DOI^2

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We discuss some seemingly unrelated observations on integers, whose close or farther away neighbors show a complex of combinatorial, ordering, arithmetical or probabilistic properties, emphasizing puzzlement in more common expectations.

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The unusual short title 'DOI^2' requires an explanation. It is a recursive acronym of the parity adjectives odd and even (the sound of pronunciation) or just the uppercase writing of *doi*, the Romanian word for two. In our manuscript, 2 is the main character, or the *red string* of the 1st of March celebrating symbol named in Romanian '*mărţişor*' (little March), which is embraced by the *white string* embodying the parity of '*noted divisors*'. Their double bow twinkles with the hanging tassel spread around the 2×2 themes we chose to present here.

1. FROM AN ELEMENTARY OLYMPIAD PROBLEM TO A WARING PROBLEM QUESTION

As sometimes happens in a matter intended to avoid any involvement of chance, at one of the many local math competitions, which is of major importance for the young beginners and their supporters, at the first stage of The National Mathematical Olympiad, Prahova county Romania, February 24, 2019, the written problem looked different from what the author probably wanted to ask the participants.

Problem 2b (G. Achim, ONM Prahova 2019). *Find the non-negative distinct integers m, n, p, q knowing that*

$$m^3 + n^3 + p^3 + q^3 = 10^{2021}. \tag{1}$$

Leaving aside the necessity on the solutions to have distinct components, condition that does not oversimplify any approach to the problem, the straight meaning of the requirement is to find all solutions of (1). This is a classic Waring problem for cubes, with a particularly large constant term $N = 10^{2021}$.

Originally, in 1770, Edward Waring, in his *Meditationes Arithmeticae* (see [1] for a recent cover), summarized his experiments, and adventured to state that any natural number can be written as a sum of at most $g(k)$ positive integers that are k-th powers. Thus, any natural number can be expressed as a sum of 4 squares, or 9 cubes, or 19 fourth powers, and so on, he wrote. A text in a nutshell that developed into a lot of dreams and endeavors for centuries. A major step towards understanding the problems raised by Waring was made by Hilbert, who proved in 1909 the existence of $g(k)$ for all k. Precisely, he showed that for any $k \geqslant 2$, there is a positive integer $g = g(k)$ such that any integer $n \geqslant 0$ can be represented as $n = a_1^k + \cdots + a_g^k$, for some natural numbers $a_1, \ldots, a_g$. As an aside, we mention that one might find it interesting to know more about the peaking events of that time in the mathematical life of Göttingen, as described in the fascinating biography written by Reid [43, XIV Space, Time and Number], including the touching fate of Minkowski, in whose memory Hilbert dedicated his work on Waring's problem. Furthermore, the interested reader might start consulting the survey of Vaughan and Wooley [48] and the articles of Deshouillers et al. [19], Pollack [40] and Siksek [47] for a summary of the results and methods developed over the years. Two examples from the latest results that are valid for all n except just a few particular cases are the following. Siksek [47, Theorem 1] showed that any $n \geqslant 455$ can be written as a sum of seven cubes, making effective the same statement proved by Linnik [28] to hold true for n sufficiently large, and Deshouillers et al. [19, Conjecture 1,2] who concluded their findings by conjecturing that there are exactly 113 936 676 positive integers that cannot be written as a sum of four nonnegative integral cubes and the largest of them is the title of their article.

Turning back to Problem 2b and knowing the limited time the children have had available during the competition, while looking at how big N is, one is driven to pinpoint two catches. First, there should be a concise way to write any solution and second, it should not be too difficult to enumerate all the solutions. But instead, only an epsilon part of this is possible, unless one accepts as an answer to the Gordian Knot problem a Gödelian-type presentation of all solutions in less than 100 words, without even a single one being able to be extracted explicitly.

Finding a solution is not too difficult using the hint in Problem 2a of the mentioned olympiad, which implied immediately that $1^3 + 2^3 + 3^3 + 4^3 = 10^2$.

Then, observing that $2021 - 2 = 2019$ is divisible by 3, one finds that the equality

$$10^{3\cdot 673}\left(1^3 + 2^3 + 3^3 + 4^3\right) = 10^{2019}\cdot 10^2$$

is equivalent to

$$(2)\qquad \left(10^{673}\right)^3 + \left(2\cdot 10^{673}\right)^3 + \left(3\cdot 10^{673}\right)^3 + \left(4\cdot 10^{673}\right)^3 = 10^{2021}\,,$$

that is, $S = \left(10^{673}, 2\cdot 10^{673}, 3\cdot 10^{673}, 4\cdot 10^{673}\right)$ is a solution of equation (1). But next, finding other solutions or proving that S is unique is not an easy task at all. Actually the facts are as follows.

Let $\nu(n)$ denote the number of representations of $n \in \mathbb{N}$ as a sum of four cubes. The problem raised later, after Waring, asked to show that $\nu(n) > 0$ for $n > n_0$. For the upper bound of $\nu(n)$, using the large sieve and bounds of exponential sums in several variables, Hooley [24] showed that

$$\nu(n) = O\left(n^{11/18+\epsilon}\right),$$

while the expectation is that the true order of magnitude is around $O\big(n^{1/3}\big)$. Further, for the lower margin, Hooley [23] has showed that $\nu(n)$ is not of order $o\big(n^{1/3}\big(\log\log n\big)^4\big)$. Moreover, common experiments in Waring problems prove that $\nu(n)$ is within the above margins even if n is relatively small. Therefore, no question of uniqueness of the solution of (1), but the number of solutions is huge, around 10^{673}. Therefore, how could someone face the endeavor to write so many solutions during the allocated time of 2 hours (reduced from 3 hours, as it was in the old days), by comparing with some worldwide margins:

- The number of atoms in the observable universe is $\approx 10^{81}$;
- The number of atoms in a A4-paper is $\ll 10^{23}$;
- (Assume all atoms are grouped in A4-papers.) The number of A4-papers in universe $\ll 10^{58}$.

Therefore, one needs to write about $10^{673}/10^{58} = 10^{615}$ solutions on each A4-paper, which is quite a number compared with the number of characters of the King James authorized Bible, which is $3116480 < 10^7$.

In spite of these big numbers, let us see that the idea of deducing solution (2) can be exploited to find a *tower of solutions* of (1). Let $a, b \geqslant 0$ with $a + b = 673$ be integers and let $(m_a, n_a, p_a, q_a) \in \mathbb{N}^4$ be a solution of

$$(3)\qquad \sum m_a^3 = 10^{2+3a}.$$

Then $\sum m_a^3{\cdot}10^{3b} = 10^{2+3a+3b} = 10^{2021}$, so that, $\left(m_a 10^{3b}, n_a 10^{3b}, p_a 10^{3b}, q_a 10^{3b}\right)$ is a solution of $\sum m^3 = 10^{2021}$. This means that finding tuples (m_a, n_a, p_a, q_a)

that are *root solutions* of (3), equations in which the constant term is smaller, allows to find solutions of equation (1), where the constant term is large. The smallest root solutions whose components are ordered lexicographically are the following:

For $a = 0,\ b = 673$, $1^3 + 2^3 + 3^3 + 4^3 = 10^2$.

For $a = 1,\ b = 672$, there are three root solutions, for which:

$$6^3 + 24^3 + 34^3 + 36^3 = 10^5,$$
$$10^3 + 20^3 + 30^3 + 40^3 = 10^5,$$
$$12^3 + 16^3 + 34^3 + 38^3 = 10^5\,.$$

For $a = 2,\ b = 671$ there are 43 ordered root solutions listed in Table 1.

(0,196,312,396)	(44,64,250,438)	(92,136,240,436)	(155,309,322,322)
(4,122,295,417)	(44,100,160,456)	(92,244,256,408)	(156,176,244,424)
(4,302,304,354)	(54,151,288,417)	(100,200,300,400)	(193,267,299,361)
(14,58,106,462)	(58,134,256,432)	(100,256,272,396)	(200,210,295,385)
(18,107,220,445)	(58,159,337,386)	(107,184,213,436)	(204,256,292,368)
(18,200,232,430)	(58,188,319,393)	(114,147,277,420)	(216,260,298,358)
(22,263,316,369)	(60,240,340,360)	(114,170,274,418)	(225,295,300,330)
(28,44,358,378)	(64,65,255,436)	(120,160,340,380)	(230,288,295,337)
(32,124,148,456)	(67,92,352,381)	(128,172,292,408)	(240,244,256,380)
(37,65,75,463)	(70,183,198,441)	(145,170,340,375)	(260,265,274,351)
(41,57,79,463)	(72,195,277,414)	(151,282,288,369)	

Table 1 – The solutions of the equation $x^3 + y^3 + z^3 + v^3 = 10^8$ ordered lexicographically.

Let us remark that the root solutions, whose components are ordered lexicographically, generate $1 \times 4! = 24$ solutions (with no restrictions on the order of the non-negative integer components) of equation (3) with $a = 0$ and $3\times 4! = 72$ solutions of equation (3) with $a = 1$. If $a = 2$, excluding the 12 permutations that would have been counted twice, since $(155, 309, 322, 322)$ has tow equal components, we have $43 \times 4! - 12 = 1020$ solutions of equation (3) with $a = 2$. Also, compare with the expected order of magnitude in the last case, which is $\approx 10^{8/3} = 464.15\ldots$

Much harder is to find by brute force the next root solutions. For $a = 3$, $b = 670$, the first ordered solutions are:

$$0^3 + 1960^3 + 3120^3 + 3960^3 = 10^{11},$$
$$3^3 + 649^3 + 1775^3 + 4549^3 = 10^{11}\,.$$

Notice in the lists the towers of solutions being formed: (1,2,3,4); $(10, 20, 30, 40)$; $(100, 200, 300, 400)$ and $(6, 24, 34, 36)$; $(60, 240, 340, 360)$ and $(0, 196, 312, 396)$; $(0, 1960, 3120, 3960)$.

As the power of the constant terms increases, it is more and more difficult to find corresponding new root solutions, despite their number increases fast. These solutions produce solutions of the original equation (1) with fewer and fewer ending zeros. Thus, the unwished but inspiring formulation of Problem 2b raises the question of finding particular solutions of equation (1), since there are so many solutions, around 10^{673}, but the probability to find one is very small, about $10^{673}/10^{2021} = 10^{-1348}$.

PROBLEM 1. *Find a solution of equation $m^3 + n^3 + p^3 + q^3 = 10^{2021}$ whose components do not end with no zeros. More generally, find solutions of general Waring problems whose components are not tower-wise related to their constant term.*

2. A COVERING PROBLEM AND A QUESTION: IS IT TRUE THAT PRIME NUMBERS ARE SUPERABUNDANT?

In this section we discuss a generalization of a problem of Pillai [38] regarding finite sequences of consecutive integers with no member relatively prime to all the others. The problem appears in a question about the representation of a product of consecutive integers as a perfect power. Pillai treats the problem in a series of papers [38], [39] and Brauer [5] is the first to prove completely the main conjecture of Pillai. Later, Eggeleton [21] revisits Pillai's problem introducing a new language in terms of graphs and gives new proofs to the results of Pillai and Brauer. More recently, Ghorpade and Ram [46] generalize the results for arithmetical progressions in integral domains. The theme was also met in several other contexts, whose links can be found by starting with the references within the cited papers.

Let us consider the Eratosthenes sieve as a reverse process. Most of the discussion here can take place in a broader context, with numbers being not necessarily prime, but for brevity and simplicity we stick to the primes case. Thus, let us say we have a large basket $\mathcal{P}$ containing all prime numbers and an endless sequence of equal boxes situated on a straight line $\mathcal{L}$, the analogue of the set of all integers. Further, let us think of any prime number as a sequence of pearls arranged on a wire at distances equal to its size. The boxes of $\mathcal{L}$ are supposed to be tall enough to fit any tower of pearls resulting from successive placement of primes on $\mathcal{L}$. During a placement of a prime p, when the left-right positioning of a prime is considered to be the suitable, the wire is dematerialized and the pearls of p are attracted by gravity to the bottom. In Figure 1 one can see two such ongoing arrangements.

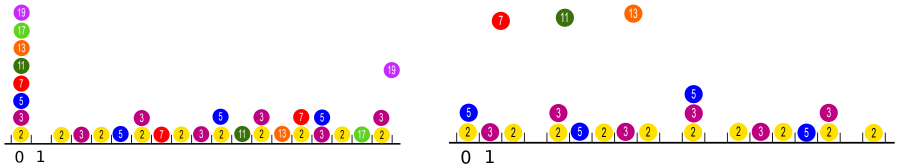

Figure 1 – Two ways to fill the boxes of $\mathcal{L}$ by pearls of primes. On the left, all primes are placed with a pearl in the zero box, exactly as in the sequence of integers, where the box one always remains unoccupied, while on the covering on the right side, primes 2 and 5 are also placed in the zero box, 3 is shifted to the right and three more boxes are ready to fit effectively the remaining primes $7, 11, 13$ to generate the perfect minimal complete covering of the segment of length 17.

We call *segment* any finite sequence of consecutive boxes of $\mathcal{L}$. The length of a segment $\mathcal{S}$, which we denote by $l(\mathcal{S})$, is equal to the number of boxes of $\mathcal{S}$.

Our interest will be to place *effectively* a finite number of primes on $\mathcal{L}$ so that all boxes on a certain segment $\mathcal{S}$ are filled. The *requirement of effectiveness* demands that the pearls of any prime that is used in such an arrangement occupy simultaneously at least two boxes of $\mathcal{S}$. This means that either p is short enough, roughly less than half of $l(\mathcal{S})$, or p is placed sufficiently close to the end points of $\mathcal{S}$, so that two boxes of $\mathcal{S}$ are filled by the pearls of p.

We say that an arrangement of a set of primes $\mathcal{M} \subset \mathcal{P}$ *covers* a segment $\mathcal{S} \subset \mathcal{L}$ if in each box of $\mathcal{S}$ there is a pearl of a prime in $\mathcal{M}$. More precisely, in arithmetic language, identifying a segment of boxes filled by pearls of primes with a finite sequence of consecutive integers, we say that the segment is *covered completely* if no element of the sequence is relatively prime with the product of all the others.

The description above is a natural solitary PL-game in which the player places prime pearls in straightly aligned boxes. The objective of the player is to fill effectively with pearls of primes as many boxes as possible of a certain segment. The player wins if he succeeds to cover completely any segment of his choice, and his performance is the better the longer is the length of the covered segment.

A natural question to ask is whether there actually exist and how long are such segments that can be covered completely. The short answer is that there are no segments of length $\leqslant 16$ that can be completely covered and any segment of length $\geqslant 17$ can be completely covered (see [39], [5]).

Let us remark that the arrangements of primes that generate the sequence of all integers (call it the $\mathbb{Z}$-arrangement) is close, but always fails to cover completely a segment with and endpoint at 0 and, more generally, relatively close to 0. In the $\mathbb{Z}$-arrangement all primes are placed calibrated with a pearl in the zero box, forming an infinitely high tower. Although, the neighbor boxes ± 1 remained forever unfilled. At the other endpoint, if one wishes to extend any such almost completely covered segment, even the scantiest possible boxes always contain a marble of a prime with another marble in the zero box.

A successful arrangement of primes on a completely covered segment of length 17 is described by a sequence of 17 consecutive integers $a, a+1, a+2, \dots a+16$, where a is a solution of the following system of congruences:

$$\begin{cases} a \equiv 0 & \pmod{2, 5, 11} \\ a \equiv -1 & \pmod{3} \\ a \equiv -2 & \pmod{7} \\ a \equiv -3 & \pmod{13}. \end{cases} \tag{4}$$

By the Chinese Remainder Theorem, the system 4 has infinitely many solutions $a = 27830 + 30030k,\ k \in \mathbb{Z}$. Similarly, any arrangement of a segment $\mathcal{S}$ is reproduced infinitively many times on $\mathbb{Z}$, at places equally spaced in an endless arithmetic progression.

An exhaustive analysis of all possibilities shows that the shortest complete covering is a segment of length 17, and the arrangement is unique, except for its mirror, whose primes are placed in reversed order. The reversed covering actually has an initial positive solution of the system of congruences analogue to (4) that is closer to zero. This solution gives the completely covered segment with boxes labeled: $2184, 2185, \dots, 2200$, and the periodicity of the solutions of the reversed covering is the same: $30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

The fact that the complete covering 4 of a segment of length 17 can be extended naturally to longer segments, both to the left and to the right. This can be easily seen immediately in the list of the prime factorizations of the integers in the initial solution. In Figure 2 one sees the barriers at boxes $\mathrm{b} = 0$ and $\mathrm{b} = 17$, which are occupied by large non-effective primes. But, we have the prime 17 available, which can be placed in any of the two boxes to obtain a complete covering of length 18. Furthermore, if for example, we place 17 in box labeled $\mathrm{b} = 17$, we are offered for free complete coverings of segments of length $18, 19, 20, 21$. Then, the new barrier can be overcome by placing $p = 19$, which appears perfectly fit at our disposal. And the process can be continued finding sufficiently many available primes.

For example, continuing this *first come first served* arrangement process, one fills box $\mathrm{b} = 999$ by prime $p = 647$ and has 50 available primes remaining, and at a farther place one arrives to fill box b $= 9191$ by prime $p = 6043$ and remains with 351 spare ones.

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b = -3   n = 27827 = 27827                  b = 10   n = 27840 = 2^6 * 3 * 5 * 29
b = -2   n = 27828 = 2^2 * 3^2 * 773        b = 11   n = 27841 = 11 * 2531
b = -1   n = 27829 = 17 * 1637              b = 12   n = 27842 = 2 * 13921
-------------------------------------       b = 13   n = 27843 = 3 * 9281
b =  0   n = 27830 = 2 * 5 * 11^2 * 23      b = 14   n = 27844 = 2^2 * 6961
b =  1   n = 27831 = 3 * 9277               b = 15   n = 27845 = 5 * 5569
b =  2   n = 27832 = 2^3 * 7^2 * 71         b = 16   n = 27846 = 2 * 3^2 * 7 * 13 * 17
b =  3   n = 27833 = 13 * 2141              ----------------------------------------
b =  4   n = 27834 = 2 * 3 * 4639           b = 17   n = 27847 = 27847
b =  5   n = 27835 = 5 * 19 * 293           b = 18   n = 27848 = 2^3 * 59^2
b =  6   n = 27836 = 2^2 * 6959             b = 19   n = 27849 = 3 * 9283
b =  7   n = 27837 = 3^3 * 1031             b = 20   n = 27850 = 2 * 5^2 * 557
b =  8   n = 27838 = 2 * 31 * 449           b = 21   n = 27851 = 27851
b =  9   n = 27839 = 7 * 41 * 97            b = 22   n = 27852 = 2^2 * 3 * 11 * 211
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Figure 2 – The arrangement of prime pearls in the 17 long completely covered segment. The factorization of the integers in the boxes b situated inside and near the minimal solution of the covering.

The question is: for how long this extension can be continued to the right by the first come first served rule of placing the smallest available prime in the new empty barrier box?

Conjecture 1. *Starting with the minimal covering of length* 17 *and applying the 'first come first served rule', in which the first free box is filled by the smallest available prime, the extension extension of completely covered segments can be done indefinitely. Also, a similar fact should occur if one follows other more or less regular rules and by starting with different completely or almost completely covered segments.*

The 'first come first served rule' can be applied in different ways: always to the right of the segment to be extended, or always to its left. Also, other choices of the first empty box to fill, such as selecting alternatively from the right and from the left. Some other more complex rules of selection of the first box to fill seem, experimentally, to be even more efficient. We remark that in the process, the longer the segment, one may find several distinct complete coverings for it. Almost always, these coverings may be taken as the starting point of Conjecture 1, with still the same endless extension effect. This phenomenon is similar to the generation of the Euclid-Mullin sequences (see Mullin [36] and [14]), but it offers easier more tangible results.

Furthermore, when the segment becomes long enough, we find seemingly that we can even skip some of the available odd primes by keeping them out for good of the arrangement process and without being too much disturbed in it. We don't know to prove that an infinite extension is possible by the "first come first served' rule for any starting complete covering, but we can show that there is a pattern of germ-coverings that produce completely covered segments of any size. Moreover, these patterns create complete coverings of segments even when keeping out of the processes any finite set of odd primes. The lengths of completely covered segments are all integers greater than a certain size depending only on the set of primes left aside.

THEOREM 1 ([7]). *Let $k \geqslant 1$ and let $\mathcal{M} = \{q_1, \ldots, q_k\}$ be a finite set of odd primes. Then there exists $n_{\mathcal{M}} \in \mathbb{N}$ such that for any $N \geqslant n_{\mathcal{M}}$ there exists a sequence of N consecutive integers such that no one of them is relatively prime to the product of all the others. Moreover, the greatest common divisor of any number in the sequence and the product of all the others is divisible by a prime that is different to any $q \in \mathcal{M}$.*

Theorem 1 is effective, meaning that the bound $n_{\mathcal{M}}$ can be calculated explicitly, and it contains the conjecture of Pillai as a particular case.

COROLLARY 1 (Brauer [5]). *For any $N \geqslant 17$, there exists a sequence of N consecutive integers such that no one of them is relatively prime to the product of all the others.*

As an example of a choice of $\mathcal{M}$ in Theorem 1, if the the first odd prime, $q = 3$, is excluded from the operation of filling the boxes, all segments of lengths greater than some integer smaller than 1300 can be completely filled.

COROLLARY 2. *For any $N \geqslant 1300$, there exists a sequence of N consecutive integers such that no one of them is relatively prime to the product of all the others, and moreover, the greatest common divisor of any integer in the sequence and the product of all the others divided by the largest power of 3 in its decomposition is $\geqslant 1$.*

Using a different initial arrangement and taking account of the larger number of boxes that might not be filled by the pearls of the even prime $q = 2$, one can still show that Theorem 1 holds true even with the restriction imposing the primes to be odd is removed.

THEOREM 2 ([7]). *Let $k \geqslant 1$ and let $\mathcal{M} = \{q_1, \ldots, q_k\}$ be a set of prime numbers. Then there exists $n_{\mathcal{M}} \in \mathbb{N}$ such that for any $N \geqslant n_{\mathcal{M}}$ there exists a sequence of N consecutive integers such that none of them is relatively prime to the product of all the others. Moreover, the greatest common divisor of any number in the sequence and the product of all the others is divisible by a prime that is different to any $q \in \mathcal{M}$.*

From Theorem 2 it follows that the chances of a player to win remain intact, no matter his initial choices for a finite number of steps. The only possible inconvenience might be just the extension of the length of the segment that would ensure his win.

COROLLARY 3. *A player of the solitary* PL-*game has a strategy to win no matter what choices he followed for a finite number of moves.*

3. CONSTANT DIGIT NUMBERS IN THE SEQUENCE OF SELF-POWERS

Usually, according to the law of large numbers, one expects that two sparse sequences of integers have few points of intersections. This happens even when there is some regularity in the definition of the two sequences, but their nature is different. Is it possible that the number of points of incidence is infinite? In the following the two sequences we will consider are the sequence of self-powers and the sequence of integers whose representation in base b (we restrict to the case $b = 10$) has all digits equal.

For any non-negative integer n, let $l(n)$ denote the number of digits of n. We say that $n \in \mathbb{N}$ is a *constant word number* if $l(n)$ has all digits equal. Our startling example was noticed two years ago [6], [16] with the following match:

$$l\left(2017^{2017}\right) = 6666, \tag{5}$$

so 2017^{2017} is a constant word number in base 10. Remark that one needs some space to write the digits of 2017^{2017}, since a dense A4 sheet may hardly contain 6000 characters.

All small numbers are constant word numbers, since $l(n) = 1$ for all $n \leqslant 999\,999\,999$. The next constant word number is 10^{10}, which is the first member of a group that ends with $10^{11} - 1$. They all have 11 digits. Constant word numbers appear in groups that are longer and longer but farther and farther apart, like in a sort of generalized geometric progression. These groups are composed of numbers that have 1, 11, $22, \ldots, 99$, $111, 222, \ldots, 999$, $1111, 2222, \ldots$ digits, respectively.

In view of example (5), one might wonder if there are other special years, for which their self-power is a constant word number. And the answer is that there are. The previous one occurred 300 years before, since 1717^{1717} has 5555 digits. The next close misses are 2312 and 2602, since $l\left(2312^{2312}\right) = 7778$ and $l\left(2602^{2602}\right) = 8887$. For the next real special years, we have to wait till 2889 and 3173, for which $l\left(2889^{2889}\right) = 9999$ and $l\left(3173^{3173}\right) = 11111$.

The list of the constant word numbers starts with $1, 2, \ldots, 9, 10, 35, 46, 51, 194, 234, 273, 349, 423$ (see Table 2 for the number of digits of their self powers).

n	1-9	10	35	46	51	194	234	273	349	386	423	1411	1717	2017	2889
$l(n^n)$	1	11	55	77	88	444	555	666	888	999	1111	4444	5555	6666	9999

Table 2 – The first 23 self powers constant word numbers.

And there are more such self power constant word numbers, for example $n = 631\,296\,394$, for which $l(n^n)$ has $5\,555\,555\,555$ digits.

Question 1. Given the following two sequences: $\mathcal{S}_1$ of the constant word numbers, and $\mathcal{S}_2$ of the number of digits of self powers, that is,

$$\mathcal{S}_1 : 1, 2, \ldots, 9, 11, 22, 33, \ldots, 99, 111, 222, \ldots, 999, 1111, 2222, \ldots$$
$$\mathcal{S}_2 : l(n^n), \text{ for } n \geqslant 1,$$

how many common points do they have?

Checking the gaps, we find that two type of gaps that increase exponentially combine to separate the elements of $\mathcal{S}_1$, while the average gap between the elements of $\mathcal{S}_2$ is asymptotically equal to $e \cdot n^{n+1}$, for $n \geqslant 1$.

We can also consider pairs of constant word self powers. We say that m and n are *amicable constant word self powers* if $l(m^n)$ and $l(n^m)$ are constant word numbers. For example, such amicable pairs are: $(4, 368)$ since 4^{368} has 222 digits and 368^4 has 11 digits; $(39, 698)$ since 39^{698} has 1111 digits and 698^{39} has 111 digits; and $(48, 66)$ since 48^{66} has 111 digits and 66^{48} has 88 digits.

More generally, we look at any size analogue of amicable pairs. Thus, for any $k \geqslant 1$, we say that a tuple of positive integers $(m_1, \ldots, m_k)$ is a tuple of *amicable constant word of self powers* if each of the numbers $l(m_1^{m_2})$, $l(m_2^{m_3})$,..., $l\left(m_{k-1}^{m_k}\right)$, and $l\left(m_k^{m_1}\right)$ are written with only one digit. If $k = 1$ the amicable k-tuples coincide with self powers constant word numbers. Here are some examples of amicable tuples:

- $(26, 62, 49)$: $26^{62} \leftsquigarrow$ 88 digits; $62^{49} \leftsquigarrow$ 88 digits; $49^{26} \leftsquigarrow$ 44 digits
- $(49, 39, 62)$: $49^{39} \leftsquigarrow$ 66 digits; $39^{62} \leftsquigarrow$ 99 digits; $62^{49} \leftsquigarrow$ 88 digits.

- $(26, 31, 22, 49)$: $26^{31} \leftsquigarrow$ 44 digits; $31^{22} \leftsquigarrow$ 33 digits; $22^{49} \leftsquigarrow$ 66 digits; $49^{26} \leftsquigarrow$ 44 digits
- $(66, 54, 25, 47)$: $66^{54} \leftsquigarrow$ 99 digits; $54^{25} \leftsquigarrow$ 44 digits; $25^{47} \leftsquigarrow$ 66 digits; $47^{66} \leftsquigarrow$ 111 digits.

Counting k amicable tuples with components less than a given fixed margin, we found that the chances to find an amicable k tuple decrease with k, although the total number of k amicable tuples increases significantly with k. Then, again, the basic question is whether there are really an infinite number of such amicable tuples.

Question 2. How many amicable tuples of constant word self powers exist?

4. CONTRASTING SHAPES OF STURMIAN WORDS

A common preconception at that initial shallow contact with a problem of a probabilistic nature regarding the parity of the involved objects is the expectation of a fifty-fifty occurrence of odds and evens. We present a case in which there is a clear bias between odds and evens, in fact the reflection of a plainly wide spread phenomenon regarding the parity of the divisors of the terms of sequences belonging to some quite different classes [17].

Our object here are the 'simplest' non-finally periodic binary sequences. These are known as Sturmian words and the simplicity condition for a binary word w is the requirement that $p_w(n) = n + 1$ for all $n \geqslant 1$. Here $p_w(n)$ is the complexity function of w, which by definition counts the number of distinct sub-words of length n. There is a gap from Sturmian words to the class of ultimately periodic sequence, whose characteristic is the fact that their complexity function is bounded.

Since their introduction by Hedlund and Morse [35], Sturmian words were intensely studied by different authors within a broad area of interests (see [2–4, 27, 29–33, 37, 44]).

Following the ideas presented in [16] and [18], we wish to show the two contrasting faces of Sturmian words, the more regular fractal-type face and the the random looking one. The asymptotic bias slope of the parity of the divisors function [16] and [18] is a type of phenomenon seen also in other contexts, such as those discussed in the following works: [8–13, 15, 26, 45].

4.1. Fractal face of Sturmian words

A classic example of Sturmian words is the Fibonacci word, generalized by Dumaine [20], and Ramírez et al. [41], [42]. They are generated using the concatenation operation in their defining recursive formula.

A general convenient way to define Sturmian words is using the *rotation function* with two parameters modulo one. Denote $R_\theta(\varphi) = \varphi + \theta$

(mod 1), where $\varphi \in [0,1)$ and suppose θ is irrational. Then define the word $w = w_1 w_2 \cdots$, with letters $w_n = a$ if $R_\theta^{(n)}(\varphi) \in [0,\theta)$ and $w_n = b$, else. For example:

$$
\begin{array}{ll}
R_{\sqrt{7}/7}(0.2) \text{ generates } w = abbababbabbababbababbabbababbabbab\ldots \\
R_{\pi/8}(0.2) \quad \text{generates } w = abbababbababbababbababbababbabbaba\ldots
\end{array}
\tag{6}
$$

Notice that both words in (6) contain exactly three distinct sub-words of length 2 and aa is not a sub-word, verifying the particular case of the complexity condition $p_w(2) = 3$.

A standard drawing rule named the *odd-even drawing rule*, was used originally to draw Fibonacci fractals [20], [22]. Starting at the origin and looking up, the curve associated to the Sturmian word $w = w_1 w_2 w_3 \ldots$ are constructed as follows:

- ∘ $w_n = a$: walk one step forward;
- ∘ $w_n = b$ and n even: walk one step forward and turn left 90°;
- ∘ $w_n = b$ and n odd: walk one step forward and turn right 90°.

Applying the odd-even drawing rule to arbitrary Sturmian words we found a great variety of fractal-type shapes. Two examples are shown in Figures 3 and 4. Let us mention that in various experiments we have noticed that there are many unexpected contrasting situations related to the complexity of the curves and that of the generators. For example, it is not rare to find simpler curves if the rotation parameter θ is transcendent than in the case where θ is algebraic.

4.2. **Random walks of the divisor parity slope on Sturmian trajectories**

Fix the alphabet $\mathcal{A} = \{a, b\}$ and let $w = w_1 w_2 \cdots$ be a word, with letters $w_j \in \mathcal{A}$. We define the following counters of the parity of the divisors ranks:

$$
\begin{aligned}
o_w(n) &:= |\{j \in \mathbb{N} : j \text{ divides } n, w_j = b \text{ and } n/j \text{ is odd }\}|, \\
e_w(n) &:= |\{j \in \mathbb{N} : j \text{ divides } n, w_j = b \text{ and } n/j \text{ is even }\}|\,.
\end{aligned}
$$

To illustrate the meaning of the above parity function, let $n \geqslant 1$, suppose $n = r \cdot s$ and make the association of the rank to the pair of the divisors: $n \rightsquigarrow (r, s)$. Then, the r–flag checks the letter w_r (is it a or b?) and, if the flag is right, the parity of s increments correspondingly either $o_w(n)$ or $e_w(n)$. The calculation of $o_w(n)$ and $e_w(n)$ ends after all the pairs of the divisors of n are evaluated.

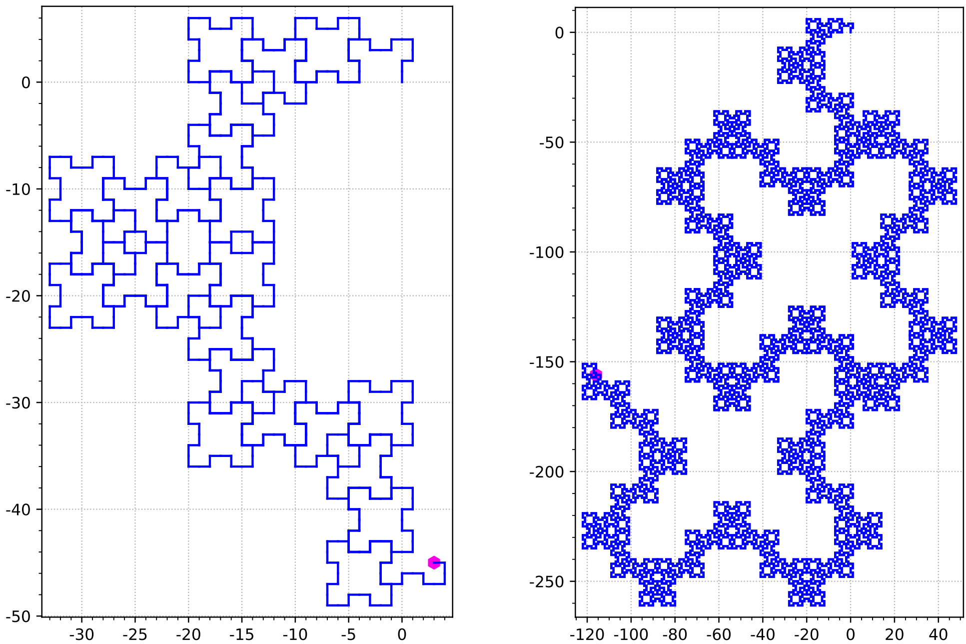


Figure 3 – Fractal type curves generated by the odd-even drawing rule applied to the Sturmian word defined by the rotation $R_{\sqrt{7}/7}(0.2)$. The image on the left represents the trajectory after 1000 steps and the one on the right after 20000 steps (so the image on the left side can be found embedded in image on the right side).

Examples: Let $w = abbaa\,bbbaa\,ba\ldots$. Then, we have:

$$\begin{array}{llll}
n = 8 \rightsquigarrow (1,8) & w_1 = a \ \checkmark & n = 9 \rightsquigarrow (1,9) & w_1 = a \ \checkmark \\
n = 8 \rightsquigarrow (2,4) & w_2 = b \ \checkmark,\ 4 \text{ even} & n = 9 \rightsquigarrow (3,3) & w_3 = b \ \checkmark, 3 \text{ odd} \\
n = 8 \rightsquigarrow (4,2) & w_4 = a \ \checkmark & n = 9 \rightsquigarrow (9,1) & w_9 = a \ \checkmark \\
n = 8 \rightsquigarrow (8,1) & w_8 = b \ \checkmark,\ 1 \text{ odd} & &
\end{array}$$

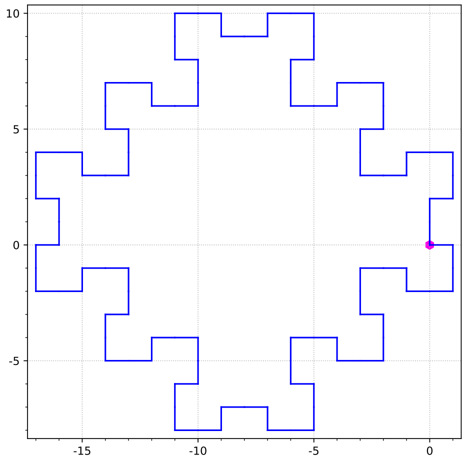
Therefore:

$$o_w(8) = 1,\ e_w(8) = 1 \quad \text{and} \quad o_w(9) = 1,\ e_w(9) = 0.$$

The parity functions $e_w(n)$ and $o_w(n)$ are quite irregular and we calculate their difference:

$$D_w(n) = o_w(n) - e_w(n)\,.$$

The difference of the divisors parity functions preserves the degree of irregularity of $e_w(n)$ and $o_w(n)$, taking negative, zero and positive values. For each Sturmian word, $D_w(n)$ generates a path drawn by a rule similar to the odd-even drawing rule used to draw the face-type fractal of w. Here, instead, we replace



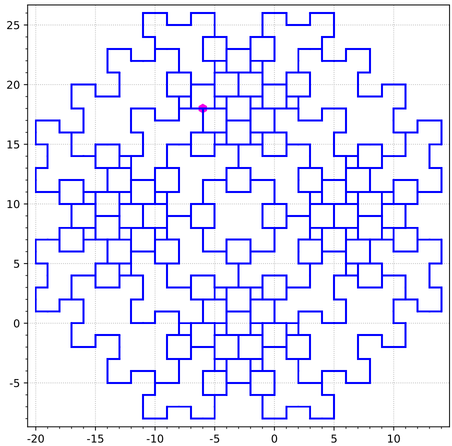


Figure 4 – Fractal curves generated by the odd-even drawing rule applied to the Sturmian word defined by the rotation $R_{\pi/8}(0.2)$. The image on the left side represents the trajectory after 200 steps and the one on the right side is the continuation for a total of 20000 steps.

the odd-even conditions by the negative-positive sign of $D_w(n)$, respectively, and just draw a red dot at the point reached on the path if $D_w(n) = 0$. The aspect of the path is always of a random walk. Two representative examples are shown in Figure 5.

In order to draw information on such a irregular function, the authors of [16] and [18] estimated the mollified average of $D_w(n)$, which is defined by

$$M_w(x) := \sum_{n \leqslant x} \left(1 - \frac{x}{n}\right) D_w(n) \,.$$

In the following we present a sketch of the estimation of $M_w(x)$, which shows the asymptotic prevalence of the odd divisors over the even ones on any Sturmian word (see [16] and [18] for complete details of the proof). It turns out that $M_w(x)$ has an asymptotic behavior whose limit depends on the density:

$$\beta_w := \lim_{n\to\infty} \frac{|\{1 \leqslant j \leqslant n : w(j) = b\}|}{n} \,,$$

which does exists for any Sturmian words. As an example, the density of the Fibonacci word [20] is $\beta_w = \frac{3-\sqrt{5}}{2} = 0.3819660\ldots$.

The estimation of $M_w(x)$ is made by first translating the word w into an analytic language as the coefficients of the Dirichlet series

$$F(H, w, s) := \sum_{n=1}^{\infty} \frac{H(w(n))}{n^s} \,,$$

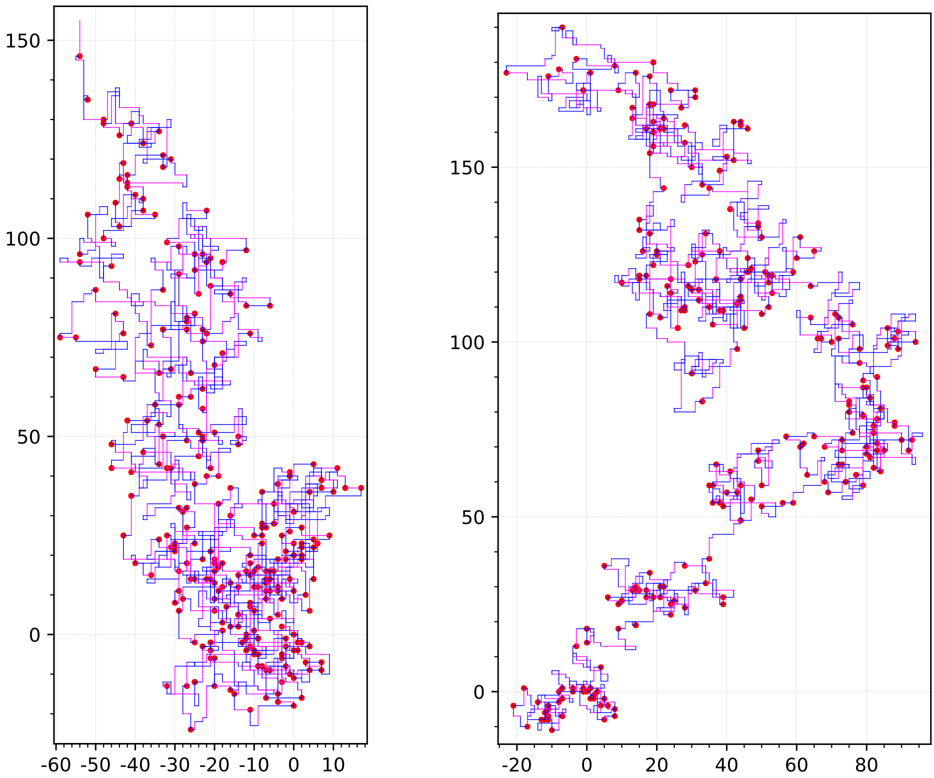


Figure 5 – The first 2000 steps of two random walks generated by the parity divisors slope of Sturmian words defined by rotations $R_\theta(\varphi)$, with $\varphi = 0.2$ and $\theta = \sqrt{7}/7$ (left side) and $\theta = \pi/8$ (right side). The red dots are drawn to indicate the zero-length steps (places where the odd and even divisor parity functions are equal).

where $H(a) = 0$ and $H(b) = 1$. The series $F(H, w, s)$ is absolutely convergent in the half-plane $Re(s) > 1$ and its coefficients are related to β_w through the limit

$$\beta_w = \lim_{n\to\infty} \frac{1}{n} \sum_{1\leqslant j\leqslant n} H(w(j)) \,.$$

Then, by applying the Perron 2nd formula [25], $M_w(x)$ can be expressed as a complex integral:

$$M_w(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s)F(H,w,s)x^s}{s(s+1)}\, ds, \quad x \geqslant 1,\ c > 1 \,.$$

The integral is estimated by moving the path of integration to the left of the pole at $s = 1$ and applying the residue theorem [49].

THEOREM 3. *For any Sturmian word w and any $\delta > 0$, we have*

$$M_w(x) = \frac{\beta_w \log 2}{2}\, x + O_\delta\left(x^{\frac{1}{3}+\delta}\right) . \tag{7}$$

In conclusion, since the main term in the estimate (7) becomes positive for x large enough, it follows that there is a significant quantifiable bias towards the odd divisors of any Sturmian word.

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