Explicit Logics of Knowledge and Conservativity

Melvin Fitting
Lehman College, CUNY, 250 Bedford Park Boulevard West, Bronx, NY 10468-1589
CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016

Dedicated to Victor Marek
on his 65th birthday

Abstract

Several *justification logics* have evolved, starting with the logic LP, (Artemov 2001). These can be thought of as explicit versions of modal logics, or logics of knowledge or belief, in which the unanalyzed necessity (knowledge, belief) operator has been replaced with a family of explicit justification terms. Modal logics come in various strengths. For their corresponding justification logics, differing strength is reflected in different vocabularies. What we show here is that for justification logics corresponding to modal logics extending T, various familiar extensions are actually conservative with respect to each other. Our method of proof is very simple, and general enough to handle several justification logics not directly corresponding to distinct modal logics. Our methods do not, however, allow us to prove comparable results for justification logics corresponding to modal logics that do not extend T. That is, we are able to handle explicit logics of knowledge, but not explicit logics of belief. This remains open.

1 Introduction

Let me begin with the obvious. In the sequence of modal logics T, S4, S5, each is stronger than the one before. They have the same vocabulary, so it does not make sense to ask if each is conservative over its predecessor. But each of these logics has an *explicit* counterpart. These are logics in which, instead of formulas of the form $\Box X$, we have formulas of the form $t{:}X$, read "t is an explicit justification, or reason, or proof of X." These explicit justifications come equipped with certain machinery, and there is a small calculus involving this machinery. The first such logic was LP, an explicit counterpart of S4, introduced by Sergei Artemov in a series of papers culminating in (Artemov 2001). The syntax for justification terms in LP allows a 'bang' operator, !. Dropping it produces a logic often called $\mathsf{LP}(\mathsf{T})$, an explicit counterpart of T. Adding an operator ? produces a logic often called $\mathsf{LP}(\mathsf{S5})$, an explicit counterpart of S5. (All this will be presented more formally below.) What it means to say these are explicit counterparts of the well-known modal logics is embodied in the *Realization Theorem*, a fundamental result first proved (constructively) for LP by Artemov, see (Artemov 2001). First a definition, then the theorem.

Definition 1.1 (Realization) Let X be a modal formula. An LP *realization* of X is a formula in the language of LP that results by replacing each occurrence of $\Box$ with some explicit justification, t. A realization is *normal* if negative occurrences of $\Box$ are replaced with distinct variables (which are always part of the language of explicit justification logics).

Theorem 1.2 (LP Realization Theorem) *If X is a theorem of S4 there is some normal realization of X that is a theorem of LP. Conversely (and much simpler), if some realization of X is a theorem of LP then X is a theorem of S4.*

Here is an example, taken from (Artemov 2001). Axiomatic properties of the various operators will be introduced in the next section, after which the example should take on more significance. The following is a theorem of S4:

$$(\Box A \vee \Box B) \supset \Box(\Box A \vee \Box B) \tag{1}$$

And here is a normal realization of (1), provable in LP.

$$(x{:}A \vee y{:}B) \supset (a\cdot!x + b\cdot!y){:}(x{:}A \vee y{:}B) \tag{2}$$

In (2) the role of the constants is as follows (see the discussion of the Axiom Necessitation rule in the next section). Constant a is an unanalyzed justification of the classical axiom $x{:}A \supset (x{:}A \vee y{:}B)$ and b is an unanalyzed justification of the classical axiom $y{:}B \supset (x{:}A \vee y{:}B)$.

There are analogs of the S4 Realization Theorem connecting $\mathsf{LP}(\mathsf{T})$ with T, and $\mathsf{LP}(\mathsf{S5})$ with S5. Now, each of $\mathsf{LP}(\mathsf{T})$, $\mathsf{LP} = \mathsf{LP}(\mathsf{S4})$, and $\mathsf{LP}(\mathsf{S5})$ is an extension of its predecessor, vocabularies are different, and in fact each is a *conservative* extension of its predecessor.

In this paper we will show the conservativity result just stated, as part of a broader family of similar results. This will be done using a very simple proof theoretic approach. Unfortunately, the approach has its limits, so there are open problems at the end. The present paper had its origins in a technical report, (Fitting 2007a).

2 Justification Logics

It is reasonable to assume a reader of this paper is familiar with the 'standard' modal logics: K, T, K4, S4, S5. No other modal logics will be involved here. Justification logics are much less familiar, however, so we first introduce the language, then the axiomatic characterizations for several of them. We also introduce a non-standard system of designating them, which is particularly handy here. Of course we supply the names that are standard in the literature as well.

2.1 Language

We begin with the family of *justification terms*. (These were called *proof terms* in (Artemov 2001), for important reasons that are not part of our concern here.) Justification terms are built up from *variables*: x_1, x_2, $\ldots$; and *constant symbols*: c_1, c_2, $\ldots$. They are built up using the following *operation symbols*: $+$ and $\cdot$, both binary, and $!$ and $?$, both unary. These are used as infix and prefix, respectively.

This is not the place for an elaborate discussion of the intended meaning of these operations. See (Fitting 2007b) for something of a history of the subject. But here is a brief outline. The operation $\cdot$ is an application operation. The intention is, if t is a justification of $X \supset Y$ and u is a justification of X then $t \cdot u$ is a justification of Y. The operation $+$ combines justifications, $t + u$ justifies whatever t justifies and also whatever u justifies. $!$ is a kind of positive verifier, if t justifies X then $!t$ justifies the fact that t justifies X. And $?$ is a negative verifier, if t does not justify X then $?t$ justifies that fact.

Formulas are built up from *propositional letters*: P_1, P_2, $\ldots$, and a *falsehood constant*, $\bot$, using $\supset$, in the usual way, together with an additional rule of formation, $t{:}X$ is a formula provided t is a justification term and X is a formula.

We will be interested in sub-languages, and so the following notation will be used. If S is any subset of $\{+, \cdot, !, ?\}$ then $L(S)$ is that part of the language described above in which all justification operations come from the set S.

2.2 Axiomatics

Axiom systems for justification logics evolved from one for LP, either by removing or by adding machinery. To begin with, here is a list of axioms from which we will pick and choose; more properly these are axiom schemes.

Classical Axioms:	all tautologies
Truth Axioms:	$t{:}X \supset X$
	$(X \supset Y) \supset (t{:}X \supset Y)$
	$t{:}(X \supset Y) \supset (u{:}X \supset Y)$
$+$ Axioms:	$t{:}X \supset (t + u){:}X$
	$u{:}X \supset (t + u){:}X$
$\cdot$ Axiom:	$t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y)$
! Axiom:	$t{:}X \supset !t{:}t{:}X$
? Axiom:	$\neg t{:}X \supset ?t{:}\neg t{:}X$

The Truth Axioms include two that are not standard. The last two are, in fact, easy consequences of the first Truth Axiom and the other axioms. Likewise, as Classical Axioms we assume all tautologies though a finite set of schemes would be sufficient. Both of these peculiarities arise for the same reason, and have to do with the role of constants in LP. Further discussion is postponed until after they have been introduced.

For rules, of course we have the standard one.

$$\textit{Modus Ponens}: \frac{X \quad X \supset Y}{Y}$$

Finally there is a version of the modal necessitation rule, and here there is some non-uniformity. Constant symbols are intended to serve as justifications for truths that we cannot further analyze, but our ability to analyze is dependent on available machinery. Consequently, we have three different versions, about which more will be said below.

Definition 2.1 The following are versions of a *Constant Necessitation* rule.

Axiom Necessitation If X is an axiom and c is a constant, then $c{:}X$ is a theorem.

Iterated Axiom Necessitation If X is an axiom and c_1, c_2, $\ldots$, c_n are constants, then $c_1{:}c_2{:}\ldots c_n{:}X$ is theorem.

Theorem Necessitation If X is a theorem and c_1, c_2, $\ldots$, c_n are constants, then $c_1{:}c_2{:}\ldots c_n{:}X$ is theorem.

An important feature of justification logics is *internalization*: if X is a theorem then for some justification term t, $t{:}X$ is a theorem. Typically the term t can be constructed from a proof of X, and thus justification logics internalize their own proof theory. But the construction of t requires a certain minimal amount of machinery. Axioms themselves are never the result of elaborate proofs—we simply assume them. This is embodied in the *Axiom Necessitation* rule above, the weakest of the three versions. Suppose we have this rule and we are working with a justification logic with ! available, that is, we have the ! axiom. Then if X is an axiom it has a constant justification, so we have $c{:}X$, this in turn has a justification, $!c{:}c{:}X$, this has its justification, $!!c{:}!c{:}c{:}X$, and so on. But if ! is not part of the machinery we cannot take this route, and so *Iterated Axiom Necessitation* will be assumed instead. Finally if we have a really weak justification logic, not containing $\cdot$, we lack machinery to analyze anything complex, and the *Theorem Necessitation* version will be assumed—everything provable has a justification, about which nothing very interesting can be said.

Clearly the role of constants has much to do with the choice of axioms. Say the version of Constant Necessitation being used is Axiom Necessitation. Then a *constant specification* $\mathcal{C}$ is an assignment of axioms to constants (there is a similar notion for Iterated Axiom Necessitation). A proof meets constant specification $\mathcal{C}$ provided that whenever $c{:}X$ is introduced using the Axiom Necessitation rule, then X is a formula that $\mathcal{C}$ assigns to constant c. A constant specification can be given ahead of time, or created during the course of a proof. A constant specification is *injective* if at most one formula is associated with each constant. Replacing axioms with equivalent versions changes the use of constants in both the Axiom Necessitation and the Interated Axiom Necessitation rules—it alters the constant specification. It is not simple to say, then, what it means to have equivalent axiomatizations of a justification logic—it does not simply mean they have the same set of theorems. This issue will be partially addressed below, in Section 4. For the time being, our adoption of extra Truth Axioms is so that the behavior of constants is simple to describe, as proofs are manipulated in the ways we will consider below. Again, Section 4 examines ways to get rid of this mildly non-standard item.

Now we can properly specify the family of justification logics we will be considering.

Definition 2.2 Let S be a subset of $\{+, \cdot, !, ?\}$. We define two justification logics whose language is $L(S)$. They are denoted $K(S)$ (with K for knowledge) and $B(S)$ (with B for belief). These have axioms and rules specified as follows.

1. For axioms, both $K(S)$ and $B(S)$ have the Classical Axioms. $K(S)$ assumes the Truth Axioms, while $B(S)$ does not. Finally, both assume the $+$ axiom if $+$ is in S, and similarly for $\cdot$, !, and ?.
2. For rules, both have *Modus Ponens*. If both $\cdot$ and ! are in S, $K(S)$ and $B(S)$ have the *Axiom Necessitation* rule. If $\cdot$ is in S but ! is not, both have the *Iterated Axiom Necessitation* rule. Finally, if $\cdot$ is not in S, both have the *Theorem Necessitation* rule.

The primary utility of the notation introduced here is that it makes it very easy to state our main results compactly. Since our nomenclature is not standard, here are some correspondences with the literature. Besides these logics, there are others that have been considered in the literature, and there are also systems that can be characterized in present terms, for example $K(\{!\})$, that have not been considered in the literature. (It's probably not very interesting.)

Standard Name	Name Used Here	Origin	Modal Version
LP(K)	$B(\{+, \cdot\})$	(Brezhnev 1999)	K
LP^-(K)	$B(\{\cdot\})$	(Fitting 2005)	
LP(T)	$K(\{+, \cdot\})$	(Brezhnev 1999)	T
LP^-(T)	$K(\{\cdot\})$	(Fitting 2005)	
LP(K4)	$B(\{+, \cdot, !\})$	(Brezhnev 1999)	K4
LP^-(K4)	$B(\cdot, !\})$	(Fitting 2005)	
LP	$K(\{+, \cdot, !\})$	(Artemov 2001)	S4
LP^-	$K(\{\cdot, !\})$	(Fitting 2005)	
LP(S5)	$K(\{+, \cdot, !, ?\})$	(Pacuit 2005) (Rubtsova 2006)	S5

All the logics considered here have two fundamental properties common to justification logics. Since these will be needed in Section 4, they are stated now for the record.

Proposition 2.3 (Substitution Closure) *For every $S \subseteq \{+, \cdot, !, ?\}$, both $K(S)$ and $B(S)$ are closed under substitution. That is, if X is a theorem of one of these logics, and X' is the result of replacing all occurrences of a variable x with a justification term t, then X' is also a theorem.*

The proof for this is standard. It is true for axioms, since they are specified by axiom schemes. Then one shows it is true for each line of a proof by induction on proof length. Generally the constant specification needed for a theorem and for a substitution instance of it will be different.

Proposition 2.4 (Internalization) *For every operator set $S \subseteq \{+, \cdot, !, ?\}$, both $K(S)$ and $B(S)$ have the internalization property: if X is a theorem so is $t{:}X$ for some ground (that is, variable free) justification term t.*

If S contains $\cdot$, this proposition has a proof due to Artemov, (Artemov 2001). If S does not contain $\cdot$, the proposition defaults to the Theorem Necessitation rule.

3 Results

Theorem 3.1 *Let $S_1, S_2 \subseteq \{+, \cdot, !, ?\}$ and suppose $S_1 \subsetneq S_2$. Then $K(S_2)$ is a conservative extension of $K(S_1)$.*

The proof for this Theorem shows how to convert proofs from logic extensions back into proofs in the logic being extended. It does this by eliminating operator symbols. The rest of the section is devoted to giving the argument.

Definition 3.2 Let o be one of $+$, $\cdot$, !, or ?. If X is a formula of $L(\{+, \cdot, !, ?\})$, by X^o we mean the result of eliminating all justification terms containing o. More precisely, we have the following recursive characterization. For propositional letters $P^o = P$, and also $\bot^o = \bot$. Of course $(X \supset Y)^o = (X^o \supset Y^o)$. And finally:

$$(t{:}X)^o = \begin{cases} X^o & \text{if } o \text{ occurs in } t \\ t{:}X^o & \text{if } o \text{ does not occur in } t \end{cases}$$

The central part of the proof of Theorem 3.1 is contained in the following Proposition. Note that its proof is constructive (and simple).

Proposition 3.3 (Operator Elimination) *Assume $S \subseteq \{+, \cdot, !, ?\}$ and let o be one of the operation symbols in S. If Z is one of the axioms of $K(S)$, then Z^o is an axiom of $K(S - \{o\})$.*

Proof There are several cases and subcases, depending on choice of axiom and choice of operation symbol. The argument in each case is straightforward. It might be simpler to construct your own argument rather than reading mine. Here are the cases.

Classical Axiom: If Z is a tautology, so is Z^o.

Truth Axiom: Z **is** $t{:}X \supset X$ **or** $(X \supset Y) \supset (t{:}X \supset Y)$
There are two simple subcases

- o does not occur in t. Then Z^o is again a Truth Axiom, of the same kind.
- o occurs in t. Then Z^o is a Classical Axiom.

Truth Axiom: $Z = t{:}(X \supset Y) \supset (u{:}X \supset Y)$ Again there are simple subcases.

- o does not occur in t or in u. Then Z^o is again a Truth Axiom, of the same kind.
- o occurs in u but not in t. Then Z^o is $t{:}(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a different kind of Truth Axiom.
- o occurs in t but not in u. Then Z^o is $(X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, again a different kind of Truth Axiom.
- o occurs in both t and u. Then Z^o is $(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a Classical Axiom.

$+$ **Axiom:** $Z = t{:}X \supset (t+u){:}X$ The other $+$ axiom is similar so only this one is considered.

- o occurs in t. Z^o is $X^o \supset X^o$, a Classical Axiom.
- o occurs in u but not in t. Z^o is $t{:}X^o \supset X^o$, a Truth Axiom.
- o occurs in neither t nor u, and o is not $+$. Z^o is $t{:}X^o \supset (t+u){:}X^o$, another $+$ axiom.
- o occurs in neither t nor u, and o is $+$. Z^o is $t{:}X^o \supset X^o$, a Truth Axiom.

$\cdot$ **Axiom:** $Z = (t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y))$ The subcases are as follows.

- o occurs in both t and u. In this case Z^o is $(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a Classical Axiom.
- o occurs in u but not in t. Then Z^o is $t{:}(X^o \supset Y^o) \supset (X^o \supset Y^o)$, an instance of the first Truth Axiom.
- o occurs in t but not in u. Then Z^o is $(X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, an instance of the second Truth Axiom.
- o occurs in neither t nor u, and o is not $\cdot$. Then Z^o is $t : (X^o \supset Y^o) \supset (u{:}X^o \supset (t \cdot u){:}Y^o)$, an instance of the $\cdot$ Axiom.
- o occurs in neither t nor u, and o is $\cdot$. Then Z^o is $t : (X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, an instance of the third Truth Axiom.

! **Axiom:** $Z = t{:}X \supset !t{:}t{:}X$ The cases are as follows.

- o occurs in t. Z^o is $X^o \supset X^o$, a Classical Axiom.
- o does not occur in t, and o is not !. Z^o is $t{:}X^o \supset !t{:}t{:}X^o$, a ! Axiom.
- o does not occur in t, and o is !. Z^o is $t{:}X^o \supset t{:}X^o$, a Classical Axiom.

? **Axiom:** $Z = \neg t{:}X \supset ?t{:}\neg t{:}X$ This case is similar to the ! case.

■

Finally there is very little left to do.

Proof of Theorem 3.1 Suppose $S_1 \subsetneq S_2$, where both are subsets of $\{+, \cdot, !, ?\}$. Assume S_2 contains a single operation symbol o that is missing from S_1. (The case of multiple operation symbols is handled by iterating the single operator case.) Let X be a theorem of $K(S_2)$, where X does not contain any occurrence of o. We show X is a theorem of $K(S_1)$.

Consider a proof of X in $K(S_2)$. Replace each line, Z, of that proof with Z^o. Each axiom of $K(S_2)$ is replaced with an axiom of $K(S_1)$, by Proposition 3.3. Applications of *modus ponens* turn into other applications of *modus ponens*. Also, applications of Constant Necessitation in $K(S_2)$ turn into applications of Constant Necessitation in $K(S_1)$, because $K(S_2)$ axioms turn into $K(S_1)$ axioms. Thus the entire proof converts to one in $K(S_1)$. Finally, since X did not contain o, it is still the last line of the proof, hence X is provable in $K(S_1)$. ■

4 Embedding and Equivalence

The role of constants in justification logics imposes certain peculiar complications. For instance consider LP, or $K(\{+, \cdot, !\})$ in present terminology. There is much flexibility possible in its axiomatization. For one thing we need an underpinning of classical logic, but that could be axiomatized in several ways—infinitely many different ways, in fact. But a choice of axiomatization affects applications of the *Constant Necessitation* rule. If, say, $X \supset X$ is an axiom, we can conclude $c{:}(X \supset X)$ for a constant c. If we have a different axiomatization of classical logic in which $X \supset X$ is not an axiom, nonetheless it will be a theorem, but then *Constant Necessitation* does not apply to it. We do, however, have the internalization feature to appeal to, Proposition 2.4: for some justification term t, $t{:}(X \supset X)$ will be a theorem. In some sense these differences shouldn't matter very much—what is basic in one axiomatization (and so has a constant justification) is subject to proof in the other (and so has a more complex justification). In this section we address the issue. Our treatment is not as general as might be desired. It assumes $\cdot$ and ! are present, so the version of Constant Necessitation used is Axiom Necessitation. And it assumes an *injective* constant specification (defined below) is used. Some of this can be relaxed, but the technical details become more complex. What is given here is enough to 'justify' the presence of three Truth Axiom schema in Section 2.2, instead of the customary single one.

Recall that in this section we are assuming Axiom Necessitation is the version of Constant Necessitation we use.

Definition 4.1 We say one justification logic, J_1, *embeds in another*, J_2, provided there is a mapping from constants of J_1 to justification terms of J_2 that converts each theorem of J_1 into a theorem of J_2.

We say two justification logics are *equivalent* if each embeds in the other.

Here is a basic result concerning these notions. As noted earlier, this is not as general as it might be.

Theorem 4.2 (Embedding) *Let J_1 and J_2 be two justification logics in the same language $L(S)$, where $\{\cdot, !\} \subseteq S \subseteq \{+, \cdot, !, ?\}$. We assume the rules of infererence for J_1 and J_2 are* modus ponens *and Axiom Necessitation, as given in Section 2.2, but the choice of axioms may be entirely different. Suppose the following conditions are met.*

1. *An injective constant specification $\mathcal{C}$ is used for proofs in J_1.*
2. *J_1 is axiomatized using axiom schemes.*
3. *J_2 satisfies Substitution Closure (see Proposition 2.3).*
4. *J_2 satisfies Internalization (see Proposition 2.4).*
5. *Every axiom of J_1 is a theorem of J_2.*

Then J_1 embeds in J_2.

Proof We must create a mapping from constants of $L(S)$ to terms. If c is a constant that the constant specification $\mathcal{C}$ does not assign any J_1 axiom to, we simply map c to itself. Now suppose $\mathcal{C}$ does assign J_1 axiom A to c; we specify which term t the constant c maps to. Since $\mathcal{C}$ is injective, this axiom A is uniquely determined by c. Still, complications can arise due to the fact that A may contain an occurrence of

c itself. If this happens, c is said to be *self-referential*, and it was shown in (Kuznets 2006) that such self-referentiality is essential. (Thanks to Sergei Artemov for suggestions on how to handle this.) Suppose $c_1 (= c)$, c_2, …, c_n are all the constants occurring in A (in some standard order). For simplicity we write $A(c, c_2, \ldots, c_n)$ for A. Let x_1, x_2, …, x_n be distinct variables not occurring in A (again in some standard order). Since A is an axiom of J_1, which is axiomatized using axiom schemes, then $A(x_1, x_2, \ldots, x_n)$ will also be an axiom. Then by hypothesis, $A(x_1, x_2, \ldots, x_n)$ is a theorem of J_2. Since J_2 satisfies Internalization, there is some ground justification term t such that $t{:}A(x_1, x_2, \ldots, x_n)$ is a theorem of J_2, if there is more than one such term, say we choose the first in some standard enumeration. Now, we map the constant c to the justification term t.

For each constant c, let c' be the term that was assigned to c above. For each formula Z in the language $L(S)$, let Z' be the formula that results when each constant c is replaced by the justification term c'.

Suppose Z_1, Z_2, …, Z_n is a proof in the logic J_1, meeting constant specification $\mathcal{C}$. The sequence Z_1', Z_2', …, Z_n' is not, itself, a proof in J_2, but each item in it is a theorem of J_2. This has a straightforward proof by induction. If Z_i is an axiom of J_1, since J_1 is axiomatized by schemes, Z_i' will also be an axiom, and hence a theorem of J_2 by hypothesis 5. If Z_i follows from earlier terms Z_j and $Z_j \supset Z_i$ by *modus ponens*, Z_i' also follows from Z_j' and $(Z_j \supset Z_i)' = (Z_j' \supset Z_i')$ by *modus ponens*. Finally we have the Axiom Necessitation case. Suppose Z_i is $c{:}A$ where A is a J_1 axiom. Say $A = A(c, c_2, \ldots, c_n)$, where all the constants of A are explicitly displayed. If $x_1, x_2, \ldots, x_n$ are variables not occurring in A, as above, there is a ground term $t = c'$ such that $t{:}A(x_1, x_2, \ldots, x_n)$ is a theorem of J_2. Since J_2 satisfies Internalization, $t{:}A(c', c_2', \ldots, c_n')$ is also a theorem, and this is $c'{:}A' = [c{:}A]' = Z_i'$.

It now follows that if X is any theorem of J_1 then X' will be a theorem of J_2. ■

Usually in the literature, the Truth Axiom is given by a single schema: $t:X \supset X$. We assumed two additional schemas: $(X \supset Y) \supset (t:X \supset Y)$ and $t:(X \supset Y) \supset (u{:}X \supset Y)$. It is an easy consequence of Theorem 4.2 that, for any S with $\{\cdot, !\} \subseteq S$, if we had axiomatized $K(S)$ with the usual single Truth schema instead of the way we did, the resulting logic would have been equivalent to the version we used, in the sense of Definition 4.1. Similarly we could have used “enough” tautologies instead of taking all of them as Classical Axioms, and that would have given equivalent logics as well. We made the choices we did because then the manipulations involved in the proof of Proposition 3.3 always turned axioms into axioms, and hence the behavior of Constant Necessitation was simple to describe.

5 Conclusion

The main thing left undone is quite obvious: there is no analog of Theorem 3.1 for logics of belief instead of knowledge—of the form $B(S)$ instead of $K(S)$. The methods of proof used here clearly do not extend to explicit logics of belief. Many of the cases involved in the proof of Proposition 3.3 yield an instance of a Truth Axiom. Without the Truth Axioms, present methods cannot succeed. Nonetheless, either a belief analog of Theorem 3.1 holds, or it does not. A result either way would be of interest. The desirable conjecture is that it holds, but a proof is left to others.

One other item was left unfinished, but it is of lesser importance. Theorem 4.2 needed the presence of both $\cdot$ and !. Producing a version not needing ! is straightforward, but a bit messy to state. A version without $\cdot$ probably has little intrinsic interest. Also, giving an appropriate version (if there is one) not requiring an *injective* constant specification is still a problem. But this too is left to others.

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