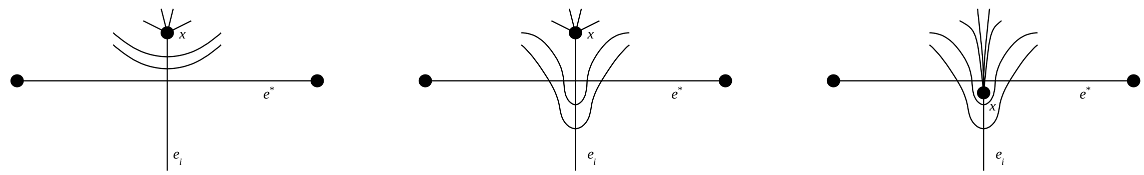

Figure 8: Make room, then pull x across e^*.

We conclude that u and v are the only vertices (remaining after the contractions). Redraw e^* near u by adding an extra twist (see Fig. 9). This changes the parity of crossings of e^* with each non-loop e_j incident to u, which is impossible since it would reduce the odd crossing number of the drawing; so there are no non-loops in E' incident to u. By the same argument this is true for v as well. In other words, the only edge that is not a loop is e^*.

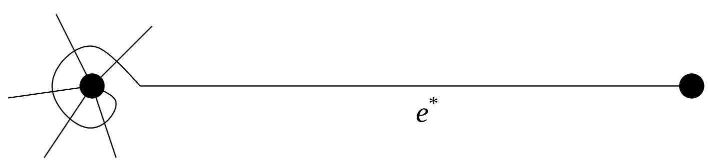

Figure 9: Add an extra twist near u.

Each loop at u can be contracted within the sphere $S^2 - \{v\}$ to a small neighborhood of u (preserving rotation at u), without changing the parity of any pair's crossings (since we do not cross v). The same argument holds for v. See Fig. 10 for an illustration of the resulting drawing. But now we can flip all loops as shown in the right side of Fig. 10 to free e^* of all intersections without changing any other pair's parity of crossings). This reduces the odd crossing number contradicting our assumption of starting with an ocr-optimal drawing. □

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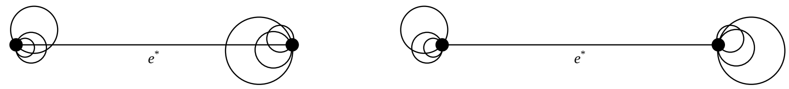

Figure 10: Flip all loops.

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