Funkcialaj Ekvacioj, 46 (2003) 121–171

On a Class of Algebraic Solutions to the Painlevé VI Equation, Its Determinant Formula and Coalescence Cascade

By

Tetsu MASUDA

(Kobe University, Japan)

Abstract. A determinant formula for a class of algebraic solutions to the Painlevé VI equation (P_{VI}) is presented. This expression is regarded as a special case of the universal characters. The entries of the determinant are given by the Jacobi polynomials. Degeneration to the rational solutions of P_V and P_{III} is discussed, using the coalescence procedure. The relationship between Umemura polynomials associated with P_{VI} and our formula is also discussed.

1. Introduction

Enlarging the work by Yablonskii and Vorob'ev for P_{II} [28] and Okamoto for P_{IV} [23], Umemura has introduced special polynomials associated with a class of algebraic (or rational) solutions to each of the Painlevé equations P_{III}, P_V and P_{VI} [27]. These polynomials are generated by the Toda equation that arises from the Bäcklund transformations of each Painlevé equation. It has been also found that the coefficients of the polynomials admit mysterious combinatorial properties [15, 26].

It is remarkable that some of these polynomials are expressed as a specialization of the Schur functions. Yablonskii-Vorob'ev polynomials are expressible by 2-core Schur functions, and Okamoto polynomials by 3-core Schur functions [7, 8, 16]. It is now recognized that these structures reflect the affine Weyl group symmetry, as groups of the Bäcklund transformations [29]. The determinant formulas of Jacobi-Trudi type for Umemura polynomials of P_{III} and P_V resemble each other. In both cases, they are expressed by 2-core Schur functions, and entries of the determinant are given by the Laguerre polynomials [5, 17].

Furthermore, in a recent work, it has been revealed that the entire families of the characteristic polynomials for rational solutions of P_V, which include Umemura polynomials for P_V as a special case, admit more general structures [12]. Namely, they are expressed in terms of the universal characters that are a generalization of the Schur functions. The latter are the characters of the irreducible polynomial representations of $GL(n)$, while the former were introduced to describe the irreducible rational representations [11].

What kind of determinant structures do Umemura polynomials for P_{VI} admit? Recently, Kirillov and Taneda have introduced a generalization of Umemura polynomials for P_{VI} in the context of combinatorics and have shown that their polynomials degenerate to the special polynomials for P_V in some limit [9, 10]. This result suggests that the special polynomials associated with a class of algebraic solutions to P_{VI} are also expressible by the universal characters.

In this paper, we consider P_{VI}

$$\begin{aligned}\frac{d^2y}{dt^2}&=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\&\quad+\frac{y(y-1)(y-t)}{2t^2(t-1)^2}\left[\kappa_\infty^2-\kappa_0^2\frac{t}{y^2}+\kappa_1^2\frac{t-1}{(y-1)^2}+(1-\theta^2)\frac{t(t-1)}{(y-t)^2}\right],\end{aligned}\tag{1.1}$$

where $\kappa_\infty, \kappa_0, \kappa_1$ and θ are parameters. As is well known [21], P_{VI} (1.1) is equivalent to the Hamilton system

$$S_{VI}:\quad q'=\frac{\partial H}{\partial p},\qquad p'=-\frac{\partial H}{\partial q},\qquad '=t(t-1)\frac{d}{dt},\tag{1.2}$$

with the Hamiltonian

$$\begin{aligned}H&=q(q-1)(q-t)p^2-[\kappa_0(q-1)(q-t)+\kappa_1 q(q-t)\\&\quad+(\theta-1)q(q-1)]p+\kappa(q-t),\qquad \kappa=\frac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2-\frac{1}{4}\kappa_\infty^2.\end{aligned}\tag{1.3}$$

In fact, the equation for $y=q$ is nothing but P_{VI} (1.1).

The aim of this paper is to investigate a class of algebraic solutions to P_{VI} (or S_{VI}) that originate from the fixed points of the Bäcklund transformations corresponding to Dynkin automorphisms and, then to present its explicit determinant formula.

Let us remark on the terminology of "algebraic solutions". P_{VI} admits several classes of algebraic solutions [1, 2, 3, 13, 14], and the classification has not yet been established. In this paper, we concentrate our attention to the above restricted class of algebraic solutions.

This paper is organized as follows. In Section 2, we first present a determinant formula for a family of algebraic solutions to P_{VI} (or S_{VI}). This expression is also a specialization of the universal characters, and the entries of the determinant are given by the Jacobi polynomials. The symmetry of P_{VI} is described by the affine Weyl group of type $D_4^{(1)}$. In Section 3, as a preparation for constructing special solutions, we present a symmetric description of Bäcklund transformations for P_{VI} [6, 20]. We also derive several sets of bilinear equations for the τ-functions. In Section 4, starting from a seed solution on fixed points of a Dynkin automorphism, we construct a family of algebraic solutions to P_{VI} (or S_{VI}) by application of Bäcklund transformations. A family of special polynomials is extracted as the non-trivial factor of the τ-function, and our algebraic solutions are expressed by a ratio of these polynomials. A proof of our result is given in Section 5.

As is well known, P_{VI} degenerates to $P_V, \dots, P_I$ by successive limiting procedures [25, 4]. In Section 6, we show that the family of algebraic solutions to P_{VI} given in Section 2 degenerate to rational solutions to P_V and P_{III} with the same determinant structures. Section 7 is devoted to discussing the relationship to the original Umemura polynomials for P_{VI}.

2. A determinant formula

Definition 2.1. Let $p_k = p_k^{(c,d)}(x)$ and $q_k = q_k^{(c,d)}(x)$, $k \in \boldsymbol{Z}$, be two sets of polynomials defined by

$$\sum_{k=0}^{\infty} p_k^{(c,d)}(x)\lambda^k = G(x;c,d;\lambda), \qquad p_k^{(c,d)}(x) = 0 \quad \text{for } k < 0, \tag{2.1}$$

$$q_k^{(c,d)}(x) = p_k^{(c,d)}(x^{-1}),$$

respectively, where the generating function $G(x;c,d;\lambda)$ is given by

$$G(x;c,d;\lambda) = (1-\lambda)^{c-d}(1+x\lambda)^{-c}. \tag{2.2}$$

For $m,n \in \boldsymbol{Z}_{\geq 0}$, we define a family of polynomials $R_{m,n} = R_{m,n}(x;c,d)$ by

$$R_{m,n}(x;c,d) \tag{2.3}$$

$$= \begin{vmatrix} q_1 & q_0 & \cdots & q_{-m+2} & q_{-m+1} & \cdots & q_{-m-n+3} & q_{-m-n+2} \\ q_3 & q_2 & \cdots & q_{-m+4} & q_{-m+3} & \cdots & q_{-m-n+5} & q_{-m-n+4} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1} & q_{2m-2} & \cdots & q_m & q_{m-1} & \cdots & q_{m-n+1} & q_{m-n} \\ p_{n-m} & p_{n-m+1} & \cdots & p_{n-1} & p_n & \cdots & p_{2n-2} & p_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4} & p_{-n-m+5} & \cdots & p_{-n+3} & p_{-n+4} & \cdots & p_2 & p_3 \\ p_{-n-m+2} & p_{-n-m+3} & \cdots & p_{-n+1} & p_{-n+2} & \cdots & p_0 & p_1 \end{vmatrix}.$$

For $m,n \in \boldsymbol{Z}_{<0}$, we define $R_{m,n}$ by

$$R_{m,n} = (-1)^{m(m+1)/2} R_{-m-1,n}, \qquad R_{m,n} = (-1)^{n(n+1)/2} R_{m,-n-1}. \tag{2.4}$$

Theorem 2.2. *We set*

$$R_{m,n}(x;c,d)=S_{m,n}(x;a,b), \tag{2.5}$$

with

$$c=a+b+n-\frac{1}{2},\qquad d=2b-m+n. \tag{2.6}$$

Then, for the parameters

$$\kappa_\infty=b,\qquad \kappa_0=b-m+n,\qquad \kappa_1=a+m+n,\qquad \theta=a, \tag{2.7}$$

we have a family of algebraic solutions of the Hamilton system $\mathrm{S_{VI}}$,

$$q=x\frac{S_{m,n-1}(x;a+1,b)S_{m-1,n}(x;a+1,b)}{S_{m-1,n}(x;a+1,b-1)S_{m,n-1}(x;a+1,b+1)}, \tag{2.8}$$

$$p=\frac{1}{2}\left(a+b+n-\frac{1}{2}\right)$$
$$\times x^{-1}\frac{S_{m-1,n}(x;a+1,b-1)S_{m,n-1}(x;a+1,b+1)S_{m,n-1}(x;a,b)}{S_{m,n}(x;a,b)S_{m-1,n-1}(x;a+1,b)S_{m,n-1}(x;a+1,b)},$$

with $x^2=t$.

This theorem means that a class of algebraic solutions of $\mathrm{P_{VI}}$ is expressed in terms of the universal characters [11], which also appear in the expression for a class of rational solutions to $\mathrm{P_V}$ [12]. Note that the entries p_k and q_k are essentially the Jacobi polynomials, namely,

$$p_k^{(c,d)}(x)=P_k^{(d-1,c-d-k)}(-1-2x). \tag{2.9}$$

Applying some Bäcklund transformations, which can include the outer transformations given in (7.14), to the above solutions, we can get other families of algebraic solutions of $\mathrm{P_{VI}}$. Some examples are presented in Corollaries 6.3 and 6.9, so we omit any details here.

3. A symmetric description of the Painlevé VI equation

Noumi and Yamada have introduced the symmetric form of the Painlevé equations [18, 19, 16]. This formulation provides us with a clear description of symmetry structures of Bäcklund transformations and a systematic tool to construct special solutions.

In this section, we present a symmetric description for the Bäcklund transformations of $\mathrm{P_{VI}}$ [20]. After introducing the τ-functions via Hamiltonians, we derive several sets of bilinear equations.

3.1. Bäcklund transformations of P_{VI}

We set

$$f_0 = q - t, \qquad f_3 = q - 1, \qquad f_4 = q, \qquad f_2 = p, \tag{3.1}$$

and

$$\alpha_0 = \theta, \qquad \alpha_1 = \kappa_\infty, \qquad \alpha_3 = \kappa_1, \qquad \alpha_4 = \kappa_0. \tag{3.2}$$

Then, the Hamiltonian (1.3) is written as

$$H = f_2^2 f_0 f_3 f_4 - [(\alpha_0 - 1) f_3 f_4 + \alpha_3 f_0 f_4 + \alpha_4 f_0 f_3] f_2 + \alpha_2(\alpha_1 + \alpha_2) f_0, \tag{3.3}$$

with

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \tag{3.4}$$

and the Hamilton equation (1.2) is written as

$$\begin{aligned} f_4' &= 2 f_2 f_0 f_3 f_4 - (\alpha_0 - 1) f_3 f_4 - \alpha_3 f_0 f_4 - \alpha_4 f_0 f_3, \\ f_2' &= -(f_0 f_3 + f_0 f_4 + f_3 f_4) f_2^2 \\ &\quad + [(\alpha_0 - 1)(f_3 + f_4) + \alpha_3 (f_0 + f_4) + \alpha_4 (f_0 + f_3)] f_2 - \alpha_2(\alpha_1 + \alpha_2). \end{aligned} \tag{3.5}$$

The Bäcklund transformations of P_{VI} are described as follows [6, 20],

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \qquad (i, j = 0, 1, 2, 3, 4) \tag{3.6}$$

$$s_2(f_i) = f_i + \frac{\alpha_2}{f_2}, \qquad s_i(f_2) = f_2 - \frac{\alpha_i}{f_i}, \qquad (i = 0, 3, 4) \tag{3.7}$$

$$\begin{aligned} &s_5: \ \alpha_0 \leftrightarrow \alpha_1, \quad \alpha_3 \leftrightarrow \alpha_4, \\ &f_2 \to -\frac{f_0(f_2 f_0 + \alpha_2)}{t(t-1)}, \quad f_0 \to \frac{t(t-1)}{f_0}, \quad f_3 \to (t-1)\frac{f_4}{f_0}, \quad f_4 \to t\frac{f_3}{f_0}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &s_6: \ \alpha_0 \leftrightarrow \alpha_3, \quad \alpha_1 \leftrightarrow \alpha_4, \\ &f_2 \to -\frac{f_4(f_4 f_2 + \alpha_2)}{t}, \quad f_0 \to -t\frac{f_3}{f_4}, \quad f_3 \to -\frac{f_0}{f_4}, \quad f_4 \to \frac{t}{f_4}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &s_7: \ \alpha_0 \leftrightarrow \alpha_4, \quad \alpha_1 \leftrightarrow \alpha_3, \\ &f_2 \to \frac{f_3(f_3 f_2 + \alpha_2)}{t-1}, \quad f_0 \to -(t-1)\frac{f_4}{f_3}, \quad f_3 \to -\frac{t-1}{f_3}, \quad f_4 \to \frac{f_0}{f_3}, \end{aligned} \tag{3.10}$$

where $A = (a_{ij})_{i,j=0}^4$ is the Cartan matrix of type $D_4^{(1)}$:

$$(3.11)\qquad a_{ii}=2 \quad (i=0,1,2,3,4), \qquad a_{2j}=a_{j2}=-1, \qquad (j=0,1,3,4),$$
$$a_{ij}=0 \quad (\text{otherwise}).$$

These transformations commute with the derivation $'$, and satisfy the following relations

$$(3.12)\qquad s_i^2=1 \quad (i=0,\dots,7), \qquad s_is_2s_i=s_2s_is_2 \quad (i=0,1,3,4),$$
$$s_5s_{\{0,1,2,3,4\}}=s_{\{1,0,2,4,3\}}s_5, \qquad s_6s_{\{0,1,2,3,4\}}=s_{\{3,4,2,0,1\}}s_6,$$
$$s_7s_{\{0,1,2,3,4\}}=s_{\{4,3,2,1,0\}}s_7,$$
$$s_5s_6=s_6s_5, \qquad s_5s_7=s_7s_5, \qquad s_6s_7=s_7s_6.$$

This means that transformations s_i $(i=0,\dots,4)$ generate the affine Weyl group $W(D_4^{(1)})$, and s_i $(i=0,\dots,7)$ generate its extension including the Dynkin diagram automorphisms.

3.2. The τ-functions and bilinear equations

We add a correction term to the Hamiltonian (3.3) as follows,

$$(3.13)\qquad H_0=H+\frac{t}{4}[1+4\alpha_1\alpha_2+4\alpha_2^2-(\alpha_3+\alpha_4)^2]$$
$$+\frac{1}{4}[(\alpha_1+\alpha_4)^2+(\alpha_3+\alpha_4)^2+4\alpha_2\alpha_4].$$

This modification gives a simpler behavior of the Hamiltonian with respect to the Bäcklund transformations. From the corrected Hamiltonian (3.13), we introduce a family of Hamiltonians h_i $(i=0,1,2,3,4)$ as

$$(3.14)\qquad h_0=H_0+\frac{t}{4}, \qquad h_1=s_5(H_0)-\frac{t-1}{4}, \qquad h_3=s_6(H_0)+\frac{1}{4},$$
$$h_4=s_7(H_0), \qquad h_2=h_1+s_1(h_1).$$

Then, we have

$$(3.15)\qquad s_i(h_j)=h_j, \qquad (i\neq j, i,j=0,1,2,3,4),$$

$$(3.16)\qquad s_0(h_0)=h_0-\alpha_0(t-1)\frac{f_4}{f_0}, \qquad s_1(h_1)=h_1-\alpha_1f_3,$$
$$s_3(h_3)=h_3+\alpha_3\frac{t-1}{f_3}, \qquad s_4(h_4)=h_4+\alpha_4\frac{f_0}{f_4}.$$

Moreover, from (3.14), (3.16) and the equations (3.5), we obtain

$$(3.17) \qquad [s_i(h_i)+h_i]-[s_1(h_1)+h_1]=\frac{f_i'}{f_i}, \qquad (i=0,3,4)$$

$$[s_2(h_2)+h_2]-(h_0+h_1+h_3+h_4)=\frac{f_2'}{f_2}-\frac{1}{2}(t-1).$$

Next, we also introduce τ-functions τ_i $(i=0,1,2,3,4)$ by

$$(3.18) \qquad h_i=\frac{\tau_i'}{\tau_i}.$$

Imposing the condition that the action of the s_i's on the τ-functions also commute with the derivation $'$, one can lift the Bäcklund transformations to the τ-functions. From (3.15) and (3.17), we get

$$(3.19) \qquad s_i(\tau_j)=\tau_j, \qquad (i\neq j, i,j=0,1,2,3,4),$$

and

$$(3.20) \qquad s_0(\tau_0)=f_0\frac{\tau_2}{\tau_0}, \qquad s_1(\tau_1)=\frac{\tau_2}{\tau_1}, \qquad s_2(\tau_2)=t^{-1/2}f_2\frac{\tau_0\tau_1\tau_3\tau_4}{\tau_2},$$

$$s_3(\tau_3)=f_3\frac{\tau_2}{\tau_3}, \qquad s_4(\tau_4)=f_4\frac{\tau_2}{\tau_4},$$

respectively. The action of the diagram automorphisms s_5, s_6 and s_7 are derived from (3.14) as follows,

$$(3.21) \qquad s_5\colon \quad \tau_0\to[t(t-1)]^{1/4}\tau_1, \qquad \tau_1\to[t(t-1)]^{-1/4}\tau_0,$$

$$\tau_3\to t^{-1/4}(t-1)^{1/4}\tau_4, \qquad \tau_4\to t^{1/4}(t-1)^{-1/4}\tau_3,$$

$$\tau_2\to[t(t-1)]^{-1/2}f_0\tau_2,$$

$$(3.22) \qquad s_6\colon \quad \tau_0\to it^{1/4}\tau_3, \qquad \tau_3\to -it^{-1/4}\tau_0, \qquad \tau_1\to t^{-1/4}\tau_4,$$

$$\tau_4\to t^{1/4}\tau_1, \qquad \tau_2\to t^{-1/2}f_4\tau_2,$$

$$(3.23) \qquad s_7\colon \quad \tau_0\to(-1)^{-3/4}(t-1)^{1/4}\tau_4, \qquad \tau_4\to(-1)^{3/4}(t-1)^{-1/4}\tau_0,$$

$$\tau_1\to(-1)^{3/4}(t-1)^{-1/4}\tau_3, \qquad \tau_3\to(-1)^{-3/4}(t-1)^{1/4}\tau_1,$$

$$\tau_2\to -i(t-1)^{-1/2}f_3\tau_2.$$

The algebraic relations of the s_i's are preserved in this lifting except for the following modifications:

$$(3.24) \qquad s_is_2(\tau_2)=-s_2s_i(\tau_2) \qquad (i=5,6,7),$$

and

$$s_5s_6\tau_{\{0,1,2,3,4\}} = \{i,-i,-1,-i,i\}s_6s_5\tau_{\{0,1,2,3,4\}}, \tag{3.25}$$

$$s_5s_7\tau_{\{0,1,2,3,4\}} = \{i,-i,-1,i,-i\}s_7s_5\tau_{\{0,1,2,3,4\}},$$

$$s_6s_7\tau_{\{0,1,2,3,4\}} = \{-i,-i,-1,i,i\}s_7s_6\tau_{\{0,1,2,3,4\}}.$$

Note that one can regard (3.20) as the multiplicative formulas for f_i in terms of the τ-functions,

$$f_0 = \frac{\tau_0 s_0(\tau_0)}{\tau_1 s_1(\tau_1)}, \quad f_3 = \frac{\tau_3 s_3(\tau_3)}{\tau_1 s_1(\tau_1)}, \quad f_4 = \frac{\tau_4 s_4(\tau_4)}{\tau_1 s_1(\tau_1)}, \quad f_2 = t^{1/2}\frac{\tau_1 s_1(\tau_1)s_2s_1(\tau_1)}{\tau_0\tau_3\tau_4}. \tag{3.26}$$

From these formulas, it is possible to derive various bilinear equations for the τ-functions. First, the constraints for the f-variables

$$f_0 = f_4 - t, \qquad f_3 = f_4 - 1, \tag{3.27}$$

yield

$$\tau_1 s_1(\tau_1) + \tau_3 s_3(\tau_3) - \tau_4 s_4(\tau_4) = 0, \tag{3.28}$$

$$\tau_0 s_0(\tau_0) + t\tau_1 s_1(\tau_1) - \tau_4 s_4(\tau_4) = 0,$$

$$\tau_1 s_2s_1(\tau_1) + \tau_3 s_2s_3(\tau_3) - \tau_4 s_2s_4(\tau_4) = 0,$$

$$\tau_0 s_2s_0(\tau_0) + t\tau_1 s_2s_1(\tau_1) - \tau_4 s_2s_4(\tau_4) = 0.$$

The Bäcklund transformations (3.7) lead to the following sets of bilinear equations,

$$\alpha_0 t^{-1/2}\tau_3\tau_4 - s_0(\tau_0)s_2s_1(\tau_1) + \tau_0 s_0s_2s_1(\tau_1) = 0, \tag{3.29}$$

$$\alpha_0 t^{-1/2}(t-1)\tau_1\tau_4 - s_0(\tau_0)s_2s_3(\tau_3) + \tau_0 s_0s_2s_3(\tau_3) = 0,$$

$$\alpha_0 t^{1/2}\tau_1\tau_3 - s_0(\tau_0)s_2s_4(\tau_4) + \tau_0 s_0s_2s_4(\tau_4) = 0,$$

$$\alpha_1 t^{-1/2}\tau_3\tau_4 + s_1(\tau_1)s_2s_0(\tau_0) - \tau_1 s_1s_2s_0(\tau_0) = 0, \tag{3.30}$$

$$\alpha_1 t^{-1/2}\tau_0\tau_4 + s_1(\tau_1)s_2s_3(\tau_3) - \tau_1 s_1s_2s_3(\tau_3) = 0,$$

$$\alpha_1 t^{-1/2}\tau_0\tau_3 + s_1(\tau_1)s_2s_4(\tau_4) - \tau_1 s_1s_2s_4(\tau_4) = 0,$$

$$\alpha_3 t^{-1/2}\tau_0\tau_4 - s_3(\tau_3)s_2s_1(\tau_1) + \tau_3 s_3s_2s_1(\tau_1) = 0, \tag{3.31}$$

$$\alpha_3 t^{-1/2}(1-t)\tau_1\tau_4 - s_3(\tau_3)s_2s_0(\tau_0) + \tau_3 s_3s_2s_0(\tau_0) = 0,$$

$$\alpha_3 t^{-1/2}\tau_0\tau_1 - s_3(\tau_3)s_2s_4(\tau_4) + \tau_3 s_3s_2s_4(\tau_4) = 0,$$

$$\alpha_4 t^{-1/2}\tau_0\tau_3 - s_4(\tau_4)s_2s_1(\tau_1) + \tau_4 s_4s_2s_1(\tau_1) = 0, \tag{3.32}$$

$$-\alpha_4 t^{1/2}\tau_1\tau_3 - s_4(\tau_4)s_2s_0(\tau_0) + \tau_4 s_4s_2s_0(\tau_0) = 0,$$

$$-\alpha_4 t^{-1/2}\tau_0\tau_1 - s_4(\tau_4)s_2s_3(\tau_3) + \tau_4 s_4s_2s_3(\tau_3) = 0,$$

$$\alpha_2 t^{-1/2}\tau_3\tau_4 - s_1(\tau_1)s_2s_0(\tau_0) + s_0(\tau_0)s_2s_1(\tau_1) = 0, \tag{3.33}$$
$$\alpha_2 t^{-1/2}\tau_0\tau_4 - s_1(\tau_1)s_2s_3(\tau_3) + s_3(\tau_3)s_2s_1(\tau_1) = 0,$$
$$\alpha_2 t^{-1/2}\tau_0\tau_3 - s_1(\tau_1)s_2s_4(\tau_4) + s_4(\tau_4)s_2s_1(\tau_1) = 0,$$
$$\alpha_2 t^{-1/2}\tau_0\tau_1 - s_4(\tau_4)s_2s_3(\tau_3) + s_3(\tau_3)s_2s_4(\tau_4) = 0,$$
$$\alpha_2 t^{1/2}\tau_1\tau_3 - s_4(\tau_4)s_2s_0(\tau_0) + s_0(\tau_0)s_2s_4(\tau_4) = 0,$$
$$\alpha_2 t^{-1/2}(t-1)\tau_1\tau_4 - s_3(\tau_3)s_2s_0(\tau_0) + s_0(\tau_0)s_2s_3(\tau_3) = 0.$$

3.3. The τ-functions on the weight lattice of type D_4

Let us define the following translation operators

$$T_{03} = s_3s_0s_2s_4s_1s_2s_6, \qquad T_{14} = s_4s_1s_2s_3s_0s_2s_6, \tag{3.34}$$
$$\hat{T}_{34} = s_3s_2s_0s_1s_2s_3s_5, \qquad T_{34} = s_4s_3s_2s_1s_0s_2s_5,$$

which act on parameters α_i as

$$T_{03}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (1,0,-1,1,0), \tag{3.35}$$
$$T_{14}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,1,-1,0,1),$$
$$\hat{T}_{34}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,0,0,1,-1),$$
$$T_{34}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,0,-1,1,1),$$

and generate the weight lattice of type D_4. It is possible to derive Toda and Toda-like equations.

Proposition 3.1. *We have*

(3.36)
$$T_{03}(\tau_0)T_{03}^{-1}(\tau_0) = t^{-1/2}\left[(t-1)\frac{d}{dt}(\log\tau_0)' - (\log\tau_0)' + \frac{1}{4}(1-\alpha_0-\alpha_3)^2 + \frac{1}{2}\right]\tau_0^2,$$
$$T_{14}(\tau_0)T_{14}^{-1}(\tau_0) = -t^{-1/2}\left[(t-1)\frac{d}{dt}(\log\tau_0)' - (\log\tau_0)' + \frac{1}{4}(\alpha_1+\alpha_4)^2 + \frac{1}{2}\right]\tau_0^2,$$
$$\hat{T}_{34}(\tau_0)\hat{T}_{34}^{-1}(\tau_0) = \left(\frac{t-1}{t}\right)^{1/2}\left[\frac{d}{dt}(\log\tau_0)' + \frac{1}{4}(\alpha_3-\alpha_4)^2 - \frac{1}{2}\right]\tau_0^2,$$
$$T_{34}(\tau_0)T_{34}^{-1}(\tau_0) = \left(\frac{t-1}{t}\right)^{1/2}\left[\frac{d}{dt}(\log\tau_0)' + \frac{1}{4}(\alpha_3+\alpha_4)^2 - \frac{1}{2}\right]\tau_0^2.$$

Proof. Note that

$$\frac{d}{dt}h_0 = -f_2^2f_3f_4 + (\alpha_3f_4+\alpha_4f_3)f_2 - \frac{1}{4}(\alpha_3+\alpha_4)^2 + \frac{1}{2}. \tag{3.37}$$

Using (3.6), (3.7), (3.20), (3.22) and (3.34), we have

$$(3.38)\quad T_{03}(\tau_0)T_{03}^{-1}(\tau_0) = t^{-1/2}\left[(t-1)\frac{d}{dt}h_0 - h_0 + \frac{1}{4}(1-\alpha_0-\alpha_3)^2 + \frac{1}{2}\right]\tau_0^2,$$

which gives the first equation in (3.36). The other equations are obtained in a similar way. ■

Due to the algebraic relations (3.24) and (3.25), the action of these translation operators on the τ-functions is not commutative. For example, we have

$$(3.39)\quad \begin{aligned} &T_{03}T_{14}(\tau_0) = -T_{14}T_{03}(\tau_0), \qquad T_{03}\hat{T}_{34}(\tau_0) = i\hat{T}_{34}T_{03}(\tau_0),\\ &T_{03}T_{34}(\tau_0) = iT_{34}T_{03}(\tau_0), \qquad T_{14}\hat{T}_{34}(\tau_0) = -i\hat{T}_{34}T_{14}(\tau_0),\\ &T_{14}T_{34}(\tau_0) = -iT_{34}T_{14}(\tau_0), \qquad \hat{T}_{34}T_{34}(\tau_0) = T_{34}\hat{T}_{34}(\tau_0). \end{aligned}$$

We introduce τ-functions on the weight lattice of type D_4 as

$$(3.40)\quad \tau_{k,l,m,n} = T_{34}^n\hat{T}_{34}^m T_{14}^l T_{03}^k(\tau_0), \qquad k,l,m,n \in \boldsymbol{Z}.$$

In terms of this notation, the 24 τ-functions in the bilinear equations (3.28)–(3.33) are expressed as follows,

$$(3.41)\quad \begin{aligned} &\tau_{0,0,0,0} = \tau_0, \qquad \tau_{1,-1,-1,0} = -[t(t-1)]^{1/4}\tau_1,\\ &\tau_{1,0,-1,-1} = it^{1/4}\tau_3, \qquad \tau_{1,0,0,-1} = i(t-1)^{1/4}\tau_4,\\ &\tau_{2,0,-1,-1} = -s_0(\tau_0), \qquad \tau_{1,1,0,-1} = [t(t-1)]^{1/4}s_1(\tau_1),\\ &\tau_{1,0,0,0} = it^{1/4}s_3(\tau_3), \qquad \tau_{1,0,-1,0} = i(t-1)^{1/4}s_4(\tau_4),\\ &\tau_{1,-1,-1,-1} = s_2s_0(\tau_0), \qquad \tau_{0,0,0,-1} = -[t(t-1)]^{1/4}s_2s_1(\tau_1),\\ &\tau_{0,-1,0,0} = -it^{1/4}s_2s_3(\tau_3), \qquad \tau_{0,-1,-1,0} = i(t-1)^{1/4}s_2s_4(\tau_4), \end{aligned}$$

$$(3.42)\quad \begin{aligned} &\tau_{1,1,0,-2} = -s_1s_2s_0(\tau_0), \quad \tau_{1,-1,0,0} = -s_3s_2s_0(\tau_0), \quad \tau_{1,-1,-2,0} = s_4s_2s_0(\tau_0),\\ &\tau_{2,0,-1,-2} = [t(t-1)]^{1/4}s_0s_2s_1(\tau_1), \quad \tau_{0,0,1,0} = [t(t-1)]^{1/4}s_3s_2s_1(\tau_1),\\ &\tau_{0,0,-1,0} = -[t(t-1)]^{1/4}s_4s_2s_1(\tau_1),\\ &\tau_{2,-1,-1,-1} = -it^{1/4}s_0s_2s_3(\tau_3), \qquad \tau_{0,1,1,-1} = it^{1/4}s_1s_2s_3(\tau_3),\\ &\tau_{0,-1,-1,1} = -it^{1/4}s_4s_2s_3(\tau_3),\\ &\tau_{2,-1,-2,-1} = i(t-1)^{1/4}s_0s_2s_4(\tau_4), \quad \tau_{0,1,0,-1} = -i(t-1)^{1/4}s_1s_2s_4(\tau_4),\\ &\tau_{0,-1,0,1} = i(t-1)^{1/4}s_3s_2s_4(\tau_4). \end{aligned}$$

The Toda equations (3.36) yield

$$(3.43)\quad \tau_{k+1,l,m,n}\tau_{k-1,l,m,n}$$

$$= t^{-1/2}\left[(t-1)\frac{d}{dt}(\log \tau_{k,l,m,n})' - (\log \tau_{k,l,m,n})' + \frac{(1-\alpha_0-\alpha_3-2k-m-n)^2}{4} + \frac{1}{2}\right]\tau_{k,l,m,n}^2,$$

$$\tau_{k,l+1,m,n}\tau_{k,l-1,m,n}$$

$$= -t^{-1/2}\left[(t-1)\frac{d}{dt}(\log \tau_{k,l,m,n})' - (\log \tau_{k,l,m,n})' + \frac{(\alpha_1+\alpha_4+2l-m+n)^2}{4} + \frac{1}{2}\right]\tau_{k,l,m,n}^2,$$

$$\tau_{k,l,m+1,n}\tau_{k,l,m-1,n} = \left(\frac{t-1}{t}\right)^{1/2}\left[\frac{d}{dt}(\log \tau_{k,l,m,n})' + \frac{(\alpha_3-\alpha_4+k-l+2m)^2}{4} - \frac{1}{2}\right]\tau_{k,l,m,n}^2,$$

$$\tau_{k,l,m,n+1}\tau_{k,l,m,n-1} = \left(\frac{t-1}{t}\right)^{1/2}\left[\frac{d}{dt}(\log \tau_{k,l,m,n})' + \frac{(\alpha_3+\alpha_4+k+l+2n)^2}{4} - \frac{1}{2}\right]\tau_{k,l,m,n}^2.$$

It is easy to see from (3.26), (3.40) and (3.41) that we have

$$(3.44)\quad T_{34}^n\hat{T}_{34}^m T_{14}^l T_{03}^k(f_0) = t^{1/2}(t-1)^{1/2}\frac{\tau_{k,l,m,n}\tau_{k+2,l,m-1,n-1}}{\tau_{k+1,l-1,m-1,n}\tau_{k+1,l+1,m,n-1}},$$

$$T_{34}^n\hat{T}_{34}^m T_{14}^l T_{03}^k(f_3) = (t-1)^{1/2}\frac{\tau_{k+1,l,m-1,n-1}\tau_{k+1,l,m,n}}{\tau_{k+1,l-1,m-1,n}\tau_{k+1,l+1,m,n-1}},$$

$$T_{34}^n\hat{T}_{34}^m T_{14}^l T_{03}^k(f_4) = t^{1/2}\frac{\tau_{k+1,l,m,n-1}\tau_{k+1,l,m-1,n}}{\tau_{k+1,l-1,m-1,n}\tau_{k+1,l+1,m,n-1}},$$

$$T_{34}^n\hat{T}_{34}^m T_{14}^l T_{03}^k(f_2) = -(t-1)^{-1/2}\frac{\tau_{k+1,l-1,m-1,n}\tau_{k+1,l+1,m,n-1}\tau_{k,l,m,n-1}}{\tau_{k,l,m,n}\tau_{k+1,l,m-1,n-1}\tau_{k+1,l,m,n-1}}.$$

4. Construction of a family of algebraic solutions

It is known that one can get an algebraic solution of Painlevé equations by considering the fixed points with respect to a Bäcklund transformation corresponding to a Dynkin automorphism [27, 16]. Iteration of Bäcklund transformations to the seed solution gives a family of algebraic solutions, which are expressed by a ratio of some characteristic polynomials, such as Yablonskii-Vorob'ev, Okamoto and Umemura polynomials. These polynomials are defined as the non-trivial factors of τ-functions and are generated by Toda type recursion relations.

In this section, we construct a family of algebraic solutions to the symmetric form of P_{VI} by following the above recipe.

4.1. A seed solution

Consider the Dynkin diagram automorphism s_6 to get a seed solution. By (3.9), the fixed solution is derived from

$$\alpha_0 = \alpha_3, \qquad \alpha_1 = \alpha_4, \qquad f_4 = \frac{t}{f_4}, \qquad f_2 = -\frac{f_4(f_4 f_2 + \alpha_2)}{t}. \tag{4.1}$$

Then, we obtain

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(a, b, \frac{1}{2} - a - b, a, b\right), \tag{4.2}$$

$$f_0 = x - x^2, \qquad f_3 = x - 1, \qquad f_4 = x, \qquad f_2 = \frac{1}{2}\left(a + b - \frac{1}{2}\right)x^{-1}, \qquad x^2 = t,$$

as a seed solution, which is equivalent to the following algebraic solution of S_{VI},

$$q = x, \qquad p = \frac{1}{2}\left(a + b - \frac{1}{2}\right)x^{-1}, \tag{4.3}$$

for the parameters

$$\kappa_\infty = b, \qquad \kappa_0 = b, \qquad \kappa_1 = a, \qquad \theta = a. \tag{4.4}$$

Remark 4.1. One can choose another diagram automorphism to get a seed solution. Such a solution can be transformed to (4.2) by some Bäcklund transformations. The seed solution (4.2) is the simplest one.

Remark 4.2. Hitchin has discussed some algebraic solutions of P_{VI} in [3]. The seed solution (4.2), with special values for the parameters, appears as one of them. Also, part of the algebraic solutions derived by Andreev and Kitaev [1] is expressible by rational functions of $\sqrt{t}$ and $\sqrt{t-1}$, and can be transformed to (4.2) by some Bäcklund transformations.

Under the specialization of (4.2), the Hamiltonians h_i and τ-functions τ_i are calculated as

$$
\begin{aligned}
(4.5)\quad h_0 &= \frac{7}{16}x^2 + \frac{1}{8}(2a+2b-1)(2a-2b-1)x + \frac{1}{16}(8a^2-8a+3+8b^2),\\
h_1 &= -\frac{1}{16}x^2 + \frac{1}{8}(2a+2b-1)(2a-2b+1)x + \frac{1}{16}(8a^2+8b^2-8b+7),\\
h_2 &= -\frac{1}{8}x^2 + \frac{1}{4}(4a^2-4b^2-1)x + \frac{1}{8}(8a^2+8b^2+7),\\
h_3 &= \frac{3}{16}x^2 + \frac{1}{8}(2a+2b-1)(2a-2b-1)x + \frac{1}{16}(8a^2-8a+7+8b^2),\\
h_4 &= \frac{3}{16}x^2 + \frac{1}{8}(2a+2b-1)(2a-2b+1)x + \frac{1}{16}(8a^2+8b^2-8b+3),
\end{aligned}
$$

and

$$
\begin{aligned}
(4.6)\quad \tau_0 &= (x-1)^{a^2-a+3/4}x^{-a^2+a-b^2-3/8}(x+1)^{b^2+1/2},\\
\tau_1 &= (x-1)^{a^2+1/4}x^{-a^2-b^2+b-7/8}(x+1)^{b^2-b+1/2},\\
\tau_2 &= (x-1)^{2a^2+1/2}x^{-2a^2-2b^2-7/4}(x+1)^{2b^2+1},\\
\tau_3 &= (x-1)^{a^2-a+3/4}x^{-a^2+a-b^2-7/8}(x+1)^{b^2+1/2},\\
\tau_4 &= (x-1)^{a^2+1/4}x^{-a^2-b^2+b-3/8}(x+1)^{b^2-b+1/2},
\end{aligned}
$$

up to multiplication by some constants, respectively.

Using the multiplicative formulas (3.26) and the bilinear equations (3.28)–(3.33), we get the 24 τ-functions in (3.41) and (3.42). These are expressed in the form

$$
\begin{aligned}
(4.7)\quad \tau_{k,l,m,n} &= \sigma_{k,l,m,n}(x-1)^{\tilde{a}^2+1/2}\\
&\quad\times x^{-\tilde{a}^2-\tilde{b}^2-(1/2)m(m+1)-(1/2)n(n+1)-1/8}(x+1)^{\tilde{b}^2+1/2},\\
\tilde{a} &= a+k+\frac{m+n-1}{2}, \qquad \tilde{b} = b+l+\frac{-m+n}{2},
\end{aligned}
$$

where $\sigma_{k,l,m,n}$ are given as follows,

$$
\begin{aligned}
(4.8)\quad &\sigma_{0,0,0,0} = 1, \qquad \sigma_{1,0,0,0} = i, \qquad \sigma_{0,-1,0,0} = -\frac{i}{2}\left(\frac{1}{2}-a-b\right),\\
&\sigma_{1,-1,0,0} = -\frac{1}{2}\left(\frac{1}{2}+a-b\right),
\end{aligned}
$$

$$\sigma_{0,0,-1,0}=\frac{1}{2}\left(\frac{1}{2}-a+b\right),\qquad \sigma_{0,-1,-1,0}=\frac{i}{2}\left(\frac{1}{2}-a-b\right),\tag{4.9}$$

$$\sigma_{1,0,-1,0}=i,\qquad \sigma_{1,-1,-1,0}=-1,$$

$$\sigma_{0,0,0,-1}=\frac{1}{2}\left(\frac{1}{2}-a-b\right),\qquad \sigma_{0,1,0,-1}=-\frac{i}{2}\left(\frac{1}{2}-a+b\right),\tag{4.10}$$

$$\sigma_{1,0,0,-1}=i,\qquad \sigma_{1,1,0,-1}=1,$$

$$\sigma_{1,0,-1,-1}=i,\qquad \sigma_{2,0,-1,-1}=1,\qquad \sigma_{1,-1,-1,-1}=\frac{1}{2}\left(\frac{1}{2}-a-b\right),\tag{4.11}$$

$$\sigma_{2,-1,-1,-1}=\frac{i}{2}\left(\frac{1}{2}+a-b\right),$$

$$\sigma_{0,-1,0,1}=\frac{i}{2}\left[\left(\frac{1}{2}-a-b\right)x-\left(\frac{1}{2}+a-b\right)\right],\tag{4.12}$$

$$\sigma_{0,0,1,0}=-\frac{1}{2}\left[\left(\frac{1}{2}+a-b\right)x-\left(\frac{1}{2}-a-b\right)\right],$$

$$\sigma_{0,1,1,-1}=\frac{i}{2}\left[\left(\frac{1}{2}-a+b\right)x+\left(\frac{1}{2}-a-b\right)\right],\tag{4.13}$$

$$\sigma_{0,-1,-1,1}=-\frac{i}{2}\left[\left(\frac{1}{2}-a-b\right)x+\left(\frac{1}{2}-a+b\right)\right],$$

$$\sigma_{1,1,0,-2}=-\frac{1}{2}\left[\left(\frac{1}{2}-a-b\right)x+\left(\frac{1}{2}-a+b\right)\right],\tag{4.14}$$

$$\sigma_{1,-1,-2,0}=\frac{1}{2}\left[\left(\frac{1}{2}-a+b\right)x+\left(\frac{1}{2}-a-b\right)\right],$$

$$\sigma_{2,-1,-2,-1}=-\frac{i}{2}\left[\left(\frac{1}{2}+a-b\right)x-\left(\frac{1}{2}-a-b\right)\right],\tag{4.15}$$

$$\sigma_{2,0,-1,-2}=\frac{1}{2}\left[\left(\frac{1}{2}-a-b\right)x-\left(\frac{1}{2}+a-b\right)\right].$$

4.2. Application of Bäcklund transformations

Assume that $\tau_{k,l,m,n}$ are expressed as in (4.7) for any $k,l,m,n\in \boldsymbol{Z}$. Substituting $\alpha_0=\alpha_3=a$, $\alpha_1=\alpha_4=b$ and (4.7) into (3.43), we obtain the Toda equations for $\sigma_{k,l,m,n}$. The first two equations yield

$$(4.16)\qquad 4\sigma_{k+1,l,m,n}\sigma_{k-1,l,m,n} = [(x+1)^2\mathscr{D}^2 - (\tilde{a}+\tilde{b})(\tilde{a}-\tilde{b})]\sigma_{k,l,m,n}\cdot\sigma_{k,l,m,n},$$
$$4\sigma_{k,l+1,m,n}\sigma_{k,l-1,m,n} = -[(x-1)^2\mathscr{D}^2 - (\tilde{a}+\tilde{b})(\tilde{a}-\tilde{b})]\sigma_{k,l,m,n}\cdot\sigma_{k,l,m,n},$$

where we denote

$$(4.17)\qquad \mathscr{D}^2\sigma\cdot\sigma = x(\ddot{\sigma}\sigma - \dot{\sigma}^2) + \dot{\sigma}\sigma, \qquad \dot{\sigma} = \frac{d\sigma}{dx}.$$

The others are reduced to

$$(4.18)\qquad 4\sigma_{k,l,m+1,n}\sigma_{k,l,m-1,n} = (x-1)x(x+1)(\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} - \dot{\sigma}^2_{k,l,m,n})$$
$$+ (3x^2-1)\dot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n}$$
$$+ \left\{\left[\left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)\left(\hat{a}-\hat{b}+3m+\frac{1}{2}\right) - n(n+1)\right]x\right.$$
$$\left. + \left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)\left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)\right\}\sigma^2_{k,l,m,n},$$
$$4\sigma_{k,l,m,n+1}\sigma_{k,l,m,n-1} = (x-1)x(x+1)(\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} - \dot{\sigma}^2_{k,l,m,n})$$
$$+ (3x^2-1)\dot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n}$$
$$+ \left\{\left[\left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)\left(\hat{a}+\hat{b}+3n+\frac{1}{2}\right) - m(m+1)\right]x\right.$$
$$\left. + \left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)\left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)\right\}\sigma^2_{k,l,m,n},$$

with

$$(4.19)\qquad \hat{a} = a+k, \qquad \hat{b} = b+l.$$

Toda equations (4.16) and (4.18) with the initial data (4.8)–(4.15) generate $\sigma_{k,l,m,n} = \sigma_{k,l,m,n}(x;a,b)$ for $k,l,m,n \in \boldsymbol{Z}$. From (3.44) and (4.7), we see that the ratio of $\sigma_{k,l,m,n}$ gives a family of algebraic solutions to the symmetric form of P_{VI}.

Noticing that we have

$$(4.20)\qquad T_{14}^{l}T_{03}^{k}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\hat{a},\hat{b},\frac{1}{2}-\hat{a}-\hat{b},\hat{a},\hat{b}\right),$$

under the specialization (4.2), we see that the action of T_{03} and T_{14} on the parameter space is absorbed by a shift of the parameters a and b, respectively. This suggests that we do not need to consider the translations T_{03} and T_{14} in order to get a family of algebraic solutions of P_{VI}.

To verify this, we put

$$\sigma_{k,l,m,n} = \omega_{k,l,m,n} V_{k,l,m,n}, \qquad \omega_{k,l,m,n} = \omega_{k,l,m,n}(a,b), \qquad k,l,m,n \in \mathbf{Z}. \tag{4.21}$$

The constants $\omega_{k,l,m,n}$ are determined by recurrence relations as follows. With respect to the indices k and l, $\omega_{k,l,i,j}$ with $(i,j) = (-1,-1), (-1,0), (0,-1), (0,0)$ are subject to

$$\begin{aligned} 4\omega_{k+1,l,i,j}\omega_{k-1,l,i,j} &= -\left(\hat{a} - \hat{b} + i - \frac{1}{2}\right)\left(\hat{a} + \hat{b} + j - \frac{1}{2}\right)\omega_{k,l,i,j}^2, \\ 4\omega_{k,l+1,i,j}\omega_{k,l-1,i,j} &= \left(\hat{a} - \hat{b} + i - \frac{1}{2}\right)\left(\hat{a} + \hat{b} + j - \frac{1}{2}\right)\omega_{k,l,i,j}^2. \end{aligned} \tag{4.22}$$

The initial conditions are given by

$$\omega_{0,0,0,0} = 1, \qquad \omega_{1,0,0,0} = i \tag{4.23}$$

$$\begin{aligned} &\omega_{0,-1,0,0} = -\frac{i}{2}\left(\frac{1}{2} - a - b\right), \qquad \omega_{1,-1,0,0} = -\frac{1}{2}\left(\frac{1}{2} + a - b\right), \\ &\omega_{0,0,-1,0} = \frac{1}{2}\left(\frac{1}{2} - a + b\right), \qquad \omega_{0,-1,-1,0} = \frac{i}{2}\left(\frac{1}{2} - a - b\right), \\ &\omega_{1,0,-1,0} = i, \qquad \omega_{1,-1,-1,0} = -1, \end{aligned} \tag{4.24}$$

$$\omega_{1,0,0,-1} = i, \qquad \omega_{1,1,0,-1} = 1 \tag{4.25}$$

$$\omega_{0,0,0,-1} = \frac{1}{2}\left(\frac{1}{2} - a - b\right), \qquad \omega_{0,1,0,-1} = -\frac{i}{2}\left(\frac{1}{2} - a + b\right),$$

$$\omega_{1,0,-1,-1} = i, \qquad \omega_{2,0,-1,-1} = 1 \tag{4.26}$$

$$\omega_{1,-1,-1,-1} = \frac{1}{2}\left(\frac{1}{2} - a - b\right), \qquad \omega_{2,-1,-1,-1} = \frac{i}{2}\left(\frac{1}{2} + a - b\right).$$

Note that these imply

$$V_{k,l,-1,-1} = V_{k,l,-1,0} = V_{k,l,0,-1} = V_{k,l,0,0} = 1, \qquad k,l \in \mathbf{Z}. \tag{4.27}$$

For the indices m and n, we set

$$\begin{aligned} 8\omega_{k,l,m+1,n}\omega_{k,l,m-1,n} &= \left(\hat{a} - \hat{b} + m - \frac{1}{2}\right)\omega_{k,l,m,n}^2, \\ 8\omega_{k,l,m,n+1}\omega_{k,l,m,n-1} &= \left(\hat{a} + \hat{b} + n - \frac{1}{2}\right)\omega_{k,l,m,n}^2. \end{aligned} \tag{4.28}$$

Thus, $\omega_{k,l,m,n}$ are determined for any $k,l,m,n \in \boldsymbol{Z}$. As a result, the functions $V_{k,l,m,n} = V_{k,l,m,n}(x;a,b)$ introduced in (4.21) have a symmetry described in the following lemma.

Lemma 4.3. *We have*

$$V_{k,l,m,n}(x;a,b) = V_{0,0,m,n}(x;a+k,b+l). \tag{4.29}$$

Proof. The Toda equations (4.18) are reduced to

$$\begin{aligned}
&\frac{1}{2}\left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)V_{k,l,m+1,n}V_{k,l,m-1,n} \\
&\quad = (x-1)x(x+1)(\ddot{V}_{k,l,m,n}V_{k,l,m,n} - \dot{V}^2_{k,l,m,n}) \\
&\quad + (3x^2-1)\dot{V}_{k,l,m,n}V_{k,l,m,n} \\
&\quad + \left\{\left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)\left[\left(\hat{a}-\hat{b}+3m+\frac{1}{2}\right)x + \left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)\right]\right. \\
&\quad \left. - n(n+1)x\right\}V^2_{k,l,m,n},
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&\frac{1}{2}\left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)V_{k,l,m,n+1}V_{k,l,m,n-1} \\
&\quad = (x-1)x(x+1)(\ddot{V}_{k,l,m,n}V_{k,l,m,n} - \dot{V}^2_{k,l,m,n}) \\
&\quad + (3x^2-1)\dot{V}_{k,l,m,n}V_{k,l,m,n} \\
&\quad + \left\{\left(\hat{a}+\hat{b}+n-\frac{1}{2}\right)\left[\left(\hat{a}+\hat{b}+3n+\frac{1}{2}\right)x + \left(\hat{a}-\hat{b}+m-\frac{1}{2}\right)\right]\right. \\
&\quad \left. - m(m+1)x\right\}V^2_{k,l,m,n}.
\end{aligned}$$

Then, we see that $V_{k,l,m,n}(x;a,b)$ satisfy the same Toda equations as $V_{0,0,m,n}(x;a+k,b+l)$. Since the initial conditions are given by (4.27), we obtain (4.29). ■

On the other hand, we have from (3.44), (4.7) and (4.21)

$$\begin{aligned}
T_{34}^n \hat{T}_{34}^m T_{14}^l T_{03}^k(f_0) &= x(x-1)\frac{\sigma_{k,l,m,n}\sigma_{k+2,l,m-1,n-1}}{\sigma_{k+1,l-1,m-1,n}\sigma_{k+1,l+1,m,n-1}} \\
&= x(x-1)\frac{\omega_{k,l,m,n}\omega_{k+2,l,m-1,n-1}}{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}}\frac{V_{k,l,m,n}V_{k+2,l,m-1,n-1}}{V_{k+1,l-1,m-1,n}V_{k+1,l+1,m,n-1}}.
\end{aligned} \tag{4.31}$$

The ratio of the ω's is calculated as

$$\frac{\omega_{k,l,m,n}\omega_{k+2,l,m-1,n-1}}{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}} = -1. \tag{4.32}$$

Similarly, for f_3, f_4 and f_2, we get

$$\frac{\omega_{k+1,l,m-1,n-1}\omega_{k+1,l,m,n}}{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}} = 1, \qquad \frac{\omega_{k+1,l,m,n-1}\omega_{k+1,l,m-1,n}}{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}} = 1, \tag{4.33}$$

$$\frac{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}\omega_{k,l,m,n-1}}{\omega_{k,l,m,n}\omega_{k+1,l,m-1,n-1}\omega_{k+1,l,m,n-1}} = -\frac{1}{2}\left(\hat{a} + \hat{b} + n - \frac{1}{2}\right).$$

The above discussion means that, for our purpose, we can set $k = l = 0$ without loss of generality. Hence, we denote $V_{0,0,m,n} = V_{m,n}$.

Moreover, we observe that $V_{m,n} = V_{m,n}(x; a, b)$ $(m, n \in \boldsymbol{Z})$ are polynomials in a, b and x with coefficients in $\boldsymbol{Z}$. We will show this in the next section by presenting the explicit expressions.

Proposition 4.4. *Let $V_{m,n} = V_{m,n}(x; a, b)$ $(m, n \in \boldsymbol{Z})$ be polynomials generated by Toda equations,*

$$\begin{aligned}
&\frac{1}{2}\left(a - b + m - \frac{1}{2}\right) V_{m+1,n} V_{m-1,n} \\
&\quad = (x-1)x(x+1)(\ddot{V}_{m,n} V_{m,n} - \dot{V}_{m,n}^2) + (3x^2 - 1)\dot{V}_{m,n} V_{m,n} \\
&\qquad + \left\{\left(a - b + m - \frac{1}{2}\right)\left[\left(a - b + 3m + \frac{1}{2}\right)x + \left(a + b + n - \frac{1}{2}\right)\right]\right. \\
&\qquad \left. - n(n+1)x\right\} V_{m,n}^2,
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
&\frac{1}{2}\left(a + b + n - \frac{1}{2}\right) V_{m,n+1} V_{m,n-1} \\
&\quad = (x-1)x(x+1)(\ddot{V}_{m,n} V_{m,n} - \dot{V}_{m,n}^2) + (3x^2 - 1)\dot{V}_{m,n} V_{m,n} \\
&\qquad + \left\{\left(a + b + n - \frac{1}{2}\right)\left[\left(a + b + 3n + \frac{1}{2}\right)x + \left(a - b + m - \frac{1}{2}\right)\right]\right. \\
&\qquad \left. - m(m+1)x\right\} V_{m,n}^2,
\end{aligned}$$

with the initial conditions,

$$V_{-1,-1} = V_{-1,0} = V_{0,-1} = V_{0,0} = 1. \tag{4.35}$$

Then,

$$(4.36)\quad f_0 = x(1-x)\frac{V_{m,n}(x;a,b)V_{m-1,n-1}(x;a+2,b)}{V_{m-1,n}(x;a+1,b-1)V_{m,n-1}(x;a+1,b+1)},$$

$$f_3 = (x-1)\frac{V_{m-1,n-1}(x;a+1,b)V_{m,n}(x;a+1,b)}{V_{m-1,n}(x;a+1,b-1)V_{m,n-1}(x;a+1,b+1)},$$

$$f_4 = x\frac{V_{m,n-1}(x;a+1,b)V_{m-1,n}(x;a+1,b)}{V_{m-1,n}(x;a+1,b-1)V_{m,n-1}(x;a+1,b+1)},$$

$$f_2 = \frac{1}{2}\left(a+b+n-\frac{1}{2}\right)$$
$$\times x^{-1}\frac{V_{m-1,n}(x;a+1,b-1)V_{m,n-1}(x;a+1,b+1)V_{m,n-1}(x;a,b)}{V_{m,n}(x;a,b)V_{m-1,n-1}(x;a+1,b)V_{m,n-1}(x;a+1,b)},$$

satisfy the symmetric form of P_{VI} *for the parameters*

$$(4.37)\quad (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(a,b,\frac{1}{2}-a-b-n,a+m+n,b-m+n\right).$$

Furthermore, the bilinear relations for $V_{m,n}$ are derived from (3.28)–(3.33).

Proposition 4.5. *The polynomials* $V_{m,n}(x;a,b)$ *satisfy the following bilinear relations,*

$$(4.38)\quad V^{(1,-1)}_{m-1,n}V^{(1,1)}_{m,n-1} + (x-1)V^{(1,0)}_{m-1,n-1}V^{(1,0)}_{m,n} - xV^{(1,0)}_{m,n-1}V^{(1,0)}_{m-1,n} = 0,$$

$$(x-1)V^{(0,0)}_{m,n}V^{(2,0)}_{m-1,n-1} - xV^{(1,-1)}_{m-1,n}V^{(1,1)}_{m,n-1} + V^{(1,0)}_{m,n-1}V^{(1,0)}_{m-1,n} = 0,$$

$$V^{(1,-1)}_{m-1,n}V^{(0,0)}_{m,n-1} - (x+1)V^{(1,0)}_{m-1,n-1}V^{(0,-1)}_{m,n} + xV^{(1,0)}_{m,n-1}V^{(0,-1)}_{m-1,n} = 0,$$

$$(x+1)V^{(0,0)}_{m,n}V^{(1,-1)}_{m-1,n-1} - xV^{(1,-1)}_{m-1,n}V^{(0,0)}_{m,n-1} - V^{(1,0)}_{m,n-1}V^{(0,-1)}_{m-1,n} = 0,$$

$$(4.39)\quad 4aV^{(1,0)}_{m-1,n-1}V^{(1,0)}_{m,n-1} + 2\nu_n(x-1)V^{(2,0)}_{m-1,n-1}V^{(0,0)}_{m,n-1} - V^{(0,0)}_{m,n}V^{(2,0)}_{m-1,n-2} = 0,$$

$$2aV^{(1,-1)}_{m-1,n}V^{(1,0)}_{m,n-1} - \nu_nV^{(2,0)}_{m-1,n-1}V^{(0,-1)}_{m,n} - \mu_{m+1}V^{(0,0)}_{m,n}V^{(2,-1)}_{m-1,n-1} = 0,$$

$$4axV^{(1,-1)}_{m-1,n}V^{(1,0)}_{m-1,n-1} - 2\nu_n(x-1)V^{(2,0)}_{m-1,n-1}V^{(0,-1)}_{m-1,n} - V^{(0,0)}_{m,n}V^{(2,-1)}_{m-2,n-1} = 0,$$

$$(4.40)\quad 4bV^{(1,0)}_{m-1,n-1}V^{(1,0)}_{m,n-1} - 2\nu_n(x+1)V^{(1,1)}_{m,n-1}V^{(1,-1)}_{m-1,n-1} + V^{(1,-1)}_{m-1,n}V^{(1,1)}_{m,n-2} = 0,$$

$$4bxV^{(0,0)}_{m,n}V^{(1,0)}_{m,n-1} - 2\nu_n(x+1)V^{(1,1)}_{m,n-1}V^{(0,-1)}_{m,n} + V^{(1,-1)}_{m-1,n}V^{(0,1)}_{m+1,n-1} = 0,$$

$$2bV^{(0,0)}_{m,n}V^{(1,0)}_{m-1,n-1} - \nu_nV^{(1,1)}_{m,n-1}V^{(0,-1)}_{m-1,n} + \mu_mV^{(1,-1)}_{m-1,n}V^{(0,1)}_{m,n-1} = 0,$$

(4.41)

$$4(a+m+n)xV^{(0,0)}_{m,n}V^{(1,0)}_{m,n-1}-2\nu_n(x-1)V^{(1,0)}_{m,n}V^{(0,0)}_{m,n-1}-V^{(1,0)}_{m-1,n-1}V^{(0,0)}_{m+1,n}=0,$$
$$2(a+m+n)V^{(1,-1)}_{m-1,n}V^{(1,0)}_{m,n-1}-\nu_nV^{(1,0)}_{m,n}V^{(1,-1)}_{m-1,n-1}-\mu_{m+1}V^{(1,0)}_{m-1,n-1}V^{(1,-1)}_{m,n}=0,$$
$$4(a+m+n)V^{(0,0)}_{m,n}V^{(1,-1)}_{m-1,n}+2\nu_n(x-1)V^{(1,0)}_{m,n}V^{(0,-1)}_{m-1,n}-V^{(1,0)}_{m-1,n-1}V^{(0,-1)}_{m,n+1}=0,$$

(4.42)

$$2(b-m+n)V^{(0,0)}_{m,n}V^{(1,0)}_{m-1,n-1}-\nu_nV^{(1,0)}_{m-1,n}V^{(0,0)}_{m,n-1}+\mu_mV^{(1,0)}_{m,n-1}V^{(0,0)}_{m-1,n}=0,$$
$$4(b-m+n)xV^{(1,-1)}_{m-1,n}V^{(1,0)}_{m-1,n-1}-2\nu_n(x+1)V^{(1,0)}_{m-1,n}V^{(1,-1)}_{m-1,n-1}+V^{(1,0)}_{m,n-1}V^{(1,-1)}_{m-2,n}=0,$$
$$4(b-m+n)V^{(0,0)}_{m,n}V^{(1,-1)}_{m-1,n}-2\nu_n(x+1)V^{(1,0)}_{m-1,n}V^{(0,-1)}_{m,n}+V^{(1,0)}_{m,n-1}V^{(0,-1)}_{m-1,n+1}=0,$$

(4.43)

$$2V^{(1,0)}_{m-1,n-1}V^{(1,0)}_{m,n-1}-(x+1)V^{(1,1)}_{m,n-1}V^{(1,-1)}_{m-1,n-1}+(x-1)V^{(2,0)}_{m-1,n-1}V^{(0,0)}_{m,n-1}=0,$$
$$2xV^{(0,0)}_{m,n}V^{(1,0)}_{m,n-1}-(x+1)V^{(1,1)}_{m,n-1}V^{(0,-1)}_{m,n}-(x-1)V^{(1,0)}_{m,n}V^{(0,0)}_{m,n-1}=0,$$
$$2V^{(0,0)}_{m,n}V^{(1,0)}_{m-1,n-1}-V^{(1,1)}_{m,n-1}V^{(0,-1)}_{m-1,n}-V^{(1,0)}_{m-1,n}V^{(0,0)}_{m,n-1}=0,$$
$$2V^{(0,0)}_{m,n}V^{(1,-1)}_{m-1,n}-(x+1)V^{(1,0)}_{m-1,n}V^{(0,-1)}_{m,n}+(x-1)V^{(1,0)}_{m,n}V^{(0,-1)}_{m-1,n}=0,$$
$$2xV^{(1,-1)}_{m-1,n}V^{(1,0)}_{m-1,n-1}-(x+1)V^{(1,0)}_{m-1,n}V^{(1,-1)}_{m-1,n-1}-(x-1)V^{(2,0)}_{m-1,n-1}V^{(0,-1)}_{m-1,n}=0,$$
$$2V^{(1,-1)}_{m-1,n}V^{(1,0)}_{m,n-1}-V^{(1,0)}_{m,n}V^{(1,-1)}_{m-1,n-1}-V^{(2,0)}_{m-1,n-1}V^{(0,-1)}_{m,n}=0,$$

where we denote

$$\mu_m=a-b+m-\frac{1}{2},\qquad \nu_n=a+b+n-\frac{1}{2}, \tag{4.44}$$

$$V^{(k,l)}_{m,n}=V_{m,n}(x;a+k,b+l). \tag{4.45}$$

Conversely, by solving the bilinear relations (4.38)–(4.43) with the initial conditions (4.35), one can get the family of algebraic solutions (4.36) with (4.37). Even though these bilinear relations are overdetermined systems, thier consistency is guaranteed by construction. In order to show Theorem 2.2, we will prove not the Toda equations (4.34) but the bilinear relations (4.38)–(4.43).

5. Proof of the determinant formula

In this section, we give a proof of Theorem 2.2.

Proposition 5.1. *We have*

$$V_{m,n}(x;a,b)=(-2x)^{m(m+1)/2}(-2)^{n(n+1)/2}\xi_m\xi_nS_{m,n}(x;a,b),\qquad m,n\in\mathbf{Z}, \tag{5.1}$$

where $S_{m,n} = S_{m,n}(x; a, b)$ *is defined in Theorem* 2.2 *and* ξ_n *is the factor determined by*

$$\xi_{n+1}\xi_{n-1} = (2n+1)\xi_n^2, \qquad \xi_{-1} = \xi_0 = 1. \tag{5.2}$$

From this proposition, it is easy to verify that the $V_{m,n}(x; a, b)$ are indeed polynomials in a, b and x with coefficients in $\boldsymbol{Z}$.

Substituting (5.1) into (4.36), we find that Theorem 2.2 is a direct consequence of Proposition 5.1. Taking (2.5), (2.6) and (5.1) into account, we obtain the bilinear relations for $R_{m,n}$.

Proposition 5.2. *Let* $R_{m,n}$ *be a family of polynomials given in Definition* 2.1. *Then, we have*

$$\begin{gathered}
R^{(0,-1)}_{m-1,n}R^{(1,1)}_{m,n-1} + (x-1)R^{(0,0)}_{m-1,n-1}R^{(1,0)}_{m,n} - xR^{(0,-1)}_{m,n-1}R^{(1,1)}_{m-1,n} = 0,\\
(x-1)R^{(0,0)}_{m,n}R^{(1,0)}_{m-1,n-1} - xR^{(0,-1)}_{m-1,n}R^{(1,1)}_{m,n-1} + R^{(0,-1)}_{m,n-1}R^{(1,1)}_{m-1,n} = 0,\\
R^{(1,0)}_{m-1,n}R^{(0,0)}_{m,n-1} - (x+1)R^{(1,1)}_{m-1,n-1}R^{(0,-1)}_{m,n} + xR^{(1,0)}_{m,n-1}R^{(0,0)}_{m-1,n} = 0,\\
(x+1)R^{(1,1)}_{m,n}R^{(0,-1)}_{m-1,n-1} - xR^{(1,0)}_{m-1,n}R^{(0,0)}_{m,n-1} - R^{(1,0)}_{m,n-1}R^{(0,0)}_{m-1,n} = 0,
\end{gathered} \tag{5.3}$$

$$\begin{gathered}
(2c-d-m-n)R^{(0,0)}_{m-1,n}R^{(0,-1)}_{m,n} + c(x-1)R^{(1,0)}_{m-1,n}R^{(-1,-1)}_{m,n}\\
+ (2n+1)R^{(0,0)}_{m,n+1}R^{(0,-1)}_{m-1,n-1} = 0,\\
(2c-d-m-n)R^{(0,0)}_{m-1,n}R^{(0,0)}_{m,n-1} - cR^{(1,1)}_{m-1,n-1}R^{(-1,-1)}_{m,n}\\
- (c-d)R^{(0,1)}_{m,n}R^{(0,-1)}_{m-1,n-1} = 0,\\
(2c-d-m-n)xR^{(0,-1)}_{m,n}R^{(0,0)}_{m,n-1} - c(x-1)R^{(1,0)}_{m,n-1}R^{(-1,-1)}_{m,n}\\
+ (2m+1)xR^{(0,0)}_{m+1,n}R^{(0,-1)}_{m-1,n-1} = 0,
\end{gathered} \tag{5.4}$$

$$\begin{gathered}
(d+m-n)R^{(0,1)}_{m-1,n}R^{(0,0)}_{m,n} - c(x+1)R^{(1,2)}_{m,n}R^{(-1,-1)}_{m-1,n}\\
- (2n+1)R^{(0,0)}_{m-1,n+1}R^{(0,1)}_{m,n-1} = 0,\\
(d+m-n)xR^{(0,0)}_{m,n}R^{(0,-1)}_{m,n-1} - c(x+1)R^{(1,1)}_{m,n-1}R^{(-1,-2)}_{m,n}\\
- (2m+1)xR^{(0,-1)}_{m-1,n}R^{(0,0)}_{m+1,n-1} = 0,\\
(d+m-n)R^{(0,0)}_{m,n}R^{(0,0)}_{m-1,n-1} - cR^{(1,1)}_{m,n-1}R^{(-1,-1)}_{m-1,n} + (c-d)R^{(0,-1)}_{m-1,n}R^{(0,1)}_{m,n-1} = 0,
\end{gathered} \tag{5.5}$$

$$(5.6)\qquad \begin{aligned} &(2c-d+m+n)xR^{(0,1)}_{m,n}R^{(0,0)}_{m,n-1}-c(x-1)R^{(1,1)}_{m,n}R^{(-1,0)}_{m,n-1}\\ &\quad+(2m+1)xR^{(0,1)}_{m-1,n-1}R^{(0,0)}_{m+1,n}=0,\\ &(2c-d+m+n)R^{(0,0)}_{m-1,n}R^{(0,0)}_{m,n-1}-cR^{(1,1)}_{m,n}R^{(-1,-1)}_{m-1,n-1}\\ &\quad-(c-d)R^{(0,1)}_{m-1,n-1}R^{(0,-1)}_{m,n}=0,\\ &(2c-d+m+n)R^{(0,1)}_{m,n}R^{(0,0)}_{m-1,n}+c(x-1)R^{(1,1)}_{m,n}R^{(-1,0)}_{m-1,n}\\ &\quad+(2n+1)R^{(0,1)}_{m-1,n-1}R^{(0,0)}_{m,n+1}=0,\end{aligned}$$

$$(5.7)\qquad \begin{aligned} &(d-m+n)R^{(0,0)}_{m,n}R^{(0,0)}_{m-1,n-1}-cR^{(1,1)}_{m-1,n}R^{(-1,-1)}_{m,n-1}+(c-d)R^{(0,-1)}_{m,n-1}R^{(0,1)}_{m-1,n}=0,\\ &(d-m+n)xR^{(0,0)}_{m,n}R^{(0,1)}_{m,n-1}-c(x+1)R^{(1,2)}_{m,n}R^{(-1,-1)}_{m,n-1}\\ &\quad-(2m+1)xR^{(0,0)}_{m+1,n-1}R^{(0,1)}_{m-1,n}=0,\\ &(d-m+n)R^{(0,0)}_{m,n}R^{(0,-1)}_{m-1,n}-c(x+1)R^{(1,1)}_{m-1,n}R^{(-1,-2)}_{m,n}\\ &\quad-(2n+1)R^{(0,-1)}_{m,n-1}R^{(0,0)}_{m-1,n+1}=0,\end{aligned}$$

$$(5.8)\qquad \begin{aligned} &2R^{(0,0)}_{m-1,n}R^{(0,-1)}_{m,n}-(x+1)R^{(1,1)}_{m,n}R^{(-1,-2)}_{m-1,n}+(x-1)R^{(1,0)}_{m-1,n}R^{(-1,-1)}_{m,n}=0,\\ &2xR^{(0,0)}_{m,n}R^{(0,-1)}_{m,n-1}-(x+1)R^{(1,1)}_{m,n-1}R^{(-1,-2)}_{m,n}-(x-1)R^{(1,0)}_{m,n}R^{(-1,-1)}_{m,n-1}=0,\\ &2R^{(0,0)}_{m,n}R^{(0,0)}_{m-1,n-1}-R^{(1,1)}_{m,n-1}R^{(-1,-1)}_{m-1,n}-R^{(1,1)}_{m-1,n}R^{(-1,-1)}_{m,n-1}=0,\\ &2R^{(0,0)}_{m,n}R^{(0,-1)}_{m-1,n}-(x+1)R^{(1,1)}_{m-1,n}R^{(-1,-2)}_{m,n}+(x-1)R^{(1,0)}_{m,n}R^{(-1,-1)}_{m-1,n}=0,\\ &2xR^{(0,-1)}_{m,n}R^{(0,0)}_{m,n-1}-(x+1)R^{(1,1)}_{m,n}R^{(-1,-2)}_{m,n-1}-(x-1)R^{(1,0)}_{m,n-1}R^{(-1,-1)}_{m,n}=0,\\ &2R^{(0,0)}_{m-1,n}R^{(0,0)}_{m,n-1}-R^{(1,1)}_{m,n}R^{(-1,-1)}_{m-1,n-1}-R^{(1,1)}_{m-1,n-1}R^{(-1,-1)}_{m,n}=0,\end{aligned}$$

where we denote

$$(5.9)\qquad R^{(i,j)}_{m,n}=R_{m,n}(c+i,d+j).$$

From the above discussion, we see that the proof of Theorem 2.2 is reduced to that of Proposition 5.2.

It is possible to reduce the number of bilinear relations to be proved by the following symmetries of $R_{m,n}$:

Lemma 5.3. *We have the following relations for* $m,n\in \boldsymbol{Z}_{\geq 0}$*:*

(5.10) $$R_{n,m}(x^{-1}) = R_{m,n}(x),$$

(5.11) $$R_{m,n}(-c,-d) = (-1)^{m(m+1)/2+n(n+1)/2} R_{m,n}(c,d),$$

(5.12) $$R_{m,n}(-x;c,2c-d) = (-1)^{m(m+1)/2+n(n+1)/2} R_{m,n}(x;c,d).$$

Proof. The first relation (5.10) is easily obtained from Definition 2.1. To verify the second relation (5.11), we introduce two sets of polynomials $\bar{p}_k = \bar{p}_k^{(c,d)}(x)$ and $\bar{q}_k = \bar{q}_k^{(c,d)}(x)$, $k \in \boldsymbol{Z}$, by

(5.13) $$\sum_{k=0}^{\infty} \bar{p}_k^{(c,d)}(x)\lambda^k = G(x;-c,-d;-\lambda), \qquad \bar{p}_k^{(c,d)}(x) = 0 \quad \text{for } k < 0,$$

$$\bar{q}_k^{(c,d)}(x) = \bar{p}_k^{(c,d)}(x^{-1}),$$

where G is the generating function (2.2). Since we have

(5.14) $$\frac{G(x;-c,-d;-\lambda)}{G(x;c,d;\lambda)} = (1-\lambda^2)^{-c+d}(1-x^2\lambda^2)^c,$$

we see that

(5.15) $$\bar{p}_k(x) = p_k(x) + \sum_{j=1}^{\infty} \rho_j(x) p_{k-2j}(x), \quad \bar{q}_k(x) = q_k(x) + \sum_{j=1}^{\infty} \rho_j(x^{-1}) q_{k-2j}(x),$$

where $\rho_j(x) = \rho_j(x;c,d)$ are some functions. Therefore, $R_{m,n}$ for $m,n \in \boldsymbol{Z}_{\geq 0}$, can be expressed in terms of the same determinant as (2.3) with the entries p_k and q_k replaced by $\bar{p}_k$ and $\bar{q}_k$, respectively. Noticing that

(5.16) $$\bar{p}_k^{(c,d)}(x) = (-1)^k p_k^{(-c,-d)}(x), \qquad \bar{q}_k^{(c,d)}(x) = (-1)^k q_k^{(-c,-d)}(x),$$

we obtain the relation (5.11). The third relation (5.12) is verified similarly. ■

By the symmetries of $R_{m,n}$ described by (2.4) and Lemma 5.3, it is sufficient to prove the following bilinear relations for $m,n \in \boldsymbol{Z}_{\geq 0}$,

(5.17) $$(x+1)R_{m,n}^{(1,1)} R_{m-1,n-1}^{(0,-1)} - xR_{m-1,n}^{(1,0)} R_{m,n-1}^{(0,0)} - R_{m,n-1}^{(1,0)} R_{m-1,n}^{(0,0)} = 0,$$

(5.18) $$(d+m-n)R_{m-1,n}^{(0,1)} R_{m,n}^{(0,0)} - c(x+1)R_{m,n}^{(1,2)} R_{m-1,n}^{(-1,-1)} - (2n+1)R_{m-1,n+1}^{(0,0)} R_{m,n-1}^{(0,1)} = 0,$$

(5.19) $$(d+m-n)R_{m,n}^{(0,0)} R_{m-1,n-1}^{(0,0)} - cR_{m,n-1}^{(1,1)} R_{m-1,n}^{(-1,-1)} + (c-d)R_{m-1,n}^{(0,-1)} R_{m,n-1}^{(0,1)} = 0,$$

(5.20) $$2R_{m,n}^{(0,0)} R_{m-1,n-1}^{(0,0)} - R_{m,n-1}^{(1,1)} R_{m-1,n}^{(-1,-1)} - R_{m-1,n}^{(1,1)} R_{m,n-1}^{(-1,-1)} = 0,$$

(5.21) $$2R_{m-1,n}^{(0,0)} R_{m,n-1}^{(0,0)} - R_{m,n}^{(1,1)} R_{m-1,n-1}^{(-1,-1)} - R_{m-1,n-1}^{(1,1)} R_{m,n}^{(-1,-1)} = 0,$$

(5.22) $$2R_{m-1,n}^{(0,0)} R_{m,n}^{(0,-1)} - (x+1)R_{m,n}^{(1,1)} R_{m-1,n}^{(-1,-2)} + (x-1)R_{m-1,n}^{(1,0)} R_{m,n}^{(-1,-1)} = 0.$$

From the symmetry (5.12) and the bilinear relation (5.17), we have

$$(5.23)\qquad (x-1)R^{(0,0)}_{m,n}R^{(1,0)}_{m-1,n-1}-xR^{(0,-1)}_{m-1,n}R^{(1,1)}_{m,n-1}+R^{(0,-1)}_{m,n-1}R^{(1,1)}_{m-1,n}=0.$$

Then, it is possible to derive the bilinear relation (5.22) as follows,

$$(5.24)\qquad R^{(0,0)}_{m-1,n}\times(5.20)|_{d\to d-1}-R^{(-1,-2)}_{m-1,n}\times(5.17)+R^{(1,0)}_{m-1,n}\times(5.23)|_{c\to c-1,d\to d-1}$$
$$=R^{(0,-1)}_{m-1,n-1}\times(5.22).$$

Therefore, the bilinear relations we need to prove are (5.17)–(5.21).

In the following, we show that these bilinear relations (5.17)–(5.21) reduce to Jacobi's identity for determinants. Let D be an $(m+n+1)\times(m+n+1)$ determinant and $D\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k\end{bmatrix}$ the minor that is obtained by deleting the rows with indices $i_1,\ldots,i_k$ and the columns with indices $j_1,\ldots,j_k$. Then, we have Jacobi's identities

(5.25)
$$D\cdot D\begin{bmatrix} m & m+1 \\ 1 & m+n+1\end{bmatrix}=D\begin{bmatrix} m \\ 1\end{bmatrix}D\begin{bmatrix} m+1 \\ m+n+1\end{bmatrix}-D\begin{bmatrix} m \\ m+n+1\end{bmatrix}D\begin{bmatrix} m+1 \\ 1\end{bmatrix},$$

$$(5.26)\qquad D\cdot D\begin{bmatrix} m & m+1 \\ 1 & 2\end{bmatrix}=D\begin{bmatrix} m \\ 1\end{bmatrix}D\begin{bmatrix} m+1 \\ 2\end{bmatrix}-D\begin{bmatrix} m \\ 2\end{bmatrix}D\begin{bmatrix} m+1 \\ 1\end{bmatrix},$$

(5.27)
$$D\cdot D\begin{bmatrix} 1 & m+1 \\ 2 & m+n+1\end{bmatrix}=D\begin{bmatrix} 1 \\ 2\end{bmatrix}D\begin{bmatrix} m+1 \\ m+n+1\end{bmatrix}-D\begin{bmatrix} 1 \\ m+n+1\end{bmatrix}D\begin{bmatrix} m+1 \\ 2\end{bmatrix}.$$

First, we give the proof of the bilinear relations (5.17)–(5.19). We have the following lemmas.

Lemma 5.4. *Set*

(5.28)
$$D=\begin{vmatrix}
-q_1^{(c-m-n,d-m-n)} & q_1^{(c-m-n+1,d-m-n)} & \cdots & q_{-m-n+3}^{(c-1,d-2)} & q_{-m-n+2}^{(c,d-1)} \\
-q_3^{(c-m-n,d-m-n)} & q_3^{(c-m-n+1,d-m-n)} & \cdots & q_{-m-n+5}^{(c-1,d-2)} & q_{-m-n+4}^{(c,d-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-1}^{(c-m-n+1,d-m-n)} & \cdots & q_{m-n+1}^{(c-1,d-2)} & q_{m-n}^{(c,d-1)} \\
x^{-m-n}p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{2n}^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_{2n}^{(c-1,d-2)} & p_{2n}^{(c,d-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-m-n}p_2^{(c-m-n,d-m-n)} & x^{-m-n+1}p_2^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_2^{(c-1,d-2)} & p_2^{(c,d-1)} \\
x^{-m-n}p_0^{(c-m-n,d-m-n)} & x^{-m-n+1}p_0^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_0^{(c-1,d-2)} & p_0^{(c,d-1)}
\end{vmatrix}.$$

Then, we have

$$(5.29)\qquad D=(-1)^m(1+x^{-1})^{m+n}R^{(0,0)}_{m,n},\quad D\begin{bmatrix} m & m+1\\ 1 & m+n+1\end{bmatrix}=x^{-n}R^{(-1,-2)}_{m-1,n-1},$$

$$D\begin{bmatrix} m\\ 1\end{bmatrix}=R^{(0,-1)}_{m-1,n},\quad D\begin{bmatrix} m+1\\ m+n+1\end{bmatrix}=(-1)^m x^{-n}(1+x^{-1})^{m+n-1}R^{(-1,-1)}_{m,n-1},$$

$$D\begin{bmatrix} m\\ m+n+1\end{bmatrix}=(-1)^{m-1}x^{-n-1}(1+x^{-1})^{m+n-1}R^{(-1,-1)}_{m-1,n},\quad D\begin{bmatrix} m+1\\ 1\end{bmatrix}=R^{(0,-1)}_{m,n-1}.$$

Lemma 5.5. *Set*

(5.30)

$$D=\begin{vmatrix}
\tilde q_1^{(c-m-n,d-m-n)} & q_1^{(c-m-n,d-m-n-1)} & \cdots & q_{-m-n+3}^{(c-2,d-3)} & q_{-m-n+2}^{(c-1,d-2)}\\
\tilde q_3^{(c-m-n,d-m-n)} & q_3^{(c-m-n,d-m-n-1)} & \cdots & q_{-m-n+5}^{(c-2,d-3)} & q_{-m-n+4}^{(c-1,d-2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\tilde q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-1}^{(c-m-n,d-m-n-1)} & \cdots & q_{m-n+1}^{(c-2,d-3)} & q_{m-n}^{(c-1,d-2)}\\
\hat p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{2n+1}^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_{2n+1}^{(c-2,d-3)} & p_{2n+1}^{(c-1,d-2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\hat p_{2}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{3}^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_{3}^{(c-2,d-3)} & p_{3}^{(c-1,d-2)}\\
\hat p_{0}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{1}^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_{1}^{(c-2,d-3)} & p_{1}^{(c-1,d-2)}
\end{vmatrix},$$

with

$$(5.31)\qquad \hat p_{2k}^{(c-m-n,d-m-n)}=\frac{x^{-m-n}}{2k+1}p_{2k}^{(c-m-n,d-m-n)},$$

$$\tilde q_{2k-1}^{(c-m-n,d-m-n)}=\frac{q_{2k-1}^{(c-m-n,d-m-n)}}{(d-m-n+2k-2)x}.$$

Then, we have

$$(5.32)\qquad D=(-1)^{m+n}(1+x)^{m+n}x^{-m}\frac{\prod\limits_{j=1}^{m+n}(c-m-n+j-1)}{\prod\limits_{i=1}^{m}(d-m-n+2i-2)\prod\limits_{k=0}^{n}(2k+1)}R^{(0,0)}_{m,n},$$

$$D\begin{bmatrix} m\\ m+n+1\end{bmatrix}=(-1)^{m+n-1}(1+x)^{m+n-1}$$

$$\times\, x^{-m-n}\frac{\prod\limits_{j=1}^{m+n-1}(c-m-n+j-1)}{\prod\limits_{i=1}^{m-1}(d-m-n+2i-2)\prod\limits_{k=0}^{n}(2k+1)}R^{(-1,-1)}_{m-1,n},$$

$$D\begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} = (-1)^{m+n-1}(1+x)^{m+n-1}$$

$$\times x^{-m-n}\frac{\prod_{j=1}^{m+n-1}(c-m-n+j-1)}{\prod_{i=1}^{m}(d-m-n+2i-2)\prod_{k=0}^{n-1}(2k+1)}R_{m,n-1}^{(-1,-1)},$$

$$D\begin{bmatrix} m \\ 1 \end{bmatrix} = R_{m-1,n+1}^{(-1,-2)}, \quad D\begin{bmatrix} m+1 \\ 1 \end{bmatrix} = R_{m,n}^{(-1,-2)}, \quad D\begin{bmatrix} m & m+1 \\ 1 & m+n+1 \end{bmatrix} = x^{-n}R_{m-1,n}^{(-2,-3)}.$$

Lemma 5.6. *Set*

(5.33)

$$D = \begin{vmatrix} \tilde{q}_1^{(c,d-m-n)} & q_1^{(c-1,d-m-n-1)} & \cdots & q_{-m-n+3}^{(c-1,d-3)} & q_{-m-n+2}^{(c-1,d-2)} \\ \tilde{q}_3^{(c,d-m-n)} & q_3^{(c-1,d-m-n-1)} & \cdots & q_{-m-n+5}^{(c-1,d-3)} & q_{-m-n+4}^{(c-1,d-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{q}_{2m-1}^{(c,d-m-n)} & q_{2m-1}^{(c-1,d-m-n-1)} & \cdots & q_{m-n+1}^{(c-1,d-3)} & q_{m-n}^{(c-1,d-2)} \\ \hat{p}_{2n}^{(c,d-m-n)} & (-1)^{m+n-1}p_{2n+1}^{(c-1,d-m-n-1)} & \cdots & (-1)^1 p_{2n+1}^{(c-1,d-3)} & p_{2n+1}^{(c-1,d-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{p}_2^{(c,d-m-n)} & (-1)^{m+n-1}p_3^{(c-1,d-m-n-1)} & \cdots & (-1)^1 p_3^{(c-1,d-3)} & p_3^{(c-1,d-2)} \\ \hat{p}_0^{(c,d-m-n)} & (-1)^{m+n-1}p_1^{(c-1,d-m-n-1)} & \cdots & (-1)^1 p_1^{(c-1,d-3)} & p_1^{(c-1,d-2)} \end{vmatrix},$$

with

$$(5.34) \quad \hat{p}_{2k}^{(c,d-m-n)} = (-1)^{m+n}\frac{p_{2k}^{(c,d-m-n)}}{2k+1}, \quad \tilde{q}_{2k-1}^{(c,d-m-n)} = \frac{q_{2k-1}^{(c,d-m-n)}}{(d-m-n+2k-2)x}.$$

Then, we have

$$(5.35) \quad D = (-1)^{m+n}(1+x)^{m+n}x^{-m}\frac{\prod_{j=1}^{m+n}(c-d+m+n-j+1)}{\prod_{i=1}^{m}(d-m-n+2i-2)\prod_{k=0}^{n}(2k+1)}R_{m,n}^{(0,0)},$$

$$D\begin{bmatrix} m \\ m+n+1 \end{bmatrix} = (-1)^{m}(1+x)^{m+n-1}$$

$$\times x^{-m+1}\frac{\prod_{j=1}^{m+n-1}(c-d+m+n-j+1)}{\prod_{i=1}^{m-1}(d-m-n+2i-2)\prod_{k=0}^{n}(2k+1)}R_{m-1,n}^{(0,-1)},$$

$$D\begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} = (-1)^{m-1}(1+x)^{m+n-1}$$

$$\times x^{-m} \frac{\prod_{j=1}^{m+n-1}(c-d+m+n-j+1)}{\prod_{i=1}^{m}(d-m-n+2i-2)\prod_{k=0}^{n-1}(2k+1)} R^{(0,-1)}_{m,n-1},$$

$$D\begin{bmatrix} m \\ 1 \end{bmatrix} = R^{(-1,-2)}_{m-1,n+1}, \qquad D\begin{bmatrix} m+1 \\ 1 \end{bmatrix} = R^{(-1,-2)}_{m,n},$$

$$D\begin{bmatrix} m & m+1 \\ 1 & m+n+1 \end{bmatrix} = (-1)^n R^{(-1,-3)}_{m-1,n}.$$

It is easy to see that the bilinear relations (5.17) and (5.18) follow immediately from Jacobi's identity (5.25) by using Lemmas 5.4 and 5.5, respectively. By Lemma 5.6, Jacobi's identity (5.25) is reduced to

$$(5.36) \qquad (d+m-n)xR^{(1,1)}_{m-1,n}R^{(0,0)}_{m,n} + (c-d)(x+1)R^{(1,2)}_{m,n}R^{(0,-1)}_{m-1,n} + (2n+1)R^{(0,0)}_{m-1,n+1}R^{(1,1)}_{m,n-1} = 0.$$

Then, the bilinear relation (5.19) is derived as follows,

$$(5.37) \qquad R^{(0,1)}_{m,n-1} \times (5.36) + R^{(1,1)}_{m,n-1} \times (5.18) + (d+m-n)R^{(0,0)}_{m,n} \times (5.17)|_{d\to d+1} = (x+1)R^{(1,2)}_{m,n} \times (5.19).$$

The proof of Lemmas 5.4~5.6 is given in Appendix B.

Next, we prove the bilinear relations (5.20) and (5.21). We have the following lemmas.

Lemma 5.7. *Set*

(5.38)

$$D = \begin{vmatrix} -x^{-1}q_1^- & x^{-1}q_1^+ & q_1 & \cdots & q_{-m-n+4} \\ -x^{-1}(q_3^- + x^{-2}q_1^-) & x^{-1}q_3^+ & q_3 & \cdots & q_{-m-n+6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{-1}(q_{2m-1}^- + \cdots + x^{-2m+2}q_1^-) & x^{-1}q_{2m-1}^+ & q_{2m-1} & \cdots & q_{m-n+2} \\ p_{n-m+1}^- + \cdots + x^{2n-2}p_{-n-m+3}^- & p_{n-m+1}^+ & p_{n-m+2} & \cdots & p_{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+5}^- + x^2 p_{-n-m+3}^- & p_{-n-m+5}^+ & p_{-n-m+6} & \cdots & p_3 \\ p_{-n-m+3}^- & p_{-n-m+3}^+ & p_{-n-m+4} & \cdots & p_1 \end{vmatrix},$$

with $p_k^{\pm} = p_k^{(c\pm1,d\pm1)}$. *Then, we have*

$$(5.39)\qquad D\begin{bmatrix} m \\ 1 \end{bmatrix} = x^{-m+1}R^{(1,1)}_{m-1,n}, \qquad D\begin{bmatrix} m+1 \\ 1 \end{bmatrix} = x^{-m}R^{(1,1)}_{m,n-1},$$

$$D\begin{bmatrix} m \\ 2 \end{bmatrix} = (-x)^{-m+1}R^{(-1,-1)}_{m-1,n}, \qquad D\begin{bmatrix} m+1 \\ 2 \end{bmatrix} = (-x)^{-m}R^{(-1,-1)}_{m,n-1},$$

$$D\begin{bmatrix} m & m+1 \\ 1 & 2 \end{bmatrix} = R^{(0,0)}_{m-1,n-1},$$

and

$$(5.40)\qquad D = 2(-1)^{-m}x^{-2m+1}R^{(0,0)}_{m,n}.$$

Lemma 5.8. *Set*

(5.41)

$$D = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -x^{-1}q_1^- & x^{-1}q_1^+ & q_1 & \cdots & q_{-m-n+3} \\ -x^{-1}(q_3^- + x^{-2}q_1^-) & x^{-1}q_3^+ & q_3 & \cdots & q_{-m-n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{-1}(q_{2m-1}^- + \cdots + x^{-2m+2}q_1^-) & x^{-1}q_{2m-1}^+ & q_{2m-1} & \cdots & q_{m-n+1} \\ p_{n-m}^- + \cdots + x^{2n-2}p_{-n-m+2}^- & p_{n-m}^+ & p_{n-m+1} & \cdots & p_{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+4}^- + x^2p_{-n-m+2}^- & p_{-n-m+4}^+ & p_{-n-m+5} & \cdots & p_3 \\ p_{-n-m+2}^- & p_{-n-m+2}^+ & p_{-n-m+3} & \cdots & p_1 \end{vmatrix}.$$

Then, we have

$$(5.42)\qquad D = x^{-m}R^{(1,1)}_{m,n}, \qquad D\begin{bmatrix} 1 & m+1 \\ 2 & m+n+1 \end{bmatrix} = (-x)^{-m+1}R^{(-1,-1)}_{m-1,n-1},$$

$$D\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-x)^{-m}R^{(-1,-1)}_{m,n}, \qquad D\begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} = x^{-m+1}R^{(1,1)}_{m-1,n-1},$$

$$D\begin{bmatrix} m+1 \\ 2 \end{bmatrix} = R^{(0,0)}_{m-1,n}, \qquad D\begin{bmatrix} 1 \\ m+n+1 \end{bmatrix} = 2(-1)^{-m}x^{-2m+1}R^{(0,0)}_{m,n-1}.$$

From Lemma 5.7, Jacobi's identity (5.26) leads to the bilinear relation (5.20). Lemma 5.8 and Jacobi's identity (5.27) give the bilinear relation (5.21). We also give the proof of Lemmas 5.7 and 5.8 in Appendix B.

6. Degeneration of algebraic solutions

It is well known that, starting from P_{VI}, one can obtain $P_V, \dots, P_I$ by successive limiting procedures in the following diagram [25, 4],

$$
\begin{array}{ccccc}
P_{VI} & \longrightarrow & P_V & \longrightarrow & P_{III} & & \\
 & & \downarrow & & \downarrow & & \\
 & & P_{IV} & \longrightarrow & P_{II} & \longrightarrow & P_I.
\end{array}
\tag{6.1}
$$

It is also known that each Painlevé equation, except for P_I, admits particular solutions expressed by special functions, and that the coalescence diagram of these special functions is given as

$$
\begin{array}{ccccc}
\text{hypergeometric} & \longrightarrow & \text{confluent hypergeometric} & \longrightarrow & \text{Bessel} \\
 & & \downarrow & & \downarrow \\
 & & \text{Hermite-Weber} & \longrightarrow & \text{Airy.}
\end{array}
\tag{6.2}
$$

What is the degeneration diagram of algebraic (or rational) solutions that originate from the fixed points of Dynkin automorphisms? In this section, we show that, starting from the family of algebraic solutions to P_{VI} given in Theorem 2.2, we can obtain rational solutions to P_V, P_{III} and P_{II} by degeneration in the following diagram,

$$
\begin{array}{ccc}
P_{VI} & \longrightarrow & P_V \\
\downarrow & & \downarrow \\
P_{III} & \longrightarrow & P_{II}.
\end{array}
\tag{6.3}
$$

Remark 6.1. It seems that the rational solutions of P_V cannot degenerate to those of P_{III}, and that there is no way to include the rational solutions of P_{IV} expressed in terms of Okamoto polynomials in the above diagram (6.3).

6.1. Degeneration from P_{VI} to P_V

As is known [22], P_V

(6.4)
$$
\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{2t^2}\left(\kappa_\infty^2 y - \frac{\kappa_0^2}{y}\right) - (\theta+1)\frac{y}{t} - \frac{y(y+1)}{2(y-1)}
$$

is equivalent to the Hamilton system

$$
S_V: \quad q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}, \qquad ' = t\frac{d}{dt},
\tag{6.5}
$$

with the Hamiltonian

$$H = q(q-1)^2p^2 - [\kappa_0(q-1)^2 + \theta q(q-1) + tq]p + \kappa(q-1), \tag{6.6}$$

$$\kappa = \frac{1}{4}(\kappa_0+\theta)^2 - \frac{1}{4}\kappa_\infty^2.$$

This system can be derived from S_{VI} by degeneration [4]. The Hamilton equation (1.2) with the Hamiltonian (1.3) is reduced to (6.5) with (6.6) by putting

$$t \to 1 - \varepsilon t, \qquad \kappa_1 \to \varepsilon^{-1} + \theta + 1, \qquad \theta \to -\varepsilon^{-1}, \tag{6.7}$$

and taking the limit as $\varepsilon \to 0$.

On rational solutions of S_V, we have the following proposition [12].

Proposition 6.2. *Let $p_k = p_k^{(r)}(z)$ and $q_k = q_k^{(r)}(z)$, $k \in \boldsymbol{Z}$, be two sets of polynomials defined by*

$$\sum_{k=0}^{\infty} p_k^{(r)}\lambda^k = (1-\lambda)^{-r}\exp\left(-\frac{z\lambda}{1-\lambda}\right), \qquad p_k^{(r)} = 0 \quad \text{for } k<0, \tag{6.8}$$

$$q_k^{(r)}(z) = p_k^{(r)}(-z).$$

We define the polynomials $R_{m,n} = R_{m,n}^{(r)}(z)$ by

(6.9)

$$R_{m,n}^{(r)}(z) = \begin{vmatrix} q_1^{(r)} & q_0^{(r)} & \cdots & q_{-m+2}^{(r)} & q_{-m+1}^{(r)} & \cdots & q_{-m-n+3}^{(r)} & q_{-m-n+2}^{(r)} \\ q_3^{(r)} & q_2^{(r)} & \cdots & q_{-m+4}^{(r)} & q_{-m+3}^{(r)} & \cdots & q_{-m-n+5}^{(r)} & q_{-m-n+4}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1}^{(r)} & q_{2m-2}^{(r)} & \cdots & q_m^{(r)} & q_{m-1}^{(r)} & \cdots & q_{m-n+1}^{(r)} & q_{m-n}^{(r)} \\ p_{n-m}^{(r)} & p_{n-m+1}^{(r)} & \cdots & p_{n-1}^{(r)} & p_n^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4}^{(r)} & p_{-n-m+5}^{(r)} & \cdots & p_{-n+3}^{(r)} & p_{-n+4}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} \\ p_{-n-m+2}^{(r)} & p_{-n-m+3}^{(r)} & \cdots & p_{-n+1}^{(r)} & p_{-n+2}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)} \end{vmatrix},$$

for $m, n \in \boldsymbol{Z}_{\geq 0}$ and by

$$R_{m,n} = (-1)^{m(m+1)/2}R_{-m-1,n}, \qquad R_{m,n} = (-1)^{n(n+1)/2}R_{m,-n-1}, \tag{6.10}$$

for $m, n \in \boldsymbol{Z}_{<0}$, respectively. Then, setting

$$R_{m,n}^{(r)}(z) = S_{m,n}(t,s), \tag{6.11}$$

with

$$z = \frac{t}{2}, \qquad r = 2s - m + n, \tag{6.12}$$

we see that

$$q = -\frac{S_{m,n-1}(t,s)S_{m-1,n}(t,s)}{S_{m-1,n}(t,s-1)S_{m,n-1}(t,s+1)}, \tag{6.13}$$

$$p = -\frac{2n-1}{4}\frac{S_{m-1,n}(t,s-1)S_{m,n-1}(t,s+1)S_{m-1,n-2}(t,s)}{S^2_{m-1,n-1}(t,s)S_{m,n-1}(t,s)},$$

give a family of rational solutions to the Hamilton system S_V *for the parameters*

$$\kappa_\infty = s, \qquad \kappa_0 = s - m + n, \qquad \theta = m + n - 1. \tag{6.14}$$

Let us consider the degeneration of the algebraic solutions of S_{VI}. Applying the Bäcklund transformation s_0 to the solutions in Theorem 2.2, we obtain the following corollary.

Corollary 6.3. *Let* $S_{m,n} = S_{m,n}(x;a,b)$ *be polynomials given in Theorem 2.2. Then, for* $m, n \in \boldsymbol{Z}$,

$$q = x\frac{S^{(1,0)}_{m,n-1}S^{(1,0)}_{m-1,n}}{S^{(1,-1)}_{m-1,n}S^{(1,1)}_{m,n-1}}, \qquad p = \frac{2n-1}{2x(1-x)}\frac{S^{(1,-1)}_{m-1,n}S^{(1,1)}_{m,n-1}S^{(2,0)}_{m-1,n-2}}{S^{(1,0)}_{m-1,n-1}S^{(1,0)}_{m,n-1}S^{(2,0)}_{m-1,n-1}}, \tag{6.15}$$

where we denote $S^{(k,l)}_{m,n} = S_{m,n}(x;a+k,b+l)$, *satisfy* S_{VI} *for the parameters*

$$\kappa_\infty = b, \qquad \kappa_0 = b - m + n, \qquad \kappa_1 = a + m + n, \qquad \theta = -a, \tag{6.16}$$

with $x^2 = t$.

It is easy to see that by putting

$$t \to 1 - \varepsilon t, \qquad a = \varepsilon^{-1}, \tag{6.17}$$

S_{VI} with (6.16) is reduced to S_V with (6.14) in the limit as $\varepsilon \to 0$.

Next, we investigate the degeneration of $R^{(i,j)}_{m,n}$ given in Definition 2.1. Putting

$$x \to -(1-\varepsilon t)^{1/2}, \qquad c = \varepsilon^{-1} + s + n - \frac{1}{2}, \qquad d = 2s - m + n, \tag{6.18}$$

we see that the generating function (2.2) degenerates as

$$G = (1-\lambda)^{-d}\exp\{c[\log(1-\lambda) - \log(1+x^{\pm 1}\lambda)]\} \tag{6.19}$$

$$= (1-\lambda)^{-r}\exp\left(\mp\frac{z\lambda}{1-\lambda} + O(\varepsilon)\right),$$

where we use (6.12). Then, we have

$$\lim_{\varepsilon\to 0} p_k^{(c,d)}(x) = p_k^{(r)}(z), \qquad \lim_{\varepsilon\to 0} q_k^{(c,d)}(x) = q_k^{(r)}(z), \tag{6.20}$$

which gives

$$\lim_{\varepsilon\to 0} R_{m,n}^{(i,j)}(x) = R_{m,n}^{(r+j)}(z). \tag{6.21}$$

Finally, it is easy to see that (6.15) yields (6.13).

Remark 6.4. As we mentioned in Section 1, Kirillov and Taneda have introduced "generalized Umemura polynomials" for P_{VI} in the context of combinatorics and have shown that these polynomials degenerate to $S_{m,n} = S_{m,n}(t,s)$ defined in Proposition 6.2 in some limit [9, 10].

Remark 6.5. The polynomials $p_k^{(r)}$ (and $q_k^{(r)}$) defined in (6.8) are essentially the Laguerre polynomials, namely, $p_k^{(r)}(z) = L_k^{(r-1)}(z)$. The above degeneration corresponds to that from the Jacobi polynomials to the Laguerre polynomials.

6.2. Degeneration from P_{VI} to P_{III}

Next, we consider P_{III}

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} - \frac{4}{t}[\eta_\infty\theta_\infty y^2 + \eta_0(\theta_0+1)] + 4\eta_\infty^2 y^3 - \frac{4\eta_0^2}{y}, \tag{6.22}$$

which is equivalent to the Hamilton system [24]

$$S_{III}: \quad q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}, \qquad ' = t\frac{d}{dt}, \tag{6.23}$$

with the Hamiltonian

$$H = 2q^2p^2 - [2\eta_\infty tq^2 + (2\theta_0+1)q + 2\eta_0 t]p + \eta_\infty(\theta_\infty+\theta_0)tq. \tag{6.24}$$

This system can be also derived from S_{VI}, directly, by degeneration. This process is achieved by putting

$$t \to \varepsilon^2 t^2, \qquad q \to \varepsilon tq, \qquad p \to \varepsilon^{-1}t^{-1}p, \tag{6.25}$$

$$\kappa_\infty \to \eta_\infty\varepsilon^{-1} + \theta_\infty^{(1)}, \qquad \kappa_0 \to -\eta_0\varepsilon^{-1} + \theta_0^{(1)} + 1, \tag{6.26}$$

$$\kappa_1 \to -\eta_\infty\varepsilon^{-1} + \theta_\infty^{(2)}, \qquad \theta \to \eta_0\varepsilon^{-1} + \theta_0^{(2)},$$

$$H \to -\frac{1}{2}(H + qp), \tag{6.27}$$

and taking the limit as $\varepsilon \to 0$. In fact, the system (1.2) with the Hamiltonian (1.3) is reduced to (6.23) with (6.24) by this procedure, where we set

$$\theta_\infty = \theta_\infty^{(1)} + \theta_\infty^{(2)}, \qquad \theta_0 = \theta_0^{(1)} + \theta_0^{(2)}. \tag{6.28}$$

On rational solutions of $\mathrm{S_{III}}$, we have the following proposition [5].

Proposition 6.6. *Let $p_k = p_k^{(r)}(t)$, $k \in \boldsymbol{Z}$, be polynomials defined by*

$$\sum_{k=0}^{\infty} p_k^{(r)} \lambda^k = (1+\lambda)^r \exp(-t\lambda), \qquad p_k^{(r)} = 0 \quad \textit{for } k < 0. \tag{6.29}$$

We define a family of polynomials $R_n^{(r)} = R_n^{(r)}(t)$ by

$$R_n^{(r)}(t) = \begin{vmatrix} p_n^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ p_{-n+4}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} \\ p_{-n+2}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)} \end{vmatrix}, \tag{6.30}$$

for $n \in \boldsymbol{Z}_{\geq 0}$ and by

$$R_n = (-1)^{n(n+1)/2} R_{-n-1}, \tag{6.31}$$

for $n \in \boldsymbol{Z}_{<0}$, respectively. Then,

$$q = \frac{R_{n-1}^{(r+1)} R_n^{(r)}}{R_n^{(r+1)} R_{n-1}^{(r)}}, \qquad p = -\frac{2n-1}{2} \frac{R_n^{(r+1)} R_{n-2}^{(r+1)}}{R_{n-1}^{(r+1)} R_{n-1}^{(r+1)}} \tag{6.32}$$

give a family of rational solutions to $\mathrm{S_{III}}$ for the parameters

$$\theta_\infty = r + \frac{1}{2} + n, \qquad \theta_0 + 1 = -r - \frac{1}{2} + n, \tag{6.33}$$

with

$$\eta_\infty = \eta_0 = \frac{1}{2}. \tag{6.34}$$

Before discussing the degeneration to the rational solutions of $\mathrm{S_{III}}$, we slightly rewrite the determinant expression in Definition 2.1, for convenience.

Lemma 6.7. *Let $\bar{p}_k = \bar{p}_k^{(\bar{c},\bar{d})}(x)$ and $\bar{q}_k = \bar{q}_k^{(\bar{c},\bar{d})}(x)$, $k \in \boldsymbol{Z}$, be two sets of polynomials defined by*

$$\sum_{k=0}^{\infty} \bar{p}_k^{(\bar{c},\bar{d})}(x) \lambda^k = \bar{G}(x; \bar{c}, \bar{d}; \lambda), \qquad \bar{p}_k^{(\bar{c},\bar{d})}(x) = 0 \quad \textit{for } k < 0, \tag{6.35}$$

$$\bar{q}_k^{(\bar{c},\bar{d})}(x) = \bar{p}_k^{(\bar{c},\bar{d})}(x^{-1}),$$

respectively, where the generating function $\bar{G}(x;\bar{c},\bar{d};\lambda)$ *is given by*

$$\bar{G}(x;\bar{c},\bar{d};\lambda)=(1-\lambda)^{\bar{d}-1}(1+x\lambda)^{-\bar{c}}. \tag{6.36}$$

Define $\bar{R}_{m,n}=\bar{R}_{m,n}(x;\bar{c},\bar{d})$ *in terms of the same determinant as* (2.3) *with entries* p_k *and* q_k *replaced by* $\bar{p}_k$ *and* $\bar{q}_k$, *respectively. Then, we have*

$$\bar{R}_{m,n}(x;\bar{c},\bar{d})=S_{m,n}(x;a,b), \tag{6.37}$$

with

$$\bar{c}=a+b+n-\frac{1}{2},\qquad \bar{d}=a-b+m+\frac{1}{2}. \tag{6.38}$$

Remark 6.8. The polynomials $\bar{p}_k$ and $\bar{q}_k$ are also expressed by the Jacobi polynomials as

$$\bar{p}_k^{(\bar{c},\bar{d})}(x)=(-1)^k P_k^{(\bar{d}-1-k,\bar{c}-\bar{d})}(1+2x). \tag{6.39}$$

Let us consider the degeneration of the algebraic solutions of S_{VI}. Applying the Bäcklund transformation $s_1 s_0$ to the solutions in Theorem 2.2, we obtain the following corollary.

Corollary 6.9. *Let* $\bar{R}_{m,n}=\bar{R}_{m,n}(x;\bar{c},\bar{d})$ *be polynomials defined in Lemma* 6.7. *Then, for* $m,n\in\boldsymbol{Z}$,

$$q=x\frac{\bar{R}^{(0,1)}_{m,n-1}\bar{R}^{(1,0)}_{m-1,n}}{\bar{R}^{(0,1)}_{m-1,n}\bar{R}^{(1,0)}_{m,n-1}},\qquad p=\frac{2n-1}{2x(1-x)}\frac{\bar{R}^{(0,1)}_{m-1,n}\bar{R}^{(1,0)}_{m,n-1}\bar{R}^{(0,1)}_{m-1,n-2}}{\bar{R}^{(0,0)}_{m-1,n-1}\bar{R}^{(0,1)}_{m,n-1}\bar{R}^{(1,1)}_{m-1,n-1}}, \tag{6.40}$$

where we denote $\bar{R}^{(i,j)}_{m,n}=\bar{R}_{m,n}(x;\bar{c}+i,\bar{d}+j)$, *satisfy* S_{VI} *for the parameters*

$$\kappa_\infty=-b,\qquad \kappa_0=b-m+n,\qquad \kappa_1=a+m+n,\qquad \theta=-a, \tag{6.41}$$

under the setting of (6.37), (6.38) *and* $x^2=t$.

According to (6.26) and (6.41), we choose η_∞ and η_0 as in (6.34) and set

$$a=\frac{1}{2}\left(-\varepsilon^{-1}+r+\frac{1}{2}-m+\zeta\right),\qquad b=\frac{1}{2}\left(-\varepsilon^{-1}-r-\frac{1}{2}+m+\zeta\right), \tag{6.42}$$

where ζ is a quantity of $O(1)$. Then, we have

$$\theta_\infty^{(1)}=\frac{1}{2}\left(r+\frac{1}{2}-m-\zeta\right),\qquad \theta_\infty^{(2)}=\frac{1}{2}\left(r+\frac{1}{2}+m+\zeta\right)+n, \tag{6.43}$$

$$\theta_0^{(1)}+1=-\frac{1}{2}\left(r+\frac{1}{2}+m-\zeta\right)+n,\qquad \theta_0^{(2)}=-\frac{1}{2}\left(r+\frac{1}{2}-m+\zeta\right).$$

Setting as (6.25) and (6.27), we see that S_{VI} with (6.41) is reduced to S_{III} with (6.33) in the limit as $\varepsilon \to 0$. Note that m vanishes in (6.33). Then, it is possible to put $m = 0$ without losing generality in this limiting procedure.

Next, we investigate the degeneration of $\bar{R}_n^{(i,j)} = \bar{R}_{-1,n}^{(i,j)} = \bar{R}_{0,n}^{(i,j)}$. Putting

$$x \to \varepsilon t, \qquad \bar{c} = -\varepsilon^{-1} + \zeta + n - \frac{1}{2}, \qquad \bar{d} = r + 1, \tag{6.44}$$

we find that the generating function (6.36) degenerates as

$$\bar{G} = (1 - \lambda)^{\bar{d}-1} \exp[-\bar{c} \log(1 + x\lambda)] = (1 - \lambda)^r \exp[t\lambda + O(\varepsilon)]. \tag{6.45}$$

Then, we have

$$\lim_{\varepsilon \to 0} \bar{p}_k^{(\bar{c},\bar{d})}(x) = (-1)^k p_k^{(r)}(t), \tag{6.46}$$

which gives

$$\lim_{\varepsilon \to 0} \bar{R}_n^{(i,j)}(x) = (-1)^{n(n+1)/2} R_n^{(r+j)}(t). \tag{6.47}$$

Finally, it is easy to see that (6.40) leads to (6.32) in the above limit.

Remark 6.10. The polynomials $p_k^{(r)}$ defined by (6.29) are also the Laguerre polynomials, namely, $p_k^{(r)}(t) = L_k^{(r-k)}(t)$. Then, the above degeneration also corresponds to that from the Jacobi polynomials to the Laguerre polynomials.

Similarly, the rational solutions of P_V and P_{III} given in Proposition 6.2 and 6.6, respectively, degenerate to those of P_{II}. We give more details in Appendix A. Therefore, the coalescence cascade (6.3) is obtained.

7. Relationship to the original Umemura polynomials

In this section, we show that the original Umemura polynomials for P_{VI} are a special case of our polynomials $V_{m,n}(x; a, b)$ introduced in Section 4.

7.1. Umemura polynomials associated with P_{VI}

First, we briefly summarize the derivation of the original Umemura polynomials for P_{VI} [27]. Set the parameters b_i $(i = 1, 2, 3, 4)$ to

(7.1)

$$b_1 = \frac{1}{2}(\kappa_0 + \kappa_1), \quad b_2 = \frac{1}{2}(\kappa_0 - \kappa_1), \quad b_3 = \frac{1}{2}(\theta - 1 + \kappa_\infty), \quad b_4 = \frac{1}{2}(\theta - 1 - \kappa_\infty),$$

namely,

(7.2)

$$b_1 = \frac{1}{2}(\alpha_4 + \alpha_3), \quad b_2 = \frac{1}{2}(\alpha_4 - \alpha_3), \quad b_3 = \frac{1}{2}(\alpha_0 - 1 + \alpha_1), \quad b_4 = \frac{1}{2}(\alpha_0 - 1 - \alpha_1).$$

Umemura has shown that

$$q = \frac{(\alpha+\beta)^2 t \pm (\alpha^2 - \beta^2)\sqrt{t(t-1)}}{(\alpha-\beta)^2 + 4\alpha\beta t}, \qquad p = \frac{\alpha q - (\alpha+\beta)/2}{q(q-1)} \tag{7.3}$$

give an algebraic solution of the Hamilton system S_{VI} for the parameters

$$(b_1, b_2, b_3, b_4) = \left(\alpha, \beta, -\frac{1}{2}, 0\right). \tag{7.4}$$

Substituting the solution of upper sign into the Hamiltonian (1.3), one obtain

$$H = \frac{1}{4}[-(\alpha+\beta) + (\alpha+\beta)^2 + 2\alpha t - 2(\alpha^2+\beta^2)t + 2(\alpha^2 - \beta^2)\sqrt{t(t-1)}]. \tag{7.5}$$

Application of the translation

$$(b_1, b_2, b_3, b_4) \to (b_1, b_2, b_3, b_4) + n(0, 0, 1, 0), \qquad n \in \boldsymbol{Z}, \tag{7.6}$$

to the seed solution (7.3) with (7.4) generates a sequence of algebraic solutions (q_n, p_n). Let τ_n be a τ-function with respect to the solution (q_n, p_n). Okamoto has pointed out that τ_n satisfy the Toda equation [21]

$$\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = \frac{d}{dt}(\log \tau_n)' + (b_1 + b_3 + n)(b_3 + b_4 + n), \qquad ' = t(t-1)\frac{d}{dt}. \tag{7.7}$$

Define a family of functions T_n for $n \in \boldsymbol{Z}$ by

$$(\log \tau_n)' = (\log T_n)' + H - n\left(\alpha t - \frac{\alpha+\beta}{2}\right). \tag{7.8}$$

Then, the Toda equation (7.7) yields

$$T_{n+1}T_{n-1} = t(t-1)\left[\frac{d^2 T_n}{dt^2} T_n - \left(\frac{dT_n}{dt}\right)^2\right] + (2t-1)\frac{dT_n}{dt} T_n \tag{7.9}$$
$$+ \left\{\frac{1}{4}\left[-2(\alpha^2+\beta^2) + (\alpha^2-\beta^2)\frac{2t-1}{\sqrt{t(t-1)}}\right] + \left(n - \frac{1}{2}\right)^2\right\} T_n^2.$$

Moreover, introducing a new variable v as

$$v = \sqrt{\frac{t}{t-1}} + \sqrt{\frac{t-1}{t}}, \tag{7.10}$$

we find that the T_n are generated by the recurrence relation

$$(7.11)\qquad T_{n+1}T_{n-1}=\frac{1}{4}(v^2-4)\left[(v^2-4)\frac{d^2T_n}{dv^2}+v\frac{dT_n}{dv}\right]T_n-\frac{1}{4}(v^2-4)^2\left(\frac{dT_n}{dv}\right)^2 + \left\{\frac{1}{4}[-2(\alpha^2+\beta^2)+(\alpha^2-\beta^2)v]+\left(n-\frac{1}{2}\right)^2\right\}T_n^2,$$

with the initial conditions $T_0=T_1=1$. It has been shown that T_n for $n\in \mathbf{Z}_{\geq 0}$ are polynomials in α,β and v, and $\deg_v T_n = n(n-1)/2$. These polynomials are called Umemura polynomials associated with P_{VI}.

7.2. Correspondence of the seed solution

We investigate how Umemura's seed solution (7.3) with (7.4) is related to our seed solution,

$$(7.12)\qquad q=f_4=x,\qquad p=f_2=\frac{1}{2}\left(a+b-\frac{1}{2}\right)x^{-1},$$

with

$$(7.13)\qquad (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\left(a,b,\frac{1}{2}-a-b,a,b\right).$$

In addition to the Bäcklund transformations stated in Section 3, it is known that P_{VI} admits the outer symmetry as follows [21],

(7.14)

	α_0	α_1	α_2	α_3	α_4	t	f_4	f_2
σ_{01}	α_1	α_0	α_2	α_3	α_4	$1-t$	$\frac{(1-t)f_4}{f_0}$	$\frac{f_0(f_0f_2+\alpha_2)}{t(t-1)}$
σ_{03}	α_3	α_1	α_2	α_0	α_4	$\frac{1}{t}$	$\frac{f_4}{t}$	tf_2
σ_{04}	α_4	α_1	α_2	α_3	α_0	$\frac{t}{t-1}$	$\frac{f_0}{1-t}$	$(1-t)f_2$
σ_{13}	α_0	α_3	α_2	α_1	α_4	$\frac{t}{t-1}$	$\frac{f_4}{f_3}$	$-f_3(f_3f_2+\alpha_2)$
σ_{14}	α_0	α_4	α_2	α_3	α_1	$\frac{1}{t}$	$\frac{1}{f_4}$	$-f_4(f_4f_2+\alpha_2)$
σ_{34}	α_0	α_1	α_2	α_4	α_3	$1-t$	$-f_3$	$-f_2$

Proposition 7.1. *Umemura's seed solution* (7.3) *with* (7.4) *is obtained by applying the Bäcklund transformation defined by*

(7.15) $$\sigma = \sigma_{13} s_3 s_2 s_1,$$

to our (7.12) *with* (7.13), *where we set*

(7.16) $$\alpha = \frac{1}{2} - a, \qquad \beta = b.$$

Proof. First, we check for the parameters. Application of σ to (7.13) gives

(7.17) $$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{2}, -\frac{1}{2}, a, \frac{1}{2} - a - b, \frac{1}{2} - a + b\right),$$

which coincides with (7.4) by using (7.2) and (7.16).

Next, we verify the correspondence of $q = f_4$. We have

(7.18) $$\sigma(f_4) = \frac{f_2 f_4 + \alpha_1 + \alpha_2}{f_2 f_3 + \alpha_1 + \alpha_2} = \frac{\frac{1}{2} - a + b}{\left(\frac{1}{2} - a + b\right) + \left(\frac{1}{2} - a - b\right)x^{-1}}.$$

Note that x is now given by

(7.19) $$x = \mp\sqrt{\frac{t}{t-1}},$$

due to the action of σ_{13}. Thus, the expression (7.18) is equivalent to the first of (7.3). It is possible to check for $p = f_2$ in similar way. ■

7.3. Relationship to the original Umemura polynomials

The above discussion on the seed solution suggests that the family of polynomials $V_{m,n}(x; a, b)$ constructed in Section 4 corresponds to the original Umemura polynomials under the setting of (7.16) and

(7.20) $$x = -\sqrt{\frac{t}{t-1}}.$$

Notice that, from (7.9) and (7.20), the $T_n = T_n(x; \alpha, \beta)$ satisfy the recurrence relation

(7.21)
$$4T_{n+1}T_{n-1} = x^{-1}[(x^2-1)^2\mathscr{D}^2 - \alpha^2(x+1)^2 + \beta^2(x-1)^2 + (2n-1)^2 x]T_n \cdot T_n,$$

with $T_0 = T_1 = 1$.

Theorem 7.2. *We have*

$$T_n(x;\alpha,\beta)=2^{-2n(n-1)}(-x)^{-n(n-1)/2}V_{-n,-n}(x;a+n,b), \tag{7.22}$$

with (7.16).

We prove Theorem 7.2 by showing that both sides of (7.22) satisfy the same recurrence relation and initial conditions. Let $\hat{T}_{30}$ be the translation operator defined by

$$\hat{T}_{30}=T_{34}\hat{T}_{34}T_{03}^{-1}. \tag{7.23}$$

Then, we have a Toda equation

$$\hat{T}_{30}(\tau_0)\hat{T}_{30}^{-1}(\tau_0) = t^{-1/2}\left[(t-1)\frac{d}{dt}(\log\tau_0)'-(\log\tau_0)'+\frac{1}{4}(\alpha_0-\alpha_3)(\alpha_0-\alpha_3-2)+\frac{3}{4}\right]\tau_0^2. \tag{7.24}$$

For simplicity, we denote

$$\bar{\tau}_n=\hat{T}_{30}^n(\tau_0),\qquad n\in\boldsymbol{Z}, \tag{7.25}$$

namely $\bar{\tau}_n=\tau_{-n,0,n,n}$ in the notation of (3.40). The above Toda equation is expressed as

$$\bar{\tau}_{n+1}\bar{\tau}_{n-1}=t^{-1/2}\left[(t-1)\frac{d}{dt}(\log\bar{\tau}_n)'-(\log\bar{\tau}_n)' +\frac{(\alpha_0-\alpha_3-2n)(\alpha_0-\alpha_3-2n-2)}{4}+\frac{3}{4}\right]\bar{\tau}_n^2. \tag{7.26}$$

In the following, we restrict our discussion to the algebraic solutions. According to (4.7) and (4.21), we introduce $\overline{V}_n=\overline{V}_n(x;a,b)$ as

$$\bar{\tau}_n=\omega_n\overline{V}_n(x-1)^{(a-1/2)^2+1/2}x^{-(a-1/2)^2-b^2-n(n+1)-1/8}(x+1)^{b^2+1/2}, \tag{7.27}$$

where $\omega_n=\omega_{-n,0,n,n}$. Substituting (7.27) and $\alpha_0=\alpha_3=a$ into the Toda equation (7.26) and noticing

$$\omega_{n+1}\omega_{n-1}=-\frac{1}{16}\omega_n^2, \tag{7.28}$$

we find that $\overline{V}_n=\overline{V}_n(x;a,b)$ are generated by the recurrence relation

$$-\frac{1}{4}\overline{V}_{n+1}\overline{V}_{n-1} = \left[(x^2-1)^2\mathscr{D}^2-\left(a-\frac{1}{2}\right)^2(x+1)^2+b^2(x-1)^2+(2n+1)^2x\right]\overline{V}_n\cdot\overline{V}_n, \tag{7.29}$$

with the initial conditions $\bar{V}_{-1} = \bar{V}_0 = 1$. By construction, it is easy to see that

$$\bar{V}_n(x; a, b) = V_{n,n}(x; a - n, b). \tag{7.30}$$

Moreover, we introduce $\bar{T}_n = \bar{T}_n(x; a, b)$ as

$$\bar{T}_n = 2^{-2n(n-1)}(-x)^{-n(n-1)/2}\bar{V}_{-n}. \tag{7.31}$$

Then, the $\bar{T}_n$ satisfy the recurrence relation

$$4\bar{T}_{n+1}\bar{T}_{n-1} = x^{-1}\left[(x^2-1)^2\mathscr{D}^2 - \left(a - \frac{1}{2}\right)^2(x+1)^2 + b^2(x-1)^2 + (2n-1)^2 x\right]\bar{T}_n \cdot \bar{T}_n, \tag{7.32}$$

with $\bar{T}_0 = \bar{T}_1 = 1$.

Comparing (7.21) with (7.32), we find

$$T_n(x; \alpha, \beta) = \bar{T}_n(x; a, b), \tag{7.33}$$

under the setting of (7.16), which is nothing but Theorem 7.2.

Remark 7.3. The Toda equation (7.7) can be regarded as the recurrence relation with respect to the translation operator

$$T_{01} = T_{34}^{-1}T_{14}T_{03}, \tag{7.34}$$

which acts on the parameters as

$$T_{01}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, 1, -1, 0, 0). \tag{7.35}$$

Theorem 7.2, namely (7.33), is consistent with the relation

$$T_{01}\sigma = \sigma\hat{T}_{30}^{-1}. \tag{7.36}$$

From the discussion of the previous sections and (7.31), it is clear that T_n for $n \in \mathbf{Z}_{\geq 0}$ are polynomials in α, β and v, and $\deg_v T_n = n(n-1)/2$ under the setting of

$$v = -(x + x^{-1}). \tag{7.37}$$

Acknowledgment. The author would like to thank Prof. M. Noumi, Prof. Y. Yamada and Prof. K. Kajiwara for useful suggestions and discussions. Especially, he owes the initial steps of this work, including the lifting of the Bäcklund transformations of P_{VI} to the τ-functions, to discussions with them. The author would also express his sincere thanks to Prof. Kirillov and Dr. Taneda for stimulating discussions.

A. Degeneration of rational solutions

In this section, we show that the rational solutions of P_V and P_{III} given in Proposition 6.2 and Proposition 6.6, respectively, degenerate to those of P_{II},

$$\frac{d^2y}{dt^2} = 2y^3 - 4ty + 4\left(\alpha + \frac{1}{2}\right). \tag{A.1}$$

As is known [23], P_{II} (A.1) is equivalent to the Hamilton system

$$S_{II}: \quad q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}, \qquad ' = \frac{d}{dt}, \tag{A.2}$$

with the Hamiltonian

$$H = -2p^2 - (q^2 - 2t)p + \alpha q. \tag{A.3}$$

The rational solutions of S_{II} are expressed as follows [7].

Proposition A.1. *Let $q_k = q_k(t)$, $k \in \boldsymbol{Z}$, be polynomials defined by*

$$\sum_{k=0}^{\infty} q_k \lambda^k = \exp\left(t\lambda + \frac{\lambda^3}{3}\right), \qquad q_k = 0 \quad \text{for } k < 0. \tag{A.4}$$

We define $R_n = R_n(t)$ by

$$R_n = \begin{vmatrix} q_n & \cdots & q_{2n-2} & q_{2n-1} \\ \vdots & \ddots & \vdots & \vdots \\ q_{-n+4} & \cdots & q_2 & q_3 \\ q_{-n+2} & \cdots & q_0 & q_1 \end{vmatrix}, \tag{A.5}$$

for $n \in \boldsymbol{Z}_{\geq 0}$ and by

$$R_n = (-1)^{n(n+1)/2} R_{-n-1}, \tag{A.6}$$

for $n \in \boldsymbol{Z}_{<0}$, respectively. Then,

$$q = \frac{d}{dt} \log \frac{R_n}{R_{n-1}}, \qquad p = \frac{2n-1}{2} \frac{R_n R_{n-2}}{R_{n-1}^2}, \tag{A.7}$$

give the rational solutions of S_{II} for the parameters

$$\alpha = n - \frac{1}{2}. \tag{A.8}$$

A.1. From P_V to P_{II}

It is possible to derive the Hamilton system S_{II} from S_V, directly, by degeneration. Putting

$$(A.9)\qquad t \to \varepsilon^{-3}(1+2\varepsilon^2 t),\qquad q \to -1+2\varepsilon q,\qquad p \to \frac{1}{2}\varepsilon^{-1}p,$$

$$(A.10)\qquad \kappa_\infty \to \frac{\sigma}{4}\varepsilon^{-3}+\kappa_\infty^{(0)},\qquad \kappa_0 \to \frac{1}{4}\varepsilon^{-3}+\kappa_0^{(0)},\qquad \theta \to 2\theta^{(0)},$$

$$(A.11)\qquad H \to \frac{1}{2}\varepsilon^{-2}H-\frac{1}{2}\varepsilon^{-3}\alpha,\qquad \alpha = \theta^{(0)}+\frac{\kappa_0^{(0)}-\sigma\kappa_\infty^{(0)}}{2},$$

with $\sigma = \pm 1$ and taking the limit as $\varepsilon \to 0$, we find that the system (6.5) with the Hamiltonian (6.6) is reduced to (A.2) with (A.3).

We show that the rational solutions of S_V given in Proposition 6.2 degenerate to those of S_{II} in Proposition A.1. According to (A.10) and (6.14), we set $\sigma = 1$ and

$$(A.12)\qquad s = \frac{1}{4}\varepsilon^{-3},\qquad \kappa_\infty^{(0)} = 0,\qquad \kappa_0^{(0)} = -m+n,\qquad \theta^{(0)} = \frac{m+n-1}{2}.$$

Then, after the replacements (A.9) and (A.11), we find that S_V with (6.14) is reduced to S_{II} with (A.8) in the limit as $\varepsilon \to 0$. Note that m vanishes in (A.8). Then, it is possible to put $m = 0$ without loss of generality in this limiting procedure.

Next, we investigate the degeneration of $R_n^{(r)} = R_{-1,n}^{(r)} = R_{0,n}^{(r)}$. It is obvious that we have the following lemma.

Lemma A.2. *Let $\bar{p}_k = \bar{p}_k^{(r)}(z)$, $k \in \boldsymbol{Z}$, be polynomials defined by*

$$(A.13)\qquad \sum_{k=0}^{\infty}\bar{p}_k^{(r)}\lambda^k = \exp\left[\sum_{j=1}^{\infty}\left(-z+\frac{r}{j}\right)\lambda^j+\frac{r}{2}\lambda^2\right],\qquad \bar{p}_k^{(r)} = 0 \quad \text{for } k<0.$$

Then, we have

$$(A.14)\qquad R_n^{(r)}(z) = \begin{vmatrix} \bar{p}_n^{(r)} & \cdots & \bar{p}_{2n-2}^{(r)} & \bar{p}_{2n-1}^{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ \bar{p}_{-n+4}^{(r)} & \cdots & \bar{p}_2^{(r)} & \bar{p}_3^{(r)} \\ \bar{p}_{-n+2}^{(r)} & \cdots & \bar{p}_0^{(r)} & \bar{p}_1^{(r)} \end{vmatrix}.$$

Put

$$(A.15)\qquad \lambda \to -\varepsilon\lambda,\qquad \bar{q}_k^{(r)} = (-\varepsilon)^k\bar{p}_k^{(r)},$$

and

$$(A.16)\qquad z \to \frac{1}{2}\varepsilon^{-3}(1+2\varepsilon^2 t),\qquad r = \frac{1}{2}\varepsilon^{-3}+n.$$

Then, (A.13) yields

$$\text{(A.17)} \qquad \sum_{k=0}^{\infty} \bar{q}_k^{(r+j)} \lambda^k = \exp\left(t\lambda + \frac{\lambda^3}{3}\right)\left[1 - \varepsilon\left(j\lambda + n\lambda + t\lambda^2 + \frac{3}{8}\lambda^4\right) + O(\varepsilon^2)\right].$$

By using (A.4), we obtain

$$\text{(A.18)} \qquad \bar{q}_k^{(r+j)} = q_k - \varepsilon j q_{k-1} - \varepsilon\left(n q_{k-1} + t q_{k-2} + \frac{3}{8} q_{k-4}\right) + O(\varepsilon^2).$$

Since it is easy to see that

$$\text{(A.19)} \qquad \frac{dq_k}{dt} = q_{k-1},$$

we have, from (A.5),

$$\text{(A.20)} \qquad R_n^{(r+j)} = (-\varepsilon)^{-n(n+1)/2}\left[R_n - \varepsilon j \frac{dR_n}{dt} - \varepsilon Q_n + O(\varepsilon^2)\right],$$

where Q_n denotes the contribution from the third term of (A.18).

Finally, we verify the degeneration of the variables q and p. The above procedure gives

$$\text{(A.21)} \qquad -\frac{R_{n-1}^{(r-1)} R_n^{(r+1)}}{R_n^{(r-1)} R_{n-1}^{(r+1)}} = -1 + 2\varepsilon \frac{d}{dt} \log \frac{R_n}{R_{n-1}} + O(\varepsilon^2),$$

$$-\frac{R_n^{(r-1)} R_{n-1}^{(r+1)} R_{n-2}^{(r-1)}}{R_{n-1}^{(r)} R_{n-1}^{(r)} R_{n-1}^{(r-1)}} = \varepsilon^{-1} \frac{R_n R_{n-2}}{R_{n-1}^2} + O(1).$$

Thus, from (A.9), we get (A.7) in the limit as $\varepsilon \to 0$.

A.2. From P_{III} to P_{II}

It is well known that the Hamilton system S_{II} is derived from S_{III} by degeneration [4]. This process is achieved by putting

$$\text{(A.22)} \qquad t \to -\varepsilon^{-3}(1 - \varepsilon^2 t), \qquad q \to 1 + \varepsilon q, \qquad p \to \varepsilon^{-1} p,$$

$$\text{(A.23)} \qquad \theta_\infty \to -\varepsilon^{-3} + \theta_\infty^{(0)}, \qquad \theta_0 \to \varepsilon^{-3} + \theta_0^{(0)},$$

$$\text{(A.24)} \qquad H \to -\varepsilon^{-2} H - \varepsilon^{-3}\alpha, \qquad \alpha = \frac{\theta_\infty^{(0)} + \theta^{(0)}}{2},$$

and taking the limit as $\varepsilon \to 0$.

We show that the rational solutions of S_{III} given in Proposition 6.6 degenerate to those of S_{II} in Proposition A.1. From (A.23), we set

(A.25) $$r = -\varepsilon^{-3}, \qquad \theta_\infty^{(0)} = n + \frac{1}{2}, \qquad \theta_0^{(0)} = n - \frac{3}{2}.$$

Then, after replacing as (A.22) and (A.24), we see that S_{III} with (6.33) is reduced to S_{II} with (A.8) in the limit as $\varepsilon \to 0$.

Next, we investigate the degeneration of $R_n^{(r)}$ defined by (6.29) and (6.30). It is obvious that we have the following lemma.

Lemma A.3. *Let $\bar{p}_k = \bar{p}_k^{(r)}(t)$, $k \in \boldsymbol{Z}$, be polynomials defined by*

(A.26) $$\sum_{k=0}^{\infty} \bar{p}_k^{(r)} \lambda^k = \exp\left[\sum_{j=1}^{\infty} \frac{(-1)^{j-1} r}{j} \lambda^j - t\lambda + \frac{r}{2}\lambda^2\right], \qquad \bar{p}_k^{(r)} = 0 \quad \textit{for } k < 0.$$

Then, we have

(A.27) $$R_n^{(r)}(t) = \begin{vmatrix} \bar{p}_n^{(r)} & \cdots & \bar{p}_{2n-2}^{(r)} & \bar{p}_{2n-1}^{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ \bar{p}_{-n+4}^{(r)} & \cdots & \bar{p}_2^{(r)} & \bar{p}_3^{(r)} \\ \bar{p}_{-n+2}^{(r)} & \cdots & \bar{p}_0^{(r)} & \bar{p}_1^{(r)} \end{vmatrix}.$$

Put

(A.28) $$\lambda \to -\varepsilon\lambda, \qquad \bar{q}_k^{(r)} = (-\varepsilon)^k \bar{p}_k^{(r)},$$

and

(A.29) $$t \to -\varepsilon^{-3}(1 - \varepsilon^2 t), \qquad r = -\varepsilon^{-3}.$$

Then, (A.26) is written as

(A.30) $$\sum_{k=0}^{\infty} \bar{q}_k^{(r+j)} \lambda^k = \exp\left(t\lambda + \frac{\lambda^3}{3}\right)\left[1 + \varepsilon\left(-j\lambda + \frac{1}{4}\lambda^4\right) + O(\varepsilon^2)\right].$$

By using (A.4), we obtain

(A.31) $$\bar{q}_k^{(r+j)} = q_k + \varepsilon\left(-jq_{k-1} + \frac{1}{4}q_{k-4}\right) + O(\varepsilon^2).$$

Thus, we have

(A.32) $$R_n^{(r)} = (-\varepsilon)^{-n(n+1)/2}\left[R_n + \varepsilon\left(-j\frac{dR_n}{dt} + Q_n\right) + O(\varepsilon^2)\right],$$

where Q_n denotes the contribution from the term of q_{k-4} in (A.31).

Finally, it is easy to see that (6.32) is reduced to (A.7) by the above limiting procedures.

B. Proof of Lemma 5.4–5.8

We first note that the following contiguity relations hold by definition (2.1) and (2.2),

$$\text{(B.1)} \qquad p_k^{(c-1,d-1)} = p_k^{(c,d)} + x p_{k-1}^{(c,d)}, \qquad q_k^{(c-1,d-1)} = q_k^{(c,d)} + x^{-1} q_{k-1}^{(c,d)},$$

$$\text{(B.2)} \qquad p_k^{(c,d-1)} = p_k^{(c,d)} - p_{k-1}^{(c,d)}, \qquad q_k^{(c,d-1)} = q_k^{(c,d)} - q_{k-1}^{(c,d)},$$

$$\text{(B.3)} \qquad \begin{aligned} (k+1)p_{k+1}^{(c,d)} &= -(c-d)p_k^{(c,d+1)} - cx p_k^{(c+1,d+1)}, \\ (k+1)q_{k+1}^{(c,d)} &= -(c-d)q_k^{(c,d+1)} - cx^{-1} q_k^{(c+1,d+1)}. \end{aligned}$$

Let us prove Lemma 5.4. Adding the $(i+1)$-th column multiplied by x^{-1} to the i-th column of $R_{m,n}^{(0,0)}$ for $i = 1, 2, \ldots, j$, $j = m+n-1, m+n-2, \ldots, 1$, and using (B.1), we get

(B.4)

$$R_{m,n}^{(0,0)} = \begin{vmatrix} q_1^{(c-m-n+1,d-m-n+1)} & q_0^{(c-m-n+2,d-m-n+2)} & \cdots & q_{-m-n+3}^{(c-1,d-1)} & q_{-m-n+2}^{(c,d)} \\ q_3^{(c-m-n+1,d-m-n+1)} & q_2^{(c-m-n+2,d-m-n+2)} & \cdots & q_{-m-n+5}^{(c-1,d-1)} & q_{-m-n+4}^{(c,d)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1}^{(c-m-n+1,d-m-n+1)} & q_{2m-2}^{(c-m-n+2,d-m-n+2)} & \cdots & q_{m-n+1}^{(c-1,d-1)} & q_{m-n}^{(c,d)} \\ x^{-m-n+1} p_{2n-1}^{(c-m-n+1,d-m-n+1)} & x^{-m-n+2} p_{2n-1}^{(c-m-n+2,d-m-n+2)} & \cdots & x^{-1} p_{2n-1}^{(c-1,d-1)} & p_{2n-1}^{(c,d)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{-m-n+1} p_3^{(c-m-n+1,d-m-n+1)} & x^{-m-n+2} p_3^{(c-m-n+2,d-m-n+2)} & \cdots & x^{-1} p_3^{(c-1,d-1)} & p_3^{(c,d)} \\ x^{-m-n+1} p_1^{(c-m-n+1,d-m-n+1)} & x^{-m-n+2} p_1^{(c-m-n+2,d-m-n+2)} & \cdots & x^{-1} p_1^{(c-1,d-1)} & p_1^{(c,d)} \end{vmatrix}.$$

Noticing that $p_0 = 1$ and $p_k = 0$ for $k < 0$, we see that $R_{m,n}$ can be rewritten as

$$\text{(B.5)} \qquad R_{m,n} = \begin{vmatrix} q_1 & q_0 & \cdots & q_{-m-n+2} & q_{-m-n+1} \\ q_3 & q_2 & \cdots & q_{-m-n+4} & q_{-m-n+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1} & q_{2m-2} & \cdots & q_{m-n} & q_{m-n-1} \\ p_{n-m} & p_{n-m+1} & \cdots & p_{2n-1} & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+2} & p_{-n-m+3} & \cdots & p_1 & p_2 \\ p_{-n-m} & p_{-n-m+1} & \cdots & p_{-1} & p_0 \end{vmatrix}.$$

By a similar calculation, we obtain

$$
\text{(B.6)}\quad R_{m,n}^{(0,0)}
= \begin{vmatrix}
q_1^{(c-m-n,d-m-n)} & q_0^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+2}^{(c-1,d-1)} & q_{-m-n+1}^{(c,d)} \\
q_3^{(c-m-n,d-m-n)} & q_2^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+4}^{(c-1,d-1)} & q_{-m-n+3}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-2}^{(c-m-n+1,d-m-n+1)} & \cdots & q_{m-n}^{(c-1,d-1)} & q_{m-n-1}^{(c,d)} \\
x^{-m-n}p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{2n}^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_{2n}^{(c-1,d-1)} & p_{2n}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-m-n}p_2^{(c-m-n,d-m-n)} & x^{-m-n+1}p_2^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_2^{(c-1,d-1)} & p_2^{(c,d)} \\
x^{-m-n}p_0^{(c-m-n,d-m-n)} & x^{-m-n+1}p_0^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_0^{(c-1,d-1)} & p_0^{(c,d)}
\end{vmatrix}.
$$

We have from (B.1) and (B.2) that

$$
\text{(B.7)}\qquad \begin{aligned}
(1+x)p_k^{(c,d)} &= p_k^{(c-1,d-1)} + xp_k^{(c,d-1)}, \\
q_{k+1}^{(c,d-1)} + (1+x^{-1})q_k^{(c,d)} &= q_{k+1}^{(c-1,d-1)}.
\end{aligned}
$$

Subtracting the j-th column multiplied by $(1+x^{-1})^{-1}$ from the $(j+1)$-th column of (B.6) for $j = m+n, m+n-1, \dots, 1$, and using (B.7), we get

$$
\text{(B.8)}\quad R_{m,n}^{(0,0)} = (-1)^m(1+x^{-1})^{-m-n}
$$

$$
\times \begin{vmatrix}
-q_1^{(c-m-n,d-m-n)} & q_1^{(c-m-n+1,d-m-n)} & \cdots & q_{-m-n+3}^{(c-1,d-2)} & q_{-m-n+2}^{(c,d-1)} \\
-q_3^{(c-m-n,d-m-n)} & q_3^{(c-m-n+1,d-m-n)} & \cdots & q_{-m-n+5}^{(c-1,d-2)} & q_{-m-n+4}^{(c,d-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-1}^{(c-m-n+1,d-m-n)} & \cdots & q_{m-n+1}^{(c-1,d-2)} & q_{m-n}^{(c,d-1)} \\
x^{-m-n}p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{2n}^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_{2n}^{(c-1,d-2)} & p_{2n}^{(c,d-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-m-n}p_2^{(c-m-n,d-m-n)} & x^{-m-n+1}p_2^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_2^{(c-1,d-2)} & p_2^{(c,d-1)} \\
x^{-m-n}p_0^{(c-m-n,d-m-n)} & x^{-m-n+1}p_0^{(c-m-n+1,d-m-n)} & \cdots & x^{-1}p_0^{(c-1,d-2)} & p_0^{(c,d-1)}
\end{vmatrix}.
$$

From (B.6) and (B.8), we obtain Lemma 5.4.

Next, we prove Lemma 5.5. We have

$$(B.9)\qquad (k+1)p_{k+1}^{(c,d)} = dp_k^{(c,d+1)} - c(1+x)p_k^{(c+1,d+2)},$$

$$(d+k+1)q_{k+1}^{(c,d)} = dq_{k+1}^{(c,d+1)} - c(1+x^{-1})q_k^{(c+1,d+2)}.$$

Subtracting the j-th column multiplied by $\dfrac{d-m-n+j-2}{(c-m-n+j-1)(1+x^{-1})}$ from the $(j+1)$-th column of (B.6) for $j = m+n, m+n-1, \ldots, 1$, and using (B.9), we get

$$(B.10)\quad R_{m,n}^{(0,0)} = (-1)^{m+n}(1+x)^{-m-n}x^m \frac{\prod_{i=1}^{m}(d-m-n+2i-2)\prod_{k=0}^{n}(2k+1)}{\prod_{j=1}^{m+n}(c-m-n+j-1)}$$

$$\times \begin{vmatrix} \tilde{q}_1^{(c-m-n,d-m-n)} & q_1^{(c-m-n,d-m-n-1)} & \cdots & q_{-m-n+3}^{(c-2,d-3)} & q_{-m-n+2}^{(c-1,d-2)} \\ \tilde{q}_3^{(c-m-n,d-m-n)} & q_3^{(c-m-n,d-m-n-1)} & \cdots & q_{-m-n+5}^{(c-2,d-3)} & q_{-m-n+4}^{(c-1,d-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{q}_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-1}^{(c-m-n,d-m-n-1)} & \cdots & q_{m-n+1}^{(c-2,d-3)} & q_{m-n}^{(c-1,d-2)} \\ \hat{p}_{2n}^{(c-m-n,d-m-n)} & x^{-m-n+1}p_{2n+1}^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_{2n+1}^{(c-2,d-3)} & p_{2n+1}^{(c-1,d-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{p}_2^{(c-m-n,d-m-n)} & x^{-m-n+1}p_3^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_3^{(c-2,d-3)} & p_3^{(c-1,d-2)} \\ \hat{p}_0^{(c-m-n,d-m-n)} & x^{-m-n+1}p_1^{(c-m-n,d-m-n-1)} & \cdots & x^{-1}p_1^{(c-2,d-3)} & p_1^{(c-1,d-2)} \end{vmatrix}.$$

Lemma 5.5 follows from (B.4) and (B.10).

Note that we have

$$(B.11)\qquad (k+1)p_{k+1}^{(c,d)} = -dxp_k^{(c+1,d+1)} - (c-d)(1+x)p_k^{(c+1,d+2)},$$

$$(d+k+1)q_{k+1}^{(c,d)} = dq_{k+1}^{(c+1,d+1)} - (c-d)(1+x^{-1})q_k^{(c+1,d+2)}.$$

It is easy to see that Lemma 5.6 is proved similarly to Lemma 5.5 by using (B.2) and (B.11).

The proof of Lemma 5.7 is given as follows: Adding the $(j-1)$-th column multiplied by x to the j-th column of $R_{m,n}^{(1,1)}$ for $j = m+n, m+n-1, \ldots, 2$, and using (B.1), we get

$$\text{(B.12)} \qquad R_{m,n}^{(1,1)} = x^m \begin{vmatrix} x^{-1}q_1^+ & q_1 & \cdots & q_{-m-n+4} & q_{-m-n+3} \\ x^{-1}q_3^+ & q_3 & \cdots & q_{-m-n+6} & q_{-m-n+5} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{-1}q_{2m-1}^+ & q_{2m-1} & \cdots & q_{m-n+2} & q_{m-n+1} \\ p_{n-m}^+ & p_{n-m+1} & \cdots & p_{2n-2} & p_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4}^+ & p_{-n-m+5} & \cdots & p_2 & p_3 \\ p_{-n-m+2}^+ & p_{-n-m+3} & \cdots & p_0 & p_1 \end{vmatrix}.$$

We have from (B.1)

$$\text{(B.13)} \qquad \begin{aligned} p_k^{(c,d)} - x^2 p_{k-2}^{(c,d)} &= p_k^{(c-1,d-1)} - x p_{k-1}^{(c-1,d-1)}, \\ q_k^{(c,d)} - x^{-2} q_{k-2}^{(c,d)} &= q_k^{(c-1,d-1)} - x^{-1} q_{k-1}^{(c-1,d-1)}. \end{aligned}$$

Then, subtracting the $(j-1)$-th column multiplied by x from the j-th column of $R_{m,n}^{(-1,-1)}$ for $j = m+n, m+n-1, \ldots, 2$, and using (B.13), we get

(B.14)

$$R_{m,n}^{(-1,-1)} = \begin{vmatrix} q_1^- & x^{-1}q_{-1} - xq_1 & \cdots & x^{-1}q_{-m-n+1} - xq_{-m-n+3} \\ q_3^- & x^{-1}q_1 - xq_3 & \cdots & x^{-1}q_{-m-n+3} - xq_{-m-n+5} \\ \vdots & \vdots & \ddots & \vdots \\ q_{2m-1}^- & x^{-1}q_{2m-3} - xq_{2m-1} & \cdots & x^{-1}q_{m-n-1} - xq_{m-n+1} \\ p_{n-m}^- & p_{n-m+1} - x^2 p_{n-m-1} & \cdots & p_{2n-1} - x^2 p_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+4}^- & p_{-n-m+5} - x^2 p_{-n-m+3} & \cdots & p_3 - x^2 p_1 \\ p_{-n-m+2}^- & p_{-n-m+3} - x^2 p_{-n-m+1} & \cdots & p_1 - x^2 p_{-1} \end{vmatrix}.$$

Noticing that $p_k = q_k = 0$ for $k < 0$, we obtain

(B.15)

$$R_{m,n}^{(-1,-1)} = (-x)^m \begin{vmatrix} -x^{-1}q_1^- & q_1 & q_0 & \cdots & q_{-m-n+3} \\ -x^{-1}(q_3^- + x^{-2}q_1^-) & q_3 & q_2 & \cdots & q_{-m-n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{-1}(q_{2m-1}^- + \cdots + x^{-2m+2}q_1^-) & q_{2m-1} & q_{2m-2} & \cdots & q_{m-n+1} \\ p_{n-m}^- + \cdots + x^{2n-2}p_{-n-m+2}^- & p_{n-m+1} & p_{n-m+2} & \cdots & p_{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+4}^- + x^2 p_{-n-m+2}^- & p_{-n-m+5} & p_{-n-m+6} & \cdots & p_3 \\ p_{-n-m+2}^- & p_{-n-m+3} & p_{-n-m+4} & \cdots & p_1 \end{vmatrix}.$$

The first half of Lemma 5.7 is obtained from (B.12) and (B.15). Moreover, we have

(B.16)

$$D=\begin{vmatrix} -x^{-1}q_1^- & x^{-1}q_1-x^{-2}q_0 & q_1-x^{-2}q_{-1} & \cdots & q_{-m-n+4}-x^{-2}q_{-m-n+2} \\ -x^{-1}q_3^- & x^{-1}q_3-x^{-2}q_2 & q_3-x^{-2}q_1 & \cdots & q_{-m-n+6}-x^{-2}q_{-m-n+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{-1}q_{2m-1}^- & x^{-1}q_{2m-1}-x^{-2}q_{2m-2} & q_{2m-1}-x^{-2}q_{2m-3} & \cdots & q_{m-n+2}-x^{-2}q_{m-n} \\ p_{n-m+1}^- & p_{n-m+1}-xp_{n-m} & p_{n-m+2}-x^2p_{n-m} & \cdots & p_{2n-1}-x^2p_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+5}^- & p_{-n-m+5}-xp_{-n-m+4} & p_{-n-m+6}-x^2p_{-n-m+4} & \cdots & p_3-x^2p_1 \\ p_{-n-m+3}^- & p_{-n-m+3}-xp_{-n-m+2} & p_{-n-m+4}-x^2p_{-n-m+2} & \cdots & p_1-x^2p_{-1} \end{vmatrix}.$$

Subtracting the 2'nd column from the 1'st column, and using (B.1), we get

(B.17)

$$D=2\begin{vmatrix} -x^{-1}q_1 & x^{-1}q_1-x^{-2}q_0 & q_1-x^{-2}q_{-1} & \cdots & q_{-m-n+4}-x^{-2}q_{-m-n+2} \\ -x^{-1}q_3 & x^{-1}q_3-x^{-2}q_2 & q_3-x^{-2}q_1 & \cdots & q_{-m-n+6}-x^{-2}q_{-m-n+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{-1}q_{2m-1} & x^{-1}q_{2m-1}-x^{-2}q_{2m-2} & q_{2m-1}-x^{-2}q_{2m-3} & \cdots & q_{m-n+2}-x^{-2}q_{m-n} \\ xp_{n-m} & p_{n-m+1}-xp_{n-m} & p_{n-m+2}-x^2p_{n-m} & \cdots & p_{2n-1}-x^2p_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ xp_{-n-m+4} & p_{-n-m+5}-xp_{-n-m+4} & p_{-n-m+6}-x^2p_{-n-m+4} & \cdots & p_3-x^2p_1 \\ xp_{-n-m+2} & p_{-n-m+3}-xp_{-n-m+2} & p_{-n-m+4}-x^2p_{-n-m+2} & \cdots & p_1-x^2p_{-1} \end{vmatrix},$$

$$=2(-1)^{-m}x^{-2m+1}R_{m,n}^{(0,0)},$$

which is nothing but the second half of Lemma 5.7.

From the above discussion, it is easy to verify Lemma 5.8.

References

[1] Andreev, F. V. and Kitaev, A. V., Transformations $RS_4^2(3)$ of the ranks ≤ 4 and algebraic solutions of the sixth Painlevé equation, Commun. Math. Phys., **228** (2002), 151–176.

[2] Dubrovin, B. and Mazzocco, M., Monodromy of certain Painleve VI transcendents and reflection groups, Invent. Math., **141** (2000), 55–147.

[3] Hitchin, N. J., Poncelet polygons and the Painlevé equations, Geometry and analysis (Bombay, 1992) 151–185, Tata Inst. Fund. Res., Bombay, 1995.

[4] Iwasaki, K., Kimura, H., Shimomura, S. and Yoshida, M., *From Gauss to Painlevé—A Modern Theory of Special Functions*, Aspects of Mathematics E16, Vieweg, 1991.

[5] Kajiwara, K. and Masuda, T., On the Umemura polynomials for the Painlevé III equation, Phys. Lett., **A 260** (1999), 462–467.

[6] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y. and Yamada, Y., Determinant formulas for the Toda and discrete Toda equations, Funkcial. Ekvac., **44** (2001), 291–307.

[7] Kajiwara, K. and Ohta, Y., Determinant structure of the rational solutions for the Painlevé II equation, J. Math. Phys., **37** (1996), 4693–4704.

[8] Kajiwara, K. and Ohta, Y., Determinant structure of the rational solutions for the Painlevé IV equation, J. Phys. A: Math. Gen., **31** (1998), 2431–2446.

[9] Kirillov, A. N. and Taneda, M., Generalized Umemura polynomials, Rocky Mountain J. Math., **32** (2002), 691–702.

[10] Kirillov, A. N. and Taneda, M., Generalized Umemura polynomials and Hirota-Miwa equation, MathPhys Odyssey, 2001, 313–331, Prog. Math. Phys., **23**.

[11] Koike, K., On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, Adv. Math., **74** (1989), 57–86.

[12] Masuda, T., Ohta, Y. and Kajiwara, K., A determinant formula for a class of rational solutions of Painlevé V equation, Nagoya Math. J., **168** (2002), 1–25.

[13] Mazzocco, M., Picard and Chazy solutions to the Painlevé VI equation, Math. Ann., **321** (2001), 157–195.

[14] Mazzocco, M., Rational solutions of the Painlevé VI equation, J. Phys. A: Math. Gen., **34** (2001), 2281–2294.

[15] Noumi, M., Okada, S., Okamoto, K. and Umemura, H., Special polynomials associated with the Painleve equations II, In: Saito, M. H., Shimizu, Y., Ueno, K. (eds.) Proceedings of the Taniguchi Symposium, 1997, Integrable Systems and Algebraic Geometry. Singapore: World Scientific, 1998, pp. 349–372.

[16] Noumi, M. and Yamada, Y., Symmetries in the fourth Painlevé equation and Okamoto polynomials, Nagoya Math. J., **153** (1999), 53–86.

[17] Noumi, M. and Yamada, Y., Umemura polynomials for the Painlevé V equation, Phys. Lett., **A247** (1998), 65–69.

[18] Noumi, M. and Yamada, Y., Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac., **41** (1998), 483–503.

[19] Noumi, M. and Yamada, Y., Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys., **199** (1998), 281–295.

[20] Noumi, M. and Yamada, Y., A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}}(8)$, In: Kawai, T., and Fujita, K. (eds.) Microlocal Analysis and Complex Fourier Analysis, World Scientific, 2002, pp. 238–252.

[21] Okamoto, K., Studies on the Painlevé equations I, sixth Painlevé equation P_{VI}, Annali di Matematica pura ed applicata, **146** (1987), 337–381.

[22] Okamoto, K., Studies on the Painlevé equations II, fifth Painlevé equation P_V, Japan J. Math., **13** (1987), 47–76.

[23] Okamoto, K., Studies on the Painlevé equations III, second and fourth Painlevé equations, P_{II} and P_{IV}, Math. Ann., **275** (1986), 222–254.

[24] Okamoto, K., Studies on the Painlevé equations IV, third Painlevé equation P_{III}, Funkcial. Ekvac., **30** (1987), 305–332.

[25] Painlevé, P., Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris, **143** (1906), 1111–1117.

[26] Taneda, M., Polynomials associated with an algebraic solution of the sixth Painlevé equation, Jap. J. Math., **27** (2001), 257–274.

[27] Umemura, H., Special polynomials associated with the Painlevé equations I, preprint.

[28] Vorob'ev, A. P., On rational solutions of the second Painlevé equation, Diff. Uravn., 1 (1965), 58–59.

[29] Yamada, Y., Determinant formulas for the τ-functions of the Painlevé equations of type A, Nagoya Math. J., **156** (1999), 123–134.

nuna adreso:
Tetsu Masuda
Department of Mathematics
Kobe University
Rokko, Kobe 657-8501
Japan
E-mail: masuda@math.kobe-u.ac.jp

(Ricevita la 21-an de februaro, 2002)
(Reviziita la 22-an de majo, 2002)