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On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function

ABSTRACT. In this paper we derive functional relations for Takagi's continuous nowhere differentiable function T, and we give an explicit representation of T at dyadic points. As application of these functional relations we derive a limit relation at dyadic points which implies that at these points T attains locally minima. Further, T is maximal on a perfect set of Lebesgue measure zero. Though the points, where T has a locally maximum, are dense it is remarkable that there is no point where T has a *proper* maximum. Moreover, we verify the existence of the improper derivatives $T'(x) = +\infty$ or $T'(x) = -\infty$ for rational x which have an odd length of period in the binary representation. Finally we investigate one-side upper and lower derivatives.

KEY WORDS. Takagi's continuous nowhere differentiable function, functional equations, improper derivatives, upper and lower derivatives.

1 Introduction

In 1903, T. Takagi [4] discovered an example of a continuous, nowhere differentiable function that was simpler than a well-known example of K. Weierstrass. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{1.1}$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$ is an periodic function with period 1. This function T satisfies for $0 \le x \le 1$ the following system of functional equations

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{1}{2}T(x), \qquad T\left(\frac{1+x}{2}\right) = \frac{1-x}{2} + \frac{1}{2}T(x), \tag{1.2}$$

cf. [3], [2], [7]. The graph of the Takagi function is illustrated in Figure 1.

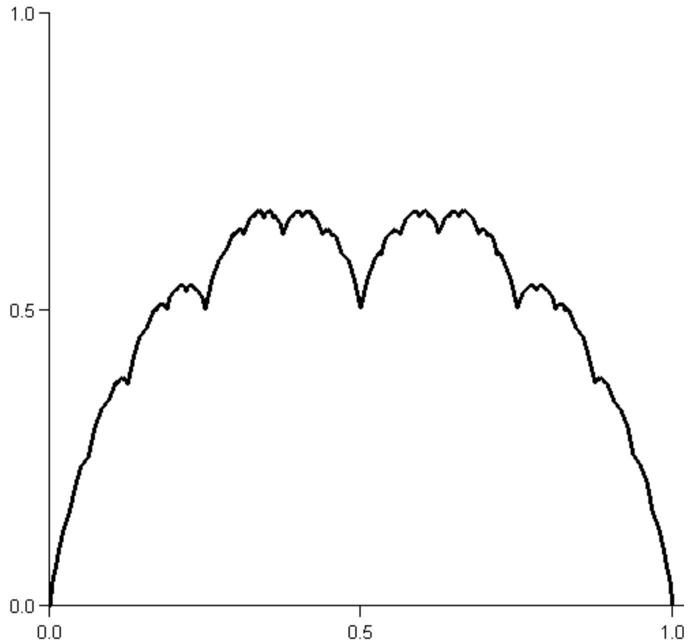

Figure 1: The graph of the Takagi function

For Takagi's function we derive functional relations and give some applications for it. First we show that at dyadic points $x = \frac{k}{2^\ell}$, $(k, \ell \in \mathbb{Z})$, there exists the limit

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{|h| \log_2 \frac{1}{|h|}} = 1. \tag{1.3}$$

Consequently, T has at all dyadic points a locally minimum, and its point out that only these points are locally minima of T (Proposition 4.1). It holds $\max T = \frac{2}{3}$ and the set M of points $x \in [0,1]$ with $T(x) = \frac{2}{3}$ is given by

$$M = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{4^k} \,:\, a_k \in \{1, 2\} \right\},$$

which is a perfect set of measure zero (Proposition 4.2). Further, the set of points where T is locally maximal is a set of first category with Lebesgue measure zero, and there is no point where T has a proper locally maximum (Proposition 4.4).

A further consequence of (1.3) is the fact that at each dyadic point x there exist the right-side improper derivative $T'_+(x) = +\infty$ and the left-side improper derivative $T'_-(x) = -\infty$. We give a simple criterion for the existence of the improper derivatives $T'(x) = +\infty$ and $T'(x) = -\infty$ (Proposition 5.3). In particular, for rational x with odd length of period in the binary representation always there exists the improper derivative (Proposition 5.4).

Moreover, we investigate the four derivatives

$$D^+(x) = \limsup_{h\to+0} \frac{T(x+h)-T(x)}{h}, \qquad D_+(x) = \liminf_{h\to+0} \frac{T(x+h)-T(x)}{h},$$
$$D^-(x) = \limsup_{h\to-0} \frac{T(x+h)-T(x)}{h}, \qquad D_-(x) = \liminf_{h\to-0} \frac{T(x+h)-T(x)}{h},$$

cf. [5], p. 354. We show that if for $x \in \mathbb{R}$ the right-side derivatives $D^+(x)$ and $D_+(x)$ are finite then

$$D^+(x) - D_+(x) \geq 2. \tag{1.4}$$

In view of the symmetry $T(1-x) = T(x)$ this is true also for the left-side derivatives. Furthermore, if all four derivatives are finite then for the upper and lower derivatives

$$\overline{D}(x) = \limsup_{h\to 0} \frac{T(x+h)-T(x)}{h} = \max\{D^+(x), D^-(x)\} \tag{1.5}$$

$$\underline{D}(x) = \liminf_{h\to 0} \frac{T(x+h)-T(x)}{h} = \min\{D_+(x), D_-(x)\} \tag{1.6}$$

it holds

$$\overline{D}(x) - \underline{D}(x) \geq 3. \tag{1.7}$$

We show that the estimates (1.4) and (1.7) are best possible.

In the textbook [3] you can find in detail investigations on Takagi's function. Unfortunately the representation contains errors which we correct in Section 7.3.

2 Functional relations

In order to derive functional relations for Takagi's function we use the binary sum-of-digit function $s(k)$ which for integers $k \geq 0$ with the dyadic representation $k = a_0 a_1 \dots a_m$, $a_j \in \{0,1\}$, is defined by

$$s(k) = \sum_{j=0}^{m} a_j \tag{2.1}$$

and which has the properties $s(2k) = s(k)$ and $s(2k+1) = s(k)+1$.

Proposition 2.1 *For $\ell \in \mathbb{N}$, $k = 0, 1, \dots, 2^\ell - 1$, $x \in [0,1]$, the Takagi function T satisfies the functional equations*

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell} x + \frac{1}{2^\ell} T(x) \tag{2.2}$$

and

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{2s(k-1) - \ell}{2^\ell} x + \frac{1}{2^\ell} T(x). \tag{2.3}$$

Moreover, for $x = \frac{n}{2^\ell}$ with $n = 0, \ldots, 2^\ell$ the function T has the representation

$$T\left(\frac{n}{2^\ell}\right) = \frac{n\ell}{2^\ell} - \frac{1}{2^{\ell-1}} \sum_{k=0}^{n-1} s(k). \tag{2.4}$$

Proof: Equation (2.2) for $\ell = 1$ turns over into (1.2). Assume that (2.2) is true for an integer $\ell \geq 1$. Replacing x by $\frac{x}{2}$ and applying (1.2) we get

$$\begin{aligned} T\left(\frac{2k+x}{2^{\ell+1}}\right) &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}\frac{x}{2} + \frac{1}{2^\ell}T\left(\frac{x}{2}\right) \\ &= T\left(\frac{2k}{2^{\ell+1}}\right) + \frac{\ell - 2s(k)}{2^{\ell+1}}x + \frac{x}{2^{\ell+1}} + \frac{1}{2^{\ell+1}}T(x). \end{aligned}$$

In view of $s(2k) = s(k)$ we obtain (2.2) with $2k$ instead of k and $\ell + 1$ instead of ℓ. If we replace x by $\frac{x+1}{2}$ in (2.2) then in view of (1.2) we obtain

$$\begin{aligned} T\left(\frac{2k+1+x}{2^{\ell+1}}\right) &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}\frac{x+1}{2} + \frac{1}{2^\ell}T\left(\frac{x+1}{2}\right) \\ &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(2k)}{2^{\ell+1}}(x+1) + \frac{1-x}{2^{\ell+1}} + \frac{1}{2^{\ell+1}}T(x). \end{aligned}$$

For $x = 0$ we find in view of $T(0) = 0$

$$T\left(\frac{2k+1}{2^{\ell+1}}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(2k)}{2^{\ell+1}} - \frac{1}{2^{\ell+1}}$$

and it follows

$$T\left(\frac{2k+1+x}{2^{\ell+1}}\right) = T\left(\frac{2k+1}{2^\ell}\right) + \frac{\ell + 1 - 2s(2k+1)}{2^{\ell+1}}x + \frac{1}{2^{\ell+1}}T(x)$$

where we have used $s(2k+1) = s(k) + 1$, so that (2.2) is proved by induction.

From (2.2) with $k - 1$ instead of k and $1 - x$ instead of x we get in view of the symmetry of T the equation

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k-1}{2^\ell}\right) + \frac{\ell - 2s(k-1)}{2^\ell}(1-x) + \frac{1}{2^\ell}T(x) \qquad (0 \leq x \leq 1). \tag{2.5}$$

It follows for $x = 0$ that

$$T\left(\frac{k}{2^\ell}\right) = T\left(\frac{k-1}{2^\ell}\right) + \frac{\ell - 2s(k-1)}{2^\ell},$$

so that (2.5) can be written as (2.3). Finally, equation (2.4) follows from (2.2) for $x = 1$ and by summation in view of $T(0) = T(1) = 0$. □

Corollary 2.2 *For $\ell \in \mathbb{N}$, $k = 1, \ldots, 2^\ell - 1$, $x \in [0,1]$, the Takagi function T satisfies*

$$T\left(\frac{k+x}{2^\ell}\right) - T\left(\frac{k-x}{2^\ell}\right) = \frac{\ell - s(k) - s(k-1)}{2^{\ell-1}} x. \tag{2.6}$$

For $k = \ell = 1$ this means the symmetry of T with respect to $\frac{1}{2}$.

It is easy to see that the partial sum

$$S_\ell(x) = \sum_{n=0}^{\ell-1} \frac{\Delta(2^n x)}{2^n} \tag{2.7}$$

of Takagi's function T from (1.1) is linear in the intervals

$$I_{k\ell} = \left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right] \tag{2.8}$$

where $\ell \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^\ell - 1\}$. Moreover, for $n \geq \ell$ and $k \in \{0, 1, \ldots, 2^\ell\}$ we have $S_n(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$ and hence also $T(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$, cf. Figure 2.

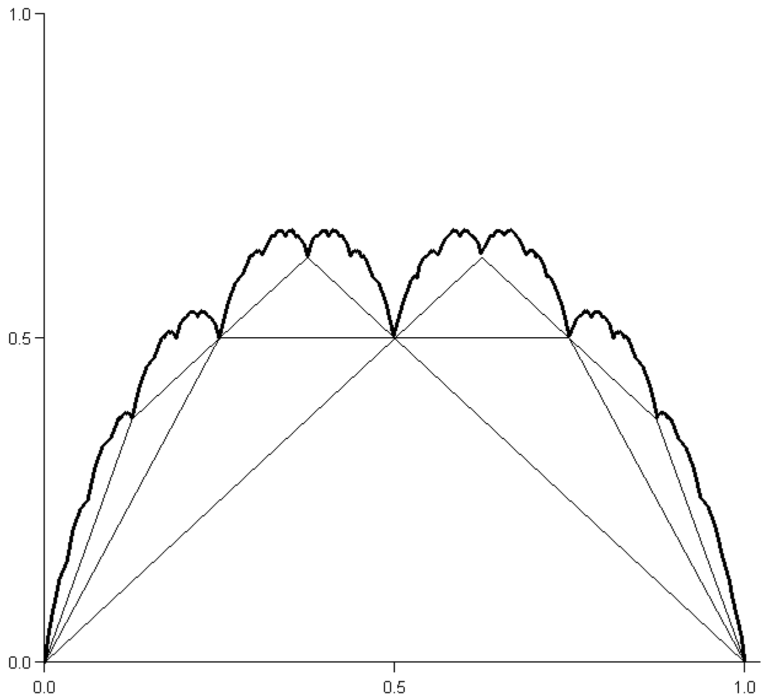

Figure 2: The partial sums S_1, S_2, S_3

Proposition 2.3 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, the partial sum (2.7) of (1.1) is linear in the interval (2.8) and it holds*

$$S_\ell\left(\frac{k+x}{2^\ell}\right) = S_\ell\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell} x \qquad (x \in [0,1]). \tag{2.9}$$

Proof: Clearly, $S_\ell(x)$ is linear in $I_{k\ell}$ so that

$$S_\ell\left(\frac{k+x}{2^\ell}\right) = S_\ell\left(\frac{k}{2^\ell}\right) + ax \qquad (x \in [0,1]).$$

In view of $T(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$ and $T(\frac{k+1}{2^\ell}) = S_\ell(\frac{k+1}{2^\ell})$ we obtain from (2.2) with $x = 1$ that

$$S_\ell\left(\frac{k+1}{2^\ell}\right) - S_\ell\left(\frac{k}{2^\ell}\right) = \frac{\ell - 2s(k)}{2^\ell}$$

which implies the assertion. □

3 A limit relation at dyadic points

In order to derive the limit relation (1.3) first we show

Lemma 3.1 *For $0 < x \le \frac{1}{2}$ the Takagi function T satisfies the estimate*

$$x \log_2 \frac{1}{x} \le T(x) \le x \log_2 \frac{1}{x} + cx \tag{3.1}$$

with a constant $c < \frac{2}{3}$.

Proof: For $0 < x \le \frac{1}{2}$ we put

$$C(x) = \frac{T(x)}{x \log_2 \frac{1}{x}}$$

and we show that for $\frac{1}{2^{\ell+1}} < x \le \frac{1}{2^\ell}$ $(\ell \in \mathbb{N})$ it holds

$$1 \le C(x) \le 1 + \frac{c}{\ell+1}. \tag{3.2}$$

Applying (1.2) we obtain

$$C\left(\frac{x}{2}\right) \log_2 \frac{2}{x} = \frac{2}{x} T\left(\frac{x}{2}\right) = 1 + C(x) \log_2 \frac{1}{x}$$

which implies

$$\left\{C\left(\frac{x}{2}\right) - 1\right\} \log_2 \frac{2}{x} = \{C(x) - 1\} \log_2 \frac{1}{x}. \tag{3.3}$$

1. First we show that $C(x) \ge 1$ for $\frac{1}{4} \le x \le \frac{1}{2}$. We use the estimate $T(x) \ge S_3(x)$ where the partial sum $S_3(x)$ from (2.7) has for $\frac{1}{4} \le x \le \frac{1}{2}$ the form

$$S_3(x) = \begin{cases} \frac{1}{2} + x & \text{for} \quad \frac{1}{4} \le x \le \frac{3}{8} \\ 1 - x & \text{for} \quad \frac{3}{8} \le x \le \frac{1}{2}\,, \end{cases}$$

cf. Figure 2. For $\frac{1}{4} < x < \frac{3}{8}$ we have for the function $f(x) = x \log_2 \frac{1}{x}$

$$f'(x) = \frac{-1 + \log \frac{1}{x}}{\log 2} < \frac{\log \frac{4}{e}}{\log 2} < 1 = S_3'(x)$$

so that from $S_3(\frac{1}{4}) = f(\frac{1}{4}) = \frac{1}{2}$ it follows $S_3(x) \geq f(x)$. For $\frac{3}{8} < x < \frac{1}{2}$ we have

$$f'(x) = \frac{-1 - \log x}{\log 2} > \frac{-\log \frac{e}{2}}{\log 2} > -1 = S_3'(x)$$

so that from $S_3(\frac{1}{2}) = f(\frac{1}{2}) = \frac{1}{2}$ it follows $S_3(x) \geq f(x)$. So we have $T(x) \geq S_3(x) \geq f(x)$ for $\frac{1}{4} \leq x \leq \frac{1}{2}$, i.e. $C(x) \geq 1$ for these x. The relation (3.3) implies that $C(x) \geq 1$ is valid for all $x \in (0, \frac{1}{2}]$.

2. Next we show that (3.2) is valid for $\ell = 1$. Since $\frac{1}{f(x)}$ is increasing for $0 < x < \frac{1}{e}$ and decreasing for $\frac{1}{e} < x$, it follows that in interval $[\frac{1}{4}, \frac{1}{2}]$ the function $\frac{1}{f(x)}$ is maximal for $x = \frac{1}{4}$ or for $x = \frac{1}{2}$. Because of $f(\frac{1}{4}) = f(\frac{1}{2}) = \frac{1}{2}$ it follows in view of $T(\frac{1}{4}) < \frac{2}{3}$ and $T(\frac{1}{2}) < \frac{2}{3}$ that $C(x) < \frac{4}{3}$ for $\frac{1}{4} \leq x \leq \frac{1}{2}$, i.e. (3.2) is true for $\ell = 1$ with a constant $c < \frac{2}{3}$. If (3.2) is true for a certain $\ell \in \mathbb{N}$ then by (3.3) we have

$$\frac{C\left(\frac{x}{2}\right) - 1}{C(x) - 1} = \frac{\log_2 \frac{1}{x}}{1 + \log_2 \frac{1}{x}} = 1 - \frac{1}{1 + \log_2 \frac{1}{x}} \leq 1 - \frac{1}{\ell + 2}$$

for $\frac{1}{2^{\ell+1}} \leq x \leq \frac{1}{2^\ell}$. This implies

$$C\left(\frac{x}{2}\right) - 1 \leq \{C(x) - 1\} \frac{\ell + 1}{\ell + 2} \leq \frac{c}{\ell + 2}$$

for $\frac{1}{2^{\ell+2}} \leq \frac{x}{2} \leq \frac{1}{2^{\ell+1}}$, i.e. (3.2) is valid also for $\ell + 1$ and hence by induction for all $\ell \in \mathbb{N}$. Finally, for $\frac{1}{2^{\ell+1}} < x$ we have $\ell + 1 > \log_2 \frac{1}{x}$, so that for the right hand side of (3.2) we get

$$C(x) \leq 1 + \frac{c}{\ell + 1} \leq 1 + \frac{c}{\log_2 \frac{1}{x}}$$

which yields the assertion. □

Proposition 3.2 *The Takagi function T satisfies at each dyadic point $x = \frac{k}{2^\ell}$ the limit relation*

$$\lim_{h \to 0} \frac{T(x + h) - T(x)}{|h| \log_2 \frac{1}{|h|}} = 1.$$

Proof: For $x = 0$ the limit relation is a consequence of Lemma 3.1. Let $x = \frac{k}{2^\ell}$ $(\ell \in \mathbb{N}, 0 \leq k \leq 2^\ell - 1)$, and $0 < h < \frac{1}{2^\ell}$. According to (2.2) we have

$$T(x + h) - T(x) = \{\ell - 2s(k)\}h + \frac{1}{2^\ell} T(2^\ell h)$$

and

$$\frac{T(x+h)-T(x)}{h\log_2\frac{1}{h}} = \frac{\ell-2s(k)}{\log_2\frac{1}{h}} + \frac{1}{2^\ell h}\frac{T(2^\ell h)}{\log_2\frac{1}{h}}$$

With $t=2^\ell h$ the last term can be written as

$$\frac{T(t)}{t\log_2\frac{2^\ell}{t}} = \frac{T(t)}{t\log_2\frac{1}{t}(1-\frac{\ell}{\log_2 t})} \to 1 \qquad (t\to +0).$$

We obtain the same limit for

$$\frac{T(x-h)-T(x)}{-h\log_2\frac{1}{h}}$$

by means of (2.3) which yields the assertion. □

4 The extreme values of Takagi's function

Clearly, since Takagi's function T is continuous and nowhere differentiable there is no interval where T is monotone. The function T has at the point x_0 a locally maximum if $T(x_0)\geq T(x)$ for all x of a certain neighbourhood U of x_0. If even $T(x_0)>T(x)$ for $x\in U$ with $x\neq x_0$ then T has at x_0 a proper locally maximum. Analogous notations are used for a proper locally minimum, cf. e.g. [3].

Proposition 4.1 *The Takagi function T attains its locally minima exactly at the dyadic points $x=\frac{k}{2^\ell}$ where all these $T(x)$ are proper minima.*

Proof: The limit relation (1.3) implies

$$\lim_{h\to 0}\frac{T(x+h)-T(x)}{|h|} = +\infty$$

so that T has at each dyadic point a proper locally minimum. Now let $x\in[0,1]$ be a nondyadic point then for arbitrary $\ell\in\mathbb{N}$ there is $k\in\{0,1,\ldots,2^\ell-1\}$ such that $\frac{k}{2^\ell}<x<\frac{k+1}{2^\ell}$, i.e. $x=t\frac{k}{2^\ell}+(1-t)\frac{k+1}{2^\ell}$ with a certain $t\in(0,1)$. For the partial sum $S_\ell(x)$ from (2.7) it holds $T(\frac{k}{2^\ell})=S_\ell(\frac{k}{2^\ell})$ and $T(\frac{k+1}{2^\ell})=S_\ell(\frac{k+1}{2^\ell})$, and $T(x)>S_\ell(x)=tS_\ell(\frac{k}{2^\ell})+(1-t)S_\ell(\frac{k+1}{2^\ell})$. This implies $T(x)>\min\{T(\frac{k}{2^\ell}),T(\frac{k+1}{2^\ell})\}$ so that T cannot have a proper minimum at x. □

Next we investigate the global maxima of Takagi's function.

Proposition 4.2 *We have $\max T=\frac{2}{3}$ and the set M of points $x\in[0,1]$ with $T(x)=\frac{2}{3}$ is given by*

$$M=\left\{x=\sum_{k=1}^{\infty}\frac{a_k}{4^k} : a_k\in\{1,2\}\right\}. \tag{4.1}$$

M is a perfect set of measure zero with $\min M=\frac{1}{3}$ and $\max M=\frac{2}{3}$.

Proof: By means of the partial sum $S_2(x)$ from (2.7) the series (1.1) can be written as

$$T(x) = \sum_{n=0}^{\infty} \frac{S_2(4^n x)}{4^n} \qquad (x \in \mathbb{R}). \tag{4.2}$$

Since $S_2(x)$ is 1-periodic and has in $[0,1]$ the form

$$S_2(x) = \begin{cases} 2x & \text{for} \quad 0 \le x < \frac{1}{4} \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} \le x < \frac{3}{4} \\ 2-2x & \text{for} \quad \frac{3}{4} \le x \le 1, \end{cases} \tag{4.3}$$

cf. Figure 2, it follows that

$$T(x) \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{2}{3}$$

and that $T(x) = \frac{2}{3}$ if and only if $S_2(4^n x) = \frac{1}{2}$ for all $n \in \mathbb{N}$. According to (4.3) this is valid for $x \in [0,1]$ exactly for $x \in M$ from (4.1). M is a perfect set since for $x \in M$ with given a_k in (4.1) also $x_n = x + \frac{3-2a_n}{4^n} \in M$ (exchange of the digits 1 and 2) and $x_n \to x$ as $n \to \infty$. Moreover, M has the measure zero, since the representations of $x \in M$ do not contain at least one digit, here 0 and 3, cf. [5], p. 329-330. From (4.1) we get $\min M = \frac{1}{3}$ and $\max M = \frac{2}{3}$. □

In order to investigate the locally maxima of Takagi's function we determine the maxima of it in the closed intervals (2.8).

Proposition 4.3 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$ the set $A_{k\ell}$ of points in $I_{k\ell}$ from (2.8), where $T(x)$ is maximal, is a perfect set of measure zero so that it is nowhere dense in $[0,1]$. For the maximum it holds*

$$\max_{x \in I_{k\ell}} T(x) = \begin{cases} T\left(\frac{k}{2^\ell}\right) + \frac{2}{3 \cdot 4^{s(k)}} & 2s(k) \ge \ell \\ T\left(\frac{k+1}{2^\ell}\right) + \frac{2}{3 \cdot 4^{\ell - s(k)}} & 2s(k) < \ell \end{cases}$$

with $s(k)$ from (2.1).

Proof: According to Proposition 2.3 in the interval $I_{k\ell}$ it holds relation (2.9), so that S_ℓ is linear in $I_{k\ell}$ with the slope $p = \ell - 2s(k)$. In case $p = 0$ we get from (2.2) that

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{1}{2^\ell} T(x) \qquad (x \in [0,1]).$$

According to Proposition 4.2 it follows that T attains its maximum in $I_{k\ell}$ on a nowhere dense perfect set $A_{k\ell}$ with measure $|A_{k\ell}| = 0$, and for the maximal value we have

$$\max_{x \in I_{k\ell}} T(x) = T\left(\frac{k}{2^\ell}\right) + \frac{2}{3} \frac{1}{2^\ell}.$$

This is the assertion in case $\ell = 2s(k)$.

In case $p < 0$ the partial sum S_ℓ is strictly decreasing in $I_{k\ell}$. The partial sum $S_{\ell+|p|}$ is decreasing in $I_{k\ell}$, where more precisely we have $S_{\ell+|p|}(x) = S_\ell(\frac{k}{2^\ell})$ for $x \in I_{2^{|p|}k,\ell+|p|} \subset I_{k\ell}$ and $S_{\ell+|p|}(x) < S_\ell(\frac{k}{2^\ell})$ for all another x in $I_{k\ell}$. Therefore, the maximum of T in $I_{k\ell}$ we find in $I_{2^{|p|}k,\ell+|p|}$ where in view of Proposition 2.1 we have

$$T\left(\frac{2^{|p|}k + x}{2^{\ell+|p|}}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{1}{2^{\ell+|p|}}T(x) \qquad (x \in [0,1]).$$

Thus, for the maximum of T in $I_{k\ell}$ we have in view of $\ell + |p| = 2s(k)$ and Proposition 4.2 that

$$\max_{x\in I_{k\ell}} T(x) = T\left(\frac{k}{2^\ell}\right) + \frac{2}{3}\frac{1}{4^{s(k)}}.$$

Finally, if $p > 0$ then the partial sum S_ℓ is strictly increasing in $I_{k\ell}$, and $S_{\ell+p}$ is increasing. Now, in this case we have $S_{\ell+p}(x) = S_\ell(\frac{k+1}{2^\ell})$ for $x \in I_{2^p(k+1)-1,\ell+p}$ and $S_{\ell+p}(x) < S_\ell(\frac{k+1}{2^\ell})$ for all another x in $I_{k\ell}$. Therefore, the maximum of T in $I_{k\ell}$ we find in $I_{2^p(k+1)-1,\ell+p}$ where in view of Proposition 2.1 we have

$$T\left(\frac{2^p(k+1) - x}{2^{\ell+p}}\right) = T\left(\frac{k+1}{2^\ell}\right) + \frac{1}{2^{\ell+p}}T(x) \qquad (x \in [0,1]).$$

As before it follows in view of $\ell + p = 2\ell - 2s(k)$ that

$$\max_{x\in I_{k\ell}} T(x) = T\left(\frac{k+1}{2^\ell}\right) + \frac{2}{3}\frac{1}{4^{\ell-s(k)}}.$$

According to Proposition 4.2 the set $A_{k\ell}$ where T is maximal in $I_{k\ell}$ is a nowhere dense set of measure zero. □

It follows from Proposition 4.3 and (2.4) that the maximum of T in $I_{k\ell}$ has the form $\frac{1}{3}\frac{m}{2^n}$ with certain integers m, n. As consequence we get

Proposition 4.4 *The set $A \subseteq [0,1]$, where T attains its locally maxima, is a set of first category, i.e. it is representable as union of at most countable many perfect nowhere dense sets. This set A has the power* **c** *and the measure zero. For $x \in A$ the values are $T(x) = \frac{1}{3}\frac{m}{2^n}$ with certain $n \in \mathbb{N}_0$ and $m \in \{1, 2, \ldots, 2^{n+1}\}$. There is no point where T has a proper maximum.*

5 Improper derivatives

As already mentioned in the introduction formula (1.3) implies that for dyadic points $x = \frac{k}{2^\ell}$ it holds

$$\lim_{h\to+0} \frac{T(x+h) - T(x)}{h} = +\infty$$

and

$$\lim_{h\to -0} \frac{T(x+h)-T(x)}{h} = -\infty.$$

Hence, T has at all dyadic points x the one-side improper derivatives $T'_+(x) = +\infty$ and $T'_-(x) = -\infty$.

For arbitrary numbers $x, y \in [0,1]$ we consider the dyadic representations

$$x = \xi_0, \xi_1\xi_2\ldots, \qquad y = \eta_0, \eta_1\eta_2\ldots \tag{5.1}$$

with $\xi_0 = \eta_0 = 0$ and $\xi_n, \eta_n \in \{0,1\}$, and we put

$$x_n = 0, \xi_n\xi_{n+1}\ldots, \qquad y_n = 0, \eta_n\eta_{n+1}\ldots \tag{5.2}$$

for $n \geq 0$.

Proposition 5.1 *Let x and y are different points in $[0,1]$ with $\xi_\nu = \eta_\nu$ for $\nu < n \in \mathbb{N}$. Then also x_n and y_n are different, and we have*

$$\frac{T(x)-T(y)}{x-y} = \sum_{\nu=0}^{n-1}(-1)^{\xi_\nu} + \frac{T(x_n)-T(y_n)}{x_n-y_n}. \tag{5.3}$$

In particular, if $\eta_\nu = 1-\xi_\nu$ for $\nu \geq n$, i.e. $x_n + y_n = 1$, then we have $|x-y| \leq \frac{1}{2^n}$ and

$$\frac{T(x)-T(y)}{x-y} = \sum_{\nu=0}^{n-1}(-1)^{\xi_\nu}. \tag{5.4}$$

Proof: We put $k_n = [2^{n-1}x]$, i.e. $k_n = \sum_{\nu=0}^{n-1} 2^{n-\nu}\xi_\nu$ then we have

$$x = \frac{2k_n + x_n}{2^n}, \qquad y = \frac{2k_n + y_n}{2^n}$$

and

$$x - y = \frac{x_n - y_n}{2^n}. \tag{5.5}$$

From equation (2.2) we get

$$T(x) = T\left(\frac{2k_n + x_n}{2^n}\right) = T\left(\frac{2k_n}{2^n}\right) + \frac{n-2s(2k_n)}{2^n}x_n + \frac{1}{2^n}T(x_n)$$

and

$$T(y) = T\left(\frac{2k_n + y_n}{2^n}\right) = T\left(\frac{2k_n}{2^n}\right) + \frac{n-2s(2k_n)}{2^n}y_n + \frac{1}{2^n}T(y_n).$$

It follows

$$\frac{T(x)-T(y)}{x-y} = n - 2s(2k_n) + \frac{T(x_n)-T(y_n)}{x_n-y_n}$$

and the relations (5.5) and $1-2\xi_\nu = (-1)^{\xi_\nu}$ for $\nu = 0,\ldots n-1$ yield the assertion (5.3). In case $y_n = 1-x_n$ we have $T(x_n) = T(1-x_n) = T(y_n)$, and hence (5.4). □

Corollary 5.2 *Formula (5.4) implies:*

1. *There is no point where T has a finite derivative since as $n \to \infty$ the right-hand side is not convergent to a finite value.*

2. *If there exists the improper derivative $T'(x) = +\infty$ then*

$$\sum_{k=0}^{\infty} (-1)^{\xi_k} = +\infty \tag{5.6}$$

and if $T'(x) = -\infty$ then

$$\sum_{k=0}^{\infty} (-1)^{\xi_k} = -\infty. \tag{5.7}$$

Proposition 5.3 *If in the dyadic representation of the number x the number of both zeros and ones which occur one after the other is bounded then (5.6) implies the existence of the improper derivative $T'(x) = +\infty$, and (5.7) implies $T'(x) = -\infty$.*

Proof: For $y \neq x$ let n be the smallest integer such that $\eta_n = \xi_n$, cf. (5.1). Then by Proposition 5.1 it holds

$$\left| \frac{T(x) - T(y)}{x - y} - \sum_{\nu=0}^{n-1} (-1)^{\xi_\nu} \right| = \left| \frac{T(x_n) - T(y_n)}{x_n - y_n} \right|$$

with x_n, y_n from (5.2). If d denotes the maximal number of equals digits ξ_ν which occur one after the other then in case $\xi_n = 1$, $\eta_n = 0$ we have $x_n > \frac{1}{2} + \frac{1}{2^{d+3}}$ and $y_n < \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2}$ so that $|x_n - y_n| > \frac{1}{2^{d+3}}$. In case $\xi_n = 0$, $\eta_n = 1$ we have $x_n < \frac{1}{4} + \ldots + \frac{1}{2^{d+2}} = \frac{1}{2} - \frac{1}{2^{d+3}}$ and $y_n \geq \frac{1}{2}$ so that $|x_n - y_n| > \frac{1}{2^{d+3}}$, too. Hence

$$\left| \frac{T(x_n) - T(y_n)}{x_n - y_n} \right| < \frac{2}{3}\, 2^{d+3}.$$

This implies the assertion. □

So for rational x we summarize

Proposition 5.4 *For the Takagi function T we have the following statements at rational points x:*

1. *If $x = \frac{k}{2^\ell}$ is a dyadic point then $T'_+(x) = +\infty$ and $T'_-(x) = -\infty$.*

2. *If $x \neq \frac{k}{2^\ell}$ has a dyadic representation with the period $\xi_{k+1} \ldots \xi_{k+p}$ then it holds:*

$$\xi_{k+1} + \xi_{k+2} + \ldots + \xi_{k+p} \begin{cases} < \frac{p}{2} \implies & T'(x) = +\infty \\ > \frac{p}{2} \implies & T'(x) = -\infty \\ = \frac{p}{2} \implies & T'(x) \text{ does not exists.} \end{cases}$$

In the last case, where p must be even, $\overline{D}(x)$ from (1.5) and $\underline{D}(x)$ from (1.6) are finite.

Remark 5.5 It follows from Proposition 5.4 that for rational x with an odd length of period in the dyadic representation always there exists the improper derivative $T'(x)$. For instance $x_1 = \frac{1}{7} = 0,001001\ldots$ has the period 001 and hence there exists the improper derivative $T'(x_1) = +\infty$, and $x_2 = \frac{6}{7} = 0,110110\ldots$ has the period 110 and hence there exists the improper derivative $T'(x_2) = -\infty$.

Remark 5.6 We know that for dyadic points $x = \frac{k}{2^\ell}$ there exists the limit (1.3). Let us mention that a similar argument as in the proof of Proposition 5.3 also yields (1.3) and moreover, that for rational $x \neq \frac{k}{2^\ell}$ it holds

$$\lim_{h\to 0} \frac{T(x+h)-T(x)}{h\log_2 \frac{1}{|h|}} = 1 - \frac{2(\xi_{k+1}+\ldots+\xi_{k+p})}{p}$$

where $\xi_{k+1}\ldots\xi_{k+p}$ is a period in the dyadic representation of x.

6 Upper and lower derivatives

Finally, we investigate the four derivatives $D^+(x)$, $D_+(x)$, $D^-(x)$, $D_-(x)$ of Takagi's function T, which are defined in the introduction. We begin with

Lemma 6.1 *For $0 < x < \frac{1}{3}$ the T satisfies the inequality $T(x) \geq 2x$ where we have equality if and only if $x = x_m$ with*

$$x_m = \sum_{\mu=1}^{m} \frac{1}{4^\mu} = \frac{4^m-1}{3\cdot 4^m} \qquad (m \in \mathbb{N}). \tag{6.1}$$

Proof: First we show by induction on m that $T(x_m) = 2x_m$. For $m = 1$ we have $x_1 = \frac{1}{4}$, and according to (2.4) it holds $T(\frac{1}{4}) = \frac{1}{2}$. Formula (6.1) implies $x_m = \frac{k_m}{4^m}$ with $k_m = 1+4+\ldots+4^{m-1}$ so that $s(k_m) = m$, cf. (2.1). Moreover, we have

$$x_{m+1} = x_m + \frac{1}{4^{m+1}} = \frac{4k_m+1}{4^{m+1}}. \tag{6.2}$$

Assume that for a fixed m it holds

$$T(x_m) = 2x_m = \frac{2(4^m-1)}{3\cdot 4^m}$$

then by (2.2) with $k = 4k_m$, $\ell = 2m+2$ and $x = 1$ we get in view of $s(4k_m) = m$ and $T(1) = 0$ that

$$\begin{aligned} T\left(\frac{4k_m+1}{2^{2m+2}}\right) &= T(x_m) + \frac{2m+2-2s(4k_m)}{2^{2m+2}} \\ &= \frac{2(4^m-1)}{3\cdot 4^m} + \frac{2}{4^{m+1}} = \frac{2(4^{m+1}-1)}{3\cdot 4^{m+1}}, \end{aligned}$$

i.e. $T(x_{m+1}) = 2x_{m+1}$. It follows that $T(x_{m+1}) = T(x_m) + 2(x_{m+1} - x_m)$. Since $x_m = \frac{4k_m}{2^{2m+2}}$ and $x_{m+1} = \frac{4k_m+1}{2^{2m+2}}$ the equation (2.2) implies in view of $T(t) > 0$ for $0 < t < \frac{1}{4^{m+1}}$ that $T(x_m + t) > T(x_m) + 2t$. □

Lemma 6.2 *The Takagi function has at the point $x = \frac{1}{3}$ the derivatives*

$$D^+\left(\frac{1}{3}\right) = 0, \quad D_+\left(\frac{1}{3}\right) = -1, \quad D^-\left(\frac{1}{3}\right) = 2, \quad D_-\left(\frac{1}{3}\right) = 1.$$

Proof: We know that $T(x) \le T(\frac{1}{3}) = \frac{2}{3}$ and that the set M of points x in $[0,1]$ with $T(x) = \frac{2}{3}$ is a perfect set. Since $\frac{1}{3} = \min M$ it follows $D^+(\frac{1}{3}) = 0$. The symmetry $T(1-x) = T(x)$ implies $D^-(\frac{2}{3}) = 0$, too.

Let x_ν be a sequence with $x_\nu \to x$ as $\nu \to \infty$. From the first equation in (1.2) with $2x$ instead of x we get for $x_\nu \ne x$ and $x_\nu, x < \frac{1}{2}$ that

$$\frac{T(x_\nu) - T(x)}{x_\nu - x} = 1 + \frac{T(2x_\nu) - T(2x)}{2(x_\nu - x)}. \tag{6.3}$$

It follows $D^-(\frac{1}{3}) = 1$ since $D^-(\frac{2}{3}) = 0$. Moreover, Lemma 6.1 implies $D_-(\frac{1}{3}) = 2$ so that $D_+(\frac{2}{3}) = -2$ since the symmetry of T. Now, (6.3) implies $D_+(\frac{1}{3}) = -1$. □

Proposition 6.3 *If for $x \in \mathbb{R}$ the right-side derivatives $D^+(x)$ and $D_+(x)$ of the Takagi function T are finite then we have*

$$D^+(x) - D_+(x) \ge 2$$

where we have equality if x has the form

$$x = \frac{k}{2^n} + \frac{1}{3 \cdot 2^n}$$

with $k, n \in \mathbb{N}_0$. Moreover, the upper and lower derivatives $\overline{D}(x)$ and $\underline{D}(x)$ of T satisfy the inequality

$$\overline{D}(x) - \underline{D}(x) \ge 3$$

where we have equality if x has above form.

Proof: For dyadic $x = \frac{k}{2^\ell}$ we know from Proposition 3.2 that $D^+T(x) = +\infty$. Let x be a nondyadic point with the representation $x = 0, \xi_1\xi_2\ldots$ and for $n \in \mathbb{N}$ let be $y = 0, \eta_1\eta_2\ldots$ with $\eta_\nu = \xi_\nu$ for $\nu \le n$ and $\eta_\nu = 1 - \xi_\nu$ for $\nu > n$. In case $\xi_{n+1} = 0$ we have $y > x$ since $y \ge 0, \xi_1\ldots\xi_n 1 > 0, \xi_1\ldots\xi_n 0\ldots = x$ and x is notdyadic. Equation (5.4) implies that $D_r(x) := D^+(x) - D_+(x) \ge 1$ where $D_r(x) \ge 2$ and the case $D_r(x) = 1$ may be only possible if there is an integer n such that for $\nu \geq 0$ it holds $\xi_{n+2\nu} = 1$ and $\xi_{n+2\nu+1} = 0$. This means that $x = x_0$ necessarily must be of the form $x_0 = 0, \xi_1 \dots \xi_n 0101 \dots$, i.e.

$$x_0 = \sum_{\nu=1}^{n} \frac{\xi_\nu}{2^\nu} + \frac{1}{2^{n+2}} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{k}{2^n} + \frac{1}{3 \cdot 2^n}$$

and according to (2.2) we get for $0 < |h| < \frac{1}{3 \cdot 2^n}$ that

$$\frac{T(x_0 + h) - T(x_0)}{h} = n - 2s(k) + \frac{T(\frac{1}{3} + 2^n h) - T(\frac{1}{3})}{2^n h}.$$

It follows $D_r(x_0) = D_r(\frac{1}{3}) = 2$ by Lemma 6.2. Consequently, for an arbitrary nondyadic point x we have $D_r(x) \geq 2$.

As before formula (5.4) implies that $S = \overline{D}(x) - \underline{D}(x) \geq 2$ where the case $S = 2$ may be only possible if there is an integer n such that for $\nu \geq 0$ it holds $\xi_{n+2\nu} = 1$ and $\xi_{n+2\nu+1} = 0$, i.e. if $x = x_0$. But for x_0 we get from Lemma 6.2 as before that $\overline{D}(x_0) - \underline{D}(x_0) = 3$. □

7 Supplements

Finally we give three supplements.

7.1. Improper derivatives at irrational points. There exists irrational points such that there exists the improper derivative. In order to give an example first we put $x = \xi_0, \xi_1\xi_2 \dots$ where ξ_k is $s(k)$ mod 2 with values from $\{0, 1\}$ which is the Morse sequence, cf. [1]. Relations $s(2k) = s(k)$ and $s(2k + 1) = s(k) + 1$ imply that $d = 2$ is the maximal number of the same digit which occur one after the other. For $k = 2^\ell + 1$ ($\ell = 1, 2, \dots$) we have $s(k) = 2$ and hence $\xi_k = 0$. We put $y = \eta_0, \eta_1\eta_2 \dots$, where $\eta_k = 1$ for $k = 2^\ell + 1$ and $\eta_k = 0$ elsewhere. We show that $z = x + y$ is irrational and that Takagi's function has at this point the improper derivative $T'(z) = -\infty$. First we show that x is irrational. Assume the representation $x = \xi_0, \xi_1\xi_2 \dots$ contains a period, i.e. there is an integer $p > 1$ such that $\xi_{k+p} = \xi_k$ for $k \geq k_0$. If $s(p) \equiv 0 \mod 2$ then for $k = 2^n \geq k_0$ we have $s(k) = 1$ but $s(kp) = s(p) \not\equiv s(k)$ mod 2 which is impossible. In case $s(p) \equiv 1 \mod 2$ we note that $\xi_{k+p'} = \xi_k$ for each multiply p' of p. In particular for $p' = (2^n + 1)p$ with $2^n > p$ we get $s(p') = 2s(p)$, and as before we get an contradiction so that x cannot be rational. Now it follows easy that also $z = \zeta_0, \zeta_1 \dots$ with $\zeta_k = \xi_k + \eta_k$ does not have a period in view of $2^{\ell+1} + 1 - (2^\ell + 1) \to \infty$ as $\ell \to \infty$. In order to apply Proposition 5.3 we have to show that

$$\sum_{k=0}^{\infty} (-1)^{\xi_k + \eta_k} = -\infty. \tag{7.1}$$

But the sequence $\sum_{k=0}^{n}(-1)^{\xi_k} \in \{0,+1,-1\}$ is bounded and $\eta_k = 1$ only for $k = 2^\ell + 1$ where $\xi_k = 0$. This implies (7.1), and by Proposition 5.3 it holds $T'(z) = -\infty$.

7.2. An example for the case $\overline{D}(x) = +\infty$ and $\underline{D}(x) = -\infty$. In order to show that the condition (5.6) is not sufficient for the existence of the improper derivative $T'(x) = +\infty$ we use the following

Lemma 7.1 *Assume that $x = \frac{k+r}{2^n}$ and $y = \frac{k-2r}{2^n}$ where k is an odd integer and $0 < r < \frac{1}{4}$. Then we have*

$$\frac{T(x)-T(y)}{x-y} = n + 2 - 2s(k) - \frac{T(r)}{3r} \tag{7.2}$$

with $s(k)$ from (2.1).

Proof: According to equation (2.2) we have

$$T(x) = T\left(\frac{k+r}{2^n}\right) = T\left(\frac{k}{2^n}\right) + \frac{n-2s(k)}{2^n}r + \frac{1}{2^n}T(r)$$

and by equation (2.3) we get

$$T(y) = T\left(\frac{k-2r}{2^n}\right) = T\left(\frac{k}{2^n}\right) + \frac{2s(k)-2-n}{2^n}2r + \frac{1}{2^n}T(2r)$$

where we have used that $s(k-1) = s(k) - 1$ since k is an odd integer. It follows

$$T(x) - T(y) = \frac{n-2s(k)}{2^n}3r + \frac{4r}{2^n} + \frac{T(r)-T(2r)}{2^n}$$

and in view of $x - y = \frac{3r}{2^n}$ we find

$$\frac{T(x)-T(y)}{x-y} = n - 2s(k) + \frac{4}{3} + \frac{T(r)-T(2r)}{3r}$$

From the first equation in (1.2) we get for $0 < r < \frac{1}{4}$ that

$$\frac{T(r)-T(2r)}{3r} = \frac{T(r)-\{2T(r)-2r\}}{3r} = -\frac{T(r)}{3r} + \frac{2}{3}$$

and hence it follows the assertion. □

Example 7.2 For

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{a_n}}$$

with $a_n \in \mathbb{N}$ such that $a_{n+1} \geq 4a_n$. Then $\sum(-1)^{\xi_\nu} = +\infty$ and hence $\overline{D}(x) = +\infty$. We show that $\underline{D}(x) = -\infty$. For this we put

$$x = \frac{k_n + r_n}{2^{a_n}}, \qquad y = \frac{k_n - 2r_n}{2^{a_n}}$$

with

$$k_n = 2^{a_n} \sum_{k=1}^{n} \frac{1}{2^{a_k}}, \qquad r_n = 2^{a_n} \sum_{k=n+1}^{\infty} \frac{1}{2^{a_k}},$$

i.e. $y = x + h_n$ with $h_n = -\frac{3}{2^{\alpha_n}} r_n$. By Lemma 7.1 we have in view of $s(k_n) = n$ that

$$\frac{T(x) - T(x+h_n)}{-h_n} = a_n - 2n - \frac{T(r_n)}{3r_n}$$

and by Proposition 3.1 it holds

$$\frac{T(r_n)}{r_n} \geq \log_2 \frac{1}{r_n} \geq a_{n+1} - a_n$$

since $\frac{1}{r_n} \geq 2^{a_{n+1}-a_n}$. In view of $a_{n+1} \geq 4a_n$ we get

$$\frac{T(x) - T(x+h_n)}{-h_n} \leq a_n - 2n - \frac{a_{n+1} - a_n}{3} \leq -2n$$

i.e. $\underline{D}(x) = -\infty$.

7.3. Some remarks to the representations in textbook [3]. The textbook [3] of K. Strubecker: "EINFÜHRUNG IN DIE HÖHERE MATHEMATIK", vol. II, contains a beautiful introduction in the foundations of the analysis. So you can find in detail a treatise on the function $f = T$ of T. Kakagi, among other things very interested investigations due to W. Wunderlich [6]. Unfortunately, in the passage on Takagi's function are misrepresentations and since it is not planed a new edition of [3], we want to make here two remarks.

1. The first remark concern the formula (56.45) in [3]:

$$D_\nu = \frac{f(x_\nu) - f(x)}{x_\nu - x} = \sum_{n=1}^{\nu} (-1)^{\tau_n} = (-1)^{\tau_1} + (-1)^{\tau_2} + \ldots + (-1)^{\tau_\nu} \tag{7.3}$$

where

$$x = 0, \tau_1 \tau_2 \ldots \tau_\nu \ldots$$

and

$$x_\nu = 0, \tau_1 \tau_2 \ldots \tau_{\nu-1} \tau'_\nu \tau_{\nu+1} \ldots$$

with $\tau'_\nu = 1 - \tau_\nu$ are the dyadic representations of x and x_ν, respectively. This formula cannot be correct as the following example shows. In case $x = \frac{2}{3} = 0,10101\ldots$ we have $\tau_{2\nu} = 0$ and $\tau_{2\nu-1} = 1$ for $\nu \geq 1$ and $x_1 = 0,0010101\ldots = \frac{1}{6}$. Now, $f(x) = \frac{2}{3}$ and $f(x_1) = \frac{1}{6} + \frac{1}{2} f(\frac{1}{3}) = \frac{1}{2}$, cf. (1.2), so that

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{1}{3}$$

but formula (7.3) yields integer values. Let us mention that instead of (7.3) it holds

$$\frac{f(x_\nu) - f(x)}{x_\nu - x} = 1 + \sum_{n=1}^{\nu-1} (-1)^{\tau_n} - \sum_{k=0}^{\infty} \frac{\tau_{\nu+k+1}}{2^k} \tag{7.4}$$

which follows from Proposition 5.1.

2. The second remark concern Satz 3 and Satz 4 on p. 255. Both theorems base on formula (7.3) which we have recognize as not correct. Moreover, x_ν is only a special sequence which converges to x so that the fact $\lim_{\nu\to\infty} D_\nu = +\infty$ does not imply the existence of (one-side) improper derivatives. Therefore the statements in Satz 3 and Satz 4 concerning the existence of (one-side) improper derivatives are not proved.

On p. 255 it says literal: "Zum Beispiel hat $f(x)$ an der Stelle

$$x = \frac{1}{7} = 0,001001001\ldots \qquad \text{(periodisch)}$$

nach (56.45) (i.e. (7.3)) die uneigentliche Ableitung $f'(x) = \lim_{\nu\to\infty} D_\nu = +\infty$ und ...". By Proposition 5.4 indeed $f'(x) = +\infty$, cf. also Remark 5.5. But this a not a consequence of $\lim_{\nu\to\infty} D_\nu = +\infty$ as Example 7.2 shows.

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